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Using Sigma theory we show that for large classes of groups G there is a subgroup H of finite index in $\mathrm{Aut}(G)$ such that for $\varphi \in H$ the Reidemeister number $R(\varphi)$ is infinite. This includes all finitely generated nonpolycyclic groups G that fall into one of the following classes: nilpotent-by-abelian groups of type FP_∞; groups G/G'' of finite Prüfer rank; groups G of type FP_2 without free nonabelian subgroups and with nonpolycyclic maximal metabelian quotient; some direct products of groups; or the pure symmetric automorphism group. Using a different argument we show that the result also holds for 1-ended nonabelian nonsurface limit groups. In some cases, such as with the generalized Thompson's groups $F_{n,0}$ and their finite direct products, $H = \mathrm{Aut}(G)$.

1. Introduction

We study the Reidemeister number $R(\varphi)$ for elements φ of subgroups of finite index in $\mathrm{Aut}(G)$ for large classes of groups G. The Reidemeister number $R(\varphi)$ counts the twisted conjugacy classes $\{ga\varphi(g)^{-1}\}_{g\in G}$ for $a \in G$. The study of $R(\varphi)$ began in the Nielsen–Reidemeister fixed point theory. Following [Taback and Wong 2007], we say a group G has the property R_∞ if $R(\varphi)$ is infinite for every automorphism φ of G. Many classes of groups have the property R_∞: nonelementary Gromov hyperbolic groups [Levitt and Lustig 2000; Fel'shtyn 2001], nonabelian generalized Baumslag–Solitar groups [Levitt 2007], saturated weakly branch groups (including the Grigorchuk group and the Gupta–Sidki group) [Fel'shtyn et al. 2008a], and the Thompson's group F [Bleak et al. 2008].

Fel'shtyn et al. [2008b] showed that for a finitely generated polyfree group G with nonzero Euler characteristic, there is a subgroup H of finite index in $\mathrm{Aut}(G)$ such that the Reidemeister number is infinite for every element of H.

In Sections 4a and 4c, we will show that to the groups in the list above that have the R_∞ property, we can add the generalized Thompson's groups $F_{n,0}$ and any finite direct product of such groups (with possibly different n).

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Furthermore we study large classes of groups G where it is possible to show that there is a subgroup H of finite index in $\mathrm{Aut}(G)$ such that $R(\varphi)$ is infinite for every $\varphi \in H$. These are all finitely generated nonpolycyclic groups G such that

(1) G is nilpotent-by-abelian of type FP_∞;

(2) G/G'' is of finite Prüfer rank but not polycyclic;

(3) G is of type FP_2 with no free nonabelian subgroups and its maximal metabelian quotient is not polycyclic;

(4) G is some direct product of groups;

(5) G is the pure symmetric automorphism group;

(6) G is a Houghton group H_n.

The proofs of these results, found in Section 4, rely on the structure of the invariant $\Sigma^1(G)$ for the groups G described above. In the preliminaries we detail the properties of this invariant. It appeared first in the study of finitely presented metabelian groups G [Bieri and Strebel 1980] and was later defined for any finitely generated group G [Bieri et al. 1987]. The geometric invariant $\Sigma^1(G)$ is invariant under the action of the automorphism group of G, which is fundamental for the proofs. In many cases we do not know the full automorphism group $\mathrm{Aut}(G)$, but using Sigma theory we can nevertheless obtain sufficient information on how $\mathrm{Aut}(G)$ can act on the abelianization of G.

In the final section we show that to the list above we can add

(7) G is any 1-ended nonabelian nonsurface limit group.

Orientable surface groups of genus at least 2 and nonorientable surface groups of genus at least 3 are Gromov hyperbolic [Gersten 1999, 3.17.5], and hence as mentioned before have R_∞. Nonorientable surface groups of genus 1 or 2 are not limit groups since in the first case they have torsion and in the second they are virtually abelian but not abelian.

Limits groups appeared in the solution of the Tarski problem by O. Kharlampovich, A. Myasnikov, and independently by Z. Sela. The definition of limit groups was suggested by Z. Sela, and it turned out that the class of limit groups coincides with the class of finitely generated, fully residually free groups studied by G. Baumslag, O. Kharlampovich, A. Myasnikov, and V. Remeslenikov. The class of limit groups includes finitely generated free groups, finite rank free abelian groups, and surface groups of Euler characteristic at most -2. Limit groups are CAT(0) [Alibegović and Bestvina 2006]. Unfortunately we cannot use $\Sigma^1(G)$ to study the Reidemeister numbers for a limit group G because the set $\Sigma^1(G)$ is empty [Kochloukova 2010]. Instead we use the structure theorem of $\mathrm{Aut}(G)$ for 1-ended limit groups G given in [Bumagin et al. 2007].

2. Preliminaries

2a. ***Sigma invariants for finitely generated groups.*** Let R be an associative ring with 1. All modules and actions in this paper are left ones. For an R-module M, we say that M has type FP_m if there is a projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

such that all projective modules P_i are finitely generated for $i \leq m$. A group G is of type FP_m if the trivial module $\mathbb{Z}$ is of type FP_m as a $\mathbb{Z}G$-module. Every group is of type FP_0, and G is of type FP_1 if and only if G is finitely generated. If G is finitely presented (that is, there is a $K(G, 1)$ with a finite 2-skeleton), then G is of type FP_2. The converse does not hold [Bestvina and Brady 1997].

The homological invariants $\Sigma^m(G, \mathbb{Z})$ were defined in their current form in [Bieri and Renz 1988], but in the case that $m = 1$ and G is metabelian, they can be traced back (under different notation) to [Bieri and Strebel 1980], where they played a crucial part in classifying finitely presented metabelian groups.

For a general finitely generated group G, define the character sphere

$$S(G) = (\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{R}) \setminus \{0\})/\sim,$$

where $\mathbb{R}$ is considered a group via addition, and $\chi_1 \sim \chi_2$ if there is a positive real number r such that $r\chi_1 = \chi_2$. Write $[\chi]$ for the class of χ in $S(G)$, and define

$$\Sigma^m(G, \mathbb{Z}) = \{[\chi] \in S(G) \mid \mathbb{Z} \text{ is of type } FP_m \text{ as a } \mathbb{Z}G_\chi\text{-module}\},$$

where G_χ is the monoid $\{g \in G \mid \chi(g) \geq 0\}$. By definition,

$$\Sigma^m(G, \mathbb{Z})^c = S(G) \setminus \Sigma^m(G, \mathbb{Z}).$$

It is known that $\Sigma^m(G, \mathbb{Z})$ is always an open subset of $S(G)$ and can be empty (as in the case of a noncyclic free group). If $\Sigma^m(G, \mathbb{Z})$ is not empty, then G is of type FP_m, but the converse does not hold (for example, when G is noncyclic free). The following theorem shows that the invariants $\Sigma^m(G, \mathbb{Z})$ are responsible for the homological type of subgroups of G that contain the commutator.

