Pacific Journal of Mathematics

EXPRESSIONS FOR CATALAN KRONECKER PRODUCTS

ANDREW A. H. BROWN,
STEPHANIE VAN WILLIGENBURG AND MIKE ZABROCKI

Volume 248 No. 1 November 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 248, No. 1, 2010

EXPRESSIONS FOR CATALAN KRONECKER PRODUCTS

ANDREW A. H. BROWN,
STEPHANIE VAN WILLIGENBURG AND MIKE ZABROCKI

We give some elementary manifestly positive formulas for the Kronecker products $s_{(d,d)} * s_{(d+k,d-k)}$. These formulas demonstrate some fundamental properties of the Kronecker coefficients, and we use them to deduce a number of enumerative and combinatorial results.

1. Introduction

A classic open problem in algebraic combinatorics is to explain in a manifestly positive combinatorial formula the Kronecker product (or internal product) of two Schur functions. This product is the Frobenius image of the internal tensor product of two irreducible symmetric group modules, or it is alternatively the characters of the induced tensor product of general linear group modules. Although for representation theoretic reasons this expression clearly has nonnegative coefficients when expanded in terms of Schur functions, it remains an open problem to provide a satisfying positive combinatorial or algebraic formula for the Kronecker product of two Schur functions.

Many attempts have been made to capture some aspect of these coefficients, for example, special cases [Bessenrodt and Behns 2004; Bessenrodt and Kleshchev 1999; Remmel and Whitehead 1994; Rosas 2001], asymptotics [Ballantine and Orellana 2005; 2005], stability [Vallejo 1999], the complexity of calculating them [Bürgisser and Ikenmeyer 2008], and conditions under which they are nonzero [Dvir 1993]. Given that the Littlewood–Richardson rule and many successors have so compactly and cleanly been able to describe the external product of two Schur functions, it seems as though some new ideas for capturing the combinatorics of Kronecker coefficients are needed.

The results in this paper were inspired by the symmetric function identity of [Garsia et al. 2009, Theorem I.1] for the Kronecker product $s_{(d,d)} * s_{(d,d)}$ of two

---

MSC2000: primary 05E05; secondary 05E10, 20C30, 20C33.

Keywords: Catalan number, Kronecker coefficient, Kronecker product, Schur function, symmetric function.

The authors were supported in part by the National Sciences and Engineering Research Council of Canada.

Schur functions. More precisely, for a subset of partitions X of $2d$, if we set $[X]=\sum_{\lambda\in X} s_\lambda$, called a *rug*, then

$$s_{(d,d)} * s_{(d,d)} = [4 \text{ parts all even or all odd}]. \tag{1-1}$$

This identity differs significantly from most published results on the Kronecker product: instead of giving a combinatorial interpretation or algorithm, it clearly states exactly which partitions have nonzero coefficients and that all of the coefficients are 0 or 1.

This computation arose in the solution to a mathematical physics problem related to resolving the interference of 4 qubits [Wallach 2005] because the sum of these coefficients is equal to the dimensions of polynomial invariants of four copies of $\mathrm{SL}(2,\mathbb{C})$ acting on $\mathbb{C}^8$. Understanding the Kronecker product of $s_{(d,d)}$ with s_λ for partitions λ with 4 parts that are all even or all odd would be useful for calculating the dimensions of invariants of six copies of $\mathrm{SL}(2,\mathbb{C})$ acting on $\mathbb{C}^{12}$, which is a measure of entanglement of 6 qubits. Ultimately we would like to be able to compute

$$CT_{a_1,a_2,\ldots,a_k}\left(\frac{\prod_{i=1}^{k}(1-a_i^2)}{\prod_{S\subseteq\{1,2,\ldots k\}}(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j)}\right)=\sum_{d\geq 0}\langle s_{(d,d)}^{*k}, s_{(2d)}\rangle q^{2d}$$

(see [Garsia et al. 2010, formulas I.4 and I.5] and [Luque and Thibon 2006] for a discussion), where CT represents the operation of taking the constant term and equations of this type are a motivation for understanding the Kronecker product with $s_{(d,d)}$ as completely as possible.

Using $s_{(d,d)} * s_{(d,d)}$ as our inspiration, we show in Corollary 3.6 that

$$s_{(d,d)} * s_{(d+1,d-1)} = [2 \text{ even parts and } 2 \text{ odd parts}]$$

and with a similar computation we also derive that

$$s_{(d,d)} * s_{(d+2,d-2)} = [4 \text{ parts, all even or all odd, but not 3 the same}] \\ + [4 \text{ distinct parts}].$$

Interestingly, this formula says that all of the coefficients in the Schur expansion of $s_{(d,d)} * s_{(d+2,d-2)}$ are either 0, 1 or 2, and that the coefficient is 2 for those Schur functions indexed by partitions with 4 distinct parts that are all even or all odd.

These and further examples suggest that the Schur function expansion of $s_{(d,d)} * s_{(d+k,d-k)}$ has the pattern of a boolean lattice of subsets, in that it can be written as the sum of $\lfloor k/2\rfloor + 1$ intersecting sums of Schur functions each with coefficient 1.

The main result of this article is Theorem 3.1, which states that

$$s_{(d,d)} * s_{(d+k,d-k)} = \sum_{i=0}^{k}[(k+i, k, i)P] + \sum_{i=1}^{k}[(k+i+1, k+1, i)P], \tag{1-2}$$

where we have used the notation γP to represent the set of partitions λ of $2d$ of length less than or equal to 4 such that $\lambda - \gamma$ (representing a vector difference) is a partition with 4 even parts or 4 odd parts. The disjoint sets of this sum can be grouped so that the sum is of only $\lfloor k/2 \rfloor + 1$ terms, which shows that the coefficients always lie in the range 0 through $\lfloor k/2 \rfloor + 1$. The most interesting aspect of this formula is that we see the lattice of subsets arising in a natural and unexpected way in a representation-theoretic setting. This is potentially part of a more general result and the hope is that this particular model will shed light on a general formula for the Kronecker product of two Schur functions, but our main motivation for computing these is to develop computational tools.

There are yet further motivations for restricting our attention to the Kronecker product of $s_{(d,d)}$ with another Schur function. The Schur functions indexed by the partition (d, d) are a special family for several combinatorial reasons, and so there is reason to believe that their behavior will be more accessible than the general case of the Kronecker product of two Schur functions. More precisely, Schur functions indexed by partitions with two parts are notable because they are the difference of two homogeneous symmetric functions, for which a combinatorial formula for the Kronecker product is known. In addition, a partition (d, d) is rectangular and hence falls under a second category of Schur functions that are often combinatorially more straightforward to manipulate than the general case.

From the hook length formula it follows that the number of standard tableau of shape (d, d) is equal to the Catalan number

$$C_d = \frac{1}{d+1}\binom{2d}{d}.$$

Therefore, from the perspective of S_{2d} representations, we may, by taking the Kronecker product with the Schur function $s_{(d,d)}$ and the Frobenius image of a module, explain how the tensor of a representation with a particular irreducible module of dimension C_d decomposes.

