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Sérgio de Moura Almaraz

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SÉRGIO DE MOURA ALMARAZ

Let (M, g) be a compact Riemannian manifold with boundary. We address the Yamabe-type problem of finding a conformal scalar-flat metric on M whose boundary is a constant mean curvature hypersurface. When the boundary is umbilic, we prove an existence theorem that finishes some of the remaining cases of this problem.

1. Introduction

J. Escobar [1992a] has studied the following Yamabe-type problem for manifolds with boundary:

Yamabe problem. Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. Is there a scalar-flat metric on M that is conformal to g and has ∂M as a constant mean curvature hypersurface?

In dimension two, the classical Riemann mapping theorem says that any simply connected, proper domain of the plane is conformally diffeomorphic to a disk. This theorem is false in higher dimensions since the only bounded open subsets of $\mathbb{R}^n$ for $n \geq 3$ that are conformally diffeomorphic to Euclidean balls are the Euclidean balls themselves. The Yamabe-type problem proposed by Escobar can be viewed as an extension of the Riemann mapping theorem for higher dimensions.

In analytical terms, this problem corresponds to finding a positive solution to

$$\begin{cases} L_g u = 0 & \text{in } M, \\ B_g u + K u^{n/(n-2)} = 0 & \text{on } \partial M \end{cases} \tag{1-1}$$

for some constant K, where $L_g = \Delta_g - \frac{1}{4}(n-2)/(n-1) R_g$ is the conformal Laplacian and $B_g = \partial/\partial\eta - \frac{1}{2}(n-2)h_g$. Here Δ_g is the Laplace–Beltrami operator, R_g is the scalar curvature, h_g is the mean curvature of ∂M and η is the inward unit normal vector to ∂M.

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The solutions of the equations (1-1) are the critical points of the functional

$$Q(u) = \frac{\int_M |\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g}{\left(\int_{\partial M} u^2(n-1)/(n-2) d\sigma_g\right)^{(n-2)/(n-1)}},$$

where dv_g and $d\sigma_g$ denote the volume forms of M and ∂M, respectively. Escobar introduced the conformally invariant Sobolev quotient

$$Q(M, \partial M) = \inf\{Q(u) : u \in C^1(M), u \not\equiv 0 \text{ on } \partial M\}$$

and proved that it satisfies $Q(M, \partial M) \leq Q(B^n, \partial B)$. Here, B^n denotes the unit ball in $\mathbb{R}^n$ endowed with the Euclidean metric.

Under the hypothesis that $Q(M, \partial M)$ is finite (which is the case when $R_g \geq 0$), he also showed that the strict inequality

$$Q(M, \partial M) < Q(B^n, \partial B) \tag{1-2}$$

implies the existence of a minimizing solution of the equations (1-1).

Notation. We denote by (M^n, g) a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M and finite Sobolev quotient $Q(M, \partial M)$.

Theorem 1.1 [Escobar 1992a]. *Assume that one of the following conditions holds*:

(1) $n \geq 6$ *and* M *has a nonumbilic point on* ∂M;

(2) $n \geq 6$, M *is locally conformally flat and* ∂M *is umbilic*;

(3) $n = 4$ *or* 5 *and* ∂M *is umbilic*;

(4) $n = 3$.

Then $Q(M, \partial M) < Q(B^n, \partial B)$ *and there is a minimizing solution to* (1-1).

The proof for $n = 6$ under condition (1) appeared later, in [Escobar 1996b].

Further existence results were obtained by F. Marques in [Marques 2005] and [Marques 2007]. Together, these results can be stated as follows:

Theorem 1.2 [Marques 2005; Marques 2007]. *Assume that one of the following conditions holds*:

(1) $n \geq 8$, $\overline{W}(x) \neq 0$ *for some* $x \in \partial M$ *and* ∂M *is umbilic*;

(2) $n \geq 9$, $W(x) \neq 0$ *for some* $x \in \partial M$ *and* ∂M *is umbilic*;

(3) $n = 4$ *or* 5 *and* ∂M *is not umbilic.*

Then $Q(M, \partial M) < Q(B^n, \partial B)$ *and there is a minimizing solution to* (1-1).

Here, W denotes the Weyl tensor of M and $\overline{W}$ the Weyl tensor of ∂M.

Our main result deals with the remaining dimensions $n = 6, 7$ and 8 when the boundary is umbilic and $W \neq 0$ at some boundary point:

Theorem 1.3. *Suppose that $n = 6, 7$ or 8, that ∂M is umbilic and that $W(x) \neq 0$ for some $x \in \partial M$. Then $Q(M, \partial M) < Q(B^n, \partial B)$ and there is a minimizing solution to the equations* (1-1).

These cases are similar to the case of dimensions 4 and 5 when the boundary is not umbilic, studied in [Marques 2007].

Other works concerning conformal deformation on manifolds with boundary include [Ahmedou 2003; Ambrosetti et al. 2002; Ben Ayed et al. 2005; Brendle 2002; Djadli et al. 2003; 2004; Escobar 1992b; 1996a; Escobar and Garcia 2004; Felli and Ould Ahmedou 2003; 2005; Han and Li 1999; 2000]

We will now discuss the strategy in the proof of Theorem 1.3. We assume that ∂M is umbilic and choose $x_0 \in \partial M$ so that $W(x_0) \neq 0$. Our proof is explicitly based on constructing a test function ψ such that

$$Q(\psi) < Q(B^n, \partial B). \tag{1-3}$$

The function ψ has support in a small half-ball around the point x_0. The usual strategy in this kind of problem (which goes back to [Aubin 1976]) is to define the function ψ in the small half-ball as one of the standard entire solutions to the corresponding Euclidean equations. In our context those are

$$U_\epsilon(x) = \left(\frac{\epsilon}{x_1^2 + \cdots + x_{n-1}^2 + (\epsilon + x_n)^2} \right)^{(n-2)/2}, \tag{1-4}$$

where $x = (x_1, \ldots, x_n)$ and $x_n \geq 0$.

The next step would be to expand the quotient of ψ in powers of ϵ and, by exploiting the local geometry around x_0, show that the inequality (1-3) holds if ϵ is small. To simplify the asymptotic analysis, we use conformal Fermi coordinates centered at x_0. This concept, introduced in [Marques 2005], plays the same role that conformal normal coordinates (see [Lee and Parker 1987]) did in the case of manifolds without boundary.

When $n \geq 9$, the strict inequality (1-3) was proved in [Marques 2005]. The difficulty arises because, when $3 \leq n \leq 8$, the first correction term in the expansion does not have the right sign. When $3 \leq n \leq 5$, Escobar proved the strict inequality by applying the positive mass theorem, a global construction originally due to Schoen [1984]. This argument does not work when $6 \leq n \leq 8$ because the metric is not sufficiently flat around the point x_0.

As we mentioned before, the situation under the hypothesis of Theorem 1.3 is quite similar to the cases of dimensions 4 and 5 when the boundary is not umbilic, cases solved by Marques [2007]. As he pointed out, the test functions U_ϵ are not optimal in these cases but the problem is still local. This phenomenon does not appear in the classical solution of the Yamabe problem for manifolds without boundary. However, perturbed test functions have been used in the works of Hebey and Vaugon [1993], Brendle [2007] and Khuri, Marques and Schoen [2009].

