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UNIQUELY PRESENTED FINITELY GENERATED COMMUTATIVE MONOIDS

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A finitely generated commutative monoid is uniquely presented if it has a unique minimal presentation. We give necessary and sufficient conditions for finitely generated, combinatorially finite, cancellative, commutative monoids to be uniquely presented. We use the concept of gluing to construct commutative monoids with this property. Finally, for some relevant families of numerical semigroups we describe the elements that are uniquely presented.

Introduction

Rédei [1965] proved that every finitely generated commutative monoid is finitely presented. Since then, the proof has been shortened drastically, and much progress has been made on the study and computation of minimal presentations of monoids, more specifically, of finitely generated subsemigroups of $\mathbb{N}^n$, known usually as affine semigroups; see for instance [Rosales 1997] and [Briales et al. 1998] or [Rosales and García-Sánchez 1999a, Chapter 9] and the references therein. For affine semigroups, the concepts of minimal presentations with respect to cardinality or set inclusion coincide, that is, any two minimal presentations have the same cardinality. This even occurs in a more general setting; see [Rosales et al. 1999].

Interest of the study of such kind of monoids and their presentations was partially motivated by their application in commutative algebra and algebraic geometry [Bruns and Herzog 1993, Chapter 6; Fulton 1993].

Recently, new applications of affine semigroups have been found in the so-called algebraic statistic. In this context, an interesting problem is to decide under which conditions such monoids have a unique minimal presentation. Roughly speaking, convenient algebraic techniques for the study of some statistical models seem to be

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more interesting for statisticians when a certain semigroup associated to the model is uniquely presented; see [Takemura and Aoki 2004].

Efforts to understand the problem of the uniqueness come from an algebraic setting and consist essentially in identifying particular minimal generators in a presentation as R-module of the semigroup algebra, where R is a polynomial ring over a field; see [Charalambous et al. 2007; Ojeda and Vigneron-Tenorio 2009]. So, whole families of uniquely presented monoids have not been determined (with the exception of some previously known cases [Ojeda 2008]) and techniques for constructing uniquely presented monoids have not been developed.

Here, we approach the uniqueness of the minimal presentations from a semigroup theoretic point of view. To begin, we recall the basic definitions and how to obtain minimal presentations of finitely generated, combinatorially finite, cancellative and commutative monoids (which include affine semigroups). In Section 2, we focus on the elements of the monoid whose factorizations yield these presentations, which we call Betti elements. Section 3 provides a necessary and sufficient condition for a monoid to be uniquely presented (Corollary 6). Some results in these sections may be also stated in combinatorial terms by using the simplicial complexes introduced by S. Eliahou in his unpublished PhD thesis (1983); see [Charalambous et al. 2007; Ojeda and Vigneron-Tenorio 2010].

In Section 4, we make extensive use of the gluing of affine semigroups, a concept defined by J. C. Rosales [1997] and used later by different authors to characterize complete intersection affine semigroup rings. In that section, given a gluing S of two affine semigroups S_1 and S_2, we show that S is uniquely presented if and only if S_1 and S_2 are uniquely presented and some extra natural condition holds where S_1 and S_2 are glued (Theorem 12). To reach this result, we need Theorem 10, which shows that the Betti elements of S are the union of the Betti elements of S_1, S_2 and the element in which S_1 and S_2 glue to produce S. We consider these two theorems to be our main results. Furthermore, Theorem 12 may be used to systematically produce uniquely presented monoids, as we show in Example 14.

Finally Section 5 identifies all uniquely presented monoids in some classical families of numerical semigroups (submonoids of $\mathbb{N}$ with finite complement in $\mathbb{N}$).

1. Preliminaries

We summarize some definitions, notations and results that will be useful later in the paper. See [Rosales and García-Sánchez 1999a] for further information.

Let S denote a *commutative monoid*, that is, a set with a binary operation that is associative, commutative and has an identity element $\mathbf{0}$. Since S is commutative, we will use additive notation. Assume that S is *cancellative*, that is, $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{a}+\mathbf{c}$ in S implies $\boldsymbol{b}=\mathbf{c}$. The monoids we study here are also *free of units*: $S\cap(-S)=\{\mathbf{0}\}$.

Some authors call these monoids *reduced* [Rosales and García-Sánchez 1999a]; others refer to this property as *positivity* [Bruns and Herzog 1993, Chapter 6]. Regardless of what we call them, their most important property is that they are *combinatorially finite*, that is, every element $\boldsymbol{a} \in S$ can be expressed only in finitely many ways as a sum $\boldsymbol{a} = \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_q$, with $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q \in S \setminus \{0\}$. See [Briales et al. 1998; Rosales et al. 1999] for a wider class of monoids where this condition still holds true. Monoids with this property are also known as FF-monoids. In [Geroldinger and Halter-Koch 2006] it is proved that mutiplicative monoids of all Krull monoids, all Dedekind domains, all orders in number fields are FF-monoids. Moreover, the binary relation on S defined by $\boldsymbol{b} \prec_S \boldsymbol{a}$ if $\boldsymbol{a} - \boldsymbol{b} \in S$ is a well-defined order on S that satisfies the descending chain condition.

All monoids considered here are finitely generated, commutative, cancellative and free of units, and thus we will omit these adjectives in what follows. Examples of monoids fulfilling these conditions are *affine semigroups*, that is, monoids isomorphic to finitely generated submonoids of $\mathbb{N}^r$ with r a positive integer ($\mathbb{N}$ denotes here the set of nonnegative integers), and in particular, *numerical semigroups* that are submonoids of the set of nonnegative integers with finite complement in $\mathbb{N}$.

We will write $S = \langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_r \rangle$ for the monoid generated by $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$, that is, $S = \boldsymbol{a}_1\mathbb{N} + \cdots + \boldsymbol{a}_r\mathbb{N}$. In such a case, $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ will be said to be a *system of generators* of S. If no proper subset of $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ generates S, the set $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ is a *minimal* system of generators of S. In our context, every monoid has a unique minimal system of generators: If $S^* = S \setminus \{0\}$, then the minimal system of generators of S is $S^* \setminus (S^* + S^*)$; see [Rosales and García-Sánchez 1999a, Chapter 3]. In particular, if S is the set of solutions of a system of linear Diophantine equations and/or inequalities, the minimal system of generators of S coincides with the so-called Hilbert basis; see for example [Sturmfels 1996, Chapter 13].

If S is a numerical semigroup minimally generated by $\{a_1 < \cdots < a_r\} \subset \mathbb{N}$, the number r is usually called the *embedding dimension* of S, and the number a_1 is the *multiplicity* of S. It is easy to show (and well known) that $\boldsymbol{a}_1 \geq r$; see [Rosales and García-Sánchez 2009, Proposition 2.10]. When $a_1 = r$, we say S is of *maximal embedding dimension*.