The paper is structured as follows. In Section 2, we review pertinent background information including necessary symmetric function notation and lemmas needed for later computation. In Section 3, we consider a generalization of formula (1-1) to an expression for $s_{(d,d)} * s_{(d+k,d-k)}$. Finally, Section 4 is devoted to combinatorial and symmetric function consequences of our results. In particular, we are able to give generating functions for the partitions that have a particular coefficient in the expression $s_{(d,d)} * s_{(d+k,d-k)}$.

2. Background

Partitions. A *partition* λ of an integer n, denoted $\lambda \vdash n$, is a finite sequence of nonnegative integers $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ whose values sum to n. The *height* or *length* of the partition, denoted $\ell(\lambda)$, is the maximum index for which $\lambda_{\ell(\lambda)} > 0$. We call the λ_i *parts* or *rows* of the partition, and if λ_i appears n_i times we abbreviate this subsequence to $\lambda_i^{n_i}$. With this in mind if $\lambda = (k^{n_k}, (k-1)^{n_{k-1}}, \ldots, 1^{n_1})$, then we define $z_\lambda = 1^{n_1} 1!\, 2^{n_2} 2! \cdots k^{n_k} k!$. The 0 parts of the partition are optional and we will assume that $\lambda_i = 0$ for $i > \ell(\lambda)$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of n. To form the *diagram* associated with λ, place a cell at each point (j, i) in matrix notation, where $1 \leq i \leq \lambda_j$ and $1 \leq j \leq \ell$. We say λ has *transpose* λ' if the diagram for λ' is given by the points (i, j) for which $1 \leq i \leq \lambda_j$ and $1 \leq j \leq \ell$.

Symmetric functions and the Kronecker product. The ring of symmetric functions is the graded subring of $\mathbb{Q}[x_1, x_2, \ldots]$ given by $\Lambda := \mathbb{Q}[p_1, p_2, \ldots]$, where $p_i = x_1^i + x_2^i + \cdots$ are the elementary power sum symmetric functions. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we define $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$. The interested reader should consult a reference such as [Macdonald 1995] for more details of the structure of this ring. It is straightforward to see that $\{p_\lambda\}_{\lambda \vdash n \geq 0}$ forms a basis for Λ. This basis is orthogonal to itself with respect to the scalar product on Λ:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$

However, our focus for this paper will be the basis $\{s_\lambda\}_{\lambda \vdash n \geq 0}$ of Λ known as the basis of *Schur functions*, which is the orthonormal basis under the scalar product:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

The *Kronecker product* is the operation

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda} \tag{2-1}$$

on symmetric functions that in terms of the Schur functions becomes

$$s_\mu * s_\nu = \sum_{\lambda \vdash |\mu|} C_{\mu\nu\lambda} s_\lambda.$$

The *Kronecker coefficients* $C_{\mu\nu\lambda}$ encode the inner tensor product of symmetric group representations. That is, if we denote the irreducible S_n module indexed by a partition λ by M^λ and if $M^\mu \otimes M^\nu$ represents the tensor of two modules with the diagonal action, then the module decomposes as

$$M^\mu \otimes M^\nu \simeq \bigoplus_\lambda (M^\lambda)^{\oplus C_{\mu\nu\lambda}}.$$

The Kronecker coefficients also encode the decomposition of GL_{nm} polynomial representations to $\mathrm{GL}_n \otimes \mathrm{GL}_m$ representations

$$\mathrm{Res}^{\mathrm{GL}_{mn}}_{\mathrm{GL}_n \otimes \mathrm{GL}_m}(V^\lambda) \simeq \bigoplus_{\mu,\nu} (V^\mu \otimes V^\nu)^{\oplus C_{\mu\nu\lambda}}.$$

It easily follows from (2-1) and the linearity of the product that these coefficients satisfy the symmetries

$$C_{\mu\nu\lambda} = C_{\nu\mu\lambda} = C_{\mu\lambda\nu} = C_{\mu'\nu'\lambda} \quad \text{and} \quad C_{\lambda\mu(n)} = C_{\lambda\mu'(1^n)} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

which we will use extensively in what follows.

We will need some symmetric function identities. Recall that for a symmetric function f, the symmetric function operator $f^\perp$ (read "eff perp") is defined to be the operator dual to multiplication with respect to the scalar product. That is,

$$\langle f^\perp g, h\rangle = \langle g, f \cdot h\rangle. \tag{2-2}$$

The perp operator can also be defined linearly by $s_\lambda^\perp s_\mu = s_{\mu/\lambda} = \sum_{\nu \vdash |\mu|-|\lambda|} c^\mu_{\lambda\nu} s_\nu$, where the $c^\mu_{\lambda\nu}$ are the Littlewood–Richardson coefficients. We will make use of the following well-known relation, which connects the internal and external products:

$$\langle s_\lambda f, g * h\rangle = \sum_{\mu,\nu \vdash |\lambda|} C_{\lambda\mu\nu} \langle f, (s_\mu^\perp g) * (s_\nu^\perp h)\rangle. \tag{2-3}$$

This formula follows because of the relationship between the internal coproduct and the scalar product. It may also be seen to hold on the power sum basis since

$$\Big\langle p_\lambda p_\mu, \frac{p_\gamma}{z_\gamma} * \frac{p_\nu}{z_\nu}\Big\rangle = \Big\langle p_\mu, \Big(p_\lambda^\perp \frac{p_\gamma}{z_\gamma}\Big) * \Big(p_\lambda^\perp \frac{p_\nu}{z_\nu}\Big)\Big\rangle, \tag{2-4}$$

which holds because both the sides are 1 if and only if $\gamma = \nu$ and are both equal to the union of the parts of λ and μ. This given, (2-3) follows by linearity.

From these two identities and the Littlewood–Richardson rule, we derive this:

Lemma 2.1. *If $\ell(\lambda) > 4$, then $C_{(d,d)(a,b)\lambda} = 0$. Otherwise it satisfies the following recurrences. If $\ell(\lambda) = 4$, then*

$$\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle = \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{\lambda-(1^4)}\rangle. \tag{2-5}$$

If $\ell(\lambda) = 3$, then

$$\begin{aligned}\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle &= \langle s_{(d-1,d-1)} * s_{(d+k-1,d-k-1)}, s_{(1)} s_{\lambda-(1^3)}\rangle \\ &\quad - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)}\rangle.\end{aligned} \tag{2-6}$$

If $\ell(\lambda)=2$, *then*

$$\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle = \begin{cases} 1 & \text{if } k \equiv \lambda_2 \ (\text{mod } 2) \text{ and } \lambda_2 \geq k, \\ 0 & \text{otherwise.} \end{cases} \tag{2-7}$$

Proof. For (2-7), see [Remmel and Whitehead 1994, Theorem 3.3] and [Rosas 2001, Corollary 1]. The result also appears in [Garsia et al. 2010, Theorem 2.2].