To prove the inequality (1-3), inspired by the ideas of Marques, we introduce

$$\phi_\epsilon(x) = \epsilon^{n-2/2} R_{ninj}(x_0) x_i x_j x_n^2 (x_1^2 + \cdots + x_{n-1}^2 + (\epsilon + x_n)^2)^{-n/2}.$$

Our test function ψ is defined as $\psi = U_\epsilon + \phi_\epsilon$ around $x_0 \in \partial M$.

In Section 2 we expand the metric g in Fermi coordinates and discuss the conformal Fermi coordinates. In Section 3 we prove Theorem 1.3 by estimating $Q(\psi)$.

Notation. Throughout, we use the index notation for tensors, with commas denoting covariant differentiation. We adopt the summation convention whenever confusion does not result. When dealing with Fermi coordinates, we will use indices $1 \le i, j, k, l, m, p, r, s \le n-1$ and $1 \le a, b, c, d \le n$. Lines over an object mean that the metric is being restricted to the boundary.

We set $\det g = \det g_{ab}$. We will denote by ∇_g or ∇ the covariant derivative and by Δ_g or Δ the Laplace–Beltrami operator. The full curvature tensor will be denoted by R_{abcd}, the Ricci tensor by R_{ab}, and the scalar curvature by R_g or R. The second fundamental form of the boundary will be denoted by h_{ij} and the mean curvature, $(n-1)^{-1}\,\mathrm{tr}(h_{ij})$, by h_g or h. We will denote the Weyl tensor by W_{abcd}.

We let $\mathbb{R}^n_+$ denote the half-space $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_n \ge 0\}$. If $x \in \mathbb{R}^n_+$ we set $\bar{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \cong \partial\mathbb{R}^n$. We will denote by $B^+_\delta(0)$ (or B^+_δ for short) the half-ball $B_\delta(0) \cap \mathbb{R}^n_+$, where $B_\delta(0)$ is the Euclidean open ball of radius $\delta > 0$ centered at the origin of $\mathbb{R}^n$. Given a subset $\mathscr{C} \subset \mathbb{R}^n_+$, we set $\partial^+\mathscr{C} = \partial\mathscr{C} \cap (\mathbb{R}^n_+ \backslash \partial\mathbb{R}^n_+)$ and $\partial'\mathscr{C} = \mathscr{C} \cap \partial\mathbb{R}^n_+$.

We denote the volume forms of M and ∂M denoted by dv_g and $d\sigma_g$, respectively. The n-dimensional sphere of radius r in $\mathbb{R}^{n+1}$ will be denoted by S^n_r. We denote the volume of the n-dimensional unit sphere S^n_1 by σ_n.

For $\mathscr{C} \subset M$, we define the energy of a function u in $\mathscr{C}$ by

$$E_{\mathscr{C}}(u) = \int_{\mathscr{C}} \Big(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \Big) dv_g + \frac{n-2}{2} \int_{\partial'\mathscr{C}} h_g u^2 d\sigma_g.$$

2. Coordinate expansions for the metric

In this section we will write expansions for the metric g in Fermi coordinates. We will also discuss the concept of conformal Fermi coordinates. The results of this section are basically proved on [Marques 2005, pages 1602–1609 and 1618].

Definition 2.1. Let $x_0 \in \partial M$. Choose geodesic normal coordinates $(x_1, \ldots, x_{n-1})$ on the boundary, centered at x_0. We say that $(x_1, \ldots, x_n)$ for small $x_n \ge 0$ are the Fermi coordinates (centered at x_0) of the point $\exp_x(x_n\eta(x)) \in M$. Here, we denote by $\eta(x)$ the inward unit vector normal to ∂M at $x \in \partial M$.

It is easy to see that in these coordinates, $g_{nn} \equiv 1$ and $g_{jn} \equiv 0$ for $j = 1, \dots, n-1$.

Fix $x_0 \in \partial M$. The existence of conformal Fermi coordinates is stated as follows:

Proposition 2.2. *For any given integer $N \geq 1$ there is a metric $\tilde{g}$, conformal to g, such that* $\det \tilde{g}(x) = 1 + O(|x|^N)$ *in $\tilde{g}$-Fermi coordinates centered at x_0. Moreover, $h_{\tilde{g}}(x) = O(|x|^{N-1})$.*

The first statement of Proposition 2.2 is [Marques 2005, Proposition 3.1]. The second follows from the equation

$$h_g = \frac{-1}{2(n-1)} g^{ij} g_{ij,n} = \frac{-1}{2(n-1)} (\log \det g)_{,n}.$$

The next three lemmas will also be used in the computations of the next section.

Lemma 2.3. *Suppose that ∂M is umbilic. Then, in conformal Fermi coordinates centered at x_0, we have $h_{ij}(x) = O(|x|^N)$, where N can be taken arbirarily large, and*

$$\begin{aligned} g^{ij}(x) = \delta_{ij} &+ \tfrac{1}{3}\overline{R}_{ikjl}x_k x_l + R_{ninj}x_n^2 + \tfrac{1}{6}\overline{R}_{ikjl;m}x_k x_l x_m + R_{ninj;k}x_n^2 x_k + \tfrac{1}{3}R_{ninj;n}x_n^3 \\ &+ \left(\tfrac{1}{20}\overline{R}_{ikjl;mp} + \tfrac{1}{15}\overline{R}_{iksl}\overline{R}_{jmsp}\right)x_k x_l x_m x_p \\ &+ \left(\tfrac{1}{2}R_{ninj;kl} + \tfrac{1}{3}\operatorname{Sym}_{ij}(\overline{R}_{iksl}R_{nsnj})\right)x_n^2 x_k x_l \\ &+ \tfrac{1}{3}R_{ninj;nk}x_n^3 x_k + \left(\tfrac{1}{12}R_{ninj;nn} + \tfrac{2}{3}R_{nins}R_{nsnj}\right)x_n^4 + O(|x|^5). \end{aligned}$$

Here, every coefficient is computed at x_0.

Lemma 2.4. *Suppose that ∂M is umbilic. Then, in conformal Fermi coordinates centered at x_0, we have these equalities at x_0:*

(i) $\overline{R}_{kl} = \operatorname{Sym}_{klm}(\overline{R}_{kl;m}) = 0$,

(ii) $R_{nn} = R_{nn;k} = \operatorname{Sym}_{kl}(R_{nn;kl}) = 0$,

(iii) $R_{nn;n} = 0$,

(iv) $\operatorname{Sym}_{klmp}(\frac{1}{2}\overline{R}_{kl;mp} + \frac{1}{9}\overline{R}_{ikjl}\overline{R}_{imjp}) = 0$,

(v) $R_{nn;nk} = 0$,

(vi) $R_{nn;nn} + 2(R_{ninj})^2 = 0$,

(vii) $R_{ij} = R_{ninj}$,

(viii) $R_{ijkn} = R_{ijkn;j} = 0$,

(ix) $R = R_{,j} = R_{,n} = 0$,

(x) $R_{,ii} = -\frac{1}{6}(\overline{W}_{ijkl})^2$,

(xi) $R_{ninj;ij} = -\frac{1}{2}R_{;nn} - (R_{ninj})^2$.

