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We investigate the composition series of the induced admissible representations of the metaplectic group $\widetilde{\mathrm{Sp}(2)}$ over a p-adic field F. In this way, we determine the nonunitary and unitary duals of $\widetilde{\mathrm{Sp}(2)}$ modulo cuspidal representations.

1. Introduction

The admissible representations of reductive groups over p-adic fields have been studied intensively by many authors, but knowledge about the unitary dual of such groups is still incomplete. Besides some results concerning specific parts of the unitary dual of some classical and exceptional groups (that is, spherical, generic [Lapid et al. 2004] and so on), there are some situations where, for some low rank groups, the complete unitary dual is described [Sally and Tadić 1993; Muić 1997; Hanzer 2006; Matić 2010].

In this paper, we completely describe the noncuspidal unitary dual of the double cover of the symplectic group of split rank two. Although this is not an algebraic group, some recent results enabled us to study this group in the same spirit as the classical split groups. More concretely, Hanzer and Muić [2009] related reducibilities of the induced representations of metaplectic groups with those of the odd orthogonal groups (using theta correspondence), while their paper [2010] describes the extension of the Jacquet module techniques of Tadić for classical groups to metaplectic groups. More specifically, Tadić's structure formula for symplectic and odd-orthogonal groups [1995] (which is a version of a geometric lemma of [Bernstein and Zelevinsky 1977]) is extended to metaplectic groups. These ingredients made the determination of the irreducible subquotients of the principal series for $\widetilde{\mathrm{Sp}(2)}$ very similar to the one obtained in [Matić $\geq$ 2010] for SO(5), but this happens to be insufficient tool in some cases. In these cases, we will use the theta correspondence to again obtain the formal similarity to the SO(5) case. This similarity was expected; see for example [Zorn 2010]. After determining complete nonunitary dual, modulo cuspidal representations, the unitary dual follows in the almost the same way as in [Matić 2010], but after discussion of some exceptional

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cases (for example, the discussion of the unitary principal series): In the case of the odd orthogonal group SO(5), the irreducibility of the unitary principal series follows from the considerations about R-groups, and in the case of $\widetilde{\mathrm{Sp}(2)}$, since the R-group theory for metaplectic groups is not available in its full generality, irreducibility is obtained using theta correspondence. In the forthcoming paper [Hanzer and Matić 2010], we extend the methods used here to prove for general n the irreducibility of unitary principal series for $\widetilde{\mathrm{Sp}(n)}$. We hope these results will have applications in the theory of automorphic forms.

We now describe the content of the paper. In Section 2, we recall the definition of the metaplectic double cover $\widetilde{\mathrm{Sp}(n)}$. We also recall the notions of parabolic subgroups, Jacquet functor, and parabolic induction in the context of metaplectic groups. We then recall the notion of the dual pair, and the lifts of an irreducible representations of one member of the pair dual to the Weil representation of the ambient metaplectic group. We recall the criteria for the square integrability and temperedness of the irreducible representations of metaplectic groups, due to Ban and Jantzen [2009] and recall the classification of the irreducible genuine representations of $\widetilde{\mathrm{Sp}(n)}$ obtained in [Hanzer and Muić 2010]. In Section 3, we analyze the principal series for $\widetilde{\mathrm{Sp}(2)}$, using both theta correspondence and Tadić's methods applied to metaplectic groups. In Section 4, we determine the unitary dual of $\widetilde{\mathrm{Sp}(2)}$ supported in the minimal parabolic subgroup. In Section 5, we describe irreducible representations of $\widetilde{\mathrm{Sp}(2)}$ supported on maximal parabolic subgroups, and the unitary dual of $\widetilde{\mathrm{Sp}(2)}$ supported on maximal parabolic subgroups.

2. Preliminaries

Let $\widetilde{\mathrm{Sp}(2)}$ be the unique nontrivial two-fold central extension of symplectic group $\mathrm{Sp}(2, F)$, where F is a non-Archimedean local field of characteristic different from two. In other words, we have maps

$$1 \to \mu_2 \to \widetilde{\mathrm{Sp}(2)} \to \mathrm{Sp}(2, F) \to 1.$$

The multiplication in $\widetilde{\mathrm{Sp}(2)}$, which is as a set given by $\mathrm{Sp}(2, F) \times \mu_2$, is given the cocycle of [Ranga Rao 1993]. The topology of $\widetilde{\mathrm{Sp}(n)}$ is explained in detail in [Hanzer and Muić 2010, Section 3.3]. There exist compact open subgroups of $\mathrm{Sp}(n)$ that split in $\widetilde{\mathrm{Sp}(n)}$. Recall that a maximal good compact subgroup $\mathrm{Sp}(O_F)$ splits if the residual characteristic of F is odd (here O_F denotes the ring of integers on F). In [Hanzer and Muić 2010], the metaplectic group $\widetilde{\mathrm{Sp}(2)}$ was denoted by $\widetilde{\mathrm{Sp}(W_2)}$. We say that the representation of $\widetilde{\mathrm{Sp}(2)}$ (or, more generally, $\widetilde{\mathrm{Sp}(n)}$) is smooth if, for every vector v in the representation space V, there exists a compact open subgroup K_1 of $\mathrm{Sp}(2)$ that splits in $\widetilde{\mathrm{Sp}(2)}$ and fixes v. The representation is admissible if for every K_1 as above, the space V^{K_1} is finite-dimensional.

Lemma 2.1. *Irreducible smooth representations of* $\widetilde{\mathrm{Sp}(n)}$ *are admissible.*

Proof. First, we prove that an irreducible, smooth, cuspidal representation of $\widetilde{\mathrm{Sp}(n)}$ is admissible. We can proceed as in the corollary on [Bernstein 1992, page 36], so we have to prove that an irreducible cuspidal representation of $\widetilde{\mathrm{Sp}(n)}$ is compact, and that was proved in proving [Hanzer and Muić 2009, Lemma 3.1]. Then, on this compact irreducible representation we can apply [Bernstein 1992, Proposition 11], which is formulated for a general totally disconnected group (so the metaplectic groups satisfy the conditions), and says that finitely generated compact representations are admissible. The claim follows since every irreducible smooth representation can be embedded in the representations parabolically induced from the cuspidal representations of Levi subgroups (and for the representations of Levi subgroups, the same reasoning as above shows that these representations are also admissible), which was proved in [Hanzer and Muić 2008, Proposition 4.4]. □

In this paper we are interested only in genuine representations of $\widetilde{\mathrm{Sp}(n)}$ (that is, those that do not factor through μ_2). So, let $R(n)$ be the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{\mathrm{Sp}(n)}$ (that is, a free abelian group over the set of all irreducible genuine representations of $\widetilde{\mathrm{Sp}(n)}$), and define $R = \bigoplus_{n\geq 0} R(n)$. By ν we denote a character of $\mathrm{GL}(k, F)$ defined by $|\det|_F$. Further, for an ordered partition $s = (n_1, n_2, \dots, n_j)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $\mathrm{Sp}(n, F)$ (consisting of block upper-triangular matrices), whose Levi factor equals $\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)\times\cdots\times \mathrm{GL}(n_j) \times \mathrm{Sp}(n - |s|, F)$, where $|s| = \sum_{i=1}^{j} n_i$. By a standard parabolic subgroup $\widetilde{P}_s$ of $\widetilde{\mathrm{Sp}(n)}$ we mean the preimage of P_s in $\widetilde{\mathrm{Sp}(n)}$. We have the analogous notation for the Levi subgroups of the metaplectic groups, and, for the completeness, we explicitly describe the structure of the parabolic and Levi subgroups, as explained in [Hanzer and Muić 2010, Section 2.2]. There is a natural splitting from the unipotent radical of N_s of the corresponding standard parabolic subgroup P_s of $\mathrm{Sp}(n, F)$ to its cover [Mœglin et al. 1987, Lemma 2.9 on page 43]; let N'_s be the image of that homomorphism. We then have $\widetilde{P}_s \cong \widetilde{M}_s \ltimes N'_s$.

We can explicitly describe $\widetilde{M}_s$ as follows. There is a natural epimorphism

$$\phi : \widetilde{\mathrm{GL}(n_1, F)} \times \cdots \times \widetilde{\mathrm{GL}(n_k, F)} \times \widetilde{\mathrm{Sp}(W_{n-|s|})} \to \widetilde{M}_s \tag{1}$$

given by

$$([g_1, \epsilon_1], \dots, [g_k, \epsilon_k], [h, \epsilon]) \mapsto [(g_1, g_2, \dots, g_k, h), \epsilon_1 \cdots \epsilon_k \epsilon \beta], \tag{2}$$

with $\beta = \prod_{i<j} (\det g_i, \det g_j)_F \big(\prod_{i=1}^{k} (\det g_i, x(h))_F\big)$, where $x(h)$ is defined in [Ranga Rao 1993, Lemma 5.1] and $(\cdot\,, \cdot)_F$ denotes the Hilbert symbol of the field F. Although $\widetilde{M}$ is not exactly the product at left in (1), it differs from it by a finite subgroup that enables us to write every irreducible representation π

of $\widetilde{M}$ in the form $\pi_1 \otimes \cdots \otimes \pi_k \otimes \pi'$, where the representations $\pi_1, \ldots, \pi_k, \pi'$ are either all genuine or none genuine. This simple property enables us to set up Tadić's machinery [Tadić 1995; Hanzer and Muić 2008] of parabolic induction and Jacquet functors. Recall that the irreducible representations in this paper, unless mentioned otherwise, are assumed to be genuine (that is, nontrivial on μ_2). Also, the cuspidality of representations is defined in the same way as for the reductive groups (because of the splitting of the unipotent radical) and characterized in terms if the support of the matrix coefficients also as for the reductive groups.

Let σ be a representation of $\widetilde{\mathrm{Sp}(2)}$. Following the notation introduced in [Hanzer and Muić 2010], we denote by $R_{\widetilde{P_{(1,1)}}}(\sigma)$ the normalized Jacquet module with respect to $\widetilde{M}_{(1,1)}$; by $R_{\widetilde{P}_1}(\sigma)$ the normalized Jacquet module with respect to $\widetilde{M}_{(1)}$; and by $R_{\widetilde{P}_2}(\sigma)$ the normalized Jacquet module with respect to $\widetilde{M}_{(2)}$.

We fix a nontrivial additive character ψ of F and let $\omega_{n,r}$ be the pullback of the Weil representation $\omega_{n(2r+1),\psi}$ of the group $\mathrm{Sp}(\widetilde{n(2r+1)})$, restricted to the dual pair $\widetilde{\mathrm{Sp}(n)} \times O(2r+1)$ [Kudla 1996, Chapter II]. Here $O(2r+1)$ denotes the split odd-orthogonal group of the split rank r, with the one-dimensional anisotropic space sitting at the bottom of the orthogonal tower [Kudla 1996, Chapter III.1]. The standard parabolic subgroups (containing the upper triangular Borel subgroup) of $O(2r+1)$ have the analogous description as the standard parabolic subgroups of $\mathrm{Sp}(n, F)$; we use the analogous notation for the normalized Jacquet functors.