Theorem 2.1 [Bieri and Renz 1988, Theorem B]. *If N is a subgroup of G that contains the commutator and G is of type FP_m, then N is FP_m if and only if $S(G, N) := \{[\chi] \in S(G) \mid \chi(N) = 0\} \subseteq \Sigma^m(G, \mathbb{Z})$.*

There is a homotopical version $\Sigma^m(G)$ of $\Sigma^m(G, \mathbb{Z})$. In the dimension $m = 1$ the invariants coincide, that is, $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$, and in general $\Sigma^m(G) \subseteq \Sigma^m(G, \mathbb{Z})$, but the inclusion can be strict for $m \geq 2$ [Meier et al. 1998]. As before, the superscript c means complement in $S(G)$.

In general it is hard to calculate all the invariants (both homological and homotopical versions for arbitrary dimension m). In this paper we apply some structural results about $\Sigma^1(G)^c$ to the theory of twisted conjugacy classes.

2b. ***A Sigma invariant for a metabelian group.*** In this section we discuss the original Bieri–Strebel invariant [1980]. This is the form that will be used in the proof of Theorem 4.6.

Let Q be a finitely generated abelian group, and let A be a finitely generated left $\mathbb{Z}Q$-module. If not otherwise stated, all modules in this paper are left ones. Then

$$\Sigma_A(Q) = \{[\chi] \in S(Q) \mid A \text{ is finitely generated as a } \mathbb{Z}Q_\chi\text{-module}\}$$

and

$$\Sigma_A^c(Q) = S(Q) \setminus \Sigma_A(Q).$$

Bieri and Strebel [1981, (1.5)] showed that $\Sigma_A^c(Q)$ is additive, that is, if

$$0 \to A_1 \to A \to A_2 \to 0$$

is a short exact sequence of finitely generated $\mathbb{Z}Q$-modules, then

$$\Sigma_A^c(Q) = \Sigma_{A_1}^c(Q) \cup \Sigma_{A_2}^c(Q).$$

It follows from the definitions that if χ is a real character of a finitely generated metabelian group G such that A is an abelian normal subgroup of G with $Q = G/A$ abelian, and if $\chi(A) \neq 0$, then $[\chi] \in \Sigma^1(G)$. If $\chi(A) = 0$, then $[\chi] \in \Sigma^1(G)^c$ if and only if $[\chi] \in \Sigma_A^c(Q)$ (here we consider χ as a character of Q since $\chi(A) = 0$).

Let I be the annihilator of A in $\mathbb{Z}Q$, that is, $I = \{\lambda \in \mathbb{Z}Q \mid \lambda A = 0\}$, and let $P_1, \ldots, P_s$ be the minimal associated primes of A as a $\mathbb{Z}Q$-module. Then we have by [Bieri and Strebel 1981, (1.12)]

$$\Sigma_A^c(Q) = \Sigma_{\mathbb{Z}Q/I}^c(Q) = \bigcup_{1 \le i \le s} \Sigma_{\mathbb{Z}Q/P_i}^c(Q).$$

2c. ***Twisted conjugacy classes.*** Let $\varphi : G \to G$ be a group endomorphism. The set of equivalence classes $\{ga\varphi(g)^{-1}\}_{g \in G}$ for $a \in G$ is denoted by $\mathcal{R}[\varphi]$. The cardinality of $\mathcal{R}[\varphi]$ is denoted by $R(\varphi)$ and called the Reidemeister number of φ.

Let H be a normal subgroup of a group G invariant under an endomorphism φ. Denote by φ' the restriction of φ on H and by $\overline{\varphi}$ the endomorphism induced by φ on $Q = G/H$. Then by [Gonçalves 1998] and [Gonçalves and Wong 2003, (2.2)], there is an exact sequence of sets

$$\mathcal{R}[\varphi'] \to \mathcal{R}[\varphi] \to \mathcal{R}[\overline{\varphi}] \to 0. \tag{2-1}$$

By [Jiang 1983, page 33], if G/H is abelian, nontrivial, torsion-free, and the map $\mathrm{id}_Q - \overline{\varphi}$ is not surjective (that is, $\overline{\varphi}$ is a linear operator with eigenvalue 1), then $R(\overline{\varphi})$ is infinite and consequently $R(\varphi)$ is infinite.

3. General results

A point $[\chi]$ of the character sphere $S(G)$ is called a discrete point (or a rational point) if $[\chi] = [\mu]$ for some character μ such that $\mathrm{Im}(\mu) = \mathbb{Z}$. In this case we call χ discrete or rational.

Lemma 3.1. *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], \ldots, [\chi_m]\}$$

is finite, nonempty, and contains only discrete points, and let φ be an automorphism of G. Let $N = \bigcap_i \mathrm{Ker}(\chi_i)$, let $V = \mathrm{Hom}_{\mathbb{Z}}(G/N, \mathbb{R})$, and let θ be the element of $\mathrm{End}_{\mathbb{Z}}(V)$ induced by φ.

Then θ permutes the images $\overline{\chi}_i$ of χ_i in V, where every χ_i is chosen so that the coordinates of $\overline{\chi}_i$ are integers with maximal common divisor 1.

Proof. Since $\Sigma^1(G)^c$ is invariant under automorphisms, we have $\theta(\overline{\chi}_i) = r_i \overline{\chi}_{\pi(i)}$, where π is some element of the symmetric group S_m and the r_i are positive real numbers. By assumption, the coordinates of the entries of $\overline{\chi}_i$ (with respect to some fixed basis of the free abelian group G/N) are coprime integers. Then every r_i is a positive integer.