Given the minimal system $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ of generators of a monoid S, consider the monoid map

$$\varphi_A : \mathbb{N}^r \to S, \quad \boldsymbol{u} = (u_1, \ldots, u_r) \mapsto \textstyle\sum_{i=1}^r u_i \boldsymbol{a}_i.$$

This map is sometimes known as the *factorization homomorphism* associated to S.

Notice that each $\boldsymbol{u} = (u_1, \ldots, u_r) \in \varphi_A^{-1}(\boldsymbol{a})$ gives a *factorization* of $\boldsymbol{a} \in S$, say $\boldsymbol{a} = \sum_{i=1}^r u_i \boldsymbol{a}_i$. Thus, $\#\varphi_A^{-1}(\boldsymbol{a})$ is the number of factorizations of $\boldsymbol{a} \in S$. This number is finite because of the combinatorial finiteness of S; see also [Rosales and García-Sánchez 1999a, Lemma 9.1].

Let $\sim_A$ be the kernel congruence of φ_A, that is, $\boldsymbol{u} \sim_A \boldsymbol{v}$ if $\varphi_A(\boldsymbol{u}) = \varphi_A(\boldsymbol{v})$ (the kernel congruence is actually a congruence, an equivalence relation compatible with addition). It follows easily that S is isomorphic to the monoid $\mathbb{N}^r/{\sim_A}$.

Given $\rho \subseteq \mathbb{N}^r \times \mathbb{N}^r$, the congruence generated by ρ is the least congruence containing ρ, that is, the intersection of all congruences containing ρ. If $\sim$ is the congruence generated by ρ, we say that ρ is a *system of generators* of $\sim$. Rédei's theorem [1965] precisely states that every congruence on $\mathbb{N}^r$ is finitely generated. A *presentation* for S is a system of generators of $\sim_A$, and a *minimal presentation* is a minimal system of generators of $\sim_A$ (in the sense that none of its proper subsets generates $\sim_A$). In our setting, all minimal presentations have the same cardinality; see for instance [Rosales et al. 1999; Rosales and García-Sánchez 1999a]. This is not the case for finitely generated monoids in general.

Next we briefly describe a procedure for finding all minimal presentations for S as presented in [Rosales et al. 1999]; in our context this description is given in [Rosales and García-Sánchez 1999a, Chapter 9].

For $\boldsymbol{u} = (u_1, \ldots, u_r)$ and $\boldsymbol{v} = (v_1, \ldots, v_r) \in \mathbb{N}^r$, we write $\boldsymbol{u} \cdot \boldsymbol{v}$ for $\sum_{i=1}^r u_i v_i$ (the dot product).

Given $\boldsymbol{a} \in S$, we define a binary relation on $\varphi_A^{-1}(\boldsymbol{a})$: For $\boldsymbol{u}, \boldsymbol{u}' \in \varphi_A^{-1}(\boldsymbol{a})$, we say $\boldsymbol{u} \,\mathcal{R}\, \boldsymbol{u}'$ if there exists a chain $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_k \in \varphi_A^{-1}(\boldsymbol{a})$ such that

(a) $\boldsymbol{u}_0 = \boldsymbol{u},\ \boldsymbol{u}_k = \boldsymbol{u}'$, and

(b) $\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1} \neq 0$ for $i \in \{0, \ldots, k-1\}$.

For every $\boldsymbol{a} \in S$, define $\rho_{\boldsymbol{a}}$ in the following way.

- If $\varphi_A^{-1}(\boldsymbol{a})$ has one $\mathcal{R}$-class, set $\rho_{\boldsymbol{a}} = \varnothing$.
- Otherwise, let $\mathcal{R}_1, \ldots, \mathcal{R}_k$ be the different $\mathcal{R}$-classes of $\varphi_A^{-1}(\boldsymbol{a})$. Choose $\boldsymbol{v}_i \in \mathcal{R}_i$ for all $i \in \{1, \ldots, k\}$ and set $\rho_{\boldsymbol{a}}$ to be any set of $k-1$ pairs of elements in $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ such that any two elements in V are connected by a sequence of pairs in $\rho_{\boldsymbol{a}}$ (or their symmetrics). For instance, we can choose $\rho_{\boldsymbol{a}} = \{(\boldsymbol{v}_1, \boldsymbol{v}_2), \ldots, (\boldsymbol{v}_1, \boldsymbol{v}_k)\}$ or $\rho_{\boldsymbol{a}} = \{(\boldsymbol{v}_1, \boldsymbol{v}_2), (\boldsymbol{v}_2, \boldsymbol{v}_3), \ldots, (\boldsymbol{v}_{k-1}, \boldsymbol{v}_k)\}$.

Then $\rho = \bigcup_{\boldsymbol{a} \in S} \rho_{\boldsymbol{a}}$ is a minimal presentation of S. In this way one can construct all minimal presentations for S. Because S is finitely presented, there are finitely many elements $\boldsymbol{a}$ in S for which $\varphi_A^{-1}(\boldsymbol{a})$ has more than one $\mathcal{R}$-class.

2. Betti elements

As we have seen above, a minimal presentation of S is a set of pairs of factorizations of some elements in S, namely, those having more than one $\mathcal{R}$-class. We say that $\boldsymbol{a} \in S$ is a *Betti element* if $\varphi_A^{-1}(\boldsymbol{a})$ has more than one $\mathcal{R}$-class.

We will say the $\boldsymbol{a} \in S$ is *Betti-minimal* if it is minimal among all the Betti elements in S with respect to $\prec_S$. Of course, Betti elements in S are not necessarily Betti-minimal. Consider, for instance, $S = \langle 4, 6, 21\rangle$ and $\boldsymbol{a} = 42$.

We will write $\mathrm{Betti}(S)$ and $\mathrm{Betti\text{-}minimal}(S)$ for the sets of Betti elements and Betti minimal elements of the monoid S, respectively.

Lemma 1. *Let* $S = \langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_r\rangle$. *If* $\boldsymbol{a} \notin \mathrm{Betti}(S)$ *and* $\#\varphi_A^{-1}(\boldsymbol{a}) \geq 2$, *there exists* $\boldsymbol{a}' \in \mathrm{Betti}(S)$ *such that* $\boldsymbol{a}' \prec_S \boldsymbol{a}$.