Let σ be an irreducible smooth genuine representation of $\widetilde{\mathrm{Sp}(n)}$. We write $\Theta(\sigma, r)$ for the smooth isotypic component of σ in $\omega_{n,r}$ (we view it as a representation of $O(2r+1)$). Denote by r_0 the smallest r such that $\Theta(\sigma, r) \neq 0$. When σ is cuspidal, we know that $\Theta(\sigma, r_0)$ is an irreducible cuspidal representation of $O(2r+1)$.

Let $\widetilde{\mathrm{GL}(n, F)}$ be a double cover of $\mathrm{GL}(n, F)$, where the multiplication is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F).$$

Here $\epsilon_i \in \mu_2$ for $i = 1, 2$ and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field F, and this cocycle on $\mathrm{GL}(n, F)$ is actually a restriction of Ranga Rao's cocyle on $\mathrm{Sp}(n, F)$ to $\mathrm{GL}(n, F)$, if we view this group as the Siegel Levi subgroup of $\mathrm{Sp}(n, F)$ [Kudla 1986, page 235]. Now we fix a character $\chi_{V,\psi}(g, \epsilon) = \chi_V(\det g)\epsilon\gamma(\det g, \psi_{1/2})^{-1}$ of $\widetilde{\mathrm{GL}(n, F)}$. Here γ denotes the Weil invariant, while χ_V is a character related to the quadratic form on $O(2r+1)$ [Kudla 1996, pages 17 and 37], and $\psi_a(x) = \psi(ax)$ for $a \in F^*$. We may suppose $\chi_V \equiv 1$ (but the arguments that follow are valid without this assumption). We write $\alpha = \chi_{V,\psi}^2$ and observe that α is a quadratic character on $\mathrm{GL}(n, F)$.

The following fact, follows directly from [Hanzer and Muić 2010], and we use it frequently while determining composition series of induced representations: For an irreducible genuine representation π of $\widetilde{\mathrm{GL}(k, F)}$ and an irreducible genuine representation σ of $\widetilde{\mathrm{Sp}(n)}$ we have $\pi \rtimes \sigma = \widetilde{\pi}\alpha \rtimes \sigma$ (in R), where $\pi \rtimes \sigma$ denotes the representation of the group $\widetilde{\mathrm{Sp}(n+k)}$ parabolically induced from the representation $\pi \otimes \sigma$ of the maximal Levi subgroup $\widetilde{M}_{(k)}$. We follow here the usual notation for parabolic induction for classical groups, adapted to the metaplectic case [Tadić 1994; Hanzer and Muić 2010]. We also freely use Zelevinsky's notation [1980] for the parabolic induction for general linear groups. We denote the Steinberg representation of the reductive algebraic group G by St_G and the trivial representation of that group by 1_G. Following [Kudla 1996], we let $\omega^+_{\psi_a,n}$ denote the even part of the Weil representation of $\widetilde{\mathrm{Sp}(n)}$ determined by the additive character ψ_a. The nontrivial character of μ_2, when we view it as a representation of $\widetilde{\mathrm{Sp}(0)}$, is denoted by ω_0.

If ζ is a quadratic character of $F^\times$, we can write $\zeta(x) = (xa)_F$ for some $a \in F^\times$. Let $sp_{\zeta,1}$ be an irreducible (square-integrable, according to the criterion for the square-integrability which we recall below) subrepresentation of $\chi_{V,\psi}\zeta\nu^{1/2} \rtimes \omega_0$. Then, as in [Kudla 1996, page 89], we have the exact sequence

$$0 \to sp_{\zeta,1} \longrightarrow \chi_{V,\psi}\zeta\nu^{1/2} \rtimes \omega_0 \longrightarrow \omega^+_{\psi_a,1} \to 0.$$

The results of [Ban and Jantzen 2009] imply that Casselman's criteria for square-integrability and temperedness hold for metaplectic groups in a similar form as for the classical groups (for example symplectic). We now recall these criteria.

Let π be an admissible irreducible (genuine) representation of $\widetilde{\mathrm{Sp}(n)}$ and let $\widetilde{P}_s$ be any standard parabolic subgroup minimal with respect to the property that $R_{\widetilde{P}_s}(\pi) \neq 0$. Write $s = (n_1, \ldots, n_k)$ and let σ be any irreducible subquotient of $R_{\widetilde{P}_s}(\pi)$. As we saw above, we can write $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \rho$, where ρ_i is an irreducible genuine cuspidal representation of some $\widetilde{\mathrm{GL}(n_i, F)}$ for $i = 1, \ldots, k$ and ρ is an irreducible genuine cuspidal representation of some $\widetilde{\mathrm{Sp}(n-l)}$. Define $e(\rho_i)$ by $\rho_i = \nu^{e(\rho_i)}\rho_i^u$, where ρ_i^u is unitary for $1 \leq i \leq n$.

Assume that the inequalities

$$\begin{aligned} n_1 e(\rho_1) &> 0, \\ n_1 e(\rho_1) + n_2 e(\rho_2) &> 0, \\ &\vdots \\ n_1 e(\rho_1) + n_2 e(\rho_2) + \cdots + n_k e(\rho_k) &> 0. \end{aligned}$$

hold for every s and σ as above. Then π is a square integrable representation. For such s and σ, these inequalities also hold if π is a square integrable representation.

The criterion for tempered representations is given by replacing every $>$ with $\geq$.

We recall the definition of a negative representation [Hanzer and Muić 2010, Definition 4.1].

Let σ be an admissible irreducible genuine representation of $\widetilde{\mathrm{Sp}(n)}$. Then σ is a strongly negative (respectively, negative) representation if and only if for every embedding $\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \rho$, where ρ_i for $1 \leq i \leq k$ and ρ are irreducible genuine supercuspidal representations of some of the $\widetilde{\mathrm{GL}}$ and of some $\widetilde{\mathrm{Sp}(n-l)}$, we have

$$\begin{aligned} n_1 e(\rho_1) &< 0 \quad \text{(respectively, } \leq 0),\\ n_1 e(\rho_1) + n_2 e(\rho_2) &< 0 \quad \text{(respectively, } \leq 0),\\ &\ \vdots \\ n_1 e(\rho_1) + n_2 e(\rho_2) + \cdots + n_k e(\rho_k) &< 0 \quad \text{(respectively, } \leq 0). \end{aligned}$$

As soon as σ as above is genuine, the ρ_i and ρ are also necessarily genuine. For notation, we recall [Hanzer and Muić 2010, Theorems 4.5 and 4.6]. Recall that, for a cuspidal representation ρ of some $\mathrm{GL}(m_\rho, F)$, a segment Δ is a set of cuspidal representations $\Delta = \{\rho, \nu\rho, \ldots, \nu^{k-1}\rho\}$ and $\langle\Delta\rangle$ is a unique irreducible subrepresentation of $\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho$. We use the same notation for genuine cuspidal representations of $\widetilde{\mathrm{GL}(m_\rho, F)}$ since the transfer from nongenuine to genuine representations in the case of $\widetilde{\mathrm{GL}(m_\rho, F)}$ is particularly simple (obtained by multiplication with the character $\chi_{V,\psi}(g, \epsilon)$ defined above). Now the two theorems above follow from the analogous results in the case of classical reductive groups of [Hanzer and Muić 2008], since the analogous calculations with Jacquet modules are possible, due to results in [Hanzer and Muić 2010].

- Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments (of genuine representations) such that $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ (we also allow $k = 0$). Let σ_{neg} be a negative (genuine) representation. Then the induced representation $\langle\Delta_1\rangle \times \langle\Delta_2\rangle \times \cdots \times \langle\Delta_k\rangle \rtimes \sigma_{\mathrm{neg}}$ has a unique irreducible subrepresentation; we denote it by $\langle\Delta_1, \ldots, \Delta_k; \sigma_{\mathrm{neg}}\rangle$.
- If σ is an irreducible admissible genuine representation of $\widetilde{\mathrm{Sp}(n)}$, then there exist a sequence of segments (of genuine representations) $\Delta_1, \ldots, \Delta_k$ such that $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ and a negative (genuine) representation σ_{neg} such that $\sigma \simeq \langle\Delta_1, \ldots, \Delta_k; \sigma_{\mathrm{neg}}\rangle$.

We can carry over Tadić's structure formula for classical groups to the metaplectic case [Hanzer and Muić 2010, Proposition 4.5], which enables us to calculate Jacquet modules of the induced representations. In more detail, let

$$R^{\mathrm{gen}} = \bigoplus_n R(\widetilde{\mathrm{GL}(n, F)})_{\mathrm{gen}},$$

where $R(\widetilde{\mathrm{GL}(n, F)})_{\mathrm{gen}}$ denotes the Grothendieck group of finite length, smooth, genuine representations of $\widetilde{\mathrm{GL}(n, F)}$. We denote by $\times$ the linear extension to

$R^{\mathrm{gen}} \otimes R^{\mathrm{gen}}$ of the parabolic induction (from a maximal parabolic subgroup). We can easily check that if σ is an irreducible genuine representation of $\widetilde{\mathrm{Sp}(W_n)}$, then $r_k(\sigma)$, the normalized Jacquet module of σ with respect to the standard maximal parabolic $\widetilde{P}_k$, is a genuine representation of $\widetilde{M_{(k)}}$ and as such can be interpreted as a (genuine) representation of $\widetilde{\mathrm{GL}(k, F)} \times \widetilde{\mathrm{Sp}(W_{n-k})}$, that is, as an element of $R^{\mathrm{gen}} \otimes R$, with R defined as above. So for irreducible genuine σ, we can introduce $\mu^*(\sigma) \in R^{\mathrm{gen}} \otimes R$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \mathrm{s.s.}(r_k(\sigma)),$$

where s.s. stands for semisimplification. We can extend μ^* linearly to the whole R. Using Jacquet modules for the maximal parabolic subgroups of $\widetilde{\mathrm{GL}(n, F)}$ we can analogously define

$$m^*(\pi) = \sum_{k=0}^{n} \mathrm{s.s.}(r_k(\pi)) \in R^{\mathrm{gen}} \otimes R^{\mathrm{gen}}$$

for a genuine, irreducible representation π of $\widetilde{\mathrm{GL}(n, F)}$ and then extend m^* linearly to the whole R^{gen}. Let $\kappa : R^{\mathrm{gen}} \otimes R^{\mathrm{gen}} \to R^{\mathrm{gen}} \otimes R^{\mathrm{gen}}$ be defined by $\kappa(x \otimes y) = y \otimes x$. We extend the contragredient $\tilde{}$ to an automorphism of R^{gen} naturally. Finally, we define

$$M^* = (m \otimes \mathrm{id}) \circ (\tilde{}\alpha \otimes m^*) \circ \kappa \circ m^*.$$

Here $\tilde{}\alpha$ means taking contragredient of a representation, and then multiplying by the character α, acting on the general linear group as $\alpha(g) = (\det g, -1)_F$.