We aim to show that $r_1 = \cdots = r_m = 1$. Indeed for $k = m!$, we have $\theta^k(\overline{\chi}_i) = \lambda_i \overline{\chi}_i$ for some positive integer λ_i divisible by r_i for every i. Now φ is invertible and N is a characteristic subgroup of G, so θ is an automorphism of V, and so θ^k is invertible. Furthermore $X = \{\overline{\chi}_i\}_{1 \le i \le m}$ spans V. Then by picking up a subset Y of X that is a basis of V, we see that the operator θ^k is diagonalizable with eigenvalues λ_i (note that a fixed $\overline{\chi}_i$ can always be included in some Y). Then $\prod_{\overline{\chi}_i \in Y} \lambda_i = 1$, so $\lambda_i = 1$ for $\overline{\chi}_i \in Y$. Then $r_i = 1$ as required. □

Theorem 3.2. *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], \ldots, [\chi_m]\}$$

is finite, nonempty, and contains only discrete points. Let $N = \bigcap_i \mathrm{Ker}(\chi_i)$, and let $V = \mathrm{Hom}_{\mathbb{Z}}(G/N, \mathbb{R})$. Suppose that the image of $\{\chi_1, \ldots, \chi_m\}$ in V is a basis of V as an $\mathbb{R}$-vector space. Then $R(\varphi)$ is infinite for every $\varphi \in \mathrm{Aut}(G)$, that is, G has the property R_∞.

Proof. By the previous lemma, we can fix representatives χ_i (remember we can multiply any χ_i with a positive real number) so that the map $\theta \in \mathrm{End}_{\mathbb{Z}}(V)$, induced

by φ, permutes the elements of $\{\overline{\chi}_1, \ldots, \overline{\chi}_m\}$, where $\{\overline{\chi}_1, \ldots, \overline{\chi}_m\}$ is the image of $\{\chi_1, \ldots, \chi_m\}$ in V. Then

$$\theta(\overline{\chi}_1 + \cdots + \overline{\chi}_m) = \overline{\chi}_1 + \cdots + \overline{\chi}_m \quad \text{and} \quad \overline{\chi}_1 + \cdots + \overline{\chi}_m \neq 0$$

in V, that is, θ has eigenvalue 1. Then $R(\overline{\varphi})$ is infinite by the preliminaries from Section 2c; hence $R(\varphi)$ is infinite. □

Theorem 3.3. *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], \ldots, [\chi_m]\}$$

is finite, nonempty, and contains only discrete points. Let $N = \bigcap_i \operatorname{Ker}(\chi_i)$ and $V = \operatorname{Hom}_{\mathbb{Z}}(G/N, \mathbb{R})$. Suppose that the image of $\{\chi_1, \ldots, \chi_m\}$ in V lies in an open half subspace. Then $R(\varphi)$ is infinite for every $\varphi \in \operatorname{Aut}(G)$, that is, G has the property R_∞.

Proof. As in the proof of the previous theorem, we can fix representatives χ_i such that $w = \overline{\chi}_1 + \cdots + \overline{\chi}_m$ is fixed by θ, where $\overline{\chi}_i$ is the image of χ_i in V. Since $\{\overline{\chi}_1, \ldots, \overline{\chi}_m\}$ lies in an open half subspace of V, we deduce that $w \neq 0$. We continue as in the proof of the previous theorem. □

Corollary 3.4. *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], \ldots, [\chi_m]\}$$

is finite, nonempty, and contains only discrete points. Then there is a subgroup of finite index H of $\operatorname{Aut}(G)$ such that $R(\varphi)$ is infinite for every $\varphi \in H$.

Proof. By Lemma 3.1 the image of $\operatorname{Aut}(G)$ in $\operatorname{End}_{\mathbb{Z}}(V)$ permutes the characters $\overline{\chi}_1, \overline{\chi}_2, \ldots, \overline{\chi}_m$, where the coordinates of $\overline{\chi}_i$ are integers with maximal common divisor 1, and $V = \operatorname{Hom}_{\mathbb{Z}}(G/N, \mathbb{R})$ is defined as before. Let H be the kernel of the action of $\operatorname{Aut}(G)$ on $\operatorname{End}_{\mathbb{Z}}(V)$. Then the elements of H fix every $\overline{\chi}_i$; in particular the map induced by φ on the maximal torsion-free abelian quotient of G has eigenvalue 1. We continue as in the proof of the previous theorem. □

Proposition 3.5. *Let G be a finitely generated nilpotent group, and let $\varphi \in \operatorname{Aut}(G)$ be such that $\varphi(g)$ is conjugate to g for some element $g \in G$ of infinite order. Then $R(\varphi)$ is infinite.*

Proof. We induct on the nilpotency length of G. The case when G is abelian is obvious and well known.

Suppose that G is nonabelian. Let $Z(G)$ be the center of G, so $\overline{G} = G/Z(G)$ is of nilpotency length strictly smaller than the nilpotency length of G.

Let $\overline{\varphi}$ be the automorphism of $\overline{G}$ induced by φ. If there is an element of $\overline{G}$ of infinite order that is sent by $\overline{\varphi}$ to a conjugate, then by induction $R(\overline{\varphi}) = \infty$, so $R(\varphi) = \infty$. We can assume that there isn't an element of $\overline{G}$ of infinite order

that is sent by $\overline{\varphi}$ to a conjugate. The conditions of [Gonçalves and Wong 2003, Lemma 2.1] then hold, and hence it suffices to show that $R(\varphi') = \infty$ to deduce that $R(\varphi) = \infty$, where φ' is the restriction of φ to N.

Let $g \in G$ be an element of infinite order such that $\varphi(g) \in g^G$. By the preceding paragraph, the image of g in $\overline{G}$ has finite order. If $g \notin Z(G)$, some nontrivial power of g is in $Z(G)$, so we can assume that $g \in Z(G)$. Then $g^G = g$ and $\varphi(g) = g$.

Consider one twisted conjugated class

$$X_h = \{ah\varphi'(a)^{-1}\}_{a \in Z(G)} \quad \text{for some } h \in Z(G).$$

Because there is an element of infinite order in $Z(G)$ that is fixed by φ', we see that $\{a\varphi'(a)^{-1}\}_{a \in Z(G)}$ is a subgroup of infinite index in $Z(G)$; hence X_h is a coset class of a subgroup of infinite index in $Z(G)$. In particular this shows that we have an infinite number of twisted conjugacy classes X_h when $h \in Z(G)$, that is, $R(\varphi') = \infty$. □

4. Applications

4a. ***On the generalized Thompson's group and some soluble groups.*** For a group G of piecewise linear transformations of the unit interval, we say that G is irreducible if G does not fix a point of $(0, 1)$. There are two distinguished real characters ρ and λ of G defined by $\lambda(f) = \log_2 f'(0)$ and $\rho(f) = \log_2 f'(1)$.