Regev [1980] proves that the maximum height of an indexing partition of the terms of $s_{(d,d)} * s_{(d+k,d-k)}$ will be at most 4; hence we can conclude that if $\ell(\lambda) > 4$, then $C_{(d,d)(a,b)\lambda} = 0$.

Assume that $\ell(\lambda) = 4$. By the Pieri rule, $s_{(1^4)} s_{\lambda-(1^4)}$ is equal to s_λ plus terms of the form s_γ, where $\ell(\gamma) > 4$. As a consequence,

$$\begin{aligned}\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle &= \langle s_{(d,d)} * s_{(d+k,d-k)}, s_{(1^4)} s_{\lambda-(1^4)}\rangle \\ &= \sum_{\mu\vdash 4} \langle (s_\mu^\perp s_{(d,d)}) * (s_{\mu'}^\perp s_{(d+k,d-k)}), s_{\lambda-(1^4)}\rangle.\end{aligned}$$

Every term in this sum is 0 unless both μ and μ' have length no more than 2. The only term for which this is true is $\mu = (2,2)$, and $s_{(2,2)}^\perp(s_{(a,b)}) = s_{(a-2,b-2)}$; hence (2-5) holds.

Assume that $\ell(\lambda) = 3$. Although there are cases to check, it follows again from the Pieri rule that $s_{(1^3)} s_{\lambda-(1^3)} - s_{(1^4)} s_{(1)}^\perp s_{\lambda-(1^3)}$ is equal to s_λ plus terms involving s_γ, where $\ell(\gamma) > 4$. Therefore,

$$\begin{aligned}\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle &= \langle s_{(d,d)} * s_{(d+k,d-k)}, s_{(1^3)} s_{\lambda-(1^3)} - s_{(1^4)} s_{(1)}^\perp s_{\lambda-(1^3)}\rangle \\ &= \sum_{\mu\vdash 3} \langle (s_\mu^\perp s_{(d,d)}) * (s_{\mu'}^\perp s_{(d+k,d-k)}), s_{\lambda-(1^3)}\rangle \\ &\qquad - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)}\rangle.\end{aligned}$$

Again, in the sum the only terms that are not equal to 0 are those such that the length of both μ and μ' less than or equal to 2, and in this case the only such partition is $\mu = (2,1)$. By the Littlewood–Richardson rule, $s_{(2,1)}^\perp(s_{(a,b)}) = s_{(1)}^\perp(s_{(a-1,b-1)})$; hence this last expression is equal to

$$\begin{aligned}&\langle (s_{(1)}^\perp s_{(d-1,d-1)}) * (s_{(1)}^\perp s_{(d+k-1,d-k-1)}), s_{\lambda-(1^3)}\rangle \\ &\qquad - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)}\rangle \\ &\qquad\qquad = \langle s_{(d-1,d-1)} * s_{(d+k-1,d-k-1)}), s_{(1)} s_{\lambda-(1^3)}\rangle \\ &\qquad\qquad\qquad - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)}\rangle. \qquad \square\end{aligned}$$

To express our main results we will use the characteristic of a boolean-valued proposition. If R is a proposition, then we denote the propositional characteristic (or indicator) function of R by

$$(\!(R)\!) = \begin{cases} 1 & \text{if proposition } R \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

3. The Kronecker product $s_{(d,d)} * s_{(d+k,d-k)}$

Let P indicate the set of partitions with four even parts or four odd parts, and let $\overline{P}$ be the set of partitions with 2 even parts and 2 odd parts (that is, the complement of P in the set of partitions of even size with at most 4 parts).

We let γP represent the set of partitions λ of $2d$ (the value of d will be implicit in the left hand side of the expression) such that $\lambda - \gamma \in P$. We also let $(\gamma \uplus \alpha)P = \gamma P \cup \alpha P$. In the cases we consider, the partitions in γP and αP are disjoint.

Theorem 3.1. *For λ a partition of $2d$,*

$$\begin{aligned} &\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle \\ &\qquad = \sum_{i=0}^{k} (\!(\lambda \in (k+i, k, i)P)\!) + \sum_{i=1}^{k} (\!(\lambda \in (k+i+1, k+1, i)P)\!). \end{aligned} \tag{3-1}$$

In the notation we have introduced, Theorem 3.1 can easily be restated:

Corollary 3.2. *For $k \geq 0$, if k is odd, then*

$$\begin{aligned} &s_{(d,d)} * s_{(d+k,d-k)} \\ &\quad = [((k,k) \uplus (k+1,k,1) \uplus (k+2,k+1,1))P] \\ &\qquad + \sum_{i=1}^{(k-1)/2} [((k+2i,k,2i) \uplus (k+2i+1,k+1,2i) \\ &\qquad\qquad \uplus (k+2i+1,k,2i+1) \uplus (k+2i+2,k+1,2i+1))P], \end{aligned}$$

and if k is even, then

$$\begin{aligned} s_{(d,d)} * s_{(d+k,d-k)} = [(k,k)P] + \sum_{i=1}^{k/2} [((k+2i-1,k,2i-1) \uplus (k+2i,k+1,2i-1) \\ \uplus (k+2i,k,2i) \uplus (k+2i+1,k+1,2i))P]. \end{aligned}$$

As a consequence, the coefficients of $s_{(d,d)} * s_{(d+k,d-k)}$ will all be less than or equal to $\lfloor k/2 \rfloor + 1$.

Remark 3.3. The upper bound on the coefficients that appear in the expressions $s_{(d,d)} * s_{(d+k,d-k)}$ are sharp in that for sufficiently large d, there is a coefficient that will be equal to $\lfloor k/2 \rfloor + 1$.

Remark 3.4. This regrouping of the rugs is not unique but is useful because there are partitions that will fall in the intersection of each of these sets. This set of rugs is also not unique in that it is possible to describe other collections of sets of partitions (for example, see (3-4)).

Remark 3.5. This case has been considered in more generality by Remmel and Whitehead [1994] and Rosas [Rosas 2001] in their study of the Kronecker product of two Schur functions indexed by two-row shapes. In relation to these results, our first (lengthy) derivation of a similar result started with the formula of Rosas, but was later replaced by the current simpler identity. Meanwhile, R. Orellana, in a personal communication, has informed us that the journal version of the earlier paper has an error in it. Our computer implementation of [Remmel and Whitehead 1994, Theorem 2.1] does not agree, for example, with direct computation for $(h,k)=(4,1)$ and $(l,m)=(3,2)$ and $\nu=(3,1,1)$. Consequently, we wanted an independent proof of Theorem 3.1, and as a result our derivation is an elementary proof that uses induction involving symmetric function identities.

We note that for $k=2$, the expression stated in the introduction does not exactly follow this decomposition, but it does follow from some manipulation.

Corollary 3.6. *For $d \geq 1$,*

$$s_{(d,d)} * s_{(d,d)} = [P], \tag{3-2}$$

$$s_{(d,d)} * s_{(d+1,d-1)} = [\overline{P}] \tag{3-3}$$

and for $d \geq 2$,

$$s_{(d,d)} * s_{(d+2,d-2)} = [P \cap \textit{at most two equal parts}] + [\textit{distinct partitions}]. \tag{3-4}$$

Proof. Note that (3-2) is just a restatement of Corollary 3.2 in the case that $k=0$ and is [Garsia et al. 2009, Theorem I.1].

