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LARGE QUANTUM CORRECTIONS IN MIRROR SYMMETRY FOR A 2-DIMENSIONAL LAGRANGIAN SUBMANIFOLD WITH AN ELLIPTIC UMBILIC

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Given the Lagrangian fibration $\mathbb{R}^4 \to \mathbb{R}^2$ and a Lagrangian submanifold, exhibiting an elliptic umbilic and supporting a flat line bundle, we study in the context of mirror symmetry the corrections necessary to solve the monodromy of the holomorphic structure of the mirror bundle on the dual fibration. This is a preliminary step towards the understanding of quantum corrections in this specific case.

1. Introduction

The first steps in the study of mirror symmetry, assuming the existence of dual torus fibrations X and $\widehat{X}$, has been undertaken in papers such as [Fukaya 2005; Arinkin and Polishchuk 2001; Leung et al. 2001; Bruzzo et al. 2001; 2002]: Under certain hypotheses, a transform is provided, defined on some subcategory of the Fukaya category of X, that maps pairs formed by a Lagrangian submanifold L and a $U(1)$-flat connection ∇ to holomorphic bundles $\widehat{E}$ over $\widehat{X}$. The caustic K of L is always assumed to be empty. The purpose of this paper is to start understanding how to remove this hypothesis.

We focus our attention on the Lagrangian fibration $\mathbb{R}^4 \to \mathbb{R}^2$ and consider a Lagrangian map $f : L \hookrightarrow \mathbb{R}^4 \to \mathbb{R}^2$. Generically, f exhibits only folds and cusps, which are singularities of codimension 1 and 2 respectively. If we restrict the fibrations and L to the subset $\mathbb{R}^2 \setminus K$, then the Lagrangian map f has no singular points, and so we can try to apply the constructions contained in the papers mentioned before. We can hope to get a holomorphic bundle $\widehat{E}$ on the dual fibration restricted to $\mathbb{R}^2 \setminus K$, and whose holomorphic structure can be extended to the whole fibration over $\mathbb{R}^2$. However this hope is in general vain (we consider the elliptic umbilic in Section 4, but see also the same example described in [Fukaya 2005, Section 5.4]): What may happen, as in the case we are going to study, is that K is a compact curve, and in the noncompact subset of $\mathbb{R}^2$ determined by K, the holomorphic structure of $\widehat{E}$ presents a monodromy when going around the caustic K, and this

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hinders us from extending the mirror bundle to K and gluing it to the mirror bundle constructed inside K, and so from producing a holomorphic bundle $\widehat{E}$ on the whole dual fibration. Some kinds of *quantum corrections* are thus required to obtain a holomorphic bundle defined on the whole dual fibration $\widehat{\mathbb{R}^4} \to \mathbb{R}^2$.

One program to perform quantum corrections is outlined in [Fukaya 2005]. The idea is that there is a "classical" or, as it is called there, "semiflat" situation, which is the one where X is a Lagrangian fibration with only smooth fibers. In this case a complex structure is easily constructed on the mirror manifold $\hat{X}$; however as soon singular fibers are allowed, the complex structure constructed on the smooth part does not extend to the whole $\hat{X}$. So there is a need for some corrections to extend the complex structure.

Computations from physics suggest that pseudoholomorphic discs provide these corrections. When a Lagrangian submanifold L is given on X, if it is not a ramified cover a holomorphic mirror bundle is obtained on $\hat{X}$. When the caustic is allowed, the situation is similar to the case of singular fibers, and the classical holomorphic structure, obtained from L away from the caustic on the mirror bundle, must be deformed to extend it to the whole mirror bundle. Again, the idea outlined in [Fukaya 2005] is that quantum corrections are provided by the instanton effect, that is, by counting pseudoholomorphic strips in $\mathbb{R}^4$ that bound L and the fiber F_x of the fibration.

In a more general framework, quantum corrections are related to the obstruction theory of well-definedness of Floer homology. As proposed in [Fukaya 2000] as a general idea and holding beyond the specific case considered there, the fiber over $x \in \mathbb{R}^2 \setminus K$ of the mirror bundle $\widehat{E}$ on $\widehat{\mathbb{R}^4}$ is constructed as the Lagrangian intersection Floer homology of L and of the Lagrangian fiber of $\mathbb{R}^4$ over x. Assume that K contains just one singular point and that this point is an elliptic umbilic. We know that in dimension 2 this singularity is neither stable nor generic; however, from [Marelli 2006a; 2006b], we know how the caustic K and the bifurcation locus B change when f is slightly perturbed to $\tilde{f}$.

According to a conjecture proposed by Fukaya [2005, Section 3.5], near K, Lagrangian intersection Floer homology is equivalent to Morse homology defined by means of the generating function f of L, which is a Morse function far from K and B. This conjecture allows us to switch from Floer to Morse homology. More precisely, Fukaya conjectured that near the caustic, the moduli space of pseudoholomorphic discs bounded by a fiber and by the Lagrangian submanifold L is isotopic to the moduli space of gradient lines of the generating function of L between the points of intersections of the fiber with L. This conjecture has been proved in [Fukaya and Oh 1997] for the case of the cotangent bundle and in some of the examples considered in [Fukaya 2005]. Its purpose is just to provide a way to simplify the computations involved in working with pseudoholomorphic discs.

In our case, we apply this for a small perturbation of the elliptic umbilic. In view of this, that is, considering gradient lines rather than pseudoholomorphic discs, our result can be only a first step in the study of quantum corrections in mirror symmetry for the example considered.

Another important fact to be remarked is that, put simply, the homological mirror symmetry conjecture establishes an equivalence between the Fukaya category on one side and the derived category of coherent sheaves on the mirror side; however here our concern is only about objects and not also about morphisms, so again this work should be considered only as a preliminary step in understanding mirror symmetry.

With all the limitations outlined above, quantum corrections (that is, rules to glue the holomorphic Morse homology bundle $\hat{\hat{E}}$, relative in our case to $\tilde{f}$) are then defined across folds that are not limit points of bifurcation lines, and across bifurcation lines far from their intersections. Fukaya [2005, Section 5.4] explained that the cancellation of monodromy, which may look accidental, is in fact related to the phenomenon of wall crossing of Floer homology. In our case, which is approximated in the sense that we use Morse homology instead of Floer homology, the caustic and bifurcation locus represent the walls in an analogous phenomenon for Morse homology. We check that in this way the holomorphic structure of $\hat{\hat{E}}$ can be extended to the codimension 2 subset of $\mathbb{R}^2$ containing the remaining points of $\tilde{K}$ and $\tilde{B}$, that is, the intersection points of bifurcation lines, folds that are limit points of bifurcation lines, and cusps. We realize however that these corrections are not enough to extend the holomorphic structure of $\hat{\hat{E}}$ to cusps. A correction of different kind is thus required, which is related to the possibility of defining a spin structure on $\tilde{L}$ or, better, a relative spin structure. This has to do with the orientation problem in Floer homology theory; see [Fukaya et al. 2000]. In this way, the monodromy around the caustic is also canceled, and so the mirror bundle $\hat{\hat{E}}$ can be endowed with a holomorphic structure defined on the whole dual fibration.

2. Preliminaries

Throughout this paper we will use results from [Marelli 2006a; 2006b], whose contents we now summarize.