Theorem 4.1. *Let G be a finitely generated, irreducible group of piecewise linear transformations of the unit interval; let ρ and λ defined above be nonzero rational nonantipodal characters such that $\lambda(\mathrm{Ker}(\rho)) = \mathrm{Im}(\lambda)$ and $\rho(\mathrm{Ker}(\lambda)) = \mathrm{Im}(\rho)$. Then $R(\varphi)$ is infinite for every $\varphi \in \mathrm{Aut}(G)$, that is, G has the property R_∞.*

Proof. By [Bieri et al. 1987, Theorem 8.1], the invariant $\Sigma^1(G)^c$ is equal to $\{[\rho], [\lambda]\}$, and by hypothesis the conditions of Theorem 3.2 are satisfied. □

The generalized Thompson's group $F_{n,0}$, where $n \geq 2$ (we use the notation of Brin and Guzmán [1998], which differs from the notation in [Brown 1987a]), is defined with the infinite presentation

$$\langle x_0, x_1, x_2, \ldots \mid x_j^{-1} x_i x_j = x_{j+n-1} \text{ for } 0 \leq j < i \rangle;$$

the case $n = 2$ is the classical Thompson's group F. The groups above were first defined and shown to be of type FP_∞ in [Brown 1987a]. Some other groups of PL functions on the interval are known to be of type FP_∞ [Stein 1992]. The automorphism group of $F_{n,0}$ is understood completely only for $n = 2$ [Brin 1996], and only some partial results are known for general $n > 2$ [Brin and Guzmán 1998]. Although we do not know much about the full automorphism group of $F_{n,0}$, we can still deduce the following result from Theorem 4.1. This answers a question

asked in [Bleak et al. 2008], where it was shown that the classical Thompson's group F has the property R_∞.

Corollary 4.2. *The generalized Thompson's group $F_{n,0}$ has the property R_∞.*

Proof. The generalized Thompson's group $F_{n,0}$ has several presentations as a group of PL functions of different intervals. In [Brin and Guzmán 1998, Lemma 2.3.1], a description of $F_{n,0}$ is given as the group of PL functions f on the interval $[0, n-1]$, where the set Y_f of break points is finite, the image of Y_f under f is inside $\mathbb{Z}[1/n]$, the slopes of all linear functions are integral powers of n, and the slopes of f at 0 and $n-1$ can be arbitrary integral powers of n. Furthermore $F_{n,0}$ does not fix a point on the open interval $(0, n-1)$. In that paper, the elements of $F_{n,0}$ act on the right of the interval, but by setting $f(x) = (x)f^{-1}$, we get an action on the left. By obvious rescaling (that is, a linear map that gives a bijection between the interval $[0, n-1]$ and $[0, 1]$), we get a description of $F_{n,0}$ as a subgroup of PL functions acting on the left of the interval $[0, 1]$, and then we can apply Theorem 4.1. □

All Σ-invariants are calculated [Bieri et al. 2010] for the original Thompson's group F, but for the generalized ones these invariants are not known except in dimension 1.

Theorem 4.3. *Let G be a nonpolycyclic nilpotent-by-abelian group of type FP_∞. Then $R(\varphi)$ is infinite for every $\varphi \in \mathrm{Aut}(G)$, that is, G has the property R_∞.*

Proof. Soluble groups of type FP_∞ are nilpotent-by-abelian-by-finite, so assuming that G is nilpotent-by-abelian is not a very strong restriction.

By the classification of soluble groups of type FP_∞, G is constructible, that is, built from the trivial group by finite and HNN extensions [Kropholler 1993].

Suppose that the commutator G' is not finitely generated; hence $\Sigma^1(G)^c$ is not empty by Theorem 2.1. By [Bieri and Strebel 1982], $\Sigma^1(G)^c$ is a finite set of discrete points $\{[\chi_1], \ldots, [\chi_m]\}$ that lies in an open hemisphere of $S(G)$. Then we can apply Theorem 3.3.

Finally if the commutator G' is finitely generated, G is polycyclic since it is nilpotent, a contradiction. □

A group G is said to be of finite Prüfer rank if there is a number d such that every finitely generated subgroup is generated by at most d elements. In particular if $1 \to A \to G \to Q \to 1$ is a short exact sequence of groups with A and Q abelian, and G finitely generated, then G has finite Prüfer rank if and only if the torsion part of A is finite and $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite-dimensional as a $\mathbb{Q}$-vector space. By the link between valuation theory and $\Sigma^c_A(Q)$ [Bieri and Groves 1984, Theorem 8.1], by the existing bijection between $\Sigma^c_A(Q)$ and $\Sigma^1(G)^c$ explained in the penultimate paragraph of Section 2b, and because G has finite Prüfer rank, $\Sigma^1(G)^c$ is a finite set $\{[\chi_1], \ldots, [\chi_m]\}$, where $\chi_1, \ldots, \chi_m$ are discrete characters.

Lemma 4.4. *Let G be a finitely generated metabelian group of finite Prüfer rank that is not polycyclic. Then there is a subgroup of finite index H of* $\mathrm{Aut}(G)$ *such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

Proof. Since G is not polycyclic, G' is not finitely generated, and so $\Sigma^1(G)^c$ is not empty by Theorem 2.1. On the other hand $\Sigma^1(G)^c$ is a finite set of discrete points. Then we can apply Corollary 3.4. □

Corollary 4.5. *Let G be a finitely generated group such that G/G'' is of finite Prüfer rank and G/G'' is not polycyclic. Then there is a subgroup of finite index H of* $\mathrm{Aut}(G)$ *such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

Proof. By the previous lemma there is a subgroup H_0 of finite index in $\mathrm{Aut}(G/G'')$ such that for every $\varphi_0 \in H_0$ the number $R(\varphi_0)$ is infinite. Then we can define H as the full preimage of H_0 in $\mathrm{Aut}(G)$. □

4b. *More on groups with metabelian quotients of type FP_2.*

Theorem 4.6. *Let G be a finitely generated metabelian group such that G is not polycyclic and $\Sigma^1(G)^c$ is not a finite union of subspheres. Then there is a subgroup of finite index H of* $\mathrm{Aut}(G)$ *such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

Proof. As before since G is not polycyclic (in this case, this is equivalent to G' not being finitely generated) $\Sigma^1(G)^c$ is not empty by Theorem 2.1.