For π in R^{gen} and σ from R, we have

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

Using this formula for the induced representations of $\widetilde{\mathrm{Sp}(2)}$, we get the following:

- Fix an admissible representation π of $\widetilde{\mathrm{GL}(2)}$, and suppose that π is of finite length. Let $m^*(\pi) = 1 \otimes \pi + \sum_i \pi_i^1 \otimes \pi_i^2 + \pi \otimes 1$, where $\sum_i \pi_i^1 \otimes \pi_i^2$ is a decomposition into a sum of irreducible representations. Now we have

$$\begin{aligned}\mu^*(\pi \rtimes \omega_0) = 1 \otimes \pi \rtimes \omega_0 + \sum_i \pi_i^1 \otimes \pi_i^2 \rtimes \omega_0 + \sum_i \alpha\widetilde{\pi}_i^2 \otimes \pi_i^1 \rtimes \omega_0 \\ + \pi \otimes \omega_0 + \alpha\widetilde{\pi} \otimes \omega_0 + \sum_i \pi_i^1 \times \alpha\widetilde{\pi}_i^2 \otimes \omega_0.\end{aligned}$$

- Fix an admissible representation π of $\widetilde{\mathrm{GL}(1)}$ and an admissible representation σ of $\widetilde{\mathrm{Sp}(1)}$. If we have

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_i \sigma_i^1 \otimes \sigma_i^2,$$

where σ_i^1 and σ_i^2 are irreducible representations, then

$$\mu^*(\pi \rtimes \sigma) = 1 \otimes \pi \rtimes \sigma + \pi \otimes \sigma + \alpha\widetilde{\pi} \otimes \sigma + \sum_i \sigma_i^1 \otimes \pi \rtimes \sigma_i^2 + \sum_i \pi \times \sigma_i^1 \otimes \sigma_i^2 + \sum_i \sigma_i^1 \times \alpha\widetilde{\pi} \otimes \sigma_i^2.$$

From now on, $\widehat{F^\times}$ denotes the set of the unitary characters of $F^\times$, while $\widetilde{F^\times}$ denotes those that are not necessarily unitary.

3. Principal series

We first state an important reducibility result that follows directly from [Hanzer and Muić 2009, Theorems 3.5. and 4.2].

Proposition 3.1. *Let $\chi \in \widehat{F^\times}$ and let $s \in \mathbb{R}$ be nonnegative. The representation $\chi_{V,\psi}\nu^s\chi \rtimes \omega_0$ of $\widetilde{\mathrm{Sp}(1)}$ reduces if and only if $\chi^2 = 1_{F^\times}$ and $s = 1/2$.*

Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. In R we have (see [Kudla 1996, page 89]

$$\chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0 = sp_{\zeta,1} + \omega^+_{\psi_a,1}.$$

The following proposition is well known and follows easily from the analogous results for the split SO(3) and SO(5).

Proposition 3.2. (1) *Let $\chi \in \widehat{F^\times}$ and suppose $s \in \mathbb{R}$ is nonnegative. The representation $\nu^s\chi \rtimes 1$ of $O(3)$ reduces if and only if $\chi^2 = 1_{F^\times}$ and $s = 1/2$. In that situation, the length of $\nu^{1/2}\chi \rtimes 1$ is two, and this representation has the unique subrepresentation that is square integrable.*

(2) *Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$. Then, the unitary principal series $\zeta_1 \times \zeta_2 \rtimes 1$ of $O(5)$ is irreducible.*

We use these two propositions in the sequel without explicitly mentioning them.

3.1. *Unitary principal series.* In this subsection we prove irreducibility of the unitary principal series $\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\chi_2 \rtimes \omega_0$, where $\chi_i \in \widehat{F^\times}$ for $i = 1, 2$.

Let Π denote the representation $\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\chi_2 \rtimes \omega_0$. Using the structure formula for $\mu^*(\Pi)$ from the end of the previous section, we get

$$R_{\widetilde{P}_1}(\Pi) = \chi_{V,\psi}\chi_1^{-1} \otimes \chi_{V,\psi}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\chi_2^{-1} \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0 + \chi_{V,\psi}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0.$$

Remark. Let π be an irreducible subrepresentation of Π. Because of irreducibility of the representations $\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\chi_2$ and $\chi_{V,\psi}\chi_i \rtimes \omega_0$ for $i = 1, 2$, we get

$$\pi \hookrightarrow \Pi \simeq \chi_{V,\psi}\chi_1^{-1} \times \chi_{V,\psi}\chi_2 \rtimes \omega_0 \simeq \chi_{V,\psi}\chi_2^{-1} \times \chi_{V,\psi}\chi_1 \rtimes \omega_0.$$

If $\chi_i \neq \chi_i^{-1}$ holds for both $i = 1, 2$ and $\chi_1 \neq \chi_2^{\pm 1}$, then Frobenius reciprocity implies that $R_{\widetilde{P}_1}(\pi) = R_{\widetilde{P}_1}(\Pi)$, so $\pi = \Pi$ and the representation Π is irreducible.

Now we prove the irreducibility of the unitary principal series for general unitary characters. Let ζ_1, ζ_2 be the unitary characters of $F^\times$. We prove irreducibility of the representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ using theta correspondence, beginning with this lemma:

Lemma 3.3. *Let π_1 be an irreducible subrepresentation of $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$. Then $\Theta(\pi_1, 2) = \zeta_1 \times \zeta_2 \rtimes 1$.*

Proof. According to the stable range condition [Kudla 1996, page 48], $\Theta(\pi_1, 4) \neq 0$ (observe that $\Theta(\pi_1, 4)$ is a smooth representation of $O(9)$). We have epimorphisms $\omega_{2,4} \to \pi_1 \otimes \Theta(\pi_1, 4)$ and $R_{P_1}(\omega_{2,4}) \to \pi_1 \otimes R_{P_1}(\Theta(\pi_1, 4))$. If τ is an irreducible quotient of $\Theta(\pi_1, 4)$, then [Kudla 1986, Corollary 2.6] implies $[\tau] = [\nu^{-3/2}, \nu^{-1/2}, \zeta_1, \zeta_2; 1]$, where $[\tau]$ denotes the cuspidal support of τ. Clearly, $R_{P_{(1,1,1,1)}}(\tau) \geq \nu^{l_1/2} \otimes \nu^{l_2/2} \otimes \zeta_1^{\pm 1} \otimes \zeta_2^{\pm 1}$ or $R_{P_{(1,1,1,1)}}(\tau) \geq \zeta_1^{\pm 1} \otimes \nu^{l_1/2} \otimes \zeta_2^{\pm 1} \otimes \nu^{l_2/2}$ (or we have some order of factors) for some $l_1, l_2 \in \{\pm 1, \pm 3\}$. If we assume that in the Jacquet module $R_{P_{(1,1,1,1)}}(\tau)$ there is an irreducible subquotient as above whose first factor consists of a unitary character, then, using [Bernstein 1992, Lemma 26] together with Frobenius reciprocity, easily follows that

$$\mathrm{Hom}(\tau, \zeta_1^{\pm 1} \times \nu^{l_1/2} \times \zeta_2^{\pm 1} \times \nu^{l_2/2} \rtimes 1) \neq 0.$$

But since $\zeta_i^{\pm 1} \times \nu^{l_i/2} \cong \nu^{l_i/2} \times \zeta_i^{\pm 1}$, we have $\mathrm{Hom}(\tau, \nu^{l_1/2} \times \zeta_1^{\pm 1} \times \zeta_2^{\pm 1} \times \nu^{l_2/2} \rtimes 1) \neq 0$. So, there is an irreducible subquotient τ' of $\zeta_1 \times \zeta_2 \times \nu^{l_2/2} \rtimes 1$ such that τ is a subrepresentation of $\nu^{l_1/2} \rtimes \tau'$. This implies that $R_{P_1}(\tau)(\nu^{l_1/2})$, the isotypic component of $R_{P_1}(\tau)$ along the generalized character $\nu^{l_1/2}$, is nonzero, as is $R_{P_1}(\Theta(\pi_1, 4))(\nu^{l_1/2})$.

Observations above imply that there is an irreducible representation τ_1 of $O(3)$ such that the mappings $R_{P_1}(\omega_{2,4}) \to \pi_1 \otimes R_{P_1}(\Theta(\pi_1, 4)) \to \pi_1 \otimes \nu^{l_1/2} \otimes \tau_1$ are epimorphisms. We denote the epimorphism $R_{P_1}(\omega_{2,4})) \to \pi_1 \otimes \nu^{l_1/2} \otimes \tau_1$ by T. Now $R_{P_1}(\omega_{2,4})$ has the filtration in which

- $I_{10} = \nu^{-3/2} \otimes \omega_{2,3}$ is the quotient and
- $I_{11} = \mathrm{Ind}_{\mathrm{GL}(1) \times \widetilde{P}_1 \times O(3)}^{M_1 \times \widetilde{\mathrm{Sp}(2)}}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,3})$ is the subrepresentation.

See [Kudla 1996, page 57] and [Hanzer and Muić 2009, Proposition 3.3], where the notation is explained in detail.

Suppose $T|_{I_{11}} \neq 0$. Because $\chi_{V,\psi}\nu^{-l_1/2}$ is the isotypic component of $\nu^{l_1/2}$ in the $\widetilde{\mathrm{GL}(1, F)} \times \mathrm{GL}(1, F)$-module $\chi_{V,\psi}\Sigma_1'$, by applying the second Frobenius we get a nonzero $\widetilde{\mathrm{GL}(1, F)} \times \mathrm{GL}(1, F) \times \widetilde{\mathrm{Sp}(1)} \times O(3)$-homomorphism

$$\nu^{l_1/2} \otimes \chi_{V,\psi}\nu^{-l_1/2} \otimes \omega_{1,3} \to \nu^{l_1/2} \otimes \tau_1 \otimes \widetilde{R_{\widetilde{P}_1}(\widetilde{\pi}_1)},$$

which implies that $\widetilde{R_{\widetilde{P}_1}(\widetilde{\pi}_1)}(\chi_{V,\psi}\nu^{-l_1/2}) \neq 0$. Because $l_1 \neq 0$, this contradicts our assumption $\pi_1 \hookrightarrow \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$; hence $T|_{I_{11}} = 0$. Therefore, we can consider T as an epimorphism $I_{10} \to \pi_1 \otimes \nu^{l_1/2} \otimes \tau_1$. Consequently, $l_1 = -3$ and there is an epimorphism $\omega_{2,3} \to \pi \otimes \tau_1$. Obviously, $\Theta(\pi_1, 3) \neq 0$.