First, by Theorem 3.1 we note that

$$C_{(d,d)(d+1,d-1)\lambda} = (\!(\lambda \in (1,1)P)\!) + (\!(\lambda \in (2,1,1)P)\!) + (\!(\lambda \in (3,2,1)P)\!).$$

If $\lambda \in P$, then $\lambda-(1,1)$, $\lambda-(2,1,1)$, $\lambda-(3,2,1)$ are all not in P, so each of the terms in that expression are 0. If λ is a partition with two even parts and two odd parts (that is, $\lambda \in \overline{P}$), then either $\lambda_1 \equiv \lambda_2$ and $\lambda_3 \equiv \lambda_4 \pmod 2$ or $\lambda_1 \equiv \lambda_3$ and $\lambda_2 \equiv \lambda_4 \pmod 2$ or $\lambda_2 \equiv \lambda_3$ and $\lambda_1 \equiv \lambda_4 \pmod 2$. In each of these three cases, exactly one of the expressions $(\!(\lambda \in (1,1)P)\!)$, $(\!(\lambda \in (2,1,1)P)\!)$ or $(\!(\lambda \in (3,2,1)P)\!)$ will be 1 and the other two will be zero. Therefore,

$$\sum_{\lambda \vdash 2d} C_{(d,d)(d+1,d-1)\lambda} s_\lambda = [\overline{P}].$$

We also have by Theorem 3.1

$$\begin{aligned}C_{(d,d)(d+2,d-2)\lambda} = (\!(\lambda \in (2,2)P)\!) + (\!(\lambda \in (4,2,2)P)\!) - (\!(\lambda \in (6,4,2)P)\!) \\ + (\!(\lambda \in (3,2,1)P)\!) + (\!(\lambda \in (4,3,1)P)\!) \\ + (\!(\lambda \in (5,3,2)P)\!) + (\!(\lambda \in (6,4,2)P)\!).\end{aligned}$$

Any distinct partition in P is also in $(6,4,2)P$. Every distinct partition in $\overline{P}$ will have two odd parts and two even parts and will be in one of $(3,2,1)P$, $(4,3,1)P$ or $(5,3,2)P$, depending on which of λ_2, λ_1 or λ_3 is equal to $\lambda_4 \pmod 2$, respectively. Therefore, we have

$$[\text{distinct partitions}] = [((3,2,1) \uplus (4,3,1) \uplus (5,3,2) \uplus (6,4,2))P]. \tag{3-5}$$

If $\lambda \in (2,2)P \cap (4,2,2)P$, then $\lambda_2 \geq \lambda_3 + 2$ because $\lambda \in (2,2)P$, and $\lambda_1 \geq \lambda_2 + 2$ and $\lambda_3 \geq \lambda_4 + 2$ because $\lambda \in (4,2,2)P$, so $\lambda \in (6,4,2)P$. Conversely, one verifies that in fact $(2,2)P \cap (4,2,2)P = (6,4,2)P$; hence

$$[(2,2)P \cup (4,2,2)P] = [(2,2)P] + [(4,2,2)P] - [(6,4,2)P].$$

If $\lambda \in P$ does not have three equal parts, then either $\lambda_2 > \lambda_3$, or $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_4$. Therefore, $\lambda \in (2,2)P \cup (4,2,2)P$ and hence $(2,2)P \cup (4,2,2)P = P \cap (\text{at most two equal parts})$. □

Proof of Theorem 3.1. Our proof proceeds by induction on the value of d and uses the Lemma 2.1. We will consider two base cases because (2-6) and (2-5) give recurrences for two smaller values of d. The exception for this is of course that λ is a partition of length 2 since it is easily verified that the two sides of (3-1) agree: the only term on the right hand side of the equation that can be nonzero is $(\!(\lambda \in (k,k)P)\!)$.

When $d = k$, the left hand side of (3-1) is $\langle s_{(k,k)} * s_{(2k)}, s_\lambda \rangle$, which is 1 if $\lambda = (k,k)$ and 0 otherwise. On the right hand side of (3-1) we have $(\!(\lambda \in (k,k)P)\!)$ is 1 if and only if $\lambda = (k,k)$ and all other terms are 0, and hence the two expressions agree.

If $d = k+1$, then $s_{(k+1,k+1)} * s_{(2k-1,1)} = s_{(k+1,k,1)} + s_{(k+2,k)}$. The only partitions λ of $2k+2$ such that the indicator functions on the right hand side of (3-1) can be satisfied are $(\!(\lambda \in (k,k)P)\!)$ when $\lambda = (k+2,k)$ and $(\!(\lambda \in (k+1,k,1)P)\!)$ when $\lambda = (k+1,k,1)$. All others must be 0 because the partitions that are subtracted off are larger than $2k+2$.

Now assume that (3-1) holds for all values strictly smaller than d. If $\ell(\lambda) = 4$, then $\lambda - \gamma \in P$ if and only if $\lambda - \gamma - (1^4) \in P$ for all partitions γ of length less than or equal to 3, so

$$\begin{aligned}\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle &= \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{\lambda-(1^4)}\rangle \\ &= \sum_{i=0}^{k} ⟪\lambda-(1^4) \in (k+i, k, i)P⟫ + \sum_{i=1}^{k} ⟪\lambda-(1^4) \in (k+i+1, k+1, i)P⟫ \\ &= \sum_{i=0}^{k} ⟪\lambda \in (k+i, k, i)P⟫ + \sum_{i=1}^{k} ⟪\lambda \in (k+i+1, k+1, i)P⟫.\end{aligned}$$

So we can now assume that $\ell(\lambda) = 3$. By (2-6) we need to consider the coefficients of the form $\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\mu\rangle$, where s_μ appears in the expansion of $s_{(1)} s_{\lambda-(1^3)}$ or $s_{(1)}^{\perp} s_{\lambda-(1^3)}$. If λ has three distinct parts and $\lambda_3 \geq 2$ then $\mu = \lambda - \delta$, where

$$\delta \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1, -1), (2, 1, 1), (1, 2, 1), (1, 1, 2)\}$$

and we can assume by induction that these expand into terms having the form $\pm⟪\lambda - \delta - \gamma \in P⟫$, where γ is a partition. However, if λ is not distinct or $\lambda_3 = 1$, then for some δ in the set, $\lambda - \delta$ will not be a partition and $⟪\lambda - \delta \in \gamma P⟫$ will be 0, and we can add these terms to our formulas so that we can treat the argument uniformly and not have to consider different possible λ.

One obvious reduction we can make to treat the expressions more uniformly is to note that $⟪\lambda - (1, 1, 1, -1) \in \gamma P⟫ = ⟪\lambda - (2, 2, 2) \in \gamma P⟫$.