- We introduced Lagrangian bundles $\pi : X \to B$, Lagrangian maps $g : L \to B$ and their generating functions, and defined the caustic K of L as the set of critical values of $\pi \circ g$;
- We recalled the classification of Lagrangian singularities. We noted that in dimension 2 only folds and cusps are stable and generic; the elliptic umbilic, which is the case considered in this paper, is stable and generic starting from dimension 3. However it can appear as unstable singularity (that is, it breaks

in folds and cusps under a small perturbation) in dimension 2; it is given by the generating function

$$f(y_1, y_2) = \tfrac{1}{3}y_1^3 - y_1 y_2^2.$$

- We showed that the elliptic umbilic in dimension 2 becomes after a small perturbation a Lagrangian map whose caustic is a tricuspoid, a curve with three edges, whose points are folds, and three cusps at vertexes.
- For f the generating function of L, we defined the family $f_x : \mathbb{R}^n \to \mathbb{R}$ of functions, where x is a point in the base of the fibration, as

$$f_x(y) = f(y) - x \cdot y$$

and considered the gradient system

$$\nabla f_x(y) = \frac{dy}{dt} \tag{1}$$

whose solutions we called gradient lines. We noted that the caustic K of L is the subset of points x where the gradient field ∇f_x exhibits a degenerate critical point. We defined B, the bifurcation locus of L, as the subset of points x where f_x is a Morse function but ∇f_x is not Morse–Smale, that is, where the phase portrait of ∇f_x features a saddle-to-saddle separatrix.
- We studied how the bifurcation locus of the elliptic umbilic, represented by three straight half-lines with a common vertex at the origin, is modified after a small perturbation. Far from the caustic, a tricuspoid, the bifurcation locus looks as that of the unperturbed elliptic umbilic with three bifurcation lines; in a neighborhood of the caustic, these are half-lines that generically have vertex at a fold point of the caustic. As for the mutual positions of the bifurcation lines and their possible intersections, see [Marelli 2006b, Theorem 4.14] and the pictures there representing the diagrams that can be expected.

3. The mirror bundle

We recall how the mirror bundle should be constructed for the trivial fibration $\mathbb{R}^4 \to \mathbb{R}^2$ and its dual (but more generally also for $\mathbb{R}^{2n} \to \mathbb{R}^n$). In [Leung et al. 2001; Bruzzo et al. 2001; 2002], it is defined by a kind of Fourier–Mukai transform that associates to each pair formed by a Lagrangian submanifold L, in the given Lagrangian fibration, and a local system ∇ on it, a vector bundle $\hat{E}$ on the dual fibration, endowed with a connection $\hat{\nabla}$. Its curvature $\hat{F}$ satisfies $\hat{F}^{0,2} = 0$ and so induces a holomorphic structure on $\hat{E}$. This is achieved under certain hypotheses, among which that L has no caustic. On the other hand, in [Fukaya 2000], the fiber of the mirror bundle $\hat{E}$ over the point (x, w) of the dual fibration (x is a coordinate on the base and w on the fiber) is defined as the Lagrangian intersection Floer homology of L and F_x:

$$\hat{E}_{(x,w)} = HF((L,\nabla),(F_x,w)),$$

where $w \in \hat{F}_x$ defines a flat connection on F_x. For the affine Lagrangian submanifolds considered in [Fukaya 2000], HF^k is nonvanishing only when k equals the dimension of the fiber. A holomorphic frame is then defined on $\hat{E}$. These two constructions are equivalent in the cases considered in the papers above, that is, when assuming at least that the fibration has no singular fibers and that L has no caustic.

Here, we will follow mainly the second construction (though sometimes we use also the Fourier–Mukai construction), since this approach seems to be more suitable as quantum corrections are provided by pseudoholomorphic discs. However, as explained in the introduction, using a conjecture by Fukaya [2005, Section 3.5], near the caustic we switch from Floer homology and pseudoholomorphic discs to Morse homology and gradient lines. So that we don't introduce notation that we don't use, we state the conjecture informally and refer to [Fukaya 2005] for the precise formulation.

Conjecture 3.1. *The moduli space of gradient lines is isotopic to the moduli space of pseudoholomorphic discs in a neighborhood of a point of the caustic.*

Partial progress towards a proof of this conjecture has been made by Floer[1988] and by Fukaya and Oh [1997], the latter in the case of the cotangent bundle.

The transfer to Morse homology is then performed as follows. Consider the trivial Lagrangian fibration $\mathbb{R}^{2n} \to \mathbb{R}^n$. To L is associated a (local) generating function $f:\mathbb{R}^n \to \mathbb{R}$. We consider as in Section 2 the family $f_x:\mathbb{R}^n \to \mathbb{R}$ and the gradient system (1). Let K and B be the caustic and the bifurcation locus of L. If $x \notin K \cup B$, with some further hypotheses on f (see [Schwarz 1993]), the Morse complex is defined over x. The space of k-chains is the free $\mathbb{C}$-module generated by critical points of Morse index k and the differential is defined counting gradient lines, that is, the solutions of the gradient system (1) joining two critical points whose Morse indexes differ by 1. The fiber of the mirror bundle is defined as the Morse homology $\hat{E}_{(x,w)} = HM(f_x)$ of the Morse complex over x, and a holomorphic frame is constructed similarly to that proposed in [Fukaya 2000; 2005]. Namely, by writing $\nabla = d + A$, a section $e(x)$ of $\hat{E}$ turns out to be holomorphic and descends on the torus fibers when multiplied by the weight

$$\exp\Bigl(2\pi\Bigl(\frac{h(x)}{2} - \frac{A(x)}{4\pi} + i\frac{\partial h}{\partial x}\cdot w\Bigr)\Bigr),$$

where h is a multivalued function on the base such that each sheet of L is locally the graph of dh. In other words, h is a set of local generating functions defined in the coordinates of the base, one for each sheet of L. The problem is to glue this bundle along the caustic K and the bifurcation locus B.

4. The monodromy of the elliptic umbilic

Consider the trivial Lagrangian torus fibration $\mathbb{R}^4 \to \mathbb{R}^2$ and a Lagrangian submanifold L whose caustic K contains an elliptic umbilic q. In a neighborhood of q, we can choose symplectic coordinates (y_1, y_2, x_1, x_2), with coordinates y_1 and y_2 on the fibers and x_1 and x_2 on the base of the fibration, such that L is given by the generating function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (y_1, y_2) \mapsto \tfrac{1}{3} y_1^3 - y_1 y_2^2 \tag{2}$$

Since we will be working in a neighborhood of q, we can use the local coordinates just introduced. This means we consider the Lagrangian fibration $\mathbb{R}^4 \to \mathbb{R}^2$ and the Lagrangian submanifold L defined by the generating function f. Associated to f, we have the caustic K and the bifurcation locus B. By hypothesis, $K = \{(0, 0)\}$, while [Marelli 2006b] shows that B is given by three half-lines from $(0, 0)$, defined by $t \to te^{i\alpha}$ for $\alpha = 0, 2\pi/3, 4\pi/3$ and $t > 0$.

Consider a line bundle E over L with a flat $U(1)$-connection ∇. The pair (L, ∇) defines an object in the Fukaya category of the symplectic manifold $\mathbb{R}^4$. On $\mathbb{R}^2 \setminus K$ the function f has no critical points, so we can apply results of [Bruzzo et al. 2002] or [Fukaya 2000], thus producing a bundle $\widehat{E}$ of rank 2 over the total space of the dual fibration restricted to $\mathbb{R}^2 \setminus K$. A hermitian connection $\widehat{\nabla}$ can be defined on $\widehat{E}$, thus inducing a holomorphic structure on $\widehat{E}$. Note that L is a 2-sheeted cover of $\mathbb{R}^2 \setminus K$. Thus for $x \in \mathbb{R}^2 \setminus K$ if $p_1(x)$ and $p_2(x)$ denote the elements of $L \cap F_x$, where F_x is the fiber of the Lagrangian fibration $\mathbb{R}^4 \to \mathbb{R}^2$ over x, and if z_1 and z_2 are coordinates along the fibers of the dual fibration, then the connection $\widehat{\nabla}$ can be written as $d + \widehat{A}$, with

$$\widehat{A}(x) = i(p_1(x)dz_1 + p_2(x)dz_2). \tag{3}$$

However, let $\Gamma \in \pi_1(\mathbb{R}^2 \setminus K)$, $\Gamma : [0, 1] \to \mathbb{R}^2$, and consider the continuous maps

$$M^i_\Gamma : [0, 1] \to \mathbb{R}^4, \quad t \mapsto p_i(\Gamma(t)) \quad \text{for } i = 1, 2. \tag{4}$$

Let $M^i_\Gamma(t)_F$ be the projection onto $F_{\Gamma(t)} \cong \mathbb{R}^2$ of $M^i_\Gamma(t)$.