Repeating the same procedure once again, we obtain $\Theta(\pi_1, 2) \neq 0$. Since the cuspidal support of each irreducible quotient of $\Theta(\pi_1, 2)$ equals $[\zeta_1, \zeta_2; 1]$, all of the irreducible quotients of $\Theta(\pi_1, 2)$ are equal to $\zeta_1 \times \zeta_2 \rtimes 1$. □

Proposition 3.4. *Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$. Then the unitary principal series representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ is irreducible.*

We present two proofs of this proposition, both based on the previous lemma. The first proof is much simpler than the second — it also uses some known results about Whittaker models for the principal series for metaplectic groups, but we have to assume that the residue characteristic of F is odd. The second proof is more technical, but it doesn't depend on the residue characteristic of F. We feel that presenting both proofs may be useful.

First proof of Proposition 3.4. We denote the representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ by Π. Suppose that the residue characteristic of F is not 2. Howe's duality conjecture and lemma then implies that the representation $\Theta(\zeta_1 \times \zeta_2 \rtimes 1, 2)$ has a unique irreducible quotient, so, by Lemma 3.3, all the irreducible subrepresentations of Π are isomorphic, that is,

$$\Pi = \pi \oplus \cdots \oplus \pi. \tag{3}$$

Now, observe that the representation Π has a unique Whittaker model [Banks 1998; Szpruch 2007]. In more words, for a nondegenerate character θ of the unipotent radical U of Borel subgroup of $\mathrm{Sp}(n)$ (observe that $\widetilde{\mathrm{Sp}(n)}$ splits over U, and the mapping $n \mapsto (n, 1)$ is the splitting) and a genuine character $\chi_{V,\psi}\zeta_1 \otimes \cdots \otimes \chi_{V,\psi}\zeta_n$ of $\widetilde{T}$ (where $\widetilde{T}$ denotes the preimage of maximal diagonal torus in $\mathrm{Sp}(n)$), we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{Sp}_n}}(\chi_{V,\psi}\zeta_1 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0, \mathrm{Ind}_U^{\widetilde{\mathrm{Sp}(n)}}(\theta)) = 1.$$

This forces that the number of copies of π in (3) to be one, and this finishes the first proof. □

Second proof of Proposition 3.4. We have already seen that there is an epimorphism $R_{\widetilde{P}_1}(\omega_{2,2}) \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi} \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1$, so

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1, R_{\widetilde{P}_1}(\omega_{2,2})) \neq 0.$$

$R_{\widetilde{P}_1}(\omega_{2,2})$ has the filtration in which

- $J_{10} = \chi_{V,\psi}\nu^{1/2} \otimes \omega_{1,2}$ is the quotient and
- $J_{11} = \mathrm{Ind}_{\widetilde{\mathrm{GL}(1)} \times P_1 \times \widetilde{\mathrm{Sp}(1)}}^{\widetilde{M_1} \times O(2)}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1})$ is the subrepresentation.

Lemma 3.5. *There is an isomorphism*

$\mathrm{Hom}_{\widetilde{M}_1}(R_{\widetilde{P}_1}(\omega_{2,2}),\, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0) \cong \mathrm{Hom}_{\widetilde{M}_1}(J_{11},\, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0)$

of vector spaces that is given by restriction (*that is*, $T \mapsto T|_{J_{11}}$).

Proof of Lemma 3.5. The map obtained by the restriction is obviously a homomorphism, while the injectivity follows directly. Surjectivity is proved as follows:

We consider the filtration $0 \subseteq W_2 \subseteq W_1 \subseteq R_{\widetilde{P}_1}(\omega_{2,2})$, where W_1 is the representation J_{11}, and $W_1/W_2 \cong \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11})$. Observe that

$$(R_{\widetilde{P}_1}(\omega_{2,2})/W_2)/(W_1/W_2) \cong R_{\widetilde{P}_1}(\omega_{2,2})/W_1 \cong J_{10}.$$

Using standard argument, it can be proved that the representation $R_{\widetilde{P}_1}(\omega_{2,2})/W_2$ is $\widetilde{\mathrm{GL}(1)}$-finite. Then, using the decomposition along the generalized central characters, which in this case coincide with the central characters because W_1/W_2 and J_{10} have different central characters, we obtain

$$R_{\widetilde{P}_1}(\omega_{2,2})/W_2 \cong W_1/W_2 \oplus J_{10}.$$

Now an element of $\mathrm{Hom}_{\widetilde{M}_1}(J_{11},\, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0)$ is trivial on W_2, so it can be extended to $R_{\widetilde{P}_1}(\omega_{2,2})$ in an obvious way and surjectivity is proved. $\square$

Using a standard relation between taking a smooth part of the isotypic component of a representation and the homomorphism functor [Hanzer and Muić 2009, page 10], it follows from Lemma 3.5 that

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, R_{\widetilde{P}_1}(\omega_{2,2})) \cong \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11}),$$

if we can prove that $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11})$ is admissible.

Lemma 3.6. *We have* $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11}) = \zeta_1 \times \zeta_2 \rtimes 1$.

Proof of Lemma 3.6. Since $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1$ is a quotient of J_{11}, there is an epimorphism $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11}) \to \zeta_1 \times \zeta_2 \rtimes 1$.

Applying [Hanzer and Muić 2009, Lemma 3.2], we have

$$\begin{aligned}
&\mathrm{Hom}_{\widetilde{M}_1 \times O(2)}(J_{11},\; \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11}))\\
&\cong \mathrm{Hom}_{\widetilde{M}_1 \times M(1)}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1},\\
&\qquad\qquad \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11}))).
\end{aligned}$$

For every intertwining map T from the first space, let T_0 be the corresponding intertwining map from the second space. Let φ be a natural homomorphism belonging to the first space.

Since $\chi_{V,\psi}\zeta_1 \otimes \zeta_1^{-1}$ (respectively, $\chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_2 \rtimes 1$) are the corresponding isotypic components in the $\widetilde{\mathrm{GL}(1, F)} \times \mathrm{GL}(1, F)$-module $\chi_{V,\psi}\Sigma_1'$(respectively, in the $\widetilde{\mathrm{Sp}(1)} \times O(3)$-module $\omega_{1,1}$), irreducibility of these isotypic components implies that the image of φ_0 is isomorphic to $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \rtimes 1$. Now, we write $\varphi_0 = \varphi'' \circ \varphi'$, where φ' is a canonical epimorphism

$$\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1} \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \rtimes 1$$

and φ'' is an inclusion of the representation

$$\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \rtimes 1$$

in $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}))$. Observe that $\mathrm{Ind}(\varphi')$ is a homomorphism

$$\mathrm{Ind}_{\widetilde{\mathrm{GL}(1)} \times P_1 \times \widetilde{\mathrm{Sp}(1)}}^{\widetilde{M}_1 \times O(2)}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1}) \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \rtimes 1.$$

Let φ_1 be an operator belonging to

$$\mathrm{Hom}(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1, \\ \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}))$$

such that $(\varphi_1)_0 = \varphi''$.

Lemma 3.7. *Under the assumptions above*, $(\varphi_1 \circ \mathrm{Ind}(\varphi'))_0 = \varphi_0$.

Proof of Lemma 3.7. We prove it much more generally. Let (π, V) be a smooth representation of some Levi subgroup M' in the parabolic P' and the opposite parabolic $\overline{P'}$ of the group G' (which is one of the groups we are considering, that is, metaplectic or odd orthogonal) and let (Π, W) be a smooth representation of G'. Then the second Frobenius isomorphism asserts

$$\mathrm{Hom}_{G'}(\mathrm{Ind}_{M'}^{G'}(\pi), \Pi) \cong \mathrm{Hom}_{M'}(\pi, R_{\overline{P'}}(\Pi)).$$

Let $\psi \hookrightarrow R_{\overline{P'}}(\mathrm{Ind}_{M'}^{G'}(\pi))$ be an embedding corresponding to the open cell $P'\overline{P'}$ in G' given in the following way:

For an open compact subgroup K of G' that has Iwahori decomposition with respect to both P' and $\overline{P'}$, and for $v \in V^{K \cap M'}$, we define

$$f_{v,K}(g) = \frac{1}{\mathrm{meas}_{\overline{N'}}(K \cap \overline{N'})} \begin{cases} 0 & \text{if } g \notin P'K \\ \delta_P^{1/2}(m)\pi(m)v & \text{if } g = mnk \\ & \quad \text{for } m \in M', n \in N', k \in K. \end{cases}$$

Then $\psi : v \mapsto f_{v,K} + \mathrm{Ind}_{M'}^{G'}(\pi)(\overline{N'})$ is independent on the choice of K.

For $\varphi \in \mathrm{Hom}_{G'}(\mathrm{Ind}_{M'}^{G'}(\pi), \Pi)$, we take φ_0 to be the corresponding element of $\mathrm{Hom}_{M'}(\pi, R_{\overline{P}}(\Pi))$. It follows that $\varphi_0(v) = \varphi(f_{v,K}) + \Pi(\overline{N'})$. Write $\varphi_0 = \varphi'' \circ \varphi'$, where φ' denotes the canonical epimorphism $\pi \to \pi/\operatorname{Ker}\varphi_0$ and φ'' denotes the embedding $\pi/\operatorname{Ker}\varphi_0 \hookrightarrow R_{\overline{P'}}$. So, we are able to construct the mapping $\operatorname{Ind}(\varphi')$: $\operatorname{Ind}_{M'}^{G'}(\pi) \to \operatorname{Ind}_{M'}^{G'}(\pi/\operatorname{Ker}\varphi_0)$. Since

$$\operatorname{Hom}_{M'}(\pi/\operatorname{Ker}\varphi_0,\, R_{\overline{P'}}(\Pi)) \cong \operatorname{Hom}_{G'}(\operatorname{Ind}_{M'}^{G'}(\pi/\operatorname{Ker}\varphi_0),\, \Pi),$$

analogously as above, we see there is an element $\varphi_1 \in \operatorname{Hom}_{G'}(\operatorname{Ind}_{M'}^{G'}(\pi/\operatorname{Ker}\varphi_0),\, \Pi)$ such that $(\varphi_1)_0 = \varphi''$.