We view $A = G'$ as a left $\mathbb{Z}Q$-module via conjugation, where $Q = G/G'$; we denote this $\mathbb{Z}Q$-action on A by $\circ$, that is, if $q = gA$ and $a \in A$, then $q \circ a = gag^{-1}$.

Let $P_1, \ldots, P_s$ be all minimal associated primes of the $\mathbb{Z}Q$-module A; see [Bourbaki 1989]. Let I be the annihilator of $\mathbb{Z}Q$ in A, that is,

$$I = \{\lambda \in \mathbb{Z}Q \mid \lambda \circ A = 0\}.$$

Then $P_1, \ldots, P_s$ are the minimal prime ideals above I. Let $\overline{\varphi}$ be the automorphism of $Q = G/G'$ induced by φ. We extend $\overline{\varphi}$ by linearity to a ring endomorphism $\hat{\varphi}$ of $\mathbb{Z}Q$ so that the restriction on $\mathbb{Z}$ is identity, and note that for $\lambda \in I$ we have $0_A = \varphi(0_A) = \varphi(\lambda \circ A) = \hat{\varphi}(\lambda) \circ \varphi(A) = \hat{\varphi}(\lambda) \circ A$, so $\hat{\varphi}(I) = I$. Then $\hat{\varphi}$ permutes the finite set of prime ideals $P_1, \ldots, P_s$. Thus by going down to a subgroup of finite index in $\mathrm{Aut}(G)$, we can assume that $\hat{\varphi}(P_i) = P_i$ for every i, and so $\hat{\varphi}$ induces an automorphism φ_i of $R_i = \mathbb{Z}Q/P_i$. By construction, φ_i is the identity on the image of $\mathbb{Z}$ in R_i.

Let $\tilde{\varphi}_i$ be the extension of φ_i to the field of fractions K_i of R_i. Let

$$Q_i = Q/T_i, \quad \text{where } T_i = \{q \in Q \mid q - 1 \in P_i\}.$$

By [Roseblade 1978, Theorem D] or [Bieri and Groves 1986, corollary on p. 426], the group of automorphisms of K_i that fixes the prime subfield k_i and permutes the elements of Q_i induces a finite group on B_i/A_i, where B_i is a minimal subgroup of

Q_i such that $k_i(B_i) \subseteq K_i$ is a purely transcendental extension with rank that equals the rank of the torsion-free group Q_i/B_i, and A_i is the subgroup of elements of Q_i that are algebraic over k_i. The group of automorphisms of K_i induces a finite group on A_i since the Galois group of a finite field extension is finite. Thus the group of automorphisms of K_i that fixes the prime subfield k_i and permutes the elements of Q_i induces a finite group on B_i.

Then by going down to a subgroup of finite index H in $\mathrm{Aut}(G)$, that is, $\varphi \in H$, we can assume that φ_i is trivial on B_i, that is, for $B_i = S_i/T_i$ with $S_i \leq Q$ we have $\varphi(s) \in sT_i$ for every $s \in S_i$. Then if B_i is infinite, the map induced by φ on $G/G' \otimes_{\mathbb{Z}} \mathbb{Q}$ has eigenvalue 1, and we can apply the observations from Section 2c. Thus we can assume henceforth that the B_i are all finite.

We claim that if B_i is finite, then

$$\Sigma^c_{\mathbb{Z}Q/P_i}(Q) = \{[\chi] \in S(Q) \mid \chi(S_i) = 0\} =: S(Q, S_i),$$

where S_i is defined as above, that is, S_i is the subgroup of Q such that $S_i/T_i = B_i$. Indeed by [Bieri and Groves 1984, Theorem 8.1], the class $[\chi]$ is in $\Sigma^c_{\mathbb{Z}Q/P_i}(Q)$ if and only if there is a real valuation of $\mathbb{Z}Q/P_i$ whose restriction on $Q_i = Q/T_i$ is induced by χ. Since K_i is a purely transcendental extension of $k_i(B_i)$ with degree exactly the rank of the torsion-free abelian group Q_i/B_i for every real valuation w of $k_i(B_i)$ and any character $\chi : Q_i \to \mathbb{R}$ such that χ and w coincide on B_i, there is a valuation of K_i that extends both w and χ. Furthermore any real valuation of $k(B_i)$ sends the finite group B_i to zero. Thus any real character of Q_i extends to a real valuation of K_i and so by restriction to a real valuation of $\mathbb{Z}Q/P_i$. Since $T_i - 1$ maps to 0 in $\mathbb{Z}Q/P_i$ and any real valuation of $\mathbb{Z}Q/P_i$ sends 0 to infinity for $[\chi] \in \Sigma^c_{\mathbb{Z}Q/P_i}(Q)$, we always have $\chi(T_i) = \chi(1) = 0$. Since B_i is finite for every real character χ such that $\chi(T_i) = 0$, we have $\chi(S_i) = 0$.

Then as mentioned in Section 2b,

$$\Sigma^1(G)^c = \Sigma^c_A(Q) = \Sigma^c_{\mathbb{Z}Q/I}(Q) = \bigcup_i \Sigma^c_{\mathbb{Z}Q/P_i}(Q).$$

That is, $\Sigma^1(G)^c$ is a finite union of subspheres $S(Q, S_i)$, a contradiction. □

Corollary 4.7. *Let G be a group of type FP_2. Suppose it does not contain a free subgroup of rank* 2, *and G/G'' is not polycyclic. Then there is a subgroup of finite index H of* $\mathrm{Aut}(G)$ *such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

Proof. By [Bieri and Strebel 1980, Theorem 5.5], the maximal metabelian quotient G/G'' of G has type FP_2, and $\Sigma^1(G/G'')^c$ does not have antipodal points. By Theorem 2.1, since G/G'' is not polycyclic, $\Sigma^1(G/G'')^c$ is not empty. Since any subsphere of $S(G)$ has antipodal points, $\Sigma^1(G/G'')^c$ does not contain a subsphere. Then by Theorem 4.6 there is a subgroup H_0 of finite index in $\mathrm{Aut}(G/G'')$ such

that $R(\varphi_0)$ is infinite for every $\varphi_0 \in H_0$. Finally define H to be the full preimage of H_0 in $\mathrm{Aut}(G)$. □

4c. ***Direct products of groups.*** The direct product of groups is a particular case of a more general construction: A graph product G of groups is constructed from a graph Δ and vertex groups G_v associated with every vertex v in the set V of vertices of Δ. Then G is the free product of all G_v factored by the normal subgroup generated by $[G_v, G_w]$, where v and w are any adjacent vertices in Δ.