Definition 4.1. The monodromy of the holomorphic structure of $\widehat{E}$ is the map

$$\mathcal{M} : \pi_1(\mathbb{R}^2 \setminus K) \to \mathrm{End}(\mathbb{R}^2), \quad \mathcal{M}(\Gamma)(M^i_\Gamma(0)_F) = M^i_\Gamma(1)_F. \tag{5}$$

Since $\Gamma(0) = \Gamma(1)$, $M^i_\Gamma(0)$ and $M^i_\Gamma(1)$ belong to the same fiber. Also, the endomorphism $\mathcal{M}(\Gamma)$ is well-defined, since $\{M^i_\Gamma(t)\}$ for $i = 1, 2$ is a basis of $F_{\Gamma(t)}$.

Lemma 4.2. *If* Γ *is a nontrivial simple loop in* $\pi_1(\mathbb{R}^2 \setminus K)$, *then the monodromy* $\mathcal{M}$ *of the holomorphic structure* $\widehat{E}$ *on* Γ *can be represented by the matrix*

$$\mathcal{M}(\Gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. This follows because the points $p_1(x)$ and $p_2(x)$ exchange when going around the origin. In fact, since L can be written as $x_1 = y_1^2 - y_2^2$ and $x_2 = -2y_1y_2$, it becomes $z = \overline{w}^2$ by writing $z = x_1 + ix_2$ and $w = y_1 + iy_2$. □

This lemma shows that $\widehat{E}$ cannot be extended to a holomorphic bundle on the whole dual fibration over $\mathbb{R}^2$. For this, some "quantum correction" must be added; see also [Fukaya 2005, Section 5.4].

5. Perturbations of the elliptic umbilic

Consider now a small perturbation $\tilde{f}$ of f. The caustic $\widetilde{K}$ and the bifurcation locus $\widetilde{B}$ of $\tilde{f}$ were studied in [Marelli 2006a] and [Marelli 2006b], respectively. There $\widetilde{K}$ was shown to be diffeomorphic to a tricuspoid, and $\widetilde{B}$, outside a disc containing $\widetilde{K}$, looks as the bifurcation locus of the unperturbed f, while inside this disc its structure can be highly complicated and bifurcation lines can intersect. See [Marelli 2006b] for pictures of the several admissible diagrams representing the reciprocal positions of $\widetilde{K}$ and $\widetilde{B}$ inside the disc. At first we restrict our attention to the subset $\mathbb{R}^2 \setminus \widetilde{K}$. Given a flat connection $\widetilde{\nabla}$ on the Lagrangian submanifold $\widetilde{L}$ defined by $\tilde{f}$, we construct a holomorphic bundle $\widehat{\widetilde{E}}$ on each of the two connected components of $\mathbb{R}^2 \setminus \widetilde{K}$, as explained in [Bruzzo et al. 2002] or in [Fukaya 2000]. As done in Section 4 for $\widehat{E}$, we can define the monodromy $\widetilde{\mathcal{M}}$ of the holomorphic structure of $\widehat{\widetilde{E}}$ and prove the following lemma:

Lemma 5.1. *If* Γ *is a nontrivial simple loop in* $\pi_1(\mathbb{R}^2 \setminus K)$, *then the monodromy* $\widetilde{\mathcal{M}}$ *of the holomorphic structure of* $\widehat{\widetilde{E}}$ *on* Γ *can be represented by the matrix*

$$\widetilde{\mathcal{M}}(\Gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Since f is perturbed on a compact subset D containing the origin, it follows that $\tilde{f}$ coincides with f outside D and that $\widetilde{K} \subset D$. So $\widetilde{\mathcal{M}}(\Gamma) = \mathcal{M}(\Gamma)$. □

Therefore, outside the caustic, the holomorphic structure of $\widehat{\widetilde{E}}$ also exhibits a monodromy.

6. Quantum corrections to perturbations of the elliptic umbilic

The problem is to solve the monodromy and extend the holomorphic structure of $\widehat{\widetilde{E}}$ across the caustic $\widetilde{K}$, gluing it with the holomorphic structure inside $\widetilde{K}$. The way to achieve this is to construct $\widehat{\widehat{E}}$ with its holomorphic structure on $\mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$, define morphism gluing this structure across $\widetilde{K}$ and $\widetilde{B}$ and check if the monodromy is solved. This is what we mean by quantum corrections. We are going to define quantum corrections on $\widehat{\widehat{E}}$. Then, since a holomorphic section is obtained by multiplying a section of $\widehat{\widehat{E}}$ by a suitable weight, we will obtain quantum corrections for holomorphic sections of $\widehat{\widehat{E}}$. If a section can be extended to $\widetilde{K} \cup \widetilde{B}$, the same will hold for a holomorphic section. The features of the set $\mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$, namely, the mutual positions of $\widetilde{K}$ and $\widetilde{B}$, are described in [Marelli 2006b, Theorem 4.14].

We explain now how to construct the mirror bundle (of Section 3) far from $\widetilde{K} \cup \widetilde{B}$ in this case. The function $\tilde{f}_x$ defined by $\tilde{f}_x(y) = \tilde{f}(y) - x \cdot y$ is a Morse function for every $x \in \mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$. As computed in [Marelli 2006b], if x lies inside the caustic, $\tilde{f}_x$ has four critical points: three saddles $s_i(x)$, the points with Morse index 1, and an unstable node $n(x)$, the point with Morse index 2. Thus the Morse complex is

$$0 \leftarrow 0 \leftarrow \bigoplus_{i=1}^{3} \mathbb{C}[s_i(x)] \xleftarrow{\partial_x} \mathbb{C}[n(x)] \leftarrow 0 \leftarrow \cdots, \tag{6}$$

where $\mathbb{C}[s_i(x)]$ and $\mathbb{C}[n(x)]$ denote the free modules over $\mathbb{C}$ generated by $s_i(x)$ and $n(x)$, respectively. The differential ∂ can be defined after an orientation is chosen on the moduli space of gradient lines from n to s_i (see [Schwarz 1993] or [McDuff and Salamon 2004] for a more detailed construction of Morse homology). In our case, ∂_x can be defined as $\partial_x n(x) = s_1(x) + s_2(x) + s_3(x)$ (anyway, the Morse complex has only two nontrivial terms, so ∂ automatically satisfies $\partial^2 = 0$); we fix this choice of orientation of gradient lines.

If x lies outside the caustic, $\tilde{f}_x$ has two saddles as critical points, so the Morse complex is simply given by

$$0 \leftarrow 0 \leftarrow \mathbb{C}[s_i(x)] \oplus \mathbb{C}[s_j(x)] \leftarrow 0 \leftarrow \cdots. \tag{7}$$

Definition 6.1. The fiber $\widehat{\widehat{E}}_x$ of $\widehat{\widehat{E}}$ over $x \in \mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$ is defined to be the Morse homology of the Morse complex (6) or (7) if x lies respectively inside or outside the caustic.

In our case, Morse homology has only one nontrivial term, so for x inside the caustic

$$\widehat{\widehat{E}}_x = \frac{\bigoplus_{i=1}^{3} \mathbb{C}[s_i(x)]}{\partial_x(\mathbb{C}[n(x)])},$$

while for x outside the caustic, $\widehat{\widehat{E}}_x = \mathbb{C}[s_i(x)] \oplus \mathbb{C}[s_j(x)]$.