To prove $(\varphi_1 \circ \operatorname{Ind}(\varphi'))_0 = \varphi_0$, it is enough to prove $(\varphi_1 \circ \operatorname{Ind}(\varphi'))_0 = (\varphi_1)_0 \circ \varphi'$. Let $v \in V$. Clearly, $\varphi'(v) = v + \operatorname{Ker}\varphi_0$. Further,

$$(\varphi_1)_0(\varphi'(v)) = \varphi_1(f_{v+\operatorname{Ker}\varphi_0,K}) + \Pi(\overline{N'}),$$
$$(\varphi_1 \circ \operatorname{Ind}(\varphi'))_0(v) = \varphi_1(\operatorname{Ind}(\varphi') f_{v,K}) + \Pi(\overline{N'}).$$

It follows easily that $f_{v+\operatorname{Ker}\varphi_0,K} = f_{v,K} + \operatorname{Ker}\varphi_0$ and $\operatorname{Ind}(\varphi') f_{v,K} = f_{v,K} + \operatorname{Ker}\varphi_0$, and the lemma follows. □

We can complete the proof of Lemma 3.6. Lemma 3.7 gives $\varphi_1 \circ \operatorname{Ind}(\varphi') = \varphi$, so the image of φ is a quotient of $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \rtimes 1$. This implies that $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11})$ is a quotient of $\zeta_1^{-1} \times \zeta_2 \rtimes 1$. Since $\zeta_1^{-1} \times \zeta_2 \rtimes 1 \simeq \zeta_1 \times \zeta_2 \rtimes 1$ is an irreducible representation,

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0,\, J_{11}) = \zeta_1 \times \zeta_2 \rtimes 1. \qquad \square$$

Lemma 3.8. *There is an epimorphism* $\Theta(\zeta_1 \times \zeta_2 \rtimes 1, 2) \to \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$.

Proof of Lemma 3.8. We have an isomorphism

$$\operatorname{Hom}_{O(2)}(\omega_{2,2},\, \zeta_1^{-1} \times \zeta_2 \rtimes 1) \cong \operatorname{Hom}(R_{P_1}(\omega_{2,2}),\, \zeta_1^{-1} \otimes \zeta_2 \rtimes 1)$$

of vector spaces, which also an isomorphism of $\widetilde{\mathrm{Sp}(2)}$ modules. By taking the smooth parts, we obtain

$$\operatorname{Hom}_{\widetilde{\mathrm{Sp}(2)} \times O(2)}(\omega_{2,2},\, \zeta_1^{-1} \times \zeta_2 \rtimes 1)_\infty \cong \operatorname{Hom}(R_{P_1}(\omega_{2,2}),\, \zeta_1^{-1} \otimes \zeta_2 \rtimes 1)_\infty,$$

so that $\Theta(\zeta_1^{-1} \times \zeta_2 \rtimes 1, 2)^{\sim} \cong \Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1,\, R_{P_1}(\omega_{2,2}))^{\sim}$.

In the same way as before, we get

$$\Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1,\, R_{P_1}(\omega_{2,2}))^{\sim} \cong \Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1,\, I_{11})^{\sim}.$$

Now, the epimorphism $I_{11} \to \zeta_1^{-1} \otimes \zeta_2 \rtimes 1 \otimes \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ gives an epimorphism $\Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1,\, I_{11})) \to \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$. Since the representations $\zeta_1^{-1} \times \zeta_2 \rtimes 1$ and $\zeta_1 \times \zeta_2 \rtimes 1$ are isomorphic, we obtain the epimorphism $\Theta(\zeta_1 \times \zeta_2 \rtimes 1, 2) \to \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$, which proves the lemma. □

Now we finish the second proof of Proposition 3.4. Suppose that the representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ reduces. Suppose also that it is the representation of length 2 and write $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0 = \pi_1 \oplus \pi_2$. Obviously, $R_{\widetilde{P}_1}(\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0) = R_{\widetilde{P}_1}(\pi_1) \oplus R_{\widetilde{P}_1}(\pi_2)$.

We have, by Lemma 3.8, an epimorphism

$$\omega_{2,2} \to \zeta_1 \times \zeta_2 \rtimes 1 \otimes \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0,$$

which leads to the epimorphisms $R_{\widetilde{P}_1}(\omega_{2,2}) \to \zeta_1 \times \zeta_2 \rtimes 1 \otimes (R_{\widetilde{P}_1}(\pi_1) \oplus R_{\widetilde{P}_1}(\pi_2))$ and $R_{\widetilde{P}_1}(\omega_{2,2}) \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes (\zeta_1 \times \zeta_2 \rtimes 1 \oplus \zeta_1 \times \zeta_2 \rtimes 1)$.

Finally, we obtain an epimorphism

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) \to \zeta_1 \times \zeta_2 \rtimes 1 \oplus \zeta_1 \times \zeta_2 \rtimes 1,$$

which contradicts Lemmas 3.5 and 3.6.

The same proof remains valid if we suppose that $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ is the representation of the length 4. □

3.2. ***Nonunitary principal series.*** First we determine the reducibility points of the representations with cuspidal support in the minimal parabolic subgroup $\widetilde{P_{(1,1)}}$.

Let $\chi_1, \chi_2 \in \widehat{F^\times}$ and $s_i \geq 0$ for $i = 1, 2$, such that $s_i > 0$ for at least one i. Define $\Pi = \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0$. We have

$$\begin{aligned}\mu^*(\Pi) = {} & \chi_{V,\psi}\nu^{s_1}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 \\ & + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0 + \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0 \\ & + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 \\ & + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 \\ & + 1 \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0.\end{aligned}$$

We prove that irreducibility of all the representations above implies irreducibility of the representation Π. We keep this assumption throughout this subsection.

First, suppose that $\nu^{s_1}\chi_1 \neq \nu^{-s_1}\chi_1^{-1}$, $\nu^{s_2}\chi_2 \neq \nu^{-s_2}\chi_2^{-1}$ and $\nu^{s_1}\chi_1 \neq \nu^{\pm s_2}\chi_2^{\pm 1}$ (that is, Jacquet modules of Π are multiplicity one).

Let τ be an irreducible subquotient of Π such that

$$\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 \leq R_{\widetilde{P}_2}(\tau).$$

From transitivity of Jacquet modules we get

$$\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \omega_0 \leq R_{\widetilde{P_{(1,1)}}}(\tau).$$

This implies

$$\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0 \leq R_{\widetilde{P}_1}(\tau).$$

We get directly that

$$R_{\widetilde{P}_2}(\tau) = \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 \\ + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0,$$

so $\tau = \Pi$ and Π is irreducible.

Now we assume that there is some i such that $\nu^{s_i}\chi_i \neq \nu^{-s_i}\chi_i^{-1}$. Without loss of generality, let $i = 1$. So, $s_1 = 0$ and $\chi_1 = \chi_1^{-1}$, that is, $\chi_1^2 = 1_{F^\times}$. We prove that in this case Π is also irreducible. Again, we start by writing corresponding Jacquet modules:

$$R_{\widetilde{P}_1}(\Pi) = 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0 \\ + \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0,$$
$$R_{\widetilde{P}_2}(\Pi) = 2\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0.$$

Let τ be an irreducible subquotient of Π such that

$$R_{\widetilde{P}_1}(\tau) \geq \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0.$$

Of course, $R_{\widetilde{P_{(1,1)}}}(\tau) \geq 2\chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \otimes \omega_0$, so

$$R_{\widetilde{P}_2}(\tau) \geq 2\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0.$$

Continuing in the same way, we get

$$R_{\widetilde{P_{(1,1)}}}(\tau) \geq 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + 2\chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \otimes \omega_0,$$
$$R_{\widetilde{P}_1}(\tau) \geq 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0.$$

Finally,

$$R_{\widetilde{P_{(1,1)}}}(\tau) \geq 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 \\ + 2\chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \otimes \omega_0,$$
$$R_{\widetilde{P}_2}(\tau) \geq 2\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 = R_{\widetilde{P}_2}(\Pi).$$

So, $\Pi = \tau$ and Π is irreducible.

If $\nu^{s_1}\chi_1 = \nu^{s_2}\chi_2$ or $\nu^{s_1}\chi_1 = \nu^{-s_2}\chi_2^{-1}$, then the irreducibility of Π follows in the same way as above. Observe that equalities $\nu^{s_1}\chi_1 = \nu^{-s_1}\chi_1^{-1}$ and $\nu^{s_2}\chi_2 = \nu^{-s_2}\chi_2^{-1}$ lead to unitary principal series.

In this way we have proved irreducibility of the principal series, with these exceptions:

- Some of the representations $\chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0$ or $\chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0$ reduce (the so-called $\widetilde{\mathrm{Sp}(1)}$ reducibility).

- Some of the representations
$$\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1}\times\chi_{V,\psi}\nu^{s_2}\chi_2,\qquad \chi_{V,\psi}\nu^{s_1}\chi_1\times\chi_{V,\psi}\nu^{-s_2}\chi_2^{-1},$$
$$\chi_{V,\psi}\nu^{s_1}\chi_1\times\chi_{V,\psi}\nu^{s_2}\chi_2,\qquad \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1}\times\chi_{V,\psi}\nu^{-s_2}\chi_2^{-1}$$
reduce (the so-called $\widetilde{\mathrm{GL}(2)}$ reducibility).

3.2.1. $\widetilde{\mathrm{Sp}(1)}$ *reducibility.* Let $\chi,\zeta\in\widehat{F^\times}$, $\zeta^2=1_{F^\times}$, and $s\geq 0$. It is well known that, in R,
$$\chi_{V,\psi}\nu^s\chi\times\chi_{V,\psi}\nu^{1/2}\zeta\rtimes\omega_0=\chi_{V,\psi}\nu^s\rtimes sp_{\zeta,1}+\chi_{V,\psi}\nu^s\rtimes\omega^+_{\psi_a,1}.$$

Let Π denote $\chi_{V,\psi}\nu^s\rtimes sp_{\zeta,1}$.

Calculating Jacquet modules, we find
$$R_{\widetilde{P_1}}(\Pi)=\chi_{V,\psi}\nu^{-s}\chi^{-1}\otimes sp_{\zeta,1}+\chi_{V,\psi}\nu^s\chi\otimes sp_{\zeta,1}+\chi_{V,\psi}\nu^{1/2}\zeta\otimes\chi_{V,\psi}\nu^s\chi\rtimes\omega_0,$$
$$R_{\widetilde{P_2}}(\Pi)=\chi_{V,\psi}\nu^{-s}\chi^{-1}\times\chi_{V,\psi}\nu^{1/2}\zeta\otimes\omega_0+\chi_{V,\psi}\nu^s\chi\times\chi_{V,\psi}\nu^{1/2}\zeta\otimes\omega_0.$$

If the representation $\chi_{V,\psi}\nu^s\chi\rtimes\omega_0$ is irreducible (that is, when $\nu^s\chi\neq\nu^{\pm1/2}\zeta_2$, where $\zeta_2^2=1_{F^\times}$), we proceed in the following way:

Let ρ be an irreducible subquotient of Π such that
$$\chi_{V,\psi}\nu^{1/2}\zeta\otimes\chi_{V,\psi}\nu^s\chi\rtimes\omega_0\leq s_1(\rho).$$

We directly get that
$$\chi_{V,\psi}\nu^{1/2}\zeta\otimes\chi_{V,\psi}\nu^s\chi\otimes\omega_0+\chi_{V,\psi}\nu^{1/2}\zeta\otimes\chi_{V,\psi}\nu^{-s}\chi^{-1}\otimes\omega_0\leq R_{\widetilde{P_{(1,1)}}}(\rho).$$

If both $\chi_{V,\psi}\nu^{-s}\chi^{-1}\times\chi_{V,\psi}\nu^{1/2}\zeta$ and $\chi_{V,\psi}\nu^s\chi\times\chi_{V,\psi}\nu^{1/2}\zeta$ are irreducible, Π is also irreducible.