The idea for proving items (i)–(vi) of Lemma 2.4 is to express g_{ij} as the exponential of a matrix A_{ij}. Then we just observe that $\mathrm{trace}(A_{ij}) = O(|x|^N)$ for any arbitrarily large integer N. Items (vii)–(xi) are applications of the Gauss and Codazzi equations and the Bianchi identity. Item (x) uses the fact that Fermi coordinates are normal on the boundary.

Lemma 2.5. *Suppose ∂M is umbilic. Then $W_{abcd}(x_0) = 0$ in conformal Fermi coordinates centered at $x_0 \in \partial M$ if and only if $R_{ninj}(x_0) = \overline{W}_{ijkl}(x_0) = 0$.*

Proof of Lemma 2.5. Recall that the Weyl tensor is defined by

$$W_{abcd} = R_{abcd} - \frac{1}{n-2}(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}) + \frac{R}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{2-1}$$

By the symmetries of the Weyl tensor, $W_{nnnn} = W_{nnni} = W_{nnij} = 0$. By the identity (2-1) and Lemma 2.4(viii), we have $W_{nijk}(x_0) = 0$. From the identity (2-1) again and from parts (ii), (vii) and (ix) of Lemma 2.4, we have

$$W_{ninj} = \frac{n-3}{n-2} R_{ninj}$$

and

$$W_{ijkl} = \overline{W}_{ijkl} - \frac{1}{n-2}(R_{nink}g_{jl} - R_{ninl}g_{jk} + R_{njnl}g_{ik} - R_{njnk}g_{il})$$

at x_0. In the last equation we also used the Gauss equation. The result follows. □

3. Estimating the Sobolev quotient

In this section, we will prove Theorem 1.3 by constructing a function ψ such that

$$Q(\psi) < Q(B^n, \partial B).$$

We first recall that the positive number $Q(B^n, \partial B)$ also appears as the best constant in the Sobolev-trace inequality

$$\left(\int_{\partial\mathbb{R}^n_+} |u|^{2(n-1)/(n-2)} d\bar{x}\right)^{(n-2)/n-1} \leq \frac{1}{Q(B^n, \partial B)} \int_{\mathbb{R}^n_+} |\nabla u|^2 dx$$

for every $u \in H^1(\mathbb{R}^n_+)$. Escobar [1988] and independently Beckner [1993] proved that the equality is achieved by the functions U_ϵ defined in (1-4). They are solutions to the boundary value problem

$$\begin{cases} \Delta U_\epsilon = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial U_\epsilon}{\partial y_n} + (n-2)U_\epsilon^{n/(n-2)} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3-1}$$

One can check, using integration by parts, that

$$\int_{\mathbb{R}^n_+} |\nabla U_\epsilon|^2 dx = (n-2) \int_{\partial\mathbb{R}^n_+} U_\epsilon^{2(n-1)/(n-2)} dx$$

and also that

$$Q(B^n, \partial B) = (n-2)\Big(\int_{\partial\mathbb{R}^n_+} U_\epsilon^{2(n-1)/(n-2)} dx\Big)^{1/(n-1)}. \tag{3-2}$$

Assumption. In what remains, we will assume that ∂M is umbilic and there is a point $x_0 \in \partial M$ such that $W(x_0) \neq 0$.

Since the Sobolev quotient $Q(M, \partial M)$ is a conformal invariant, we can use conformal Fermi coordinates centered at x_0.

Convention. Henceforth, all the curvature terms are evaluated at x_0. We fix conformal Fermi coordinates centered at x_0 and work in a half-ball $B^+_{2\delta} = B^+_{2\delta}(0) \subset \mathbb{R}^n_+$.

In particular, for any arbitrarily large N, we can write the volume element dv_g as

$$dv_g = (1 + O(|x|^N))dx. \tag{3-3}$$

Often we will use that, for any homogeneous polynomial p_k of degree k,

$$\int_{S^{n-2}_r} p_k = \frac{r^2}{k(k+n-3)} \int_{S^{n-2}_r} \Delta p_k. \tag{3-4}$$

We will now construct the test function ψ. Set

$$\phi_\epsilon(x) = \epsilon^{(n-2)/2} A R_{ninj} x_i x_j x_n^2 ((\epsilon + x_n)^2 + |\bar{x}|^2)^{-n/2}, \tag{3-5}$$

for $A \in \mathbb{R}$ to be fixed later, and

$$\phi(y) = A R_{ninj} y_i y_j y_n^2 ((1 + y_n)^2 + |\bar{y}|^2)^{-n/2}. \tag{3-6}$$

Thus, $\phi_\epsilon(x) = \epsilon^{2-(n-2)/2}\phi(\epsilon^{-1}x)$. Set $U = U_1$. Thus, $U_\epsilon(x) = \epsilon^{-(n-2)/2} U(\epsilon^{-1}x)$. Note that $U_\epsilon(x) + \phi_\epsilon(x) = (1 + O(|x|^2))U_\epsilon(x)$. Hence, if δ is sufficiently small,

$$\tfrac{1}{2} U_\epsilon \le U_\epsilon + \phi_\epsilon \le 2U_\epsilon \quad \text{in } B^+_{2\delta}.$$

Let $r \mapsto \chi(r)$ be a smooth cut-off function satisfying $\chi(r) = 1$ for $0 \le r \le \delta$, $\chi(r) = 0$ for $r \ge 2\delta$ and $0 \le \chi \le 1$ and $|\chi'(r)| \le C\delta^{-1}$ if $\delta \le r \le 2\delta$. Our test function is defined by

$$\psi(x) = \chi(|x|)(U_\epsilon(x) + \phi_\epsilon(x)).$$

3.1. ***Estimating the energy of ψ.*** The energy of ψ is given by

$$\begin{aligned}E_M(\psi) &= \int_M \Big(|\nabla_g\psi|^2+\frac{n-2}{4(n-1)}R_g\psi^2\Big)dv_g+\frac{n-2}{2}\int_{\partial M}h_g\psi^2 d\sigma_g\\ &= E_{B^+_\delta}(\psi)+E_{B^+_{2\delta}\backslash B^+_\delta}(\psi).\end{aligned}\tag{3-7}$$

Observe that

$$|\nabla_g\psi|^2\leq C|\nabla\psi|^2\leq C|\nabla\chi|^2(U_\epsilon+\phi_\epsilon)^2+C\chi^2|\nabla(U_\epsilon+\phi_\epsilon)|^2.$$