Definition 6.2. On each connected component of $\mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$, we define $\widehat{\widehat{E}}$ as the trivial bundle whose fiber at $x \in U_i$ is given by Definition 6.1.

We define now morphisms gluing the holomorphic bundle $\widehat{\widehat{E}}$ along $\widetilde{K}$ and $\widetilde{B}$. We start by considering the subset $\widetilde{K}_F$ of $\widetilde{K}$ consisting of folds that are not limit points of bifurcation lines. It is a codimension 1 subset of $\mathbb{R}^2$. Suppose U and

V are two connected components of $\mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$, lying respectively outside and inside the caustic, such that $\partial U \cap \partial V \neq \varnothing$, and let $\widetilde{K}_i \subset \partial U \cap \partial V \cap \widetilde{K}_F$ be a connected component of $\widetilde{K}_F$. For simplicity, suppose that V is inside the caustic and U is outside, so that along $\widetilde{K}_i$ the node n and the saddle s_i in V glue together and disappear in U. (The pair (n, s_i) is also called a birth/death pair.)

Definition 6.3. The isomorphism $\widehat{\widehat{E}}(U) \cong \widehat{\widehat{E}}(V)$ gluing $\widehat{\widehat{E}}$ along $\widetilde{K}_i$ is defined as the one induced in homology by the inclusion

$$\mathbb{C}[s_j] \oplus \mathbb{C}[s_k] \hookrightarrow \bigoplus_{l=1}^{3} \mathbb{C}[s_l(x)] \quad \text{for } j, k \neq i.$$

It is a good definition since the inclusion preserves kernel and image of the differential of the Morse complex.

The second group of definitions is concerned instead with gluing along the subset $\widetilde{B}_1$ of $\widetilde{B}$ consisting of points that are not intersection of bifurcation lines. It is a codimension 1 subset of $\mathbb{R}^2$.

Definition 6.4. For each $x \in \mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$ lying inside the caustic, we define the incidence matrix $I(x) = (I(x)_i) \in \mathrm{Mat}(3, 1)$ such that $I(x)_i = 0$ if there is no gradient line from $n(x)$ to $s_i(x)$, and $I(x)_i = 1$ otherwise.

Remark 6.5. Similar definitions in a different setting appear in [Igusa 2002; 1993; Igusa and Klein 1993], highlighting the relation between Morse theory and algebraic K-theory. The definition of incidence matrix also resembles that of transition matrix given in [Kokubu 2000].

The incidence matrix at x gives information about the phase portrait of the gradient vector field $\nabla \tilde{f}_x$ and is related to the Morse differential simply as

$$\partial_x n(x) = I(x)_1 s_1(x) + I(x)_2 s_2(x) + I(x)_3 s_3(x).$$

The incidence matrix is constant on each connected component of $\mathbb{R}^2 \setminus (\widetilde{K} \cup \widetilde{B})$: Indeed, the gradient vector fields $\nabla \tilde{f}_x$ are orbit equivalent for all x in the same connected component, and so the Morse complexes are isomorphic. Let U and V be two such components lying inside the caustic such that $\partial U \cap \partial V \neq \varnothing$, with incidence matrix $I(U)$ and $I(V)$, respectively. For $\tau \in \{1, 0, -1\}$, let $E_{ij}(\tau) \in \mathrm{Mat}(3, 3)$ be the triangular matrix whose (k, l)-entry is 1 if $k = l$, is τ if $k = i$ and $l = j$, and is 0 otherwise. By results in [Marelli 2006b], crossing a bifurcation line can change at most only one of the entries of the incidence matrix. Therefore either

- $I(U) \neq I(V)$ and so there is only one $k \in \{1, 2, 3\}$ such that $I(U)_k \neq I(V)_k$, or
- $I(U) = I(V)$.

Definition 6.6. Consider the transformation matrix from U to V associated to points in $\partial U \cap \partial V \cap \widetilde{B}_1$ of a bifurcation line of $\widetilde{B}$. When this bifurcation line is characterized by the appearance of a nongeneric gradient line from s_i to s_j, the transformation matrix is of the form $E_{ij}(\tau)$, with $E_{ij}(\tau)I(U) = I(V)$.

When $I(U) \neq I(V)$ it follows that $\tau = 1$ if $I(U)_j = 0$, and $\tau = -1$ if $I(U)_j = 1$. When instead $I(U) = I(V)$, there is an ambiguity in the choice of τ, which will be discussed in Example 6.8.

We give two examples to clarify the previous definition.

Example 6.7. Suppose the phase portrait of $\nabla \tilde{f}_x$ for $x \in U$ and for $x \in V$ is represented by the incidence matrices $I(U) = (1, 1, 1)$ and $I(V) = (1, 1, 0)$, respectively. There are two possible bifurcations from U to V (see [Marelli 2006a; 2006b] for further explanations and some pictures): Either the nongeneric gradient line $\gamma_{s_1 s_3}$ or the nongeneric gradient line $\gamma_{s_2 s_3}$ appears in the phase portrait of $\nabla \tilde{f}_x$ when x is the bifurcation point. The first bifurcation corresponds to the transformation matrix $E_{31}(-1)$, while the second corresponds to $E_{32}(-1)$. Instead, if crossing from V to U, the same bifurcations give the transformation matrices $E_{31}(1)$ and $E_{32}(1)$, respectively.

Example 6.8. $I(U) = I(V)$ occurs only in case (c) of the proof of Proposition 6.11 and shown in Figure 1 (which contains the notation), along the bifurcation line between δ and ϵ. The phase portraits in δ and ϵ, which are represented respectively in [Marelli 2006b, Figures 4.20 and 4.19], can be summarized here as follows. The separatrixes that connect s_1 and s_3 to n in α (the phase portrait over α is shown in [Marelli 2006b, Figure 4.17]) can form a saddle-to-saddle separatrix in ϵ, but this can not occur in δ. This can provide a criterion for the choice of τ (which cannot be justified further here) considering only the special example of the perturbed elliptic umbilic. The matrix $M(w_3)$ in the proof of Proposition 6.11 is the transformation matrix from ϵ to δ. There the choice of τ is the one that solves the monodromy.

Suppose now that U and V lie outside the caustic $\widetilde{K}$ and $\partial U \cap \partial V \cap \widetilde{B}_1$ is a subset of $\widetilde{B}_j$, one of the three bifurcation lines forming the bifurcation diagram $\widetilde{B}$, and assume $\widetilde{B}_j$ enters into $\widetilde{K}$ at a point p through the side l_j of $\widetilde{K}$, where n and s_j form a birth/death pair. Since we are working in a neighborhood of $\widetilde{K}$, we can assume that $p \in \partial U \cap \partial V$. Inside the caustic and in a neighborhood of p, we can associate a transformation matrix $E_{ik}(\tau)$ to $\widetilde{B}_j$ using Definition 6.6.

Definition 6.9. If U and V lie outside $\widetilde{K}$ and are as above, the transformation matrix from U to V associated to points in $\partial U \cap \partial V \cap \widetilde{B}_1$ of the bifurcation line $\widetilde{B}_j$ is the matrix $E_{ik}(\tau) \in \mathrm{Mat}(2, 2)$ obtained from $E_{ik}(\tau) \in \mathrm{Mat}(3, 3)$ above by deleting the j-th row and the j-th column.

The transformation matrix we associate to a bifurcation line $\widetilde{B}_j$ from U to V defines a morphism between the Morse complexes of U and V.

Definition 6.10. The isomorphism $\widehat{\widehat{E}}(U) \cong \widehat{\widehat{E}}(V)$ gluing $\widehat{\widehat{E}}$ along $\widetilde{B}_j$ is the one induced by the transformation matrix of Definition 6.6 or Definition 6.9 associated to the bifurcation line $\widetilde{B}_j$.