For the reducibility of the $\widetilde{\mathrm{Sp}(1)}$ part we still have to determine the composition factors of the representations

(i) $\chi_{V,\psi}\nu^{1/2}\zeta_1\times\chi_{V,\psi}\nu^{1/2}\zeta_2\rtimes\omega_0$,

(ii) $\chi_{V,\psi}\nu^{1/2}\zeta\times\chi_{V,\psi}\nu^{1/2}\zeta\rtimes\omega_0$, and

(iii) $\chi_{V,\psi}\nu^{3/2}\zeta\times\chi_{V,\psi}\nu^{1/2}\zeta\rtimes\omega_0$, where $\zeta^2=\zeta_1^2=\zeta_2^2=1_{F^*}$.

Thus, we have proved the following result:

Proposition 3.9. *Let $\chi\in\widehat{F^\times}$, a nonnegative $s\in\mathbb{R}$, and $\zeta\in\widehat{F^\times}$ with $\zeta^2=1_{F^\times}$. The representations $\chi_{V,\psi}\nu^s\chi\rtimes sp_{\zeta,1}$ and $\chi_{V,\psi}\nu^s\chi\rtimes\omega^+_{\psi_a,1}$ are irreducible unless $(s,\chi)=(3/2,\zeta)$ or $(1/2,\zeta_1)$, where $\zeta_1^2=1_{F^\times}$. In R, we have*
$$\chi_{V,\psi}\nu^s\chi\times\chi_{V,\psi}\nu^{1/2}\zeta\rtimes\omega_0=\chi_{V,\psi}\nu^s\chi\rtimes sp_{\zeta,1}+\chi_{V,\psi}\nu^s\chi\rtimes\omega^+_{\psi_a,1}.$$

Also, if $(s,\chi)\neq(3/2,\zeta)$ *and* $(s,\chi)\neq(1/2,\zeta_1)$, *then*

$$\chi_{V,\psi}\nu^s\chi\rtimes sp_{\zeta,1}=\begin{cases}\langle\chi_{V,\psi}\nu^{1/2}\zeta;\chi_{V,\psi}\chi\rtimes\omega_0\rangle & \text{if } s=0,\\ \langle\chi_{V,\psi}\nu^{1/2}\zeta,\chi_{V,\psi}\nu^s\chi;\omega_0\rangle & \text{if } 0<s\leq 1/2,\\ \langle\chi_{V,\psi}\nu^s\chi,\chi_{V,\psi}\nu^{1/2}\zeta;\omega_0\rangle & \text{if } s>1/2,\end{cases}$$

$$\chi_{V,\psi}\nu^s\chi\rtimes\omega^+_{\psi_a,1}=\begin{cases}\chi_{V,\psi}\chi\rtimes\omega^+_{\psi_a,1} & \text{if } s=0,\\ \langle\chi_{V,\psi}\nu^s\chi;\omega^+_{\psi_a,1}\rangle & \text{if } s>0.\end{cases}$$

3.2.2. $\widetilde{\mathrm{GL}(2)}$ *reducibility.* Let $\chi\in\widehat{F^\times}$ and $s\in\mathbb{R}$ be nonnegative. In R, we have

$$\chi_{V,\psi}\nu^{s+1/2}\chi\times\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0=\chi_{V,\psi}\nu^s\chi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\omega_0+\chi_{V,\psi}\nu^s\chi\,1_{\mathrm{GL}(2)}\rtimes\omega_0.$$

Let Π denote the representation $\chi_{V,\psi}\nu^s\chi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\omega_0$. Calculation of $\mu^*(\Pi)$ gives

$$\begin{aligned}R_{\widetilde{P}_1}(\Pi)&=\chi_{V,\psi}\nu^{s+1/2}\chi\otimes\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0\\&\qquad+\chi_{V,\psi}\nu^{1/2-s}\chi^{-1}\otimes\chi_{V,\psi}\nu^{s+1/2}\chi\rtimes\omega_0,\\ R_{\widetilde{P}_2}(\Pi)&=\chi_{V,\psi}\nu^s\chi\,\mathrm{St}_{\mathrm{GL}(2)}\otimes\omega_0+\chi_{V,\psi}\nu^{-s}\chi^{-1}\,\mathrm{St}_{\mathrm{GL}(2)}\otimes\omega_0\\&\qquad+\chi_{V,\psi}\nu^{s+1/2}\chi\times\chi_{V,\psi}\nu^{1/2-s}\chi^{-1}\otimes\omega_0.\end{aligned}$$

Looking at Jacquet modules with respect to different parabolic subgroups we can conclude, in the same way as in the $\widetilde{\mathrm{Sp}(1)}$ reducibility case, that if

$$\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0,\quad \chi_{V,\psi}\nu^{s+1/2}\chi\rtimes\omega_0,\quad \chi_{V,\psi}\nu^{s+1/2}\chi\times\chi_{V,\psi}\nu^{1/2-s}\chi^{-1}$$

are irreducible representations, then the representation Π is also irreducible.

Observe that the representation $\chi_{V,\psi}\nu^{s+1/2}\chi\rtimes\omega_0$ reduces for $(\chi,s)=(\zeta,0)$, while $\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0$ reduces for $(\chi,s)=(\zeta,1)$, where $\zeta^2=1_{F^\times}$.

The representation $\chi_{V,\psi}\nu^{s+1/2}\chi\times\chi_{V,\psi}\nu^{1/2-s}\chi^{-1}$ reduces for $(\chi,s)=(\zeta,1/2)$, where $\zeta^2=1_{F^\times}$. These observations imply this:

Proposition 3.10. *Let* $\chi\in\widehat{F^\times}$ *and* $s\in\mathbb{R}$ *be nonnegative. The representations* $\chi_{V,\psi}\nu^s\chi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes 1$ *and* $\chi_{V,\psi}\nu^s\chi\,1_{\mathrm{GL}(2)}\rtimes 1$ *are irreducible except in the cases that* $(s,\chi)=(1/2,\zeta)$, $(s,\chi)=(1,\zeta)$ *or* $(s,\chi)=(0,\zeta)$, *where* $\zeta^2=1_{F^\times}$. *In* R, *we have*

$$\chi_{V,\psi}\nu^{s+1/2}\chi\times\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0=\chi_{V,\psi}\nu^s\chi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\omega_0+\chi_{V,\psi}\nu^s\chi\,1_{\mathrm{GL}(2)}\rtimes\omega_0.$$

Also, if $\chi_{V,\psi}\nu^{s+1/2}\chi\times\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0$ *is a representation of length* 2, *then* $\chi_{V,\psi}\nu^s\chi\,1_{\mathrm{GL}(2)}\rtimes\omega_0=\langle\chi_{V,\psi}\nu^s\chi\,1_{\mathrm{GL}(2)};\omega_0\rangle$ *and*

$$\chi_{V,\psi}\nu^s\chi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\omega_0=\begin{cases}\langle\chi_{V,\psi}\nu^{s+1/2}\chi,\chi_{V,\psi}\nu^{1/2-s}\chi;\omega_0\rangle & \text{if } s<1/2,\\ \langle\chi_{V,\psi}\nu^{s+1/2}\chi;\chi_{V,\psi}\nu^{s-1/2}\chi\rtimes\omega_0\rangle & \text{if } s=1/2,\\ \langle\chi_{V,\psi}\nu^{s+1/2}\chi,\chi_{V,\psi}\nu^{s-1/2}\chi;\omega_0\rangle & \text{if } s>1/2.\end{cases}$$

For the reducibility of the $\widetilde{\mathrm{GL}(2)}$ part, we still have to determine the composition factors of the representations

(i) $\chi_{V,\psi}\nu^{1/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$,

(ii) $\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$, and

(iii) $\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$, where $\zeta^2 = 1_{F^\times}$.

Altogether, this leaves us four exceptional cases of the representations whose composition series we have to determine:

(a) $\chi_{V,\psi}\nu^{1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega_0$,

(b) $\chi_{V,\psi}\nu^{1/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$,

(c) $\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$,

(d) $\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$, where $\zeta^2 = \zeta_1^2 = \zeta_2^2 = 1_{F^\times}$ and $\zeta_1 \neq \zeta_2$.

These cases are treated in the following subsection.

3.2.3. *Exceptional cases.* All the equalities that follow are given in semisimplifications. We obtain desired composition series using case-by-case examination:

(a) Write $\chi_{V,\psi}\nu^{1/2}\zeta_i \rtimes \omega_0 = \chi_{V,\psi}\, sp_{\zeta_i,1} + \omega^+_{\psi_{a_i},1}$ for $i = 1, 2$. In R, we have

$$\begin{aligned}\chi_{V,\psi}\nu^{1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega_0 &= \chi_{V,\psi}\nu^{1/2}\zeta_2 \times \chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes \omega_0 \\ &= \chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1} + \chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1} \\ &= \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes sp_{\zeta_1,1} + \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}.\end{aligned}$$

Using standard calculations, we obtain

$$\begin{aligned}R_{\widetilde{P}_1}(\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1}) = {}& \chi_{V,\psi}\nu^{-1/2}\zeta_1 \otimes sp_{\zeta_1,1} + \chi_{V,\psi}\nu^{1/2}\zeta_1 \otimes sp_{\zeta_2,1} \\ &+ \chi_{V,\psi}\nu^{1/2}\zeta_2 \otimes sp_{\zeta_1,1} + \chi_{V,\psi}\nu^{1/2}\zeta_2 \otimes \omega^+_{\psi_{a_1},1}\end{aligned}$$

and

$$\begin{aligned}R_{\widetilde{P}_2}(\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1}) = {}& \chi_{V,\psi}\nu^{-1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \otimes \omega_0 \\ &+ \chi_{V,\psi}\nu^{1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \otimes \omega_0.\end{aligned}$$

The last equality implies that the length of $\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1}$ is no more than 2. If $\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1}$ were an irreducible representation, then it would have to be equal either to $\chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes sp_{\zeta_1,1}$ or to $\chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}$, but Jacquet modules of those two representations show that this is not the case. So, we write $\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1} = \rho_1 + \rho_2$, where ρ_1 and ρ_2 are irreducible representations such that

$$R_{\widetilde{P}_2}(\rho_1) = \chi_{V,\psi}\nu^{-1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \otimes \omega_0,$$
$$R_{\widetilde{P}_2}(\rho_2) = \chi_{V,\psi}\nu^{1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \otimes \omega_0.$$

Clearly, ρ_2 is square-integrable (since $\rho_2 = \langle \chi_{V,\psi}\nu^{1/2}\zeta_1, \chi_{V,\psi}\nu^{1/2}\zeta_2; \omega_0\rangle$) and $\rho_1 = \langle \chi_{V,\psi}\nu^{1/2}\zeta_2; \omega^+_{\psi_{a_1},1}\rangle$.