Proof. We proceed by induction on $\#\varphi_A^{-1}(\boldsymbol{a})$. If $\varphi_A^{-1}(\boldsymbol{a}) = \{\boldsymbol{u}, \boldsymbol{v}\}$ with $\boldsymbol{u} \cdot \boldsymbol{v} > 0$, consider $\boldsymbol{a}' = \boldsymbol{a} - \sum_{i=1}^{r} \min(u_i, v_i)\boldsymbol{a}_i$. Then, putting $u_i' = u_i - \min(u_i, v_i)$ and $v_i' = v_i - \min(u_i, v_i)$ for $i \in \{1, \dots, r\}$, we have $\varphi_A^{-1}(\boldsymbol{a}') = \{\boldsymbol{u}', \boldsymbol{v}'\}$, and $\boldsymbol{u}' \cdot \boldsymbol{v}' = 0$. So, $\boldsymbol{a}' \prec \boldsymbol{a}$ is Betti. Assume now that the result is true for every $\boldsymbol{a}' \in S$ such that

$$2 \leq \#\varphi_A^{-1}(\boldsymbol{a}') < \#\varphi_A^{-1}(\boldsymbol{a}).$$

Since $\boldsymbol{a}$ is not Betti, there exist unequal $\boldsymbol{u}, \boldsymbol{v} \in \varphi_A^{-1}(\boldsymbol{a})$ such that $\boldsymbol{u} \cdot \boldsymbol{v} > 0$. If $\boldsymbol{a}' = \boldsymbol{a} - \sum_{i=1}^{r} \min(u_i, v_i)\boldsymbol{a}_i$, then $2 \leq \#\varphi_A^{-1}(\boldsymbol{a}') \leq \#\varphi_A^{-1}(\boldsymbol{a})$. If the second inequality is strict, we conclude by induction hypothesis. Otherwise, if $\boldsymbol{a}'$ is not Betti, we may repeat the previous argument to produce $\boldsymbol{a}'' \prec_S \boldsymbol{a}' \prec_S \boldsymbol{a}$. The descending chain condition for $\prec_S$ guarantees that this process cannot continue indefinitely. □

Remark 2. When $S \ncong \mathbb{N}^r$, this lemma implies the existence of Betti elements in S. Otherwise, $\mathrm{Betti}(S) = \varnothing$ because φ_A is an isomorphism.

Betti-minimal elements are characterized in the following result. As we will see later, they play an important role in the study of monoids with unique presentations.

Proposition 3. *Let S be a monoid. The element* $\boldsymbol{a} \in \mathrm{Betti\text{-}minimal}(S)$ *if and only* $\varphi_A^{-1}(\boldsymbol{a})$ *has more than one* $\mathcal{R}$*-class and each* $\mathcal{R}$*-class is a singleton.*

Proof. First, observe that $\varphi_A^{-1}(\boldsymbol{a})$ has more than one $\mathcal{R}$-class and each $\mathcal{R}$-class is a singleton if and only if $\#\varphi_A^{-1}(\boldsymbol{a}) \geq 2$ and $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ for every unequal $\boldsymbol{u}, \boldsymbol{v} \in \varphi_A^{-1}(\boldsymbol{a})$.

If $\boldsymbol{a} \in \mathrm{Betti\text{-}minimal}(S)$ and there exist unequal $\boldsymbol{u}, \boldsymbol{v} \in \varphi_A^{-1}(\boldsymbol{a})$ such that $\boldsymbol{u} \cdot \boldsymbol{v} > 0$, we consider $\boldsymbol{a}' = \boldsymbol{a} - \sum_{i=1}^{r} \min(u_i, v_i)\boldsymbol{a}_i$. Since $\#\varphi_A^{-1}(\boldsymbol{a}') \geq 2$, either $\boldsymbol{a}' \prec_S \boldsymbol{a}$ is Betti or, by Lemma 1, there exist $\boldsymbol{a}'' \in \mathrm{Betti}(S)$ such that $\boldsymbol{a}'' \prec_S \boldsymbol{a}' \prec_S \boldsymbol{a}$, contradicting in both cases the Betti-minimality of $\boldsymbol{a}$. Conversely, we suppose that

$$\varphi_A^{-1}(\boldsymbol{a}) = \bigcup_{i=1}^{\#\varphi_A^{-1}(\boldsymbol{a})} \{\boldsymbol{u}^{(i)}\},$$

with $\boldsymbol{u}^{(i)} \cdot \boldsymbol{u}^{(j)} = 0$ for $i \neq j$. In particular, $\boldsymbol{a} \in \mathrm{Betti}(S)$. If $\boldsymbol{a}' \prec_S \boldsymbol{a}$, then $\#\varphi_A^{-1}(\boldsymbol{a}') = 1$; otherwise, we will find unequal i, j with $\boldsymbol{u}^{(i)} \cdot \boldsymbol{u}^{(j)} \neq 0$. Thus we conclude that $\boldsymbol{a} \in \mathrm{Betti\text{-}minimal}(S)$. □

The notion of Betti-minimal is stronger than the notion of minimal multielement given in [Aoki et al. 2008, Definition 3.2]. Concretely, $\boldsymbol{a} \in S$ is a minimal multi-element if and only if $\varphi_A^{-1}(\boldsymbol{a})$ has more than one $\mathcal{R}$-class and at least one of them is a singleton.

3. Monoids having a unique minimal presentation

According to what we have recalled and defined so far, a monoid S has a unique minimal presentation if and only if the set of factorizations of all its Betti elements have just two $\mathcal{R}$-classes, and each of which is a singleton. Moreover, if $\boldsymbol{a}$ is a Betti element of S and $\varphi_A^{-1}(\boldsymbol{a}) = \{\boldsymbol{u}, \boldsymbol{v}\}$, then either the pair $(\boldsymbol{u}, \boldsymbol{v})$ or the pair $(\boldsymbol{v}, \boldsymbol{u})$ is in any minimal presentation of S. Hence we will say that $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^r \times \mathbb{N}^r$ is *indispensable*, and that $\boldsymbol{a}$ has *unique presentation*.

Example 4. The numerical semigroup $S = \langle 6, 10, 15\rangle$ has no indispensable elements. Using the techniques explained in [Rosales and García-Sánchez 2009], one can easily see that $\operatorname{Betti}(S) = \{30\}$, and that the factorizations of 30 are $\{(0, 0, 2), (0, 3, 0), (5, 0, 0)\}$. One can also use the `GAP` package `numericalsgps` to perform this computation [Delgado et al. 2008].

Clearly, S admits a unique minimal presentation if and only if either it is isomorphic to $\mathbb{N}^r$ for some positive integer r (and thus the empty set is its unique minimal presentation) or every element in any of its minimal presentations is indispensable. If this is the case, we say that S has a *unique presentation*.

The following results are straightforward consequences of Proposition 3.

Corollary 5. *Let $\boldsymbol{a} \in S$. The following are equivalent.*

(a) *$\boldsymbol{a}$ has unique presentation.*

(b) $\boldsymbol{a} \in \operatorname{Betti}(S)$ *and* $\#\varphi_A^{-1}(\boldsymbol{a}) = 2$.

(c) $\boldsymbol{a} \in \operatorname{Betti-minimal}(S)$ *and* $\#\varphi_A^{-1}(\boldsymbol{a}) = 2$.