We have now to check that we can extend $\widehat{\widehat{E}}$ through the codimension 2 subset given by intersection points of bifurcation lines, limit points of bifurcation lines on the caustic, and the three cusps.

We start by considering intersection points of bifurcation lines. In [Marelli 2006b] we analyzed the conditions under which two bifurcation lines can intersect themselves.

Proposition 6.11. *The holomorphic bundle $\widehat{\widehat{E}}$ can be extended through intersection points of bifurcation lines.*

Proof. We check that, for all possible cases of intersection of bifurcation lines described in [Marelli 2006b] and for a chosen loop Γ around the intersection point p, the composition of the transformation matrices of bifurcation lines at intersection points with Γ is the identity.

From [Marelli 2006b], we know there are the three cases (a), (b), and (c) of Figure 1. The phase portraits in the subsets determined by bifurcation lines and of bifurcations in cases are represented in [ibid., Figures 4.7, 4.8 and 4.9 for (a), 4.11, 4.12, 4.13 and 4.14 for (b), and 4.17, 4.18, 4.19, 4.20, 4.21 and 4.22 for (c)].

In case (a), we know that the two bifurcation lines are characterized by the appearance of the same saddle-to-saddle separatrix, obtained by gluing the same pair of separatrices. So, choosing a simple loop Γ around p intersecting for simplicity the bifurcation lines at four points w_i for $i = 1, \dots, 4$, and associating to each w_i a transition matrix $M(w_i)$ according to Definition 6.6, we have

$$M(w_1) = M(w_3) = M(w_2)^{-1} = M(w_4)^{-1},$$

and thus

$$M(w_4)M(w_3)M(w_2)M(w_1) = \mathrm{Id}\,.$$

This implies that there is no monodromy around p and so the holomorphic bundle $\widehat{\widehat{E}}$ can be extended across p.

In case (b), we chose again a simple loop Γ around p intersecting the bifurcation lines at four points w_i for $i = 1, \dots, 4$. Suppose w_1 belongs to the bifurcation line from α to β, w_2 to the bifurcation line from β to δ, w_3 to the bifurcation line from δ to γ, and w_4 to the bifurcation line from γ to α. The transformation matrices

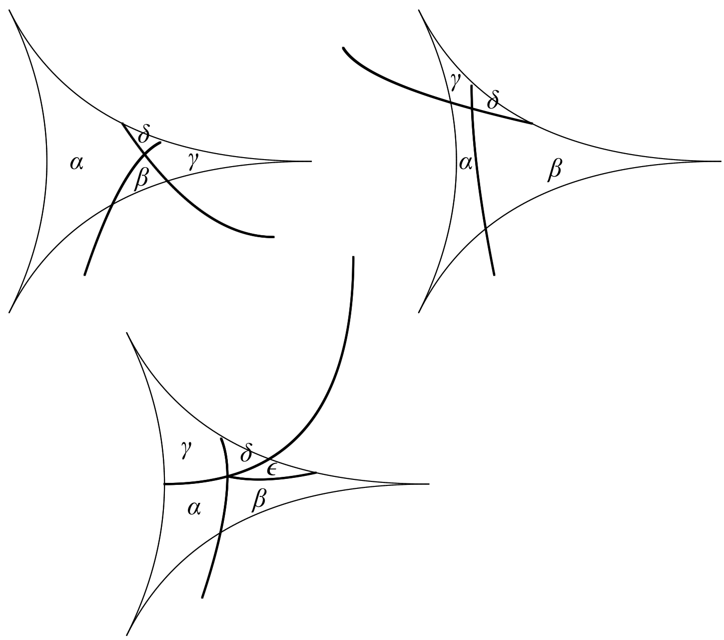

Figure 1. Clockwise from top left: Intersection of bifurcations lines in cases (a), (b), and (c)

associated by Definition 6.6 to the bifurcation lines at each w_i are given by

$$M(w_1) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M(w_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$M(w_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M(w_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $M(w_4)M(w_3)M(w_2)M(w_1) = \mathrm{Id}$, and so the holomorphic bundle $\widehat{\widehat{E}}$ can be extended across p.

In case (c) choose a simple loop Γ around p that intersects the bifurcation lines at five points w_i for $i = 1, \dots, 5$, starting from the bifurcation line from α to β and then proceeding anticlockwise. The transformation matrices are then

$$M(w_1) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M(w_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M(w_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$M(w_4) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M(w_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then $M(w_5)M(w_4)M(w_3)M(w_2)M(w_1) = \mathrm{Id}$, and so the holomorphic bundle $\widehat{\widehat{E}}$ can be extended across p. □

We analyze now the behavior of $\widehat{\widehat{E}}$ around limit points of bifurcation lines belonging to the caustic.

Proposition 6.12. *The holomorphic bundle $\widehat{\widehat{E}}$ can be extended through limit points of bifurcation lines belonging to the caustic*, *when they are not cusps.*

Proof. From [Marelli 2006b] we know there are two cases: generically, either (i) the bifurcation line $\widetilde{B}$ enters into the caustic $\widetilde{K}$ at a fold or (ii) it is a half-line with origin at a fold (and the bifurcation line $\widetilde{B}$, near its origin, lies inside $\widetilde{K}$). See Figure 2. In both cases, let us denote this fold by p.

In case (i), p is not a cusp. So at each point of the caustic $\widetilde{K}$ near p, the node n glues with a saddle, which we suppose is s_1. Suppose also that the half-line $\widetilde{B}$ has its endpoint on the side of the caustic where n glues with s_2. With α the region marked in Figure 2, suppose that the phase portrait of $\nabla \tilde{f}_x$ for $x \in \alpha$ contains all the gradient lines γ_{ns_i}. Choose a simple loop Γ around p intersecting $\widetilde{B}$ at two points w_1 and w_3, and $\widetilde{K}$ at two points w_2 and w_4. Suppose w_1 lies inside the caustic and w_4 outside. The transition matrices at w_1 and w_3, according respectively to Definitions 6.6 and 6.9, are

$$M(w_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M(w_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Consider an element $h \in \widehat{\widehat{E}}_x$ for $x \in \alpha$. Since $\widehat{\widehat{E}}_x = (\bigoplus_{i=1}^{3} \mathbb{C}[s_i(x)])/\partial_x(\mathbb{C}[n])$, we write h as an equivalence class $[(h_1, h_2, h_3)]$ in the basis (s_1, s_2, s_3) of $\mathbb{C}[s_i(x)]$,

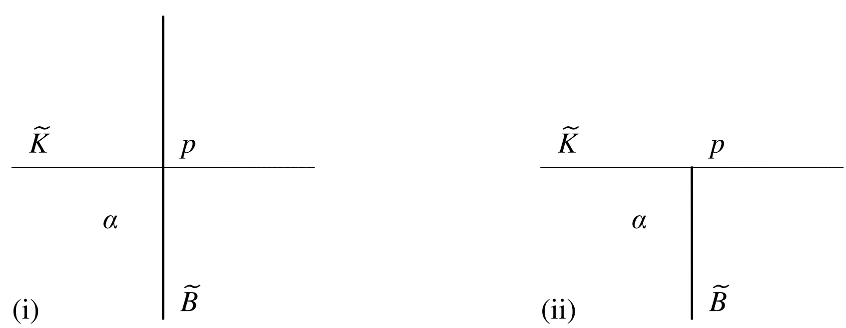

Figure 2. Mutual positions of bifurcation lines and caustic.

where $(h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3 + c)$ for every $c \in \mathbb{C}$. Moving along Γ from α into β, crossing $\widetilde{B}$ at w_1, we transform h by $M(w_1)$. At β, we have $(h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3)$ for every $c \in \mathbb{C}$, so we can write

$$[M(w_1)h] = [(h_1, h_2, -h_2 + h_3)] = [(0, h_2 - h_1, -h_2 + h_3)].$$

According to Definition 6.3, when crossing $\widetilde{K}$ at w_2 we have the gluing isomorphism

$$[(0, h_2 - h_1, -h_2 + h_3)] \cong (h_2 - h_1, -h_2 + h_3).$$

Crossing now $\widetilde{B}$ along Γ at w_3, we have

$$[M(w_3)(h_2 - h_1, -h_2 + h_3)^t] = (h_2 - h_1, h_3 - h_1).$$

Crossing $\widetilde{K}$ at w_4 and using the gluing isomorphism of Definition 6.3 we obtain

$$(h_2 - h_1, h_3 - h_1) \cong [(0, h_2 - h_1, h_3 - h_1)] = [(h_1, h_2, h_3)].$$

This shows that there is no monodromy and so $\widehat{\widehat{E}}$ can be extended through p.