Reasoning in the same way, we obtain that $\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1} = \rho_3 + \rho_4$, where ρ_3 and ρ_4 are irreducible representations such that

$$R_{\widetilde{P}_2}(\rho_3) = \chi_{V,\psi}\nu^{-1/2}\zeta_1 \times \chi_{V,\psi}\nu^{-1/2}\zeta_2 \otimes \omega_0,$$
$$R_{\widetilde{P}_2}(\rho_4) = \chi_{V,\psi}\nu^{1/2}\zeta_1 \times \chi_{V,\psi}\nu^{-1/2}\zeta_2 \otimes \omega_0.$$

So, ρ_3 is a strongly negative representation, while $\rho_4 = \langle \chi_{V,\psi}\nu^{1/2}\zeta_1; \omega^+_{\psi_{a_2},1}\rangle$. Using Jacquet modules again, we easily obtain the composition factors of the representations above. Thus, we conclude:

Proposition 3.11. *Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$ such that $\zeta_i^2 = 1_{F^\times}$ for $i = 1, 2$ (with $\zeta_1 \neq \zeta_2$). Than the representations*

$$\chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}, \quad \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes sp_{\zeta_1,1},$$
$$\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1}, \quad \chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1}$$

are reducible and $\chi_{V,\psi}\nu^{1/2}\zeta_1 \times \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega_0$ is a representation of length 4. The representations $\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1}$ and $\chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes sp_{\zeta_1,1}$ have exactly one irreducible subquotient in common; that subquotient is square-integrable, and we denote it with σ (that is, $\sigma = \langle \chi_{V,\psi}\nu^{1/2}\zeta_1, \chi_{V,\psi}\nu^{1/2}\zeta_2; \omega_0\rangle$). Also, the unique irreducible common subquotient of

$$\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1} \quad \text{and} \quad \chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}$$

is a strongly negative representation; we denote it by ρ_{sneg}. In R, we have

$$\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes sp_{\zeta_2,1} = \sigma + \langle \chi_{V,\psi}\nu^{1/2}\zeta_2; \omega^+_{\psi_{a_1},1}\rangle,$$
$$\chi_{V,\psi}\nu^{1/2}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1} = \langle \chi_{V,\psi}\nu^{1/2}\zeta_1; \omega^+_{\psi_{a_2},1}\rangle + \rho_{\mathrm{sneg}},$$
$$\chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes sp_{\zeta_1,1} = \sigma + \langle \chi_{V,\psi}\nu^{1/2}\zeta_1; \omega^+_{\psi_{a_2},1}\rangle,$$
$$\chi_{V,\psi}\nu^{1/2}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1} = \langle \chi_{V,\psi}\nu^{1/2}\zeta_2; \omega^+_{\psi_{a_1},1}\rangle + \rho_{\mathrm{sneg}}.$$

(b) In this case, we have

$$\chi_{V,\psi}\nu^{1/2}\zeta \times \chi_{V,\psi}\nu^{-1/2}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^{1/2}\zeta \rtimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^+_{\psi_a,1}$$
$$= \chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 + \chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0.$$

From Jacquet modules, we get

$$\begin{aligned}
R_{\widetilde{P}_1}(\chi_{V,\psi}\nu^{1/2}\zeta \rtimes sp_{\zeta,1}) &= 2\chi_{V,\psi}\nu^{1/2}\zeta \otimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{-1/2}\zeta \otimes sp_{\zeta,1} \\
&\qquad + \chi_{V,\psi}\nu^{1/2}\zeta \otimes \omega^+_{\psi_a,1}, \\
R_{\widetilde{P}_1}(\chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^+_{\psi_1}) &= 2\chi_{V,\psi}\nu^{-1/2}\zeta \otimes \omega^+_{\psi_a,1} + \chi_{V,\psi}\nu^{1/2}\zeta \otimes \omega^+_{\psi_a,1} \\
&\qquad + \chi_{V,\psi}\nu^{-1/2}\zeta \otimes sp_{\zeta,1}, \\
R_{\widetilde{P}_1}(\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0) &= 2\chi_{V,\psi}\nu^{1/2}\zeta \otimes sp_{\zeta,1} + 2\chi_{V,\psi}\nu^{1/2}\zeta \otimes \omega^+_{\psi_a,1}, \\
R_{\widetilde{P}_1}(\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0) &= 2\chi_{V,\psi}\nu^{-1/2}\zeta \otimes sp_{\zeta,1} + 2\chi_{V,\psi}\nu^{-1/2}\zeta \otimes \omega^+_{\psi_a,1}.
\end{aligned}$$

From preceding Jacquet modules we conclude, as in [Tadić 1994, Chapter 3], that $\chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^+_{\psi_a,1}$ and $\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0$ have an irreducible subquotient in common, which is different from both $\chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^+_{\psi_a,1}$ and $\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0$. For simplicity of the notation, we let ρ_1 stand for this subquotient. Thus $R_{\widetilde{P}_1}(\rho_1) = \chi_{V,\psi}\nu^{1/2}\zeta \otimes \omega^+_{\psi_a,1}$.

In the same way, let ρ_2 be an irreducible common subquotient that

$$\chi_{V,\psi}\nu^{1/2}\zeta \rtimes sp_{\zeta,1} \quad \text{and} \quad \chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0$$

have in common. Then $R_{\widetilde{P}_1}(\rho_2) = \chi_{V,\psi}\nu^{-1/2}\zeta \otimes sp_{\zeta,1}$.

The representations $\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \otimes \omega_0$ and $\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \omega_0$ are irreducible and unitary. The multiplicity of $\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \otimes \omega_0$ in $R_{\widetilde{P}_2}(\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0)$ is equal to 2, which implies that length of $\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0$ is 2. Analogously, the length of $\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0$ also equals 2.

Now we write $\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 = \rho_1 + \rho_3$ and $\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0 = \rho_2 + \rho_4$. Observe that

$$\begin{aligned}
R_{\widetilde{P}_1}(\rho_3) &= 2\chi_{V,\psi}\nu^{1/2}\zeta \otimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{1/2}\zeta \otimes \omega^+_{\psi_a,1}, \\
R_{\widetilde{P}_1}(\rho_4) &= \chi_{V,\psi}\nu^{-1/2}\zeta \otimes sp_{\zeta,1} + 2\chi_{V,\psi}\nu^{-1/2}\zeta \otimes \omega^+_{\psi_a,1}.
\end{aligned}$$

We immediately get this:

Proposition 3.12. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations*

$$\zeta\, 1_{\widetilde{\mathrm{GL}(2)}} \rtimes \omega_0, \quad \chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0, \quad \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^+_{\psi_a,1} \quad \chi_{V,\psi}\nu^{1/2}\zeta \rtimes sp_{\zeta,1}$$

are reducible and $\chi_{V,\psi}\nu^{1/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$ is a representation of length 4. *The representations*

$$\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 \quad \textit{and} \quad \nu^{1/2}\chi_{V,\psi}\zeta \rtimes \omega^+_{\psi_a,1}$$

(*respectively $\chi_{V,\psi}\nu^{1/2}\zeta \rtimes sp_{\zeta,1}$*) *have exactly one irreducible subquotient in common, which is tempered and denoted by τ_1* (*respectively τ_2*). *Observe that*

$$\tau_1 = \langle \chi_{V,\psi}\nu^{1/2}\zeta; \omega^+_{\psi_a,1}\rangle \quad \textit{and} \quad \tau_2 = \langle \chi_{V,\psi}\nu^{1/2}\zeta, \chi_{V,\psi}\nu^{1/2}\zeta; \omega_0\rangle.$$

Also, the unique irreducible common subquotient of

$$\chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0 \quad \textit{and} \quad \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^{+}_{\psi_a,1}$$

is a negative representation, which we denote by ρ_{neg}. *In* R, *we have*

$$\chi_{V,\psi}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 = \tau_1 + \tau_2, \qquad \chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0 = \rho_{\mathrm{neg}} + \langle \chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle$$
$$\chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega^{+}_{\psi_a,1} = \tau_1 + \rho_{\mathrm{neg}}, \quad \chi_{V,\psi}\nu^{1/2}\zeta \rtimes sp_{\zeta,1} = \tau_2 + \langle \chi_{V,\psi}\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle.$$

(c) In this case we have

$$\begin{aligned}\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0 &= \chi_{V,\psi}\nu^{3/2}\zeta \rtimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{3/2}\zeta \rtimes \omega^{+}_{\psi_a,1}\\ &= \chi_{V,\psi}\nu\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 + \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0.\end{aligned}$$

Observe that $\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$ is a regular representation. So, it is a representation of the length $2^2 = 4$ by [Tadić 1998b] (there only the techniques of Jacquet modules were used, so they can be applied in our case). Since the irreducible subquotients of $\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$ are

$$\langle \chi_{V,\psi}\nu^{3/2}\zeta, \chi_{V,\psi}\nu^{1/2}\zeta; \omega_0\rangle, \quad \langle \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle, \quad \omega^{+}_{\psi_a,2}, \quad \langle \chi_{V,\psi}\nu^{3/2}\zeta; \omega^{+}_{\psi_a,1}\rangle,$$

using Jacquet modules we easily obtain the following proposition:

Proposition 3.13. *Let* $\zeta \in \widehat{F^\times}$ *such that* $\zeta^2 = 1_{F^\times}$. *Then the representations*

$$\chi_{V,\psi}\nu^{3/2}\zeta \rtimes sp_{\zeta,1}, \qquad \chi_{V,\psi}\nu^{3/2}\zeta \rtimes \omega^{+}_{\psi_a,1},$$
$$\chi_{V,\psi}\nu\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0, \qquad \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0$$

are reducible and $\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$ *is a representation of length* 4. *The unique irreducible common subquotient of the representations* $\chi_{V,\psi}\nu^{3/2}\zeta \rtimes sp_{\zeta,1}$ *and* $\chi_{V,\psi}\nu\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0$ *is square-integrable. In* R, *we have*

$$\begin{aligned}\chi_{V,\psi}\nu^{3/2}\zeta \rtimes sp_{\zeta,1} &= \langle \chi_{V,\psi}\nu^{3/2}\zeta, \chi_{V,\psi}\nu^{1/2}\zeta; \omega_0\rangle + \langle \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle,\\ \chi_{V,\psi}\nu^{3/2}\zeta \rtimes \omega^{+}_{\psi_a,1} &= \omega^{+}_{\psi_a,2} + \langle \chi_{V,\psi}\nu^{3/2}\zeta; \omega^{+}_{\psi_a,1}\rangle,\\ \chi_{V,\psi}\nu\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 &= \langle \chi_{V,\psi}\nu^{3/2}\zeta, \chi_{V,\psi}\nu^{1/2}\zeta; \omega_0\rangle + \langle \chi_{V,\psi}\nu^{3/2}\zeta; \omega^{+}_{\psi_a,1}\rangle,\\ \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0 &= \langle \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle + \omega^{+}_{\psi_a,2}.\end{aligned}$$

(d) In this case,

$$\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^{1/2}\zeta\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 + \chi_{V,\psi}\nu^{1/2}\zeta\, 1_{\mathrm{GL}(2)} \rtimes \omega_0.$$

Since it isn't known yet if the results related to the R-groups [Goldberg 1994] also hold for metaplectic groups, this case will not be solved using only the method Jacquet modules. Tadić [1998a] used a combination of Jacquet modules techniques and knowledge about R-groups for symplectic groups to determine the composition series of the representations similar to this one (for symplectic groups). We resolve this case using theta correspondence.