Let

$$\begin{aligned}C_1 &= \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (2, 2, 2)\}, \\ C_2 &= \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}.\end{aligned}$$

By the induction hypothesis and (2-6), we have

$$\begin{aligned}&\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda\rangle \\ &\quad= \sum_{\delta\in C_1}\Big(\sum_{i=0}^{k} ⟪\lambda-\delta \in (k+i, k, i)P⟫ + \sum_{i=1}^{k} ⟪\lambda-\delta \in (k+i+1, k+1, i)P⟫\Big) \\ &\qquad - \sum_{\delta\in C_2}\Big(\sum_{i=0}^{k} ⟪\lambda-\delta \in (k+i, k, i)P⟫ + \sum_{i=1}^{k} ⟪\lambda-\delta \in (k+i+1, k+1, i)P⟫\Big).\end{aligned}$$

We notice that

$$\begin{aligned}\lambda - (2, 2, 2) - (k+i, k, i) &= \lambda - (1, 1, 2) - (k+i+1, k+1, i), \\ \lambda - (2, 1, 1) - (k+i, k, i) &= \lambda - (1, 0, 1) - (k+i+1, k+1, i), \\ \lambda - (1, 2, 1) - (k+i, k, i) &= \lambda - (0, 1, 1) - (k+i+1, k+1, i),\end{aligned}$$

so the corresponding terms always cancel. With this reduction, we are left with the terms

$$\sum_{\delta\in C_3}\sum_{i=0}^{k}(\!(\lambda-\delta \in (k+i, k, i)P)\!) + \sum_{\delta\in C_4}\sum_{i=1}^{k}(\!(\lambda-\delta \in (k+i+1, k+1, i)P)\!)$$
$$-\sum_{i=0}^{k}(\!(\lambda-(1, 1, 2) \in (k+i, k, i)P)\!) - \sum_{\delta\in C_5}\sum_{i=1}^{k}(\!(\lambda-\delta \in (k+i+1, k+1, i)P)\!)$$
$$-(\!(\lambda-(1, 2, 1) \in (k, k)P)\!) - (\!(\lambda-(2, 1, 1) \in (k, k)P)\!) + (\!(\lambda-(2, 2, 2) \in (k, k)P)\!),$$

where

$$C_3 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\},$$
$$C_4 = \{(1, 1, 0), (2, 2, 2)\},$$
$$C_5 = \{(2, 1, 1), (1, 2, 1)\}.$$

Next we notice that

$$\lambda-(1, 1, 2)-(k+i, k, i) = \lambda-(0, 1, 1)-(k+i+1, k, i+1),$$
$$\lambda-(2, 1, 1)-(k+i+1, k+1, i) = \lambda-(1, 1, 0)-(k+i+2, k+1, i+1),$$
$$\lambda-(2, 2, 2)-(k+i+1, k+1, i) = \lambda-(1, 2, 1)-(k+i+2, k+1, i+1).$$

Then by canceling these terms and joining the compositions that are being subtracted off in the sum, these sums reduce to the expression

$$\sum_{i=0}^{k}(\!(\lambda \in (k+i+1, k, i+1)P)\!) + \sum_{i=0}^{k}(\!(\lambda \in (k+i+1, k+1, i)P)\!)$$
$$+(\!(\lambda \in (k, k+1, 1)P)\!) + (\!(\lambda \in (k+3, k+2, 1)P)\!)$$
$$+(\!(\lambda \in (2k+3, k+3, k+2)P)\!) + (\!(\lambda \in (k+2, k+2, 2)P)\!)$$
$$-(\!(\lambda \in (2k+1, k+1, k+2)P)\!) - (\!(\lambda \in (2k+3, k+2, k+1)P)\!)$$
$$-(\!(\lambda \in (k+3, k+3, 2)P)\!) - (\!(\lambda \in (k+1, k+2, 1)P)\!)$$
$$-(\!(\lambda \in (k+2, k+1, 1)P)\!).$$

Since $\ell(\lambda) = 3$, if $\lambda-(a, b) \in P$, then $\lambda_1-a \geq \lambda_2-b \geq \lambda_3 \geq 2$, which is true if and only if $\lambda_1-a-2 \geq \lambda_2-b-2 \geq \lambda_3-2 \geq 0$. In particular,

$$(\!(\lambda \in (k+2, k+2, 2)P)\!) = (\!(\lambda \in (k, k)P)\!),$$
$$(\!(\lambda \in (k+3, k+3, 2)P)\!) = (\!(\lambda \in (k+1, k+1)P)\!).$$

By verifying a few conditions it is easy to check that $\lambda \in (r, s, s+1)P$ if and only if $\lambda \in (r+2, s+2, s+1)P$, and similarly $\lambda \in (s, s+1, r)P$ if and only if $\lambda \in (s+2, s+1, r)P$. With this relationship, we have these equivalences between the terms appearing in the expression above:

$$\begin{aligned}
(\!(\lambda \in (k, k+1, 1)P)\!) &= (\!(\lambda \in (k+2, k+1, 1)P)\!),\\
(\!(\lambda \in (k+1, k+2, 1)P)\!) &= (\!(\lambda \in (k+3, k+2, 1)P)\!),\\
(\!(\lambda \in (2k+1, k+1, k+2)P)\!) &= (\!(\lambda \in (2k+3, k+3, k+2)P)\!),\\
(\!(\lambda \in (2k+1, k, k+1)P)\!) &= (\!(\lambda \in (2k+3, k+2, k+1)P)\!).
\end{aligned}$$

After we cancel these terms the expression reduces to

$$\sum_{i=0}^{k-1}(\!(\lambda \in (k+i+1, k, i+1)P)\!) + \sum_{i=1}^{k}(\!(\lambda \in (k+i+1, k+1, i)P)\!) + (\!(\lambda \in (k, k)P)\!).$$

This concludes the proof by induction on d since we know the identity holds for each partition λ of length 2, 3 or 4. □

4. Combinatorial and symmetric function consequences

4.1. ***Tableaux of height less than or equal to* 4.** Since every partition of even size and of length less than or equal to 4 lies in either P or $\overline{P}$, Corollary 3.6 has the following corollary.

Corollary 4.1. *For d a positive integer*,

$$\sum_{\lambda \vdash 2d, \ell(\lambda)\le 4} s_\lambda = s_{(d,d)} * (s_{(d,d)} + s_{(d+1,d-1)}),$$

$$\sum_{\lambda \vdash 2d-1, \ell(\lambda)\le 4} s_\lambda = s_{(d,d-1)} * s_{(d,d-1)}.$$

Proof. For the sum over partitions of $2d$, (1-1) (or (3-1)) says that $s_{(d,d)} * s_{(d,d)}$ is the sum over all s_λ with $\lambda \vdash 2d$ having four even parts or four odd parts, and Corollary 3.6 says that $s_{(d,d)}*s_{(d+1,d-1)}$ is the sum over s_λ with $\lambda \vdash 2d$ where λ does not have four odd parts or four even parts. Hence, $s_{(d,d)} * s_{(d,d)} + s_{(d,d)} * s_{(d+1,d-1)}$ is the sum over s_λ where λ runs over all partitions with less than or equal to 4 parts.