Theorem 4.8 [Meinert 1995, Theorem A]. *Let G be a graph product with a finite underlying graph Δ and finitely generated vertex groups $\{G_v\}_{v\in V}$. Let $\chi : G \to \mathbb{R}$ be a nonzero homomorphism. Let Δ' be the full subgraph of Δ spanned by all vertices $V' = \{v \in V \mid \chi(G_v) \neq 0\}$. Then $[\chi] \in \Sigma^1(G)$ if and only if one of the following conditions holds*:

(1) *If $\Delta' = \{v_0\}$, then $[\chi|_{G_{v_0}}] \in \Sigma^1(G_{v_0})$, and each vertex of $V \setminus \{v_0\}$ is adjacent to v_0.*

(2) *If Δ' has at least 2 vertices, then it is connected, and each vertex of $V \setminus V'$ is adjacent to some vertex of Δ'.*

If applied to a finite graph Δ in which every pair of vertices is linked by an edge, then this theorem gives a simple formula for the Σ^1 invariant of a direct product of finitely many groups. There were some attempts to prove analogues for higher-dimensional invariants $\Sigma^m(G, \mathbb{Z})$ and $\Sigma^m(G)$ for G a direct product of groups (such analogues are called *direct product conjectures*). These conjectures turn out to be false in both homological and homotopical versions [Meier et al. 1998; Schütz 2008], but hold if we consider invariants with coefficients in a field [Bieri and Geoghegan 2010].

Corollary 4.9. *Let $H = G_1 \times \cdots \times G_m$, and let $\Sigma^1(G_i)^c$ be finite (possibly empty) and contain only rational points (if any) for all $1 \leq i \leq n$. Suppose further that $\Sigma^1(G_i)^c$ is not empty for at least one i. Then there is a subgroup of finite index H_0 of $\mathrm{Aut}(H)$ such that $R(\varphi)$ is infinite for every $\varphi \in H_0$.*

Proof. By Theorem 4.8, $\Sigma^1(G)^c$ is finite, nonempty, and contains only discrete points. Thus we can apply Corollary 3.4. □

Corollary 4.10. *Let $H = G_1 \times \cdots \times G_m$, where G_i are generalized Thompson's groups (not necessarily isomorphic). Then H has the property R_∞.*

Proof. Let ρ_i and λ_i be the characters of H that are zero when restricted to G_j for $j \neq i$, and are the classical ρ and λ from Section 4 when restricted to G_i. By Theorem 4.8, $\Sigma^1(H)^c = \{[\lambda_i], [\rho_i]\}_{1\leq i\leq m}$. Then we can apply Theorem 3.2. □

Examples. Here we exhibit groups G that have abelianization $\mathbb{Z}$ and nonempty $\Sigma^1(G)^c$. Let G be a knot group with commutator G' infinitely generated. For every knot group, we have $G/G' \simeq \mathbb{Z}$. By Theorem 2.1, since G' is infinitely generated, $\Sigma^1(G) \neq S(G)$.

Let D be the double of G, that is, we make the free amalgamated product $G *_{\mathbb{Z}^2} G$, where $\mathbb{Z}^2$ is the torus subgroup of G. Thus D is a 3-manifold group and $D/D' \simeq \mathbb{Z}$. By the results of Thurston [1986] for 3-manifold groups and their translation into algebraic language [Bieri et al. 1987, Corollary F], we have $\Sigma^1(D) = -\Sigma^1(D)$, and $\Sigma^1(D)$ is nonempty if and only if the 3-manifold fibers over S^1; in our case this means that D' is finitely generated. This obviously does not hold as G is a quotient of D, and G' is not finitely generated. Then $\Sigma^1(D)$ is empty. Since $D/D' \simeq \mathbb{Z}$, we see that $\Sigma^1(D)^c = S(D)$ has 2 elements.

4d. ***The pure symmetric automorphism group.*** In this section, G is the pure symmetric automorphism group of the free group F with basis $\{x_1, \dots, x_n\}$. It is generated by $\{\alpha_{i,j}\}_{1 \le i \neq j \le n}$, where

$$\alpha_{i,j}(x_i) = x_j^{-1} x_i x_j \quad \text{and} \quad \alpha_{i,j}(x_k) = x_k \quad \text{for } k \neq i.$$

By [McCool 1986], G is finitely presented with relations

$$[\alpha_{i,j}, \alpha_{k,l}] = 1 \qquad \text{if } i, j, k, l \text{ are different,}$$

$$[\alpha_{i,j}, \alpha_{k,j}] = 1 \quad \text{and} \quad [\alpha_{i,j}\alpha_{k,j}, \alpha_{i,k}] = 1 \qquad \text{if } i, j, k \text{ are different.}$$

By [Brady et al. 2001], G is a duality group of dimension $n-1$. By the main result of [Orlandi-Korner 2000], the class $[\chi]$ is in $\Sigma^1(G)^c$ if and only if exactly one of the following holds:

(1) $\chi(\alpha_{i,j}), \chi(\alpha_{j,i}) \in \mathbb{R}$ for some $1 \le i < j \le n$ and $\chi(\alpha_{r,s}) = 0$ for the remaining indices;

(2) $\chi(\alpha_{j,i}) = -\chi(\alpha_{k,i})$, $\chi(\alpha_{i,k}) = -\chi(\alpha_{j,k})$, and $\chi(\alpha_{k,j}) = -\chi(\alpha_{i,j})$ for some $1 \le i < j < k \le n$ and $\chi(\alpha_{r,s}) = 0$ for the remaining indices.

We write $A_{i,j}$ for the set of characters of first type and $B_{i,j,k}$ for the set of second type (in both cases including the trivial character). Note that $A_{i,j}$ and $B_{i,j,k}$ are $\mathbb{R}$-subspaces of $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{R})$ of dimensions 2 and 3, respectively.