Hence,

$$\begin{aligned}E_{B^+_{2\delta}\backslash B^+_\delta}(\psi)\leq C\int_{B^+_{2\delta}\backslash B^+_\delta}|\nabla\chi|^2U_\epsilon^2dx+C\int_{B^+_{2\delta}\backslash B^+_\delta}\chi^2|\nabla U_\epsilon|^2dx&\\ +C\int_{B^+_{2\delta}\backslash B^+_\delta}R_gU_\epsilon^2dx+C\int_{\partial' B^+_{2\delta}\backslash\partial' B^+_\delta}h_gU_\epsilon^2d\bar{x}&,\end{aligned}$$

Thus,

$$E_{B^+_{2\delta}\backslash B^+_\delta}(\psi)\leq C\epsilon^{n-2}\delta^{2-n}.\tag{3-8}$$

The first term in the right hand side of (3-7) is

$$\begin{aligned}E_{B^+_\delta}(\psi)&=E_{B^+_\delta}(U_\epsilon+\phi_\epsilon)\\ &=\int_{B^+_\delta}\Big(|\nabla_g(U_\epsilon+\phi_\epsilon)|^2+\frac{n-2}{4(n-1)}R_g(U_\epsilon+\phi_\epsilon)^2\Big)dv_g\\ &\qquad\qquad+\frac{n-2}{2}\int_{\partial' B^+_\delta}h_g(U_\epsilon+\phi_\epsilon)^2d\sigma_g\\ &=\int_{B^+_\delta}|\nabla(U_\epsilon+\phi_\epsilon)|^2dx\\ &\qquad+\int_{B^+_\delta}(g^{ij}-\delta^{ij})\partial_i(U_\epsilon+\phi_\epsilon)\partial_j(U_\epsilon+\phi_\epsilon)dx\\ &\qquad\qquad+\frac{n-2}{4(n-1)}\int_{B^+_\delta}R_g(U_\epsilon+\phi_\epsilon)^2dx+C\epsilon^{n-2}\delta.\end{aligned}\tag{3-9}$$

Here, we used the identity (3-3) for the volume term and Proposition 2.2 for the integral involving h_g.

We will treat the three integral terms in the right hand side of (3-9) in the next three lemmas.

Lemma 3.1. *We have*

$$\begin{aligned}\int_{B_\delta^+} &|\nabla(U_\epsilon+\phi_\epsilon)|^2dx\\ &\le Q(B^n,\partial B^n)\Big(\int_{\partial M}\psi^{2(n-1)/(n-2)}dx\Big)^{(n-2)/(n-1)}+C\epsilon^{n-2}\delta^{2-n}\\ &\qquad -\frac{4}{(n+1)(n-1)}\epsilon^4A^2(R_{ninj})^2\int_{B_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar y|^4}{((1+y_n)^2+|\bar y|^2)^n}dy\\ &\qquad +\frac{8n}{(n+1)(n-1)}\epsilon^4A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^3|\bar y|^4}{((1+y_n)^2+|\bar y|^2)^{n+1}}dy\\ &\qquad +\frac{12n}{(n+1)(n-1)}\epsilon^4A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar y|^4}{((1+y_n)^2+|\bar y|^2)^{n+1}}dy.\end{aligned}$$

Proof. Since $R_{nn}=0$ by Lemma 2.4(ii), we have $\int_{S_r^{n-2}}R_{ninj}y_iy_jd\sigma_r(y)=0$. Thus we see that

$$\int_{B_\delta^+}|\nabla(U_\epsilon+\phi_\epsilon)|^2dx=\int_{B_\delta^+}|\nabla U_\epsilon|^2dx+\int_{B_\delta^+}|\nabla\phi_\epsilon|^2dx. \tag{3-10}$$

Integrating by parts equations (3-1) and using the identity (3-2) we obtain

$$\begin{aligned}\int_{B_\delta^+}|\nabla U_\epsilon|^2dx &\le Q(B^n,\partial B^n)\Big(\int_{\partial' B_\delta^+}U_\epsilon^{2(n-1)/(n-2)}dx\Big)^{(n-2)/(n-1)}\\ &\le Q(B^n,\partial B^n)\Big(\int_{\partial M}\psi^{2(n-1)/(n-2)}dx\Big)^{(n-2)/(n-1)}.\end{aligned}$$

In the first inequality above we used that $\partial U_\epsilon/\partial\eta>0$ on $\partial^+B_\delta^+$, where η denotes the inward unit normal vector. In the second we used that $\phi_\epsilon=0$ on ∂M.

For the second term in the right hand side of (3-10), an integration by parts plus a change of variables gives

$$\int_{B_\delta^+}|\nabla\phi_\epsilon|^2dx\le-\epsilon^4\int_{B^+_{\epsilon^{-1}\delta}}(\Delta\phi)\phi dy+C\epsilon^{n-2}\delta^{2-n},$$

Here we have used that $\int_{\partial' B_\delta^+}(\partial\phi_\epsilon/\partial x_n)\phi_\epsilon d\bar x=0$; the term $C\epsilon^{n-2}\delta^{2-n}$ comes from the integral over $\partial^+B_\delta^+$.

Claim. *The function ϕ satisfies*

$$\begin{aligned}\Delta\phi(y)=2AR_{ninj}y_iy_j&((1+y_n)^2+|\bar y|^2)^{-n/2}\\ &-4nAR_{ninj}y_iy_jy_n((1+y_n)^2+|\bar y|^2)^{-(n+2)/2}\\ &\qquad-6nAR_{ninj}y_iy_jy_n^2((1+y_n)^2+|\bar y|^2)^{-(n+2)/2}.\end{aligned}$$

Proof. We set $Z(y)=((1+y_n)^2+|\bar{y}|^2)$. Since $R_{nn}=0$,

$$\begin{aligned}
&\Delta(R_{ninj}y_iy_jy_n^2Z^{-n/2})\\
&\quad=\Delta(R_{ninj}y_iy_jy_n^2)Z^{-n/2}+R_{ninj}y_iy_jy_n^2\Delta(Z^{-n/2})\\
&\qquad+2\partial_k(R_{ninj}y_iy_jy_n^2)\partial_k(Z^{-n/2})+2\partial_n(R_{ninj}y_iy_jy_n^2)\partial_n(Z^{-n/2})\\
&\quad=2R_{ninj}y_iy_jZ^{-n/2}+2nR_{ninj}y_iy_jy_n^2Z^{-(n+2)/2}\\
&\qquad-4nR_{ninj}y_iy_jy_n^2Z^{-(n+2)/2}-4nR_{ninj}y_iy_jy_n(y_n+1)Z^{-(n+2)/2}\\
&\quad=2R_{ninj}y_iy_jZ^{-n/2}-6nR_{ninj}y_iy_jy_n^2Z^{-(n+2)/2}\\
&\qquad-4nR_{ninj}y_iy_jy_nZ^{-(n+2)/2}. \qquad\square
\end{aligned}$$

Using this claim, we get

$$\begin{aligned}
\int_{B^+_{\delta\epsilon^{-1}}}(\Delta\phi)\phi dy=&\,2A^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n}R_{ninj}R_{nknl}y_iy_jy_ky_ly_n^2dy\\
&-4nA^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n-1}R_{ninj}R_{nknl}y_iy_jy_ky_ly_n^3dy\\
&-6nA^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n-1}R_{ninj}R_{nknl}y_iy_jy_ky_ly_n^4dy.
\end{aligned}$$