Corollary 6. *A monoid S is uniquely presented if and only if either* $\operatorname{Betti}(S) = \varnothing$ *or the number of Betti-minimal elements in S equals the cardinality of a minimal presentation of S. In particular all Betti elements of S are Betti-minimal.*

By using the close relationship between toric ideals and semigroups, one can obtain necessary and sufficient conditions for a semigroup to be uniquely presented from the results in [Charalambous et al. 2007; Ojeda and Vigneron-Tenorio 2009; Takemura and Aoki 2004].

Example 7. Corollary 6 does not hold if we remove the minimal condition. For instance, one can use `numericalsgps` to compute that $S = \langle 4, 6, 21\rangle$ has a minimal presentation with cardinality 2, and $\operatorname{Betti}(S) = \{12, 42\}$. However, 42 admits 5 different factorizations in S.

Example 8. Let $S \subset \mathbb{Z}^r$ be a monoid minimally generated by $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ for some positive integer r. If the rank of the group spanned by S is one, there exist u and $v \in \mathbb{N}$ such that $u\boldsymbol{a}_1 = v\boldsymbol{a}_2$. So, there is only one Betti element $\boldsymbol{a} = u\boldsymbol{a}_1 = v\boldsymbol{a}_2$ and $\varphi_A^{-1}(\boldsymbol{a}) = \{(u, 0), (0, v)\}$. Therefore, S is uniquely presented. In particular, embedding dimension 2 numerical semigroups are uniquely presented (the group generated by any numerical semigroup is $\mathbb{Z}$).

4. Gluings

We first fix the notation of this section. Let S be an affine semigroup generated by $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_r\} \subseteq \mathbb{Z}^n$. Let A_1 and A_2 be two proper subsets of A such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \varnothing$. Let S_1 and S_2 be the affine semigroups generated by A_1 and A_2, respectively.

Set r_1 and r_2 to be the cardinality of A_1 and A_2, respectively. After rearranging the elements of A if necessary, we may assume that $A_1 = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{r_1}\}$ and $A_2 = \{\boldsymbol{a}_{r_1+1}, \dots, \boldsymbol{a}_r\}$.

Since $\mathbb{N}^r = \mathbb{N}^{r_1} \oplus \mathbb{N}^{r_2}$, elements in $\mathbb{N}^{r_1}$ and $\mathbb{N}^{r_2}$ may be regarded as elements in $\mathbb{N}^r$ of the form $(\,\cdot\,, 0)$ and $(0, \cdot\,)$, respectively. With this in mind, subsets of $\mathbb{N}^{r_i}$ will be considered as subsets of $\mathbb{N}^r$ for $i \in \{1, 2\}$, and the elements of $\sim_{A_1}$ and $\sim_{A_2}$ are viewed inside $\sim_A$.

The monoid S is said to be the *gluing* of S_1 and S_2 if $G(S_1) \cap G(S_2) = \boldsymbol{d}\mathbb{Z}$, with $\boldsymbol{d} \in S_1 \cap S_2 \setminus \{0\}$, where G denotes the group generated by its argument.

According to [Rosales 1997, Theorem 1.4], S admits a presentation of the form $\rho_1 \cup \rho_2 \cup \{((\boldsymbol{u}, 0), (0, \boldsymbol{v}))\}$, where ρ_1 and ρ_2 are presentations of S_1 and S_2, respectively, and $\boldsymbol{u} \in \varphi_{A_1}^{-1}(\boldsymbol{d})$ and $\boldsymbol{v} \in \varphi_{A_2}^{-1}(\boldsymbol{d})$. We next explore the conditions that we must impose on S_1, S_2 and $\boldsymbol{d}$ to ensure that S has a unique minimal presentation. We start by describing the Betti elements of S, and for this we need a lemma describing the factorizations of $\boldsymbol{d}$.

Lemma 9. *Let S be the gluing of S_1 and S_2 with $G(S_1) \cap G(S_2) = \boldsymbol{d}\mathbb{Z}$. Every factorization of $\boldsymbol{d}$ in S is either a factorization of $\boldsymbol{d}$ in S_1 or a factorization of $\boldsymbol{d}$ in S_2. In particular $\boldsymbol{d} \in$* Betti(S).

Proof. By definition $\boldsymbol{d} \in S_1 \cap S_2 \setminus \{0\}$, so there exist $\boldsymbol{u} \in \mathbb{N}^{r_1}$ and $\boldsymbol{v} \in \mathbb{N}^{r_2}$ such that $\boldsymbol{d} = \sum_{i=1}^{r_1} u_i \boldsymbol{a}_i = \sum_{i=r_1+1}^{r} v_i \boldsymbol{a}_i$. If $\boldsymbol{d} = \sum_{i=1}^{r} w_i \boldsymbol{a}_i = \sum_{i=1}^{r_1} w_i \boldsymbol{a}_i + \sum_{i=r_1+1}^{r} w_i \boldsymbol{a}_i$, then

$$\boldsymbol{d} - \sum_{i=1}^{r_1} w_i \boldsymbol{a}_i = \sum_{i=1}^{r_1} u_i \boldsymbol{a}_i - \sum_{i=1}^{r_1} w_i \boldsymbol{a}_i = \sum_{i=r_1+1}^{r} w_i \boldsymbol{a}_i \in G(S_1) \cap G(S_2),$$

that is, $\boldsymbol{d} - \sum_{i=1}^{r_1} w_i \boldsymbol{a}_i = z\boldsymbol{d}$. Hence either $z = 1$ and then $w_i = 0$ for $i \in \{1, \dots, r_1\}$, or $z = 0$ and then $w_i = 0$ for $i \in \{r_1 + 1, \dots, r\}$, as claimed.

Also, we have $\varphi_A^{-1}(\boldsymbol{d}) = \varphi_{A_1}^{-1}(\boldsymbol{d}) \cup \varphi_{A_2}^{-1}(\boldsymbol{d})$ with $(\boldsymbol{u}, 0) \cdot (0, \boldsymbol{v}) = 0$ for every $\boldsymbol{u} \in \varphi_{A_1}^{-1}(\boldsymbol{d})$ and $\boldsymbol{v} \in \varphi_{A_2}^{-1}(\boldsymbol{d})$, which means that $\varphi_A^{-1}(\boldsymbol{d})$ has at least two $\mathcal{R}$-classes. Hence $\boldsymbol{d} \in \operatorname{Betti}(S)$. □

Theorem 10. *Let S be the gluing of S_1 and S_2, and $G(S_1) \cap G(S_2) = \boldsymbol{d}\mathbb{Z}$. Then*

$$\operatorname{Betti}(S) = \operatorname{Betti}(S_1) \cup \operatorname{Betti}(S_2) \cup \{\boldsymbol{d}\}.$$

Proof. By [Rosales 1997, Theorem 1.4], S admits a presentation of the form $\rho = \rho_1 \cup \rho_2 \cup \{((\boldsymbol{u}, 0), (0, \boldsymbol{v}))\}$, where ρ_1 and ρ_2 are sets of generators for $\sim_{A_1}$ and $\sim_{A_2}$, respectively, and $\varphi_{A_1}(\boldsymbol{u}) = \varphi_{A_2}(\boldsymbol{v}) = \boldsymbol{d}$. Since every system of generators of $\sim_A$ can be refined to a minimal system of generators [Rosales and García-Sánchez 1999a, Chapter 9], from the shape of ρ we deduce that the Betti elements of S are either a Betti element of S_1, a Betti element of S_2, or $\boldsymbol{d}$ itself, that is, $\operatorname{Betti}(S) \subseteq \operatorname{Betti}(S_1) \cup \operatorname{Betti}(S_2) \cup \{\boldsymbol{d}\}$.