Theorem 4.11. *Let G be the pure symmetric automorphism group of a free group of rank n. Then there is a subgroup H of finite index in* $\mathrm{Aut}(G)$ *such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

Proof. Since $\mathrm{Aut}(G)$ permutes

$$W = \Big(\bigcup_{1 \le i < j \le n} A_{i,j} \Big) \cup \Big(\bigcup_{1 \le i < j < k \le n} B_{i,j,k} \Big) \subset \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{R}),$$

and the only 3-dimensional spaces inside W are the $B_{i,j,k}$, we see that $\mathrm{Aut}(G)$ permutes the spaces $B_{i,j,k}$ and thus also the elements of $M = \bigcup_{1 \le i < j \le n} A_{i,j}$. Also, the only 2-dimensional spaces inside M are the $A_{i,j}$, so $\mathrm{Aut}(G)$ permutes the spaces $A_{i,j}$. By going down to a subgroup of finite index in $\mathrm{Aut}(G)$, we can consider only automorphisms φ of G such that φ sends every $A_{i,j}$ to $A_{i,j}$ and every $B_{i,j,k}$ to $B_{i,j,k}$.

Let $\{\alpha^*_{i,j}\}_{i,j}$ be the basis of $\mathrm{Hom}(G, \mathbb{R})$ dual to $\{\alpha_{i,j}\}_{i,j}$ and φ^* be the automorphism of $\mathrm{Hom}(G, \mathbb{R})$ induced by φ. Note that

$$B_{i,j,k} = \mathbb{R}(\alpha^*_{i,j} - \alpha^*_{k,j}) + \mathbb{R}(\alpha^*_{i,k} - \alpha^*_{j,k}) + \mathbb{R}(\alpha^*_{k,i} - \alpha^*_{j,i})$$

and $B_{i,j,k}$ is invariant under φ^*. Then

$$\varphi^*(\alpha^*_{i,j} - \alpha^*_{k,j}) \in B_{i,j,k}. \tag{4-1}$$

On the other hand, since $A_{i,j}$ and $A_{k,j}$ are invariant under φ^*, we have

$$\varphi^*(\alpha^*_{i,j}) \in \mathbb{R}\alpha^*_{i,j} + \mathbb{R}\alpha^*_{j,i} \quad \text{and} \quad \varphi^*(\alpha^*_{k,j}) \in \mathbb{R}\alpha^*_{k,j} + \mathbb{R}\alpha^*_{j,k}; \tag{4-2}$$

hence

$$\varphi^*(\alpha^*_{i,j} - \alpha^*_{k,j}) \in \mathbb{R}\alpha^*_{i,j} + \mathbb{R}\alpha^*_{j,i} + \mathbb{R}\alpha^*_{k,j} + \mathbb{R}\alpha^*_{j,k}. \tag{4-3}$$

Combining (4-1) and (4-3), we get

$$\varphi^*(\alpha^*_{i,j} - \alpha^*_{k,j}) = a\alpha^*_{i,j} - a\alpha^*_{k,j} \quad \text{for some } a \in \mathbb{R}.$$

Then by (4-2),

$$\varphi^*(\alpha^*_{i,j}) = a\alpha^*_{i,j} \quad \text{and} \quad \varphi^*(\alpha^*_{k,j}) = a\alpha^*_{k,j}.$$

In particular, φ induces a diagonal linear map $\overline{\varphi}$ on the abelianization G/G', and since φ is invertible, the product of eigenvalues is either 1 or -1, and all a are integers. Then a is either 1 or -1, and either $\overline{\varphi}$ has eigenvalue 1 and we are done by the remarks from Section 2c, or $\overline{\varphi}$ is $-\mathrm{id}_{G/G'}$. By going to a subgroup of index 2, we can avoid the last case. □

4e. ***Houghton groups.*** Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of the natural numbers, and let $M = \{1, 2, \dots, n\} \times \mathbb{N}$ for $n \in \mathbb{N}$. The Houghton group H_n consists of all permutations g of M such that there is an n-tuple $(m_1, \dots, m_n) \in \mathbb{Z}^n$ such that $g(i, r) = (i, r + m_i)$ for all $i \in \{1, \dots, n\}$ and all but finitely many $(i, r) \in M$. These groups were first introduced in [Houghton 1978]. There is an important homomorphism $f : H_n \to \mathbb{Z}^n$ sending each permutation g to the corresponding n-tuple $(m_1, \dots, m_n)$. We always have $\sum_{1 \le i \le n} m_i = 0$.

For $n \ge 2$, the group H_n is finitely generated. More generally K. Brown [1987a] has shown that H_n is of type FP_{n-1} but not of type FP_n. For $n \ge 2$, the group H_n

is generated by $\{t_2, \dots, t_n\}$, where

$$t_i(j,r) = (j,r) \quad \text{for } i \neq j,\ j \neq 1 \qquad t_i(i,1) = (1,1),$$
$$t_i(i,r) = (i,r-1) \quad \text{for } r > 1, \qquad t_i(1,r) = (1,r+1).$$

Also $H_n/[H_n, H_n] \simeq \mathbb{Z}^{n-1}$; see [Gehrke 1998]. The invariant $\Sigma^1(H_n)^c$ for $n \geq 2$ was calculated in [Brown 1987b] as

$$\Sigma^1(H_n)^c = \{[\alpha_1], \dots, [\alpha_n]\},$$

where $\alpha_i = -\pi_i f$ and $\pi_i : \mathbb{Z}^n \to \mathbb{Z}$ is the projection to the i-th coordinate. Thus $\Sigma^1(H_n)^c$ is a finite set of rational points of $S(H_n)$, so we can apply Corollary 3.4 and get the following result.

Corollary 4.12. *Let $n \geq 2$ be a natural number, and let $G = H_n$. Then there is a subgroup H of finite index in* Aut(G) *such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

5. Limit groups

In the previous section we saw that sometimes to understand the structure of $\Sigma^1(G)$, it suffices to study the twisted conjugacy classes, although we might not have a complete classification of the whole automorphism group. In this section we study twisted conjugacy classes for 1-ended limit groups G. Kochloukova [2010] proved that $\Sigma^1(G)$ is empty for any limit group G, so we have to find a different approach to study the Reidemeister number $R(\varphi)$. As noted in the introduction, the class of limit groups coincides with the class of finitely generated fully residually free groups.