In case (ii), suppose for simplicity that at p the node n and the saddle s_1 form the birth/death pair; that $\widetilde{B}$ intersects $\widetilde{K}$ at another point where n and s_2 form the birth/death pair; and that for $x \in \alpha$ the phase portrait of $\nabla \tilde{f}_x$ contains all the gradient lines γ_{ns_i}. Choose a simple loop Γ around p intersecting $\widetilde{B}$ at the point w_1, and $\widetilde{K}$ at two points w_2 and w_3. We know w_1 lies inside the caustic. The transformation matrix of Definition 6.6 at w_1 is

$$M(w_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Consider an element $h \in \widehat{\widehat{E}}_x$ for $x \in \alpha$, and write it again as in case (i) as an equivalence class $[(h_1, h_2, h_3)]$ in the basis (s_1, s_2, s_3) of $\mathbb{C}[s_i(x)]$. Going along Γ into β, crossing $\widetilde{B}$ in w_1, we transform h by $M(w_1)$. In β we have $(h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3)$ for every $c \in \mathbb{C}$, so we can write

$$[M(w_1)h] = [(h_1, h_2, -h_1 + h_3)] = [(0, h_2 - h_1, -h_1 + h_3)].$$

Now, crossing $\widetilde{K}$ at w_2 and using the gluing isomorphism of Definition 6.3, we have

$$[(0, h_2 - h_1, -h_1 + h_3)] \cong (h_2 - h_1, -h_1 + h_3).$$

Finally, entering into $\widetilde{K}$ through w_3 and using again the gluing isomorphism, we obtain in (α)

$$(h_2 - h_1, -h_1 + h_3) \cong [(0, h_2 - h_1, -h_1 + h_3)] = [(h_1, h_2, h_3)].$$

This shows that there is no monodromy and so $\widehat{\widehat{E}}$ can be extended through p. $\square$

Now we check if $\widehat{\widehat{E}}$ can be extended to cusps. To start suppose that at a cusp c the node n glues with the saddles s_2 and s_3. According to [Marelli 2006b] there are two cases: either (I) for x in a neighborhood of c, inside the caustic, the phase portrait of $\nabla \tilde{f}_x$ contains all the gradient lines γ_{ns_i}, or (II) it contains only γ_{ns_2} and γ_{ns_3}. In both cases a monodromy appears around the cusp.

Lemma 6.13. *In case* (I), *if* Γ *is a nontrivial simple loop around* c, *the monodromy of the holomorphic structure of* $\widehat{\widehat{E}}$ *along* Γ *is represented by the matrix*

$$M = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Proof. For x outside the caustic, since $\widehat{\widehat{E}}_x = \mathbb{C}[s_1] \oplus \mathbb{C}[s_j]$, we may write an element $h \in \widehat{\widehat{E}}_x$ as (h_1, h_j). Suppose for $k \in \{2, 3\}$ that l_k is the branch of the caustic where n glues with s_k. Then on l_k the gluing isomorphism of Definition 6.3 identifies s_j with the saddle different from s_k and s_1. So, entering into the caustic through l_2 we have

$$(h_1, h_j) \cong [(h_1, h_j, 0)] = [(h_1 - h_j, 0, -h_j)].$$

Now exiting from the caustic through l_3, we have

$$[(h_1 - h_j, 0, -h_j)] \cong (h_1 - h_j, -h_j),$$

which gives the expected monodromy. □

Lemma 6.14. *In case* (II), *if* Γ *is a nontrivial simple loop around* c, *the monodromy of the holomorphic structure of* $\widehat{\widehat{E}}$ *along* Γ *is represented by the matrix*

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Proof. Using the notation in the proof of the previous lemma, we have, entering into the caustic through l_2

$$(h_1, h_j) \cong [(h_1, h_j, 0)] = [(h_1, 0, -h_j)].$$

Exiting the caustic through l_3, we have

$$[(h_1, 0, -h_j)] \cong (h_1, -h_j),$$

which gives the expected monodromy. □

In both cases, the matrix M is invertible. This means that the same monodromy is associated to Γ and to its opposite Γ^{-1} in $\pi_1(L \setminus \{c\})$.

If now at c the node n glues with the saddles s_1 and s_2 we have a similar result:

Lemma 6.15. *If* Γ *is a nontrivial simple loop around* c, *the monodromy of the holomorphic structure of* $\widehat{\widehat{E}}$ *along* Γ *is represented in case* (I) *by the matrix*

$$M = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \tag{10}$$

and in case (II) *by the matrix*

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

Proof. The proof is analogous to that of Lemmas 6.13 and 6.14. □

Again observe that the matrix M is invertible, meaning that Γ and Γ^{-1} provide the same monodromy.

Finally, if the node n glues at c with the saddles s_1 and s_3, we obtain:

Lemma 6.16. *If* Γ *is a nontrivial simple loop around* c, *the monodromy of the holomorphic structure of* $\widehat{\widehat{E}}$ *along* Γ *is represented in case* (I) *by the matrix*

$$M = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \textit{or by its inverse} \quad M^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \tag{12}$$

and in case (II) *by the matrix*

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{or by its inverse} \quad M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{13}$$

Proof. The proof is similar to that of Lemma 6.13 and 6.14. □

In both cases, if Γ is associated to M (say), then Γ^{-1} is associated to M^{-1}.

To solve the monodromy around the cusps it is necessary to add a new kind of correction. It is related to the possibility of defining a spin structure on $\widetilde{L}$ and to the problem of orientation in Lagrangian intersection Floer homology. (In fact from [Fukaya et al. 2000] we know that the existence of a relative spin structure on $\widetilde{L}$ is a condition for the orientability of the moduli space of pseudoholomorphic discs.) This is suggested intuitively by what follows: Consider the composition $\pi \circ i : \widetilde{L} \hookrightarrow \mathbb{R}^4 \to \mathbb{R}^2$, where π is the projection of the fibration and i is the Lagrangian immersion, and note that a spin structure can be induced at least on the subset of $\widetilde{L}$ where $d\pi$ is invertible, that is, on $\widetilde{L} \setminus \pi^{-1}(\widetilde{K})$. This means that the caustic or a subset of it represents an obstruction to the existence of a spin structure on $\widetilde{L}$.

The following result shows that the set of cusps is actually the obstruction to the existence of a spin structure on a Lagrangian submanifold L with generating function f. It proves, in fact, that the second Stiefel–Whitney class $w_2(L) \in H^2(L, \mathbb{Z}_2)$ of L, which represents the obstruction to the existence of spin structures on L, has the set of cusps as Poincaré dual in $H_0(L, \mathbb{Z}_2)$.