For the other identity, we use (2-3) to derive

$$\langle s_{(d,d-1)} * s_{(d,d-1)}, s_\lambda\rangle = \langle s_{(d,d)} * s_{(d,d)}, s_{(1)} s_\lambda\rangle.$$

If λ is a partition of $2d-1$, then the expression is 0 if $\ell(\lambda) > 4$; if $\ell(\lambda) \le 4$ then $s_{(1)}s_\lambda$ is a sum of at most 5 terms, $s_{(\lambda_1+1,\lambda_2,\lambda_3,\lambda_4)}$, $s_{(\lambda_1,\lambda_2+1,\lambda_3,\lambda_4)}$, $s_{(\lambda_1,\lambda_2,\lambda_3+1,\lambda_4)}$, $s_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4+1)}$ and $s_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,1)}$. Because λ has exactly 3 or 1 terms that are odd, exactly one of these will have 4 even parts or 4 odd parts. □

Regev [1981], Gouyou-Beauchamps [1989], Gessel [1990] and later Bergeron, Krob, Favreau, and Gascon [Bergeron et al. 1995; Bergeron and Gascon 2000] studied tableaux of bounded height. For $y_k(n)$ equal to the number of standard tableaux of height less than or equal to k, Gessel [1990] remarks that expressions for $y_k(n)$ exist for $k = 2, 3, 4, 5$ that are simpler than the k-fold sum that one would expect to see. This is perhaps because all four of these cases have more general formulas in terms of characters.

In the case $k = 4$, Corollary 4.1 is a statement about characters indexed by partitions of bounded height. In particular, if those characters are evaluated at the identity we see a previously known result:

Corollary 4.2. $y_4(n) = C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil}$.

This follows from the hook length formula that says the number of standard tableaux of shape (d, d) is C_d, the number of standard tableaux of $(d, d-1)$ is C_d, and the number of standard tableaux of shape that are either of shape (d, d) or $(d+1, d-1)$ is C_{d+1}.

Interestingly, some known expressions for $y_2(n)$, $y_3(n)$ and $y_5(n)$ can also be explained in terms of symmetric function identities using the Pieri rule.

4.2. ***Generating functions for partitions with coefficient r in $s_{(d,d)} * s_{(d+k,d-k)}$.*** An easy consequence of Theorem 3.1 is a generating function formula for the sum of the coefficients of the expressions $s_{(d,d)} * s_{(d+k,d-k)}$.

Corollary 4.3. *For a fixed $k \geq 1$,*

$$G_k(q) := \sum_{d \geq k} \Big(\sum_{\lambda \vdash 2d} \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle \Big) q^d = \frac{q^k + q^{k+1} + q^{2k+1} + \sum_{r=k+2}^{2k} 2q^r}{(1-q)(1-q^2)^2(1-q^3)}.$$

Remark 4.4. Corollary 4.3 only holds for $k > 0$. In the case that $k = 0$, the numerator of the expression above is different and we have from Corollary 3.6

$$\begin{aligned} G_0(q) = \sum_{d \geq 0} \Big(\sum_{\lambda \vdash 2d} \langle s_{(d,d)} * s_{(d,d)}, s_\lambda \rangle \Big) q^d &= \sum_{d \geq 0} \langle s_{(d,d)} * s_{(d,d)}, s_{(d,d)} * s_{(d,d)} \rangle q^d \\ &= \sum_{d \geq 0} \langle [P], [P] \rangle q^d = \sum_{d \geq 0} |P| q^d \\ &= \frac{1}{(1-q)(1-q^2)^2(1-q^3)}. \end{aligned}$$

This last equality is the formula given in [Garsia et al. 2010, Corollary 1.2] and it follows because the generating function for partitions with even parts and length less than or equal to 4 is $1/((1-q)(1-q^2)(1-q^3)(1-q^4))$, and the generating function for the partitions of $2d$ with odd parts of length less than or equal to 4 is $q^2/((1-q)(1-q^2)(1-q^3)(1-q^4))$. The sum of these two generating functions is equal to a generating function for the number of nonzero coefficients of $s_{(d,d)} * s_{(d,d)}$.

Proof. Theorem 3.1 gives a formula for $s_{(d,d)} * s_{(d+k,d-k)}$ in terms of rugs of the form $[\gamma P]$. We can calculate that for each of the rugs that appears in this expression

$$\begin{aligned}\sum_{d\geq k}\Big(\sum_{\lambda\vdash 2d}\langle[\gamma P], s_\lambda\rangle\Big)q^d &= \sum_{d\geq k}\Big(\sum_{\lambda\vdash 2d}\langle[P], s_{\lambda-\gamma}\rangle\Big)q^d\\ &= \sum_{d\geq k}\Big(\sum_{\mu\vdash(2d-|\gamma|)}\langle[P], s_\mu\rangle\Big)q^{d+|\gamma|/2}\\ &= \frac{q^{|\gamma|/2}}{(1-q)(1-q^2)^2(1-q^3)}.\end{aligned}$$

Now $s_{(d,d)} * s_{(d+k,d-k)}$ is the sum of rugs of the form $[\gamma P]$ with γ equal to (k, k), $(k+1, k, 1)$ and $(2k+1, k+1, k)$ each contribute a term to the numerator of the form q^k, q^{k+1} and q^{2k+1}, respectively. The rugs in which γ is equal to $(k+i+1, k+1, i)$ and $(k+i+1, k, i+1)$ for $1 \leq i \leq k-1$ each contribute a term $2q^{k+i+1}$ to the numerator. □

In order to compute other generating functions of Kronecker products, we need the following very surprising theorem. It says that the partitions such that the $C_{(d,d)(d+k,d-k)\lambda}$ are of coefficient $r > 1$ are exactly those partitions $\gamma+(6, 4, 2)$ for which $C_{(d-6,d-6)(d-6+(k-2),d-6-(k-2))\gamma}$ is equal to $r-1$.

Theorem 4.5. *For $k \geq 2$, assume that $C_{(d,d)(d+k,d-k)\lambda} > 0$. Then*

$$C_{(d+6,d+6)(d+k+8,d-k+4)(\lambda+(6,4,2))} = C_{(d,d)(d+k,d-k)\lambda} + 1.$$

Lemma 4.6. *For γ a partition with $\ell(\gamma) \leq 4$, $\lambda \in \gamma P$ if and only if $\lambda + (6, 4, 2)$ is in both $(\gamma_1+2, \gamma_2+2, \gamma_3, \gamma_4)P$ and $(\gamma_1+4, \gamma_2+2, \gamma_3+2, \gamma_4)P$.*

Proof. If $\lambda \in \gamma P$, then $\lambda-\gamma$ is a partition with four even parts or four odd parts. Hence, both $\lambda-\gamma+(2, 2) = (\lambda+(6, 4, 2))-(\gamma+(4, 2, 2))$ and $\lambda-\gamma+(4, 2, 2) = (\lambda+(6, 4, 2))-(\gamma+(2, 2))$ are elements of P.