Since $\Delta^2(R_{ninj}R_{nknl}y_iy_jy_ky_l)=16(R_{ninj})^2$,

$$\int_{S_r^{n-2}}R_{ninj}R_{nknl}y_iy_jy_ky_ld\sigma_r=\frac{2\sigma_{n-2}}{(n+1)(n-1)}r^{n+2}(R_{ninj})^2.$$

Thus

$$\begin{aligned}
\int_{B^+_{\delta\epsilon^{-1}}}(\Delta\phi)\phi dy=&\,\frac{4}{(n+1)(n-1)}A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy\\
&-\frac{8n}{(n+1)(n-1)}A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\\
&-\frac{12n}{(n+1)(n-1)}A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{B^+_\delta}|\nabla\phi_\epsilon|^2dx\leq&-\frac{4}{(n+1)(n-1)}\epsilon^4A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy\\
&+\frac{8n}{(n+1)(n-1)}\epsilon^4A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy
\end{aligned}$$

$$+\frac{12n}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy + C\epsilon^{n-2}\delta^{2-n}. \qquad \square$$

Lemma 3.2. *We have*

$$\begin{aligned}
&\int_{B^+_\delta}(g^{ij}-\delta^{ij})\partial_i(U_\epsilon+\phi_\epsilon)\partial_j(U_\epsilon+\phi_\epsilon)dx \\
&\quad= \frac{(n-2)^2}{(n+1)(n-1)}\epsilon^4 R_{ninj;ij}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy \\
&\qquad+\frac{(n-2)^2}{2(n-1)}\epsilon^4 (R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^n}dy \\
&\qquad-\frac{4n(n-2)}{(n+1)(n-1)}\epsilon^4 A(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy+E_1,
\end{aligned}$$

where

$$E_1=\begin{cases} O(\epsilon^4\delta^{-4}) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8.\end{cases}$$

Proof. Observe that

$$\begin{aligned}
\text{(3-11)}\quad &\int_{B^+_\delta}(g^{ij}-\delta^{ij})\partial_i(U_\epsilon+\phi_\epsilon)\partial_j(U_\epsilon+\phi_\epsilon)dx \\
&= \int_{B^+_\delta}(g^{ij}-\delta^{ij})\partial_i U_\epsilon\partial_j U_\epsilon dx + 2\int_{B^+_\delta}(g^{ij}-\delta^{ij})\partial_i U_\epsilon\partial_j\phi_\epsilon dx \\
&\qquad+\int_{B^+_\delta}(g^{ij}-\delta^{ij})\partial_i\phi_\epsilon\partial_j\phi_\epsilon dx.
\end{aligned}$$

We will handle separately the three terms in the right side of this. The first is

$$\begin{aligned}
&\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j U_\epsilon(x)dx \\
&\quad= \int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i U(y)\partial_j U(y)dy \\
&\quad= (n-2)^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n}(g^{ij}-\delta^{ij})(\epsilon y)y_i y_j dy.
\end{aligned}$$

Hence, using Lemma A.1 we obtain

$$\begin{aligned}\int_{B^+_\delta}&(g^{ij}-\delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j U_\epsilon(x)dx\\ &=\frac{(n-2)^2}{(n+1)(n-1)}\epsilon^4 R_{ninj;ij}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy\\ &\quad+\frac{(n-2)^2}{2(n-1)}\epsilon^4(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^n}dy+E_1',\end{aligned}$$

where

$$E_1'=\begin{cases}O(\epsilon^4\delta) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8.\end{cases}$$

The second term is

$$\begin{aligned}2\int_{B^+_\delta}&(g^{ij}-\delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j\phi_\epsilon(x)dx\\ &=-2\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i\partial_j U_\epsilon(x)\phi_\epsilon(x)dx\\ &\quad-2\int_{B^+_\delta}(\partial_i g^{ij})(x)\partial_j U_\epsilon(x)\phi_\epsilon(x)dx+O(\epsilon^{n-2}\delta^{2-n})\\ &=-2\epsilon^2\int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i\partial_j U(y)\phi(y)dy\\ &\quad-2\epsilon^3\int_{B^+_{\delta\epsilon^{-1}}}(\partial_i g^{ij})(\epsilon y)\partial_j U(y)\phi(y)dy+O(\epsilon^{n-2}\delta^{2-n}).\end{aligned}\tag{3-12}$$

But

$$\begin{aligned}-2\epsilon^2&\int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i\partial_j U(y)\phi(y)dy\\ &=-2(n-2)\epsilon^2 A\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n-1}(g^{ij}-\delta^{ij})(\epsilon y)\\ &\qquad\cdot\big(ny_iy_j-((1+y_n)^2+|\bar{y}|^2)\delta_{ij}\big)R_{nknl}y_ky_ly_n^2dy\\ &=-\frac{4n(n-2)}{(n+1)(n-1)}\epsilon^4 A(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy+E_2',\end{aligned}$$

where

$$E_2'=\begin{cases}O(\epsilon^4\delta) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8\end{cases}$$

and in the last equality, we used Lemma A.2 and the fact that Lemma 2.3, together with parts (i), (ii) and (iii) of Lemma 2.4, implies

$$\int_{S_r^{n-2}} (g^{ij} - \delta^{ij})(\epsilon y)\delta_{ij} R_{nknl} y_k y_l d\sigma_r(y) = \int_{S_r^{n-2}} O(\epsilon^4 |y|^4) R_{nknl} y_k y_l d\sigma_r(y).$$

We also have, by Lemma 2.3 and Lemma 2.4(i),

$$-2\epsilon^3 \int_{B^+_{\delta\epsilon^{-1}}} (\partial_i g^{ij})(\epsilon y)\partial_j U(y)\phi(y)dy = E_3' = \begin{cases} O(\epsilon^4\delta) & \text{if } n = 6, \\ O(\epsilon^5 \log(\delta\epsilon^{-1})) & \text{if } n = 7, \\ O(\epsilon^5) & \text{if } n \geq 8. \end{cases}$$

Hence

$$\begin{aligned} 2\int_{B^+_\delta} &(g^{ij} - \delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j \phi_\epsilon(x)dx \\ &= E_2' + E_3' - \frac{4n(n-2)}{(n+1)(n-1)}\epsilon^4 A(R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4 |\bar{y}|^4}{((1+y_n)^2 + |\bar{y}|^2)^{n+1}} dy. \end{aligned}$$

Finally, the third term in the right hand side of (3-11) is written as

$$\begin{aligned} \int_{B^+_\delta} (g^{ij} - \delta^{ij})(x)\partial_i \phi_\epsilon(x)\partial_j \phi_\epsilon(x)dx &= \epsilon^4 \int_{B^+_{\delta\epsilon^{-1}}} (g^{ij} - \delta^{ij})(\epsilon y)\partial_i \phi(y)\partial_j \phi(y)dy \\ &= \begin{cases} O(\epsilon^4\delta) & \text{if } n = 6, \\ O(\epsilon^5 \log(\delta\epsilon^{-1})) & \text{if } n = 7, \\ O(\epsilon^5) & \text{if } n \geq 8. \end{cases} \end{aligned}$$