Recall that $\boldsymbol{d} \in \operatorname{Betti}(S)$ by Lemma 9. Therefore, to demonstrate the inclusion $\operatorname{Betti}(S) \supseteq \operatorname{Betti}(S_1) \cup \operatorname{Betti}(S_2) \cup \{\boldsymbol{d}\}$, it suffices to prove $\operatorname{Betti}(S_1) \cup \operatorname{Betti}(S_2) \subseteq \operatorname{Betti}(S)$. Suppose by way of contradiction that there is a $\boldsymbol{b}$ in $\operatorname{Betti}(S_1) \setminus \operatorname{Betti}(S)$ (the case where $\boldsymbol{b}$ is in $\operatorname{Betti}(S_2) \setminus \operatorname{Betti}(S)$ is argued similarly).

Since $\boldsymbol{b} \in \operatorname{Betti}(S_1)$, there exist two $\mathcal{R}$-classes in $\varphi_{A_1}^{-1}(\boldsymbol{b})$, say $\mathscr{C}_1$ and $\mathscr{C}_2$. We know because $\boldsymbol{b} \notin \operatorname{Betti}(S)$ that $\varphi_A^{-1}(\boldsymbol{b})$ has only one $\mathcal{R}$-class. Hence:

- There exist $\boldsymbol{w} \in \mathscr{C}_1$ and $\bar{\boldsymbol{w}} \in \varphi_A^{-1}(\boldsymbol{b})$ such that $\bar{\boldsymbol{w}} \cdot (\boldsymbol{w}, 0) \neq 0$ and $\boldsymbol{b} = \sum_{i=1}^{r_1} \bar{w}_i \boldsymbol{a}_i + \sum_{i=r_1+1}^{r} \bar{w}_i \boldsymbol{a}$, where $\bar{w}_i$ for $1 \leq i \leq r$ are the coordinates of $\bar{\boldsymbol{w}}$ and $\bar{w}_i \neq 0$ for some $r_1 + 1 \leq i \leq r$.
- There exist $\boldsymbol{w}' \in \mathscr{C}_2$ and $\bar{\boldsymbol{w}}' \in \varphi_A^{-1}(\boldsymbol{b})$ such that $\bar{\boldsymbol{w}}' \cdot (\boldsymbol{w}', 0) \neq 0$ and $\boldsymbol{b} = \sum_{i=1}^{r_1} \bar{w}'_i \boldsymbol{a}_i + \sum_{i=r_1+1}^{r} \bar{w}'_i \boldsymbol{a}_i$, where $\bar{w}'_i$ for $1 \leq i \leq r$ are the coordinates of $\bar{\boldsymbol{w}}'$ and $\bar{w}'_i \neq 0$ for some $r_1 + 1 \leq i \leq r$.

Since $0 \neq \boldsymbol{b} - \sum_{i=1}^{r_1} \bar{w}_i \boldsymbol{a}_i = \sum_{i=r_1+1}^{r} \bar{w}_i \boldsymbol{a}_i \in G(S_1) \cap G(S_2) = \boldsymbol{d}\mathbb{Z}$, we have

$$\boldsymbol{b} = \sum_{i=1}^{r_1} \bar{w}_i \boldsymbol{a}_i + \sum_{i=1}^{r_1} z u_i \boldsymbol{a}_i = \sum_{i=1}^{r_1} (\bar{w}_i + z u_i) \boldsymbol{a}_i \quad \text{for some } z > 0.$$

Analogously, $\boldsymbol{b} = \sum_{i=1}^{r_1} (\bar{w}'_i + z' u_i) \boldsymbol{a}_i$ for some $z' > 0$.

Let $\tilde{\boldsymbol{w}}$ and $\tilde{\boldsymbol{w}}' \in \varphi_{A_1}^{-1}(\boldsymbol{b})$ be the corresponding vectors of coordinates $\bar{w}_i + z u_i$ for $1 \leq i \leq r_1$ and $\bar{w}'_i + z' u_i$ for $1 \leq i \leq r_1$, respectively. This yields a contradiction, since $\boldsymbol{w}$ and $\boldsymbol{w}'$ are not $\mathcal{R}$-related; however $\boldsymbol{w} \cdot \tilde{\boldsymbol{w}} \neq 0$, $\tilde{\boldsymbol{w}} \cdot \tilde{\boldsymbol{w}}' \neq 0$ and $\tilde{\boldsymbol{w}}' \cdot \boldsymbol{w}' \neq 0$. □

Observe that $\varphi_A^{-1}(\boldsymbol{d}) \supseteq \{(\boldsymbol{u}, 0), (0, \boldsymbol{v})\}$, with $\varphi_{A_1}(\boldsymbol{u}) = \varphi_{A_2}(\boldsymbol{v}) = \boldsymbol{d}$, and that the equality holds if and only if $\boldsymbol{d}$ has unique presentation as an element of S.

Corollary 11. *Let S be the gluing of S_1 and S_2, and let $G(S_1) \cap G(S_2) = \boldsymbol{d}\mathbb{Z}$. Then the element $\boldsymbol{d}$ in S has unique presentation if and only if $\boldsymbol{d} - \boldsymbol{a} \notin S$ for every $\boldsymbol{a} \in \operatorname{Betti}(S_1) \cup \operatorname{Betti}(S_2)$.*

Proof. If $\boldsymbol{d}$ has unique presentation, $\boldsymbol{d}$ belongs to Betti-minimal(S) by Corollary 5. So $\boldsymbol{d}-\boldsymbol{a} \notin S$ for every $\boldsymbol{a} \in \mathrm{Betti}(S)\setminus\{\boldsymbol{d}\}$. Now $\boldsymbol{d} \notin \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$ since $\boldsymbol{d}$ has unique factorization in S_i for $i \in \{1, 2\}$. Hence $\mathrm{Betti}(S)\setminus\{\boldsymbol{d}\}=\mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$ by Theorem 10. We conclude that $\boldsymbol{d}-\boldsymbol{a} \notin S$ for every $\boldsymbol{a} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$.