Lemma 6.17. *We have* $\mathrm{PD}(w_2(L)) = A_3(f)$ *where* $A_3(f)$ *is the set of singular points of* f *of type* A_3, *that is*, *the set of cusps.*

Proof. The equality is mainly proved by using the Thom polynomials of Lagrangian singularities. The proof is essentially given in [Kazarian 2003], where it follows from other major results given there. Kazarian first demonstrates that the cohomology class $\mathrm{PD}(\Omega(f))$, the Poincaré dual to the locus $\Omega(f)$ of singularities of f of class Ω, is equal to the Thom polynomial P_Ω associated to Ω. Kazarian computes Thom polynomials (see also [Vassilyev 1988]), and in particular, when $\Omega = A_3$, shows that $P_\Omega = w_2(T^*L) = w_2(TL)$. □

Let A be an immersed 1-dimensional submanifold of $\mathbb{R}^2$ with three nonintersecting connected components, each of which is a half-line with vertex at one of the three cusps of the caustic. To solve the monodromy around the cusps it is enough to glue (for example along A) the holomorphic structure so as to cancel the monodromy. The problem is to justify this procedure, which for the moment is just an ad hoc correction. As said, the idea, coming from the orientation problem of Lagrangian intersection Floer homology and confirmed by Lemma 6.17, is that the ability to define a spin structure on some flat bundle on $\widetilde{L}$ should provide such a correction. We make the following natural definition:

Definition 6.18. Along each half-line forming the submanifold A, depending on which cusp the half-line has as vertex, we glue the holomorphic bundle $\widehat{\widehat{E}}$ using the inverse of morphism (8), (10), or (12) in case (I) and (9), (11) or (13) in case (II).

This correction is called *orientation twist* in [Fukaya 2005].

Proposition 6.19. *If* $\widehat{\widehat{E}}$ *is glued along* A *according to Definition 6.18*, *then its holomorphic structure can be extended across the cusp.*

Proof. The proof is a direct consequence of Lemma 6.13, 6.14, 6.15 and 6.16, since the corrections applied are just the inverses of what we want to cancel. □

We explain now how to justify Definition 6.18, generalizing the idea outlined in [Fukaya 2005, Section 5.4]. Before considering the case of a perturbed elliptic umbilic, let us examine for simplicity a Lagrangian submanifold L exhibiting a cusp c. In this case, A is a half-line with vertex in c. Consider a ball U containing c. Since U is contractible, L owns a spin structure over U. On the other hand, $d\pi$ is invertible over the complement of U and so induces a spin structure on L. Since by Lemma 6.17, $w_2(L)$ does not vanish because of c, it follows that the nonexistence of a spin structure on L comes from the gluing of TL along the boundary of U. The purpose now is to show how A can provide both a correction to TL, by defining a new bundle carrying a spin structure, and a correction to the flat $U(1)$-line bundle $\mathscr{L}$ on L, yielding the gluing that cancels the monodromy. Consider representations

$$\rho : \pi_1(\mathbb{R}^2 \setminus \{c\}) \to \{1, -1\} = O(1) \subset U(1)$$

defining two representations $\rho^{O(1)} := \rho_1$ and $\rho^{U(1)} := \rho_2$. According to this choice we have, respectively, a flat $O(1)$-bundle $\mathscr{L}_{\rho_1}$ or a flat $U(1)$-bundle $\mathscr{L}_{\rho_2}$ on $\mathbb{R}^2 \setminus \{c\}$. There are two possibilities for ρ. It is either the trivial or the nontrivial group homomorphism $\mathbb{Z} \to \{1, -1\}$. When ρ is the nontrivial representation, its values on a path $\Gamma \in \pi_1(\mathbb{R}^2 \setminus \{c\})$ are given by the intersection number of Γ and A. $\mathscr{L}_{\rho_1}$ is the trivial bundle when ρ is trivial but is a Möbius strip when ρ is nontrivial. The same holds for $\mathscr{L}_{\rho_2}$; the bundle $\mathscr{L}_{\rho_2}$ restricted to a generator $\Gamma \cong S^1$ of $\pi_1(\mathbb{R}^2 \setminus \{c\})$ is the flat line bundle on the torus $T^1 = S^1$ with factor of automorphy equal to either 1 or -1 according to whether ρ is trivial or not. In other words, we may think of a section of $\mathscr{L}_{\rho_2}$ over Γ as multiplied by respectively 1 or -1 at $\Gamma \cap A$ (the factor of automorphy for $U(1)$-line bundles on tori and the induced connection on the mirror bundle are explained [Bruzzo et al. 2001; 2002]). The projection of the fibration $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ and the composition $\pi \circ i$, where $i : L \hookrightarrow \mathbb{R}^4$ is the Lagrangian immersion, define respectively bundles $\mathscr{L}_\rho^{\mathbb{R}^4} = \pi^* \mathscr{L}_\rho$ on $\mathbb{R}^4$ and $\mathscr{L}_\rho^L = (\pi \circ i)^* \mathscr{L}_\rho$ on L, away from $\pi^{-1}(c)$ and $(\pi \circ i)^{-1}(c)$, respectively, where ρ can be either ρ_1 or ρ_2.

If ρ_1 is the nontrivial representation, then, since a Möbius strip has $w_1 = 1$, by setting $M = \mathscr{L}_{\rho_1}^{\mathbb{R}^4} \oplus \mathscr{L}_{\rho_1}^{\mathbb{R}^4}$, we have

$$w_1(M) = 2w_1(\mathscr{L}_{\rho_1}^{\mathbb{R}^4}) = 0 \quad \text{and} \quad w_2(M) = 2w_2(\mathscr{L}_{\rho_1}^{\mathbb{R}^4}) + w_1(\mathscr{L}_{\rho_1}^{\mathbb{R}^4}) w_1(\mathscr{L}_{\rho_2}^{\mathbb{R}^4}) = 1.$$

This implies that the bundle $TL \oplus M|_L$ over L carries a spin structure. In fact, since $\mathscr{L}_\rho^L = i^*(\pi^* \mathscr{L}_\rho) = i^*(\mathscr{L}_\rho^{\mathbb{R}^4})$ and so $w_2(\mathscr{L}_\rho^L) = i^* w_2(\mathscr{L}_\rho^{\mathbb{R}^4})$, we have

$$w_2(TL \oplus M|_L) = w_2(TL) + w_1(TL) w_1(M|_L) + w_2(M|_L) = 0$$

in $H^2(L; \mathbb{Z}_2)$. This together with the facts that L has dimension 2 and that M is a real orientable vector bundle on $\mathbb{R}^4$ implies by definition that L has a relative spin structure.

Now consider the flat line bundle $\mathscr{L} \otimes \mathscr{L}_{\rho_2}^L$ over L with connection $\nabla_\rho = \nabla \otimes \nabla_{\rho_2}^L$, where $(\mathscr{L}, \nabla)$ is the given flat line bundle over L and $\nabla_{\rho_2}^L$ is the flat connection of $\mathscr{L}_{\rho_2}^L$ defined by ρ_2, and consider the effect of the connection $\hat{\nabla}_\rho$ on the transformed bundle $\hat{E}$. It induces a nontrivial gluing along A, given by multiplication by -1, which cancels the monodromy along c, given also by multiplication by -1. In fact, if s_1 and s_2 are the saddles and l_1 and l_2 are the sides of the caustic where the node n glues together with s_1 and s_2, respectively, we have according to Definition 6.3 that $(h) \cong [(h, 0)]$ along l_1. In Morse homology we have the equality $[(h, 0)] = [(0, -h)]$ along l_2. According to Definition 6.3, we have $[(0, -h)] \cong (-h)$. Finally, along A, the connection $\hat{\nabla}_\rho$ gives the gluing $(-h) \cong (h)$.

Consider now our case of a perturbed elliptic umbilic. Take a suitable ball U containing a cusp c of $\widetilde{L}$ such that $\widetilde{L} \cap \pi^{-1}(U)$ has two connected components.