Conversely, assume that $\lambda+(6, 4, 2)$ is as stated. Then $\lambda_1+6-(\gamma_1+4) \geq \lambda_2+4-(\gamma_2+2)$, $\lambda_2+4-(\gamma_2+2) \geq \lambda_3+2-\gamma_3$, and $\lambda_3+2-(\gamma_3+2) \geq \lambda_4-\gamma_4 \geq 0$. This implies that $\lambda-\gamma$ is a partition and since $\lambda-\gamma+(2, 2)$ has four even or four odd parts; then so does $\lambda-\gamma$ and hence $\lambda \in \gamma P$. □

Proof of Theorem 4.5. Consider the case where λ is a partition of $2d$ with $\lambda_2-k \equiv \lambda_4 \pmod 2$ since the case where $\lambda_2-k \not\equiv \lambda_4 \pmod 2$ is analogous and just uses different nonzero terms in the sum below. From Theorem 3.1, we have

$$C_{(d,d)(d+k,d-k)\lambda} = \sum_{i=0}^{k}(\!(\lambda \in (k+i, k, i)P)\!), \tag{4-1}$$

since the other terms are clearly zero in this case. If $\lambda_2-k \geq \lambda_3$, then the terms in this sum will be nonzero as long as $0 \leq i \leq \lambda_3-\lambda_4$ and $0 \leq k+i \leq \lambda_1-\lambda_2$

and $\lambda_3 - i \equiv \lambda_4 \pmod 2$. Consider the case where $\lambda_3 \equiv \lambda_4 \pmod 2$; then (4-1) is equal to $a+1$, where $a = \lfloor \min(\lambda_3 - \lambda_4, \lambda_1 - \lambda_2 - k, k)/2 \rfloor$ since the terms that are nonzero in this sum are $(\!(\lambda \in (k+2j, k, 2j)P)\!)$, where $0 \le j \le a$. By Lemma 4.6, these terms are true if and only if $(\!(\lambda + (6,4,2) \in (k+2+2j, k+2, 2j)P)\!)$ are true for all $0 \le j \le a+1$. But again by Theorem 3.1, in this case we also have

$$C_{(d+6,d+6)(d+k+8,d-k-4)(\lambda+(6,4,2))} = a+2 = C_{(d,d)(d+k,d-k)\lambda} + 1.$$

The case in which $\lambda_3 \equiv \lambda_4 + 1 \pmod 2$ is similar, but the terms of the form $(\!(\lambda \in (k+2j+1, k, 2j+1)P)\!)$ in (4-1) are nonzero if and only if the terms $(\!(\lambda + (6,4,2) \in (k+2+2j+1, k+2, 2j+1)P)\!)$ contribute to the expression for $C_{(d+6,d+6)(d+k+8,d-k-4)(\lambda+(6,4,2))}$ and there is exactly one more nonzero term; hence $C_{(d+6,d+6)(d+k+8,d-k-4)(\lambda+(6,4,2))} = C_{(d,d)(d+k,d-k)\lambda} + 1$. □

Now for computations it is useful to have a way of determining exactly the number of partitions of $2d$ that have a given coefficient. For integers $d, k, r > 0$, we let $L_{d,k,r}$ be the number of partitions λ of $2d$ with $\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle = r$. Theorem 3.1 has shown that $L_{d,k,r} = 0$ for $r > \lfloor k/2 \rfloor + 1$, and Theorem 4.5 says $L_{d,k,r} = L_{d-6,k-2,r-1} + 1$ for $r > 1$. These recurrences will allow us to completely determine the generating functions for the coefficients $L_{d,k,r}$. We set

$$L_{k,r}(q) = \sum_{d \ge 0} L_{d,k,r} q^d.$$

Corollary 4.7. *With the convention that $G_k(q) = 0$ for $k < 0$, we have $L_{k,r}(q) = 0$ for $r > \lfloor k/2 \rfloor + 1$, and*

$$L_{k,1}(q) = G_k(q) - 2q^6 G_{k-2}(q) + q^{12} G_{k-4}(q), \tag{4-2}$$

$$L_{k,r}(q) = q^{6r-6} L_{k-2r+2,1}(q). \tag{4-3}$$

Proof. Theorem 4.5 explains (4-3) because

$$\begin{aligned} L_{k,r}(q) &= \sum_{d \ge 0} \#\{\lambda : C_{(d,d)(d+k,d-k)\lambda} = r\} q^d \\ &= \sum_{d \ge 0} \#\{\lambda : C_{(d-6,d-6)(d+k-8,d-k+4)(\lambda+(6,4,2))} = r-1\} q^d \\ &= \sum_{d \ge 0} \#\{\lambda : C_{(d-6r+6,d-6r+6)(d+k-8r+8,d-k+4r-4)(\lambda+(6r-6,4r-4,2r-2))} = 1\} q^d \\ &= q^{6r-6} \sum_{d \ge 0} \#\{\lambda : C_{(d-6r+6,d-6r+6)(d+k-8r+8,d-k+4r-4)(\lambda+(6r-6,4r-4,2r-2))} = 1\} \\ &\qquad \cdot q^{d-6r+6} \\ &= q^{6r-6} L_{k-2r+2,1}(q). \end{aligned}$$

Now we also have by definition and Theorem 3.1 that

$$G_k(q) = \sum_{r=1}^{\lfloor k/2 \rfloor + 1} r L_{k,r}(q). \tag{4-4}$$

Hence, we can use this formula and (4-3) to define $L_{k,r}(q)$ recursively. It remains to show that the formula for $L_{k,1}(q)$ stated in (4-2) satisfies this formula, which we do by induction. Note that $L_{0,1}(q) = G_0(q)$ and $L_{1,1}(q) = G_1(q)$ and $L_{k,1}(q) = 0$ for $k < 0$. Then assuming that the formula holds for values smaller than $k > 1$, we have from (4-4)

$$\begin{aligned}
L_{k,1}(q) &= G_k(q) - \sum_{r=2}^{\lfloor k/2 \rfloor + 1} r L_{k,r}(q) \\
&= G_k(q) - \sum_{r \geq 2} r q^{6r-6} L_{k-2r+2,1}(q) \\
&= G_k(q) - \sum_{r \geq 2} r q^{6r-6} (G_{k-2r+2}(q) - 2q^6 G_{k-2r}(q) + q^{12} G_{k-2r-2}(q)) \\
&= G_k(q) - \sum_{r \geq 1} (r+1) q^{6r} G_{k-2r}(q) + \sum_{r \geq 2} 2r q^{6r} G_{k-2r}(q) \\
&\qquad\qquad - \sum_{r \geq 3} (r-1) q^{6r} G_{k-2r}(q) \\
&= G_k(q) - 2q^6 G_{k-2}(q) + q^{12} G_{k-4}(q).
\end{aligned}$$

Therefore, by induction (4-2) holds for all $k > 0$. □

References

[Ballantine and Orellana 2005] C. M. Ballantine and R. C. Orellana, "On the Kronecker product $s_{(n-p,p)} * s_\lambda$", *Electron. J. Combin.* **12** (2005), Research Paper 28, 26 pp. MR 2006g:05227 Zbl 1076.05082

[Ballentine and Orellana 2005] C. Ballentine and R. Orellana, "A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n,n-p)} * s_\lambda$", *Sém. Lothar. Combin.* **54A** (2005), Art. B54Af.