Conversely, in view of Lemma 1, we deduce that $\boldsymbol{d}$ admits a unique factorization in S_i for $i \in \{1, 2\}$, that is, $\varphi_{A_1}^{-1}(\boldsymbol{d}) = \{\boldsymbol{u}\}$ and $\varphi_{A_2}^{-1}(\boldsymbol{d}) = \{\boldsymbol{v}\}$. Since $\boldsymbol{d}$ is a Betti element by Lemma 9, we conclude that $\varphi_{A}^{-1}(\boldsymbol{d}) = \{(\boldsymbol{u}, 0), (0, \boldsymbol{v})\}$. □

Theorem 12. *Let S be the gluing of S_1 and S_2, and $G(S_1)\cap G(S_2) = \boldsymbol{d}\mathbb{Z}$. Then S is uniquely presented if and only if*

(a) *S_1 and S_2 are uniquely presented, and*

(b) *$\pm(\boldsymbol{d}-\boldsymbol{a}) \notin S$ for every $\boldsymbol{a} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$,*

Proof. By Theorem 10, $\mathrm{Betti}(S) = \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)\cup\{\boldsymbol{d}\}$. So, if S is uniquely presented, then every $\boldsymbol{a} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)\cup\{\boldsymbol{d}\}$ has unique presentation. Thus, S_1 and S_2 are uniquely presented and $\boldsymbol{d}-\boldsymbol{a} \notin S$ for every $\boldsymbol{a} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$ by Corollary 11. Finally, since, by Corollary 5, every $\boldsymbol{a} \in \mathrm{Betti}(S)$ is Betti-minimal, we conclude that $\boldsymbol{a}-\boldsymbol{d} \notin S$, for every $\boldsymbol{a} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$. (Note that $\boldsymbol{d}-\mathbf{m} \notin S$ implies $\boldsymbol{d} \neq \mathbf{m}$ for every $\mathbf{m} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$.)

Conversely, suppose that (a) and (b) hold. In particular, every $\boldsymbol{a} \in \mathrm{Betti}(S_i)$ has only two factorizations as element of S_i for $i \in \{1, 2\}$ and, by Corollary 11, $\boldsymbol{d}$ has only two factorizations in S, say $\boldsymbol{d} = \sum_{i=1}^{r_1} u_i\boldsymbol{a}_i = \sum_{i=r_1+1}^{r} v_i\boldsymbol{a}_i$. So, if $\boldsymbol{a} \in \mathrm{Betti}(S)$ has more than two factorizations in S, then $\boldsymbol{d} \neq \boldsymbol{a} \in \mathrm{Betti}(S_1)\cup\mathrm{Betti}(S_2)$. If $\boldsymbol{a} \in \mathrm{Betti}(S_1)$, then $\boldsymbol{a} = \sum_{i=1}^{r_1} w_i\boldsymbol{a}_i + \sum_{i=r_1+1}^{r} w_i\boldsymbol{a}_i$, with $w_i \neq 0$ for some i such that $r_1+1 \le i \le r$. Thus, $\boldsymbol{a} - \sum_{i=1}^{r_1} w_i\boldsymbol{a}_i = \sum_{i=r_1+1}^{r} w_i\boldsymbol{a}_i \in G(S_1)\cap G(S_2) = \boldsymbol{d}\mathbb{Z}$ and thus $\boldsymbol{a}-\boldsymbol{d} \in S$, which is impossible by hypothesis. □

The affine semigroup in the next example is borrowed from [Rosales and García-Sánchez 1999b], where the authors use it to illustrate their algorithm for checking freeness of simplicial semigroups. We use $\mathbf{e}_i \in \mathbb{N}^r$ to denote the i-th row of the identity $r\times r$ matrix.

Example 13. Let us see that $S = \langle(2, 0), (0, 3), (2, 1), (1, 2)\rangle$ is uniquely presented. On the one hand, by taking $A_1 = \{(2, 0), (0, 3), (2, 1)\}$, $A_2 = \{(1, 2)\}$, $S_1 = \langle A_1\rangle$ and $S_2 = \langle A_2\rangle$, we have $G(S_1)\cap G(S_2) = 2(1, 2)\mathbb{Z}$. On the other hand, by taking $A_{11} = \{(2, 0), (0, 3)\}$, $A_{12} = \{(2, 1)\}$, $S_{11} = \langle A_{11}\rangle$ and $S_{12} = \langle A_{12}\rangle$, we have $G(S_{11})\cap G(S_{12}) = 3(2, 1)\mathbb{Z}$. Since $S_{11} \cong \mathbb{N}^2$ and $S_{12} \cong \mathbb{N}$ are uniquely presented (because, their corresponding presentations are the empty set) and condition (b) in Theorem 12 is trivially satisfied, we are assured that S_1 is uniquely presented by $\{(3\mathbf{e}_3, 3\mathbf{e}_1+\mathbf{e}_2)\}$. Finally, since S_1 and $S_2 \cong \mathbb{N}$ are uniquely presented and the element $2(1, 2) - 3(2, 1)$ is not in S, we conclude that S is uniquely presented by $\{(3\mathbf{e}_3, 3\mathbf{e}_1+\mathbf{e}_2),\ (2\mathbf{e}_4, \mathbf{e}_2+\mathbf{e}_3)\}$.

Example 14. We may construct an infinite sequence of uniquely presented numerical semigroups. Let us start with $S_1 = \langle 2, 3\rangle$, and given S_i minimally generated by $\{a_1, \dots, a_{i+1}\}$, $i \geq 2$, set $S_{i+1} = \langle 2a_1, a_1 + a_2, 2a_2, \dots, 2a_{i+1}\rangle$. We prove by induction on i that S_{i+1} is uniquely presented by

$$\rho_{i+1} = \{(2\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3), (2\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_4), \dots, (2\mathbf{e}_i, \mathbf{e}_1 + \mathbf{e}_i), (2\mathbf{e}_{i+1}, 3\mathbf{e}_1)\}.$$

For $i = 1$, the result follows easily. Assume that $i \geq 2$ and that the result holds for S_i, and let us show it holds for S_{i+1}. Observe that S_{i+1} is the gluing of $\langle 2a_1, \dots, 2a_{i+1}\rangle = 2S_i$ and $\langle a_1 + a_2\rangle$, with $d = 2a_1 + 2a_2$, and consequently S_{i+1} is minimally generated by $\{2a_1, a_1 + a_2, 2a_2, \dots, 2a_{i+1}\}$; apply [Rosales and García-Sánchez 2009, Lemma 9.8] with $\lambda = 2$ and $\mu = a_1 + a_2$. Note that $\operatorname{Betti}(\langle a_1 + a_2\rangle) = \varnothing$ and, by induction hypothesis, $\operatorname{Betti}(2S_i) = 2\operatorname{Betti}(S_i) = \{2(2a_2), \dots, 2(2a_{i+1})\}$. Thus, by Theorem 10,

$$\operatorname{Betti}(S_{i+1}) = \{d\} \cup \operatorname{Betti}(2S_i) = \{2a_1 + 2a_2, 2(2a_2), \dots, 2(2a_{i+1})\}.$$

Now, a direct computation shows that ρ_{i+1} is a minimal presentation of S_{i+1}.