There are results about the automorphism group of 1-ended limit groups that give a description of a subgroup of finite index of Aut(G) [Bumagin et al. 2007]. In the general case (when the limit group G can be decomposed as a free product), it is known that Out(G) has a subgroup of finite index that has a finite classifying space [Guirardel and Levitt 2007, Theorem 6.5]. Here we do not need the full strength of these results, but use the existence of a canonical abelian JSJ decomposition for a 1-ended limit group G that is neither a surface group nor free abelian of finite rank [Bumagin et al. 2007, Theorem 3.13]. This means that G is the fundamental group of a finite graph of groups with finitely generated abelian edge stabilizers, that is, the group associated to each edge is abelian and there is a canonical graph of groups with these properties. There are limited possibilities for the vertex groups: abelian groups of finite rank, HQ-groups (that is, surface groups), and rigid groups [Bumagin et al. 2007].

The uniqueness of the canonical abelian JSJ decomposition shows that there is a subgroup H_0 of finite index in Aut(G) such that under an automorphism $\varphi \in H_0$ a vertex group goes to a conjugate of a vertex group. Also, up to conjugation and

generalized Dehn twists along edges (these twists are defined in [Bumagin et al. 2007, Definition 2.4] and have the property that on a fixed vertex group they act by conjugation with an element of G), an automorphism $\varphi \in H_0$ induces isomorphisms on the vertex groups that permute the adjacent edge groups up to conjugation.

The proof of the following result requires the existence of a canonical JSJ decomposition of G that has bipartite structure as defined in [Bumagin et al. 2007, Theorem 3.13]. This requires that G is a 1-ended limit group that is neither a free abelian nor a surface group.

Theorem 5.1. *Let G be a 1-ended limit group that is neither a free abelian nor a surface group. Then there is a subgroup of finite index H of $\mathrm{Aut}(G)$ such that $R(\varphi)$ is infinite for every $\varphi \in H$.*

Proof. Consider the canonical abelian JSJ decomposition of G. Then there is a subgroup of finite index H_0 of $\mathrm{Aut}(G)$ such that for every $\varphi \in H_0$ and every $t \in V(\Gamma) \cup E(\Gamma)$, we have $\varphi(G_t) = G_t^{g_t}$ for some $g_t \in G$. Note that $H_0 = \mathrm{Aut}(G)$ if the underlying graph of groups Γ of the JSJ decomposition of G does not have nontrivial symmetries that permute vertices and edges with isomorphic vertex and edge groups, respectively.

We are particularly interested in edge groups. Since the underlying graph is bipartite, every nonabelian group is linked with an abelian one. Two nonabelian groups or two abelian groups are not directly linked via an edge in the canonical JSJ decomposition. Thus if the connected graph Γ has more than one vertex, there is a nonabelian vertex group, and if Γ has just one vertex, the unique vertex group cannot be abelian as G is nonabelian. The nonabelian groups are divided into two classes: rigid and flexible.

By going down to a subgroup of finite index H_1 in H_0, we can assume that, up to conjugation and generalized Dehn twists along edges, the elements of H_1 that fix any rigid vertex group (not pointwise but as a group) are the identity on this rigid group [Kharlampovich and Myasnikov 2005, Theorem 15.1; Bumagin et al. 2007, Corollary 4.10]. Furthermore the edge groups of flexible vertices (that is, HQ vertices) are infinite cyclic [Bumagin et al. 2007, Definition 3.9], and $\mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}_2$ is finite. Then by further going down to a subgroup H of finite index in H_1, we can consider only automorphisms of G that up to conjugation and generalized Dehn twists along edges send an edge group of a flexible vertex group identically to itself. Thus

$$\varphi(g) \in g^G \text{ for every } \varphi \in H \text{ and every } g \neq 1 \text{ of an edge group.} \tag{5-1}$$

Let $\gamma_i(G)$ be the i-th term of the lower central series of G, and let $t_i(G)$ be the normal subgroup of G containing $\gamma_i(G)$ such that $G/t_i(G)$ is the maximal torsion-free quotient of $G/\gamma_i(G)$. Kochloukova [2010] showed that $\bigcap_{i\geq 1} t_i(G) = 1$, an

easy corollary of the fact that G is fully residually free. In other words, for every finite subset T of G there is a homomorphism from G to a free group that is injective on T. Then for the nontrivial element g from (5-1) we can find some i such that $g \notin t_i(G)$. Thus φ induces an automorphism $\overline{\varphi}$ on $Q = G/t_i(G)$ that sends the image $\bar{g}$ of g in Q to some conjugate of $\bar{g}$ in Q. Then $R(\overline{\varphi})$ is infinite by Proposition 3.5; hence $R(\varphi)$ is infinite. □

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PACIFIC JOURNAL OF MATHEMATICS

Volume 247 No. 2 October 2010

A family of representations of braid groups on surfaces 257
BYUNG HEE AN and KI HYOUNG KO

Parametrization of holomorphic Segre-preserving maps 283
R. BLAIR ANGLE

Chern classes on differential K-theory 313
ULRICH BUNKE

Laplacian spectrum for the nilpotent Kac–Moody Lie algebras 323
DMITRY FUCHS and CONSTANCE WILMARTH

Sigma theory and twisted conjugacy classes 335
DACIBERG GONÇALVES and DESSISLAVA HRISTOVA KOCHLOUKOVA

Properties of annular capillary surfaces with equal contact angles 353
JAMES GORDON and DAVID SIEGEL

Approximating annular capillary surfaces with equal contact angles 371
JAMES GORDON and DAVID SIEGEL

Harmonic quasiconformal self-mappings and Möbius transformations of the unit ball 389
DAVID KALAJ and MIODRAG S. MATELJEVIĆ

Klein bottle and toroidal Dehn fillings at distance 5 407
SANGYOP LEE

Representations of the two-fold central extension of $\mathrm{SL}_2(\mathbb{Q}_2)$ 435
HUNG YEAN LOKE and GORDAN SAVIN

Large quantum corrections in mirror symmetry for a 2-dimensional Lagrangian submanifold with an elliptic umbilic 455
GIOVANNI MARELLI

Crossed pointed categories and their equivariantizations 477
DEEPAK NAIDU

Painlevé analysis of generalized Zakharov equations 497
HASSAN A. ZEDAN and SALMA M. AL-TUWAIRQI

0030-8730(201010)247:2;1-D