The result now follows if we choose ϵ small enough that $\log(\delta\epsilon^{-1}) > \delta^{2-n}$. □

Lemma 3.3. *We have*

$$\begin{aligned} \frac{n-2}{4(n-1)} \int_{B^+_\delta} R_g (U_\epsilon + \phi_\epsilon)^2 dx = \\ &\frac{n-2}{8(n-1)}\epsilon^4 R_{;nn} \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2}{((1+y_n)^2 + |\bar{y}|^2)^{n-2}} dy \\ &- \frac{n-2}{24(n-1)^2}\epsilon^4 (\overline{W}_{ijkl})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{|\bar{y}|^2}{((1+y_n)^2 + |\bar{y}|^2)^{n-2}} dy + E_2, \end{aligned}$$

where

$$E_2 = \begin{cases} O(\epsilon^4\delta) & \text{if } n = 6, \\ O(\epsilon^5 \log(\delta\epsilon^{-1})) & \text{if } n = 7, \\ O(\epsilon^5) & \text{if } n \geq 8. \end{cases}$$

Proof. We first observe that

$$\int_{B_\delta^+} R_g(U_\epsilon+\phi_\epsilon)^2 dx = \int_{B_\delta^+} R_g U_\epsilon^2 dx + 2\int_{B_\delta^+} R_g U_\epsilon \phi_\epsilon dx + \int_{B_\delta^+} R_g \phi_\epsilon^2 dx. \tag{3-13}$$

We will handle each term in the right hand side of (3-13) separately. Using Lemma A.3, we see that the first is

$$\begin{aligned}\int_{B_\delta^+} R_g(x)U_\epsilon(x)^2 dx &= \epsilon^2 \int_{B^+_{\delta\epsilon^{-1}}} R_g(\epsilon y)U^2(y)dy \\ &= \tfrac{1}{2}\epsilon^4 R_{;nn}\int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy + E_4' \qquad (3\text{-}14)\\ &\quad - \frac{1}{12(n-1)}\epsilon^4(\overline{W}_{ijkl})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy,\end{aligned}$$

where

$$E_4' = \begin{cases} O(\epsilon^4\delta) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8.\end{cases}$$

By Lemma 2.4(ix), the second term is

$$\begin{aligned}2\int_{B_\delta^+} R_g(x)U_\epsilon(x)\phi_\epsilon(x)dx &= 2\epsilon^4\int_{B^+_{\delta\epsilon^{-1}}} R_g(\epsilon y)U(y)\phi(y)dy\\ &= \begin{cases} O(\epsilon^4\delta) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8\end{cases}\end{aligned}$$

and the last term is

$$\int_{B_\delta^+} R_g\phi_\epsilon^2 dx = \begin{cases} O(\epsilon^4\delta) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8.\end{cases} \qquad \square$$

Proof of Theorem 1.3. It follows from Lemmas 3.1, 3.2 and 3.3 and the identities (3-7), (3-8) and (3-9) that

$$\begin{aligned}E_M(\psi) \leq Q(B^n,\partial B^n)\Big(\int_{\partial M}\psi^{2(n-1)/(n-2)}\Big)^{(n-2)/(n-1)} + E \qquad (3\text{-}15)\\ - \epsilon^4\frac{4A^2}{(n+1)(n-1)}(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy\\ + \epsilon^4\frac{(n-2)^2}{(n+1)(n-1)}R_{ninj;ij}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^4\frac{8nA^2}{(n+1)(n-1)}(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\\
&+\epsilon^4\frac{12nA^2}{(n+1)(n-1)}(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\\
&-\epsilon^4\frac{4n(n-2)A}{(n+1)(n-1)}(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\\
&+\epsilon^4\frac{(n-2)^2}{2(n-1)}(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n}}dy\\
&+\epsilon^4\frac{n-2}{8(n-1)}R_{;nn}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy\\
&-\epsilon^4\frac{n-2}{48(n-1)^2}(\overline{W}_{ijkl})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy.
\end{aligned}
$$

where

$$
E=\begin{cases} O(\epsilon^4\delta^{-4}) & \text{if } n=6,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7,\\ O(\epsilon^5) & \text{if } n\geq 8.\end{cases}
$$

We divide the rest of the proof into two cases.

The case $n=7,8$. Set $I=\int_0^\infty r^n/(r^2+1)^n\,dr$. We will apply the change of variables $\bar{z}=(1+y_n)^{-1}\bar{y}$ and Lemmas B.1 and B.2 to compare the different integrals in the expansion (3-15).

These integrals are

$$
\begin{aligned}
I_1&=\int_{\mathbb{R}^n_+}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy_n d\bar{y}=\int_0^\infty y_n^2(1+y_n)^{3-n}dy_n\int_{\mathbb{R}^{n-1}}\frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^n}d\bar{z}\\
&=\frac{2(n+1)\sigma_{n-2}I}{(n-3)(n-4)(n-5)(n-6)},\\
I_2&=\int_{\mathbb{R}^n_+}\frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy_n d\bar{y}=\int_0^\infty y_n^3(1+y_n)^{1-n}dy_n\int_{\mathbb{R}^{n-1}}\frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^{n+1}}d\bar{z}\\
&=\frac{3(n+1)\sigma_{n-2}I}{n(n-2)(n-3)(n-4)(n-5)},\\
I_3&=\int_{\mathbb{R}^n_+}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy_n d\bar{y}=\int_0^\infty y_n^4(1+y_n)^{1-n}dy_n\int_{\mathbb{R}^{n-1}}\frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^{n+1}}d\bar{z}\\
&=\frac{12(n+1)\sigma_{n-2}I}{n(n-2)(n-3)(n-4)(n-5)(n-6)},
\end{aligned}
$$

$$I_4 = \int_{\mathbb{R}^n_+} \frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^n} dy_n d\bar{y} = \int_0^\infty y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{|\bar{z}|^2}{(1+|\bar{z}|^2)^n} d\bar{z}$$
$$= \frac{24\sigma_{n-2}I}{(n-2)(n-3)(n-4)(n-5)(n-6)},$$
$$I_5 = \int_{\mathbb{R}^n_+} \frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}} dy_n d\bar{y} = \int_0^\infty y_n^2(1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|\bar{z}|^2)^{n-2}} d\bar{z}$$
$$= \frac{8(n-2)\sigma_{n-2}I}{(n-3)(n-4)(n-5)(n-6)}.$$