[Bergeron and Gascon 2000] F. Bergeron and F. Gascon, "Counting Young tableaux of bounded height", *J. Integer Seq.* **3**:1 (2000), Art. 00.1.7. MR 2001g:05102 Zbl 0953.05079

[Bergeron et al. 1995] F. Bergeron, L. Favreau, and D. Krob, "Conjectures on the enumeration of tableaux of bounded height", *Discrete Math.* **139** (1995), 463–468. MR 96f:05188 Zbl 0833.05079

[Bessenrodt and Behns 2004] C. Bessenrodt and C. Behns, "On the Durfee size of Kronecker products of characters of the symmetric group and its double covers", *J. Algebra* **280**:1 (2004), 132–144. MR 2005d:20020 Zbl 1073.20004

[Bessenrodt and Kleshchev 1999] C. Bessenrodt and A. Kleshchev, "On Kronecker products of complex representations of the symmetric and alternating groups", *Pacific J. Math.* **190**:2 (1999), 201–223. MR 2000i:20017 Zbl 1009.20013

[Bürgisser and Ikenmeyer 2008] P. Bürgisser and C. Ikenmeyer, "The complexity of computing Kronecker coefficients", pp. 357–368 in *20th Annual International Conference on Formal Power Series and Algebraic Combinatorics* (Valparaiso-Viña del Mar, Chile, 2008), Discrete Mathematics & Theoretical Computer Science, 2008.

[Dvir 1993] Y. Dvir, "On the Kronecker product of S_n characters", *J. Algebra* **154**:1 (1993), 125–140. MR 94a:20023 Zbl 0848.20006

[Garsia et al. 2009] A. Garsia, N. Wallach, G. Xin, and M. Zabrocki, "Hilbert series of invariants, constant terms and Kostka–Foulkes polynomials", *Discrete Math.* **309**:16 (2009), 5206–5230. MR 2548921 Zbl 05648061

[Garsia et al. 2010] A. Garsia, N. Wallach, G. Xin, and M. Zabrocki, "Kronecker coefficients via symmetric functions and constant term identities", preprint, 2010, available at http://garsia.math.yorku.ca/~zabrocki/papers/KronViaSym.pdf.

[Gessel 1990] I. M. Gessel, "Symmetric functions and P-recursiveness", *J. Combin. Theory Ser. A* **53**:2 (1990), 257–285. MR 91c:05190 Zbl 0704.05001

[Gouyou-Beauchamps 1989] D. Gouyou-Beauchamps, "Standard Young tableaux of height 4 and 5", *European J. Combin.* **10**:1 (1989), 69–82. MR 90b:05013 Zbl 0672.05012

[Luque and Thibon 2006] J.-G. Luque and J.-Y. Thibon, "Algebraic invariants of five qubits", *J. Phys. A* **39**:2 (2006), 371–377. MR 2006h:81048 Zbl 1084.81020

[Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, New York, 1995. MR 96h:05207 Zbl 0824.05059

[Regev 1980] A. Regev, "The Kronecker product of S_n-characters and an $A \otimes B$ theorem for Capelli identities", *J. Algebra* **66**:2 (1980), 505–510. MR 82f:16015 Zbl 0445.16012

[Regev 1981] A. Regev, "Asymptotic values for degrees associated with strips of Young diagrams", *Adv. in Math.* **41**:2 (1981), 115–136. MR 82h:20015 Zbl 0509.20009

[Remmel and Whitehead 1994] J. B. Remmel and T. Whitehead, "On the Kronecker product of Schur functions of two row shapes", *Bull. Belg. Math. Soc. Simon Stevin* **1**:5 (1994), 649–683. MR 96h:05211 Zbl 0827.20017

[Rosas 2001] M. H. Rosas, "The Kronecker product of Schur functions indexed by two-row shapes or hook shapes", *J. Algebraic Combin.* **14**:2 (2001), 153–173. MR 2003b:05154 Zbl 0999.05098

[Vallejo 1999] E. Vallejo, "Stability of Kronecker products of irreducible characters of the symmetric group", *Electron. J. Combin.* **6** (1999), R39. MR 2001b:20022 Zbl 0944.05099

[Wallach 2005] N. R. Wallach, "The Hilbert series of measures of entanglement for 4 qubits", *Acta Appl. Math.* **86**:1-2 (2005), 203–220. MR 2006c:81020 Zbl 1097.81023

Received August 18, 2009.

ANDREW A. H. BROWN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BC V6T 1Z2
CANADA
Current address:
DEPARTMENT OF COMBINATORICS AND OPTIMIZATION
UNIVERSITY OF WATERLOO
WATERLOO, ON N2L 3G1
CANADA
a29brown@services107.math.uwaterloo.ca

STEPHANIE VAN WILLIGENBURG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BC V6T 1Z2
CANADA

steph@math.ubc.ca

MIKE ZABROCKI
DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
TORONTO, ON M3J 1P3
CANADA

zabrocki@mathstat.yorku.ca
http://garsia.math.yorku.ca/∼zabrocki/

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2010 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 248 No. 1 November 2010

An existence theorem of conformal scalar-flat metrics on manifolds with boundary 1
SÉRGIO DE MOURA ALMARAZ

Parasurface groups 23
KHALID BOU-RABEE

Expressions for Catalan Kronecker products 31
ANDREW A. H. BROWN, STEPHANIE VAN WILLIGENBURG and MIKE ZABROCKI

Metric properties of higher-dimensional Thompson's groups 49
JOSÉ BURILLO and SEAN CLEARY

Solitary waves for the Hartree equation with a slowly varying potential 63
KIRIL DATCHEV and IVAN VENTURA

Uniquely presented finitely generated commutative monoids 91
PEDRO A. GARCÍA-SÁNCHEZ and IGNACIO OJEDA

The unitary dual of p-adic $\widetilde{\mathrm{Sp}(2)}$ 107
MARCELA HANZER and IVAN MATIĆ

A Casson–Lin type invariant for links 139
ERIC HARPER and NIKOLAI SAVELIEV

Semiquandles and flat virtual knots 155
ALLISON HENRICH and SAM NELSON

Infinitesimal rigidity of polyhedra with vertices in convex position 171
IVAN IZMESTIEV and JEAN-MARC SCHLENKER

Robust four-manifolds and robust embeddings 191
VYACHESLAV S. KRUSHKAL

On sections of genus two Lefschetz fibrations 203
SINEM ÇELIK ONARAN

Biharmonic hypersurfaces in Riemannian manifolds 217
YE-LIN OU

Singular fibers and 4-dimensional cobordism group 233
OSAMU SAEKI