For simplicity, suppose that c is the cusp of the caustic where n, s_2 and s_3 glue together. Identify the critical points of the gradient system over x and the points of $\widetilde{L}$ over x. Then of the two components of $\widetilde{L} \cap \pi^{-1}(U)$, one contains s_1 and the other s_2 and s_3. Note that $T\widetilde{L}$ carries a spin structure over the first component but not over the second, where we find the same situation described above for the cusp. So choose ρ so that $\mathscr{L}^{\widetilde{L}}_{\rho_1}$ and $\mathscr{L}^{\widetilde{L}}_{\rho_2}$ are the trivial flat line bundles over the component containing s_1 and the nontrivial one over the component containing s_2 and s_3. As above, set $M = \mathscr{L}^{\widetilde{L}}_{\rho_1} \oplus \mathscr{L}^{\widetilde{L}}_{\rho_1}$. Then $T\widetilde{L} \oplus M|_L$ carries a spin structure on both the components. Moreover, the connection $\widehat{\widetilde{\nabla}}_\rho$ on the mirror bundle $\widehat{\widetilde{E}}$ induced by the connection $\widetilde{\nabla}_\rho = \widetilde{\nabla} \oplus \widetilde{\nabla}^{\widetilde{L}}_{\rho_2}$ cancels the monodromy of Lemmas 6.13 and 6.14 as we will now explain.

Consider first the case (II) described by Lemma 6.14. Since no gradient line exists from n to s_1, it can be treated as done above for the cusp. We get that the flat connection gives a gluing along A that is multiplication by 1 on chains generated by s_1 and multiplication by -1 on chains generated by s_2 or s_3; this cancels in homology the monodromy of Lemma 6.14.

Consider now case (I) described by Lemma 6.13. The gluing provided by $\widehat{\widetilde{\nabla}}_\rho$ must commute with the equivalence among cycles in Morse homology in order to define a gluing in homology, and this is not automatic as in case (II) because of the gradient line from n to s_1. In fact, the connection $\widehat{\widetilde{\nabla}}_\rho$ induces a connection on $\partial(\langle n\rangle) = \langle s_1 + s_2 + s_3\rangle$ characterized by a gluing that is multiplication by -1. On the other hand, the connection on $\sum_{i=1}^3 \mathbb{C}[s_i]$ has factor of automorphy -1 on the chains s_2 and s_3 and 1 on s_1. This means that it does not commute with the action on cycles determined by the differential ∂. Thus, to induce a connection in homology, that is, on the quotient $\sum_{i=1}^3 \mathbb{C}[s_i]/\partial(\langle n\rangle)$, the connection at the chain level, that is, on $\sum_{i=1}^3 \mathbb{C}[s_i]$, must be split into two parts. One of these, commuting with that on $\partial(\langle n\rangle)$, will induce a connection in homology. The problem is the choice of a splitting of the connection at the chains level. This is performed as follows. The gluing $(h_1, h_2, h_3) \cong (h_1, -h_2, -h_3)$ is split as $(h_1, -h_2, -h_3) = (h_1 - h_2 - h_3, -h_2, -h_3) + (h_2 + h_3, 0, 0)$ and on the quotient the gluing given by $[(h_1, -h_2, -h_3)] = [(h_1 - h_2 - h_3, -h_2, -h_3)]$ is induced. Indeed, it commutes with the Morse differential:

$$\begin{aligned}(h_1, h_2, h_3) &\cong (h_1 + g, h_2 + g, h_3 + g)\\ &\cong (h_1 + g - h_2 - g - h_3 - g, -h_2 - g, -h_3 - g)\\ &= (h_1 - h_2 - h_3 - g, -h_2 - g, -h_3 - g),\end{aligned}$$

where the first equivalence is that among cycles in Morse homology and the second is the gluing, and

$$(h_1, h_2, h_3) \cong (h_1 - h_2 - h_3, -h_2, -h_3) \cong (h_1 - h_2 - h_3 - g, -h_2 - g, -h_3 - g)$$

where now the first equivalence is the gluing and the second is that among cycles in Morse homology. The splitting we chose corresponds to a gluing, at the chain level, given by multiplication by 1 on the generator s_1, and, on the generators s_2 and s_3, by multiplication by -1, followed by a projection parallel to s_1 onto the lines generated by s_3 and s_2, respectively. A better justification for this choice requires, perhaps, considering a more general situation than that of a perturbed elliptic umbilic. Anyway, this solves the monodromy. Indeed, as in the proof of Lemma 6.13, we have along l_2 $(h_1, h_j) \cong [(h_1, h_j, 0)] = [(h_1 - h_j, 0, -h_j)]$, the connection ∇_ρ gives the gluing

$$[(h_1 - h_j, 0, -h_j)] \cong [(h_1 - h_j + h_j, 0, h_j)] = [(h_1, 0, h_j)],$$

and along l_3 we have $[(h_1, 0, h_j)] \cong (h_1, h_j)$.

It remains to check that there is no monodromy in the holomorphic structure of $\widehat{\widehat{E}}$ when going along a loop Γ such that the caustic lies in the compact region of $\mathbb{R}^2$ determined by Γ, as described in Lemma 5.1.

Theorem 6.20. *The monodromy of Lemma 5.1 is solved by the following corrections*: $\widehat{\widehat{E}}$ *is glued by means of the morphisms of Definition 6.3 along the caustic* $\widetilde{K}$, *of Definition 6.10 along the bifurcation locus* $\widetilde{B}$, *and of Definition 6.18 along the relative cycle* A.

Proof. The theorem follows from Propositions 6.11, 6.12 and 6.19. □

As an example, we write the transformation matrices associated to bifurcation lines and to half-lines forming the relative cycles A, which are met by a loop Γ as described above, and show that their composition is the identity, implying that the expected monodromy is canceled. Consider, for instance, the bifurcation diagram of Figure 3.

Assume for simplicity that Γ is directed counterclockwise. Set $a_i = A_i \cap \Gamma$ and $b_i = \widetilde{B}_i \cap \Gamma$, where A_i are the half-lines forming the relative cycle A, and $\widetilde{B}_i$ are the bifurcation lines for $i = 1, 2, 3$. Then the matrices corresponding to gluing morphisms at points a_i and b_i are

$$M(b_1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad M(b_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M(b_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$M(a_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M(a_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M(a_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe now that $M(b_3)M(a_3)M(a_2)M(b_2)M(a_1)M(b_1) = \mathrm{Id}$, which implies that the monodromy is solved.

With such corrections, the mirror bundle $\widehat{\widehat{E}}$ is endowed with a holomorphic structure that can be extended along the caustic and the bifurcation locus.

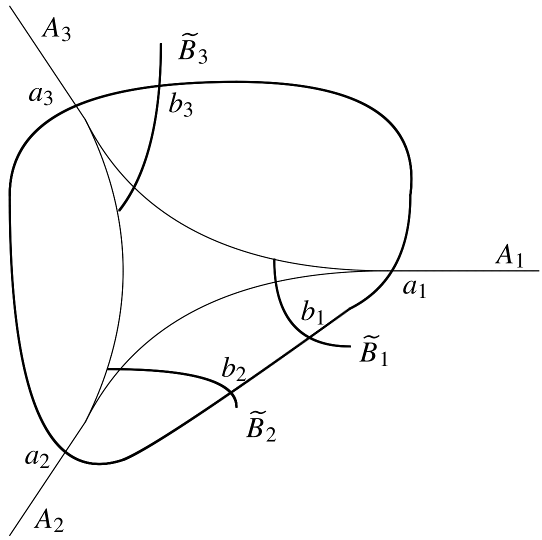

Figure 3. An allowed bifurcation diagram together with the half-cycle A and the loop Γ.

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