Thus,

$$\begin{aligned} E_M(\psi) \le\ & Q(B^n, \partial B^n)\Big(\int_{\partial M} \psi^{2(n-1)/(n-2)}\Big)^{(n-2)/(n-1)} + E' \\ & + \epsilon^4\Big(-\frac{4A^2}{(n+1)(n-1)} I_1 + \frac{8nA^2}{(n+1)(n-1)} I_2 + \frac{(n-2)^2}{2(n-1)} I_4\Big)(R_{ninj})^2 \\ & + \epsilon^4\Big(\frac{12nA^2}{(n+1)(n-1)} - \frac{4n(n-2)A}{(n+1)(n-1)}\Big) I_3 \cdot (R_{ninj})^2 \\ & + \epsilon^4 \frac{(n-2)^2}{(n+1)(n-1)} I_1 \cdot R_{ninj;ij} + \epsilon^4 \frac{n-2}{8(n-1)} I_5 \cdot R_{;nn} \\ & - \epsilon^4 \frac{n-2}{48(n-1)^2} (\overline{W}_{ijkl})^2 \int_{\mathbb{R}^n_+} \frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}} dy. \end{aligned} \tag{3-16}$$

where

$$E' = \begin{cases} O(\epsilon^5 \log(\delta\epsilon^{-1})) & \text{if } n = 7, \\ O(\epsilon^5) & \text{if } n = 8. \end{cases}$$

Using Lemma 2.4(xi) and substituting the expressions obtained for $I_1, \dots, I_5$ in the expansion (3-16), the coefficients of $R_{ninj;ij}$ and $R_{;nn}$ cancel out and we obtain

$$\begin{aligned} E_M(\psi) & \\ \le\ & Q(B^n, \partial B^n)\Big(\int_{\partial M} \psi^{2(n-1)/(n-2)}\Big)^{(n-2)/(n-1)} + E' \\ & + \epsilon^4 \sigma_{n-2} I \cdot \gamma\big(16(n+1)A^2 - 48(n-2)A + 2(8-n)(n-2)^2\big)(R_{ninj})^2 \\ & - \epsilon^4 \frac{n-2}{48(n-1)^2} (\overline{W}_{ijkl})^2 \int_{\mathbb{R}^n_+} \frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}} dy, \end{aligned} \tag{3-17}$$

where

$$1/\gamma = (n-1)(n-2)(n-3)(n-4)(n-5)(n-6).$$

Choosing $A = 1$, the term $16(n+1)A^2 - 48(n-2)A + 2(8-n)(n-2)^2$ in the expansion (3-17) is -62 for $n = 7$ and -144 for $n = 8$. Thus, for small ϵ, since $W_{abcd}(x_0) \neq 0$, the expansion (3-17) together with Lemma 2.5 implies that, in dimensions 7 and 8,

$$E_M(\psi) < Q(B^n, \partial B^n)\Big(\int_{\partial M} \psi^{2(n-1)/(n-2)}\Big)^{(n-2)/(n-1)}.$$

The case $n = 6$. We will again apply the change of variables $\bar{z} = (1 + y_n)^{-1}\bar{y}$ and Lemma B.1 to compare the different integrals in the expansion (3-15). In the next estimates we are always assuming $n = 6$.

In this case, the first integral is

$$\begin{aligned} I_{1,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n} dy_n d\bar{y} \\ &= \int_{B^+_{\delta\epsilon^{-1}} \cap \{y_n \le \delta/2\epsilon\}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n} dy_n d\bar{y} + O(1) \\ &= \int_{\mathbb{R}^n_+ \cap \{y_n \le \delta/2\epsilon\}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n} dy_n d\bar{y} + O(1). \end{aligned}$$

Hence

$$\begin{aligned} I_{1,\delta/\epsilon} &= \int_0^{\delta/2\epsilon} y_n^2(1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^n} d\bar{z} + O(1) \\ &= \log(\delta\epsilon^{-1}) \frac{n+1}{n-3} \sigma_{n-2} I + O(1). \end{aligned}$$

The second integral is

$$I_{2,\delta/\epsilon} = \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy_n d\bar{y} = O(1).$$

The others are similar to $I_{1,\delta/\epsilon}$:

$$\begin{aligned} I_{3,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy_n d\bar{y} \\ &= \int_0^{\delta/2\epsilon} y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^{n+1}} d\bar{z} + O(1) \\ &= \log(\delta\epsilon^{-1}) \frac{n+1}{2n} \sigma_{n-2} I + O(1), \\ I_{4,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n}} dy_n d\bar{y} \\ &= \int_0^{\delta/2\epsilon} y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{|\bar{z}|^2}{(1+|\bar{z}|^2)^{n}} d\bar{z} \\ &= \log(\delta\epsilon^{-1}) \sigma_{n-2} I + O(1), \end{aligned}$$

$$
\begin{aligned}
I_{5,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}} dy_n d\bar{y} \\
&= \int_0^{\delta/2\epsilon} y_n^2 (1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|\bar{z}|^2)^{n-2}} d\bar{z} + O(1) \\
&= \log(\delta\epsilon^{-1}) \frac{4(n-2)}{n-3} \sigma_{n-2} I + O(1), \\
I_{6,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}} dy_n d\bar{y} \\
&= \int_0^{\delta/2\epsilon} (1+y_n)^{5-n} dy_n \int_{\mathbb{R}^{n-1}} \frac{|\bar{z}|^2}{(1+|\bar{z}|^2)^{n-2}} d\bar{z} + O(1) \\
&= \log(\delta\epsilon^{-1}) \frac{4(n-1)(n-2)}{(n-3)(n-5)} \sigma_{n-2} I + O(1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E_M(\psi) \leq{}& Q(B^n, \partial B^n) \Big(\int_{\partial M} \psi^{2(n-1)/(n-2)} \Big)^{(n-2)/(n-1)} + O(\epsilon^4 \delta^{-4}) \\
&+ \epsilon^4 \Big(-\frac{4A^2}{(n+1)(n-1)} I_{1,\delta/\epsilon} + \frac{(n-2)^2}{2(n-1)} I_{4,\delta/\epsilon} \Big) (R_{ninj})^2 \\
&+ \epsilon^4 \Big(\frac{12nA^2}{(n+1)(n-1)} - \frac{4n(n-2)A}{(n+1)(n-1)} \Big) I_{3,\delta/\epsilon} \cdot (R_{ninj})^2 \\
&+ \epsilon^4 \frac{(n-2)^2}{(n+1)(n-1)} I_{1,\delta/\epsilon} \cdot R_{ninj;ij} + \epsilon^4 \frac{n-2}{8(n-1)} I_{5,\delta/\epsilon} \cdot R_{;nn} \\
&- \epsilon^4 \frac{n-2}{48(n-1)^2} I_{6,\delta/\epsilon} \cdot (\overline{W}_{ijkl})^2.
\end{aligned} \tag{3-18}
$$