In view of Theorem 12, it suffices to prove the uniqueness of the presentation to check that, for $b = 2(2a_j) - (2a_1 + 2a_2)$, neither b nor $-b$ belongs to S_{i+1}. Observe that $-b < 0$ since $j \geq 2$, and thus it is not in S_{i+1}. Also, if $j \neq i$, then $2(2a_j) - (2a_1 + 2a_2) = 2a_1 + 2a_{j+1} - 2a_1 - 2a_2 = 2a_{j+1} - 2a_2$. This element cannot be in S_{i+1} because $2a_{j+1}$ is one of its minimal generators. For $j = i$, we get $2(2a_{i+1}) - (2a_1 + 2a_2) = 2(3a_1) - 2a_1 - 2a_2 = 2(2a_1) - 2a_2$. If this integer belongs to S_{i+1}, then by the minimality of $2a_2$, there exists $a \in S_{i+1} \setminus \{0\}$ such that $2(2a_1) = 2a_2 + a$. But then $a \geq 2a_1$, and since $2a_2 > 2a_1$, we get a contradiction.

For every positive integer i, the numerical semigroup S_{i+1} is a free numerical semigroup in the sense of [Bertin and Carbonne 1977], and thus it is a complete intersection, that is, a numerical semigroup with minimal presentations with the least possible cardinality, the embedding dimension minus one. Some authors call these semigroups telescopic. Not all free numerical semigroups have unique minimal presentation; $\langle 4, 6, 21\rangle$ illustrates this fact (see Example 7).

5. Uniquely presented numerical semigroups

In some sense, only a few numerical semigroups have unique minimal presentation. The following sequences have been computed with the `numericalsgps` GAP package [Delgado et al. 2008]. The first contains in the i-th position the number of numerical semigroups with *Frobenius number* $i \in \{1, \dots, 20\}$, meaning that i is the largest integer not in the semigroup. The second contains those with the same condition having a unique minimal presentation.

$$(1, 1, 2, 2, 5, 4, 11, 10, 21, 22, 51, 40, 106, 103, 200, 205, 465, 405, 961, 900),$$

$$(1, 1, 1, 1, 3, 1, 5, 2, 5, 4, 8, 2, 12, 8, 6, 9, 17, 8, 20, 12).$$

Next we explore three big families of numerical semigroups and determine the elements having unique minimal presentations.

5.1. ***Numerical semigroups generated by intervals.*** Let a and x be two positive integers, and let $S = \langle a, a+1, \dots, a+x\rangle$. Since $\mathbb{N}$ is uniquely presented, we may assume that $2 \le a$. In order that $\{a, \dots, a+x\}$ becomes a minimal system of generators for S, we suppose that $x < a$.

Theorem 15. *$S = \langle a, a+1, \dots, a+x\rangle$ is uniquely presented if and only if either*

$$a=1 \text{ (\textit{that is}, } S=\mathbb{N}) \quad \textit{or} \quad x=1 \quad \textit{or} \quad x=2 \quad \textit{or} \quad x=3 \textit{ and } (a-1) \bmod x \neq 0.$$

Proof. The Betti elements in S are fully described in [García-Sánchez and Rosales 1999, Theorem 8]. If $x \ge 4$, then $m = 2(a+2)$ is a Betti element and $\#\varphi_A^{-1}(m) = 3$. Thus S is not uniquely presented for $x \ge 4$. Hence we focus on $x \in \{1, 2, 3\}$. For simplicity in the forthcoming notation, let q and r be the quotient and the remainder in the division of $a-1$ by x, that is, $a = xq + r + 1$ with $0 \le r \le x-1$. Notice that $x < a$ implies $q \ge 1$.

For $x = 1$, we get an embedding dimension two numerical semigroup that is uniquely presented; see Example 8.

For $x = 2$,

$$\operatorname{Betti}(S) = \begin{cases} \{2(a+1), qa+2(q-1)+1, qa+2(q-1)+2\} & \text{if } r=0,\\ \{2(a+1), qa+2(q-1)+2\} & \text{if } r=1.\end{cases}$$

Since the cardinality of a minimal presentation of S is $3-r$ [ibid., Theorem 8], by Corollary 6 we only must check whether they are incomparable with respect to $\prec_S$. If $r=0$, clearly $qa+2(q-1)+1$ and $qa+2(q-1)+2$ are incomparable, since $1 \notin S$. Also,

$$qa+2(q-1)+1-2(a+1) = (q-1)a+2q-1 \notin S$$

in view of [ibid., Lemma 1] (since $2q-1 > 2(q-1)$). The same argument applies to $qa+2(q-1)+2-2(a+1) = (q-1)a+2q$. If $r=1$, then

$$qa+2(q-1)+2-2(a+1) = (q-2)a+2(q-1) \notin S$$

(we use again [ibid., Lemma 1]), so we also get a (complete intersection) uniquely presented numerical semigroup. Hence every numerical semigroup of the form $\langle a, a+1, a+2\rangle$ with $a \ge 3$ is uniquely presented.

Assume that $x = 3$ (and thus $a \ge 4$).

Case: $r = 0$. In this setting, both $(q+1)(a+3)$ and $2(a+1)$ are Betti elements. However, $(q+1)(a+3) - 2(a+1) = (q-1)a + q3 + 1 = (q-1)a + (a-1) + 1 = qa \in S$. Hence $(q+1)(a+3) \notin \text{Betti-minimal}(S)$ and so, by Corollary 6, it is not uniquely presented.