Using Lemma 2.4(xi) and substituting the expressions obtained for $I_{1,\delta/\epsilon}$ through $I_{6,\delta/\epsilon}$ in expansion (3-18), we find the coefficients of $R_{ninj;ij}$ and $R_{;nn}$ cancel out and obtain

$$
\begin{aligned}
E_M(\psi) \leq{}& Q(B^n, \partial B^n) \Big(\int_{\partial M} \psi^{2(n-1)/(n-2)} \Big)^{(n-2)/(n-1)} + O(\epsilon^4 \delta^{-4}) \\
&+ \epsilon^4 \log(\delta\epsilon^{-1}) \sigma_{n-2} I \\
&\cdot \Big(\frac{6(n-3)-4}{(n-1)(n-3)} A^2 - \frac{2(n-2)}{n-1} A + \frac{(n-2)^2(n-5)}{2(n-1)(n-3)} \Big) (R_{ninj})^2 \\
&- \epsilon^4 \log(\delta\epsilon^{-1}) \sigma_{n-2} I \frac{(n-2)^2}{12(n-1)(n-3)(n-5)} (\overline{W}_{ijkl})^2.
\end{aligned} \tag{3-19}
$$

Choosing $A = 1$, the term to the left of $(R_{ninj})^2$ in the expansion (3-19) is $-2/15$ for $n = 6$. Thus, for small ϵ, since $W_{abcd}(x_0) \neq 0$, the expansion (3-19) together with Lemma 2.5 implies that, in dimension $n = 6$

$$E_M(\psi) < Q(B^n, \partial B^n)\Big(\int_{\partial M} \psi^{2(n-1)/(n-2)}\Big)^{(n-2)/(n-1)}. \qquad \square$$

Appendix A.

In this section, we will use the results of Section 2 to calculate some integrals used in the computations of Section 3. As before, all curvature coefficients are evaluated at $x_0 \in \partial M$, around which we center conformal Fermi coordinates.

Lemma A.1. *We have*

$$\int_{S_r^{n-2}} (g^{ij} - \delta^{ij})(\epsilon y) y_i y_j d\sigma_r(y) = \sigma_{n-2}\epsilon^4 \frac{y_n^2 r^{n+2}}{(n+1)(n-1)} R_{ninj;ij} + \sigma_{n-2}\epsilon^4 \frac{y_n^4 r^n}{2(n-1)} (R_{ninj})^2 + O(\epsilon^5 |(r, y_n)|^{n+5}).$$

Proof. By Lemma 2.3,

$$\begin{aligned}\int_{S_r^{n-2}} &(g^{ij} - \delta^{ij})(\epsilon y) y_i y_j d\sigma_r(y) \\ &= \epsilon^4 \int_{S_r^{n-2}} \tfrac{1}{2} R_{ninj;kl} y_i y_j y_k y_l d\sigma_r(y) + O(\epsilon^5 |(r, y_n)|^{n+5}) \\ &\quad + \epsilon^4 y_n^2 \int_{S_r^{n-2}} (\tfrac{1}{12} R_{ninj;nn} + \tfrac{2}{3} R_{nins} R_{nsnj}) y_i y_j d\sigma_r(y).\end{aligned}$$

Then we just use the identity (3-4), Lemma 2.4 and the fact that

$$\Delta^2 (R_{ninj;kl} y_i y_j y_k y_l) = 16 R_{ninj;ij}. \qquad \square$$

Lemma A.2. *We have*

$$\int_{S_r^{n-2}} (g^{ij} - \delta^{ij})(\epsilon y) R_{nknl} y_i y_j y_k y_l d\sigma_r(y) = \frac{2}{(n+1)(n-1)} \sigma_{n-2}\epsilon^2 y_n^2 r^{n+2} (R_{ninj})^2 + O(\epsilon^5 |(r, y_n)|^{n+5})$$

Proof. As in Lemma A.1, the result follows from

$$\int_{S_r^{n-2}} (g^{ij} - \delta^{ij})(\epsilon y) R_{nknl} y_i y_j y_k y_l d\sigma_r(y) = \epsilon^2 y_n^2 \int_{S_r^{n-2}} R_{ninj} R_{nknl} y_i y_j y_k y_l d\sigma_r(y) + O(\epsilon^5 |(r, y_n)|^{n+5}),$$

the fact that $\Delta^2 (R_{ninj} R_{nknl} y_i y_j y_k y_l) = 16 (R_{ninj})^2$, and the identity (3-4). $\square$

Lemma A.3. *We have*

$$\int_{S_r^{n-2}} R_g(\epsilon y)d\sigma_r(y) = \sigma_{n-2}\epsilon^2\Big(\tfrac{1}{2}y_n^2 r^{n-2}R_{;nn} - \frac{1}{12(n-1)}r^n(\overline{W}_{ijkl})^2\Big) + O(\epsilon^3|(r,y_n)|^{n+1}).$$

Proof. As in Lemma A.1, the result follows from Lemma 2.4(x), (3-4), and

$$\int_{S_r^{n-2}} R_g(\epsilon y)d\sigma_r(y) = \epsilon^2 y_n^2\int_{S_r^{n-2}}\tfrac{1}{2}R_{;nn}d\sigma_r(y) + \epsilon^2\int_{S_r^{n-2}}\tfrac{1}{2}R_{;ij}y_iy_jd\sigma_r(y) + O(\epsilon^3|(r,y_n)|^{n+1}). \qquad \square$$

Appendix B.

Finally, we prove results used in the computations of Section 3.

Lemma B.1. *We have*

(a) $\displaystyle\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m}{\alpha+1}\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m+1}}$ *for* $\alpha+1<2m$;

(b) $\displaystyle\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m}{2m-\alpha-1}\int_0^\infty \frac{s^{\alpha}ds}{(1+s^2)^{m+1}}$ *for* $\alpha+1<2m$;

(c) $\displaystyle\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m-\alpha-3}{\alpha+1}\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m}}$ *for* $\alpha+3<2m$.

Proof. Integrating by parts, we get

$$\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m+1}} = \int_0^\infty s^{\alpha+1}\frac{s\,ds}{(1+s^2)^{m+1}} = \frac{\alpha+1}{2m}\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m}$$

for $\alpha+1<2m$, which proves item (a).

Item (b) follows from (a) and from

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \int_0^\infty \frac{s^\alpha(1+s^2)}{(1+s^2)^{m+1}}ds = \int_0^\infty \frac{s^\alpha ds}{(1+s^2)^{m+1}} + \int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m+1}}.$$

To prove (c), observe that by (a),

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^{m-1}} = \frac{2(m-1)}{\alpha+1}\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^m},$$

for $\alpha+3<2m$. But by (b), we have

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^{m-1}} = \frac{2(m-1)}{2(m-1)-\alpha-1}\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m}. \qquad \square$$

Lemma B.2. $\displaystyle\int_0^\infty \frac{t^k}{(1+t)^m}dt = \frac{k!}{(m-1)(m-2)\cdots(m-1-k)}$ *for* $m>k+1$.

Proof. The proof follows straightforwardly from treating the integral

$$\int_0^\infty \frac{t^{k-1}}{(1+t)^{m-1}} dt$$

two ways: first, by integrating it by parts, and second by borrowing a factor of $(1+t)$ from its denominator and dividing that integral into two. □

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SÉRGIO DE MOURA ALMARAZ
INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL FLUMINENSE (UFF)
RUA MÁRIO SANTOS BRAGA S/N
24020-140 NITERÓI, RJ
BRAZIL

almaraz@impa.br

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