Case: $r \neq 0$. In this case,

$$\text{Betti}(S) = \begin{cases} \{2(a+1),\ (a+1)+(a+2), 2(a+2), \\ \quad qa+3(q-1)+2,\ qa+3(q-1)+3\} & \text{if } r=1, \\ \{2(a+1),\ (a+1)+(a+2), \\ \quad 2(a+2),\ qa+3(q-1)+3\} & \text{if } r=2. \end{cases}$$

Since the cardinality of a minimal presentation of S is $6-r$ [García-Sánchez and Rosales 1999, Theorem 8], by Corollary 6 we only must check whether they are incomparable with respect to $\prec_S$. Observe that

$$qa + (q-1)3 + j - 2a - i = (q-2)a + (q-1)3 + j - i \notin S$$

if and only if $q + j + 1 > i$ [ibid., Lemma 1]. Since in our case $i \in \{2, 3, 4\}$, $j \in \{2, 3\}$ and $q \geq 1$, we obtain that these elements are incomparable. Thus, S is uniquely presented. □

5.2. ***Embedding dimension three numerical semigroups.*** As we have pointed out above, the Frobenius number of a numerical semigroup is the largest integer not belonging to it. A numerical semigroup S with Frobenius number f is *symmetric* if $f - x \in S$ for every $x \in \mathbb{Z} \setminus S$. For embedding dimension three numerical semigroups, it is well known that the concepts of symmetric and complete intersection numerical semigroups coincide. (In the embedding dimension three case, the concept of free also coincides with that of symmetric and complete intersection; see for instance [Rosales and García-Sánchez 2009, Chapter 9] or [Herzog 1970].) Nonsymmetric numerical semigroups with embedded dimension three are uniquely presented [Herzog 1970]. Thus, we will focus on the symmetric case, which is the free case, and as Delorme [1976] proved, these semigroups are the gluing of an embedding dimension two numerical semigroup and $\mathbb{N}$; see [Rosales 1997] for a proof using the concept of gluing. So every symmetric numerical semigroup with embedding dimension three can be described as follows.

Proposition 16 [Rosales and García-Sánchez 2009, Theorem 10.6]. *Let m_1 and m_2 be two relatively prime integers greater than one. Let a, b and c be non-negative integers with $a \geq 2$, $b + c \geq 2$ and* $\gcd(a, bm_1 + cm_2) = 1$. *Then $S = \langle am_1, am_2, bm_1 + cm_2 \rangle$ is a symmetric numerical semigroup with embedding dimension three. Every embedding dimension three symmetric numerical semigroup is of this form.*

Our main result is now just a special case of what we have seen in Section 4.

Theorem 17. *In the notation of Proposition 16, S is a symmetric numerical semigroup uniquely presented with embedding dimension three if and only if $0 < b < m_2$ and $0 < c < m_1$.*

Lemma 18. *Let m_1 and m_2 be two relatively prime integers greater than one. Then, $m_1m_2 = \alpha m_1 + \beta m_2$ for some $\alpha \geq 0$ and $\beta \geq 0$ if and only if $\alpha = m_2$ and $\beta = 0$, or $\alpha = 0$ and $\beta = m_1$.*

Proof. We have $m_1m_2 = \alpha m_1 + \beta m_2$ for some $\alpha \geq 0$ and $\beta \geq 0$ if and only if $(m_2 - \alpha)m_1 = \beta m_2$ for some $\alpha \geq 0$ and $\beta \geq 0$. Since $\gcd(m_1, m_2) = 1$, it follows that $(m_2 - \alpha)m_1 = \beta m_2$ for some $\alpha \geq 0$ and $\beta \geq 0$, if and only if $m_2 - \alpha = \gamma m_2$ and $\beta = \gamma m_1$ for some $\gamma \geq 0$, if and only if $\alpha = (1 - \gamma)m_2$ and $\beta = \gamma m_1$ for some $0 \leq \gamma \leq 1$, if and only if $\alpha = m_2$ and $\beta = 0$ or $\alpha = 0$ and $\beta = m_1$. □

Proof of Theorem 17. S is the gluing of $S_1 = \langle am_1, am_2\rangle$ and $S_2 = \langle bm_1 + cm_2\rangle$ with $d = a(bm_1 + cm_2)$. Also $\mathrm{Betti}(S_1) = am_1m_2$ and $\mathrm{Betti}(S_2) = \varnothing$. Therefore, by Theorem 10, $\mathrm{Betti}(S) = \{am_1m_2, a(bm_1 + cm_2)\}$. Thus, by Theorem 12, S is uniquely presented if and only if $\pm(am_1m_2 - a(bm_1 + cm_2)) \notin S$.

By direct computation, one can check that $a(bm_1 + cm_2) - am_1m_2 \in S$ if and only if $b \geq m_2$ or $c \geq m_1$. Also, $am_1m_2 - a(bm_1 + cm_2) \in S$ if and only if

$$m_1m_2 = ((\alpha_3 + 1)b + \alpha_1)m_1 + ((\alpha_3 + c)c + \alpha_1)m_2$$

for some $\alpha_i \geq 0$, with $i \in \{1, 2, 3\}$. In view of Lemma 18, this is equivalent to $((\alpha_3 + 1)b + \alpha_1) = 0$ and $((\alpha_3 + c)c + \alpha_1) = m_1$ or $((\alpha_3 + 1)b + \alpha_1) = m_2$ and $((\alpha_3 + c)c + \alpha_1) = 0$ for some $\alpha_i \geq 0$ with $i \in \{1, 2, 3\}$. This holds if and only if $b = 0$ and $c \leq m_1$ or $b \leq m_2$ and $c = 0$.

Therefore, $\pm(am_1m_2 - a(bm_1 + cm_2)) \notin S$ if and only if $0 < b < m_2$ and $0 < c < m_1$. □

5.3. *Maximal embedding dimension numerical semigroups.*

Theorem 19. *A numerical semigroup S minimally generated by $a_1 < a_2 < \cdots < a_r$ with $a_1 = r$ is uniquely presented if only if $r = 3$.*

Proof. For $r = 3$, we obtain numerical semigroups of the form $\langle 3, a, b\rangle$, with a and b not multiples of 3 and thus coprime with 3. It follows easily that these semigroups do not have the shape given in Proposition 16, and thus are not symmetric. Consequently, they are uniquely presented.

We now prove that S cannot be uniquely presented if $a_1 = r \geq 4$. According to [Rosales 1996], $\mathrm{Betti}(S) = \{a_i + a_j \mid i, j \in \{2, \ldots, r\}\}$. All the elements in $\{0, a_2, \ldots, a_r\}$ belong to different classes modulo a_1, and there are precisely a_1 of them. Thus $2a_r$ can be uniquely be written as $ba_1 + a_i$ for some $i \in \{2, \ldots, r-1\}$ and b a positive integer.

Let f be the Frobenius number of S. It is well known that $f = a_r - a_1$ in this setting; see for instance [Rosales and García-Sánchez 2009]. Since $2a_r - a_i = a_r + (a_r - a_i) > a_r - a_1 = f$ for all i, it follows that $2a_r - a_i \in S$. Hence $2a_r = a_i + m_i$ for some $m_i \in S$ for every $i \in \{1, \ldots, r\}$. Take $i \neq k$. Then $2a_r$ admits at least three expressions: $2a_r$, $ba_1 + a_k$ and $a_i + m_i$. By Corollary 5, S cannot have a unique minimal presentation. □

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