Lemma 3.14. *The following equalities hold*:

(1) $\Theta(\zeta\nu \otimes \zeta \rtimes 1, R_{P_1}(\omega_{2,2})) = \chi_{V,\psi}\nu^{-1}\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$,

(2) $\Theta(\zeta\nu^{-1} \otimes \zeta \rtimes 1, R_{P_1}(\omega_{2,2})) = \chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$,

(3) $\Theta(\zeta \otimes \zeta\nu \rtimes 1, R_{P_1}(\omega_{2,2})) = \chi_{V,\psi}\zeta \times \chi_{V,\psi}\nu\zeta \rtimes \omega_0$,

(4) $\Theta(\chi_{V,\psi}\nu\zeta \otimes \chi_{V,\psi}\zeta \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) = \zeta\nu^{-1} \times \zeta \rtimes 1$,

(5) $\Theta(\chi_{V,\psi}\nu^{-1}\zeta \otimes \chi_{V,\psi}\zeta \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) = \zeta\nu \times \zeta \rtimes 1$,

(6) $\Theta(\chi_{V,\psi}\zeta \otimes \chi_{V,\psi}\nu\zeta \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) = \zeta \times \zeta\nu \rtimes 1$.

Proof. Recall that $R_{P_1}(\omega_{2,2})$ has the filtration in which

- $I_{10} = \nu^{1/2} \otimes \omega_{2,1}$ is the quotient, and
- $I_{11} = \mathrm{Ind}_{\mathrm{GL}(1)\times\widetilde{P}_1\times O(3)}^{M_1\times\widetilde{\mathrm{Sp}(2)}}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1})$ is the subrepresentation.

We will prove (1); the proofs of (2)–(6) are analogous. In the same way as in the second proof of Proposition 3.4, we get

$$\Theta(\zeta\nu \otimes \zeta \rtimes 1, R_{P_1}(\omega_{2,2})) = \Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11}),$$

so it is sufficient to show $\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11}) = \chi_{V,\psi}\nu^{-1}\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$. It can be seen easily that there is an $\mathrm{GL}(1) \times \widetilde{M}_1 \times O(3)$-invariant epimorphism

$$\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1} \to \chi_{V,\psi}\zeta\nu^{-1} \otimes \chi_{V,\psi}\zeta \rtimes \omega_0 \otimes \zeta\nu \otimes \zeta \rtimes 1.$$

Consequently, we get an $M_1 \times \widetilde{\mathrm{Sp}(2)}$-invariant epimorphism

$$I_{11} = \mathrm{Ind}_{\mathrm{GL}(1)\times\widetilde{P}_1\times O(3)}^{M_1\times\widetilde{\mathrm{Sp}(2)}}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1}) \to \zeta\nu \otimes \zeta \rtimes 1 \otimes \chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0,$$

so we conclude that $\chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0$ is a quotient of $\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11})$.

We prove that $\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11})$ is also a quotient of $\chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0$. Let $\varphi \in \mathrm{Hom}(I_{11}, \zeta\nu \otimes \zeta \rtimes 1 \otimes \Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11}))$. Using the second Frobenius reciprocity, as before, we get

$$\begin{aligned}&\mathrm{Hom}(I_{11}, \zeta\nu \otimes \zeta \rtimes 1 \otimes \Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11}))\\ &\qquad\cong \mathrm{Hom}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1}, \zeta\nu \otimes \zeta \rtimes 1 \otimes R_{\overline{\widetilde{P}}_1}(\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11})));\end{aligned}$$

let φ_0 be an element corresponding to φ. Since the representations $\zeta \rtimes 1$ and $\chi_{V,\psi}\zeta \rtimes \omega_0$ are irreducible, the image of φ_0 equals $\zeta\nu \otimes \zeta \rtimes 1 \otimes \chi_{V,\psi}\zeta\nu^{-1} \otimes \chi_{V,\psi}\zeta \rtimes \omega_0$. Reasoning as before, we get that the image of φ is a quotient of $\zeta\nu \otimes \zeta \rtimes 1 \otimes \chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0$. Finally, $\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11})$ is a quotient of $\chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0$. Hence $\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11}) = \chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0$. □

Proposition 3.15. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations*

$$\chi_{V,\psi}\nu^{1/2}\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 \quad \textit{and} \quad \chi_{V,\psi}\nu^{1/2}\zeta\,1_{\mathrm{GL}(2)} \rtimes \omega_0$$

are irreducible and $\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$ is a representation of length 2. Also

$$\chi_{V,\psi}\nu^{1/2}\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu\zeta; \chi_{V,\psi}\zeta \rtimes \omega_0\rangle,$$
$$\chi_{V,\psi}\nu^{1/2}\zeta\,1_{\mathrm{GL}(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu^{1/2}\zeta\,1_{\mathrm{GL}(2)}; \omega_0\rangle.$$

Proof. Suppose on the contrary that the representation $\chi_{V,\psi}\nu^{1/2}\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0$ reduces. Jacquet modules imply that length of this representation is at most 2. Choose π_1 and π_2 so the equality $\chi_{V,\psi}\nu^{1/2}\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0 = \pi_1 + \pi_2$ holds in R. Also suppose $R_{\widetilde{P}_1}(\pi_1) = \chi_{V,\psi}\zeta \otimes \chi_{V,\psi}\zeta\nu \rtimes \omega_0$ and $R_{\widetilde{P}_1}(\pi_2) = \chi_{V,\psi}\zeta\nu \otimes \chi_{V,\psi}\zeta \rtimes \omega_0$. Frobenius reciprocity implies

$$\mathrm{Hom}(\omega_{2,2}, \pi_1 \otimes \zeta \times \zeta\nu^{-1} \rtimes 1) \cong \mathrm{Hom}(R_{P_1}(\omega_{2,2}), \pi_1 \otimes \zeta \otimes \zeta\nu^{-1} \rtimes 1).$$

Using Lemma 3.14 we obtain

$$\mathrm{Hom}(R_{P_1}(\omega_{2,2}), \pi_1 \otimes \zeta \otimes \zeta\nu^{-1} \rtimes 1) \cong \mathrm{Hom}(\chi_{V,\psi}\zeta \times \chi_{V,\psi}\nu^{-1}\zeta \rtimes \omega_0, \pi_1) \neq 0,$$

because π_1 is a quotient of $\chi_{V,\psi}\zeta \times \chi_{V,\psi}\nu\zeta \rtimes \omega_0$. So, $\Theta(\pi_1, 2) \neq 0$.

The representation $\chi_{V,\psi}\zeta \otimes \chi_{V,\psi}\zeta\nu \rtimes \omega_0 \otimes \Theta(\pi_1, 2)$ is a quotient of $R_{\widetilde{P}_1}(\omega_{2,2})$. Lemma 3.14 implies that $\Theta(\pi_1, 2)$ is a quotient of $\zeta \times \zeta\nu \rtimes 1$. Listing quotients of $\zeta \times \zeta\nu \rtimes 1$ we get the possibilities

(a) $\Theta(\pi_1, 2) = \zeta \times \zeta\nu \rtimes 1$,

(b) $\Theta(\pi_1, 2) = \nu^{1/2}\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes 1$,

(c) $\Theta(\pi_1, 2) = \nu^{-1/2}\zeta\,1_{\mathrm{GL}(2)} \rtimes 1$.

Suppose that (a) holds. Obviously, $\pi_1 \otimes \zeta\nu^{-1} \otimes \zeta \rtimes 1$ is then a quotient of $R_{P_1}(\omega_{2,2})$, since it is a quotient of $\pi_1 \otimes R_{P_1}(\zeta \times \zeta\nu \rtimes 1)$. This implies that π_1 is a quotient of $\chi_{V,\psi}\zeta\nu \times \chi_{V,\psi}\zeta \rtimes \omega_0$ and $R_{\widetilde{P}_1}(\pi_1)$ contains $\chi_{V,\psi}\zeta\nu^{-1} \otimes \chi_{V,\psi}\zeta \rtimes \omega_0$. This contradicts our assumption on π_1.

Similarly, using Jacquet modules, we obtain contradiction with (b) and (c). So, $\chi_{V,\psi}\nu^{1/2}\zeta\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \omega_0$ is irreducible.

Irreducibility of $\chi_{V,\psi}\nu^{1/2}\zeta\,1_{\mathrm{GL}(2)} \rtimes \omega_0$ can be proved in the same way. □

4. Unitary dual supported in minimal parabolic subgroup

Let π be an irreducible genuine admissible representation of $\widetilde{\mathrm{Sp}(n)}$. We recall that the contragredient representation is denoted by $\widetilde{\pi}$. We write $\bar{\pi}$ for the complex conjugate representation of the representation π. The representation π is called Hermitian if $\pi \simeq \widetilde{\bar{\pi}}$. It is well known that every unitary representation is Hermitian. For a deeper discussion, we refer the reader to [Muić and Tadić 2007].

Suppose that $\Delta_1, \dots, \Delta_k$ is a sequence of segments such that $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$, let σ_{neg} be a negative representation of some $\widetilde{\mathrm{Sp}(n')}$. From [Hanzer and Muić 2010, Theorem 4.5(v)], we directly get

$$\overline{\langle \Delta_1, \dots, \Delta_k; \sigma_{\text{neg}} \rangle} = \langle \overline{\Delta}_1, \dots, \overline{\Delta}_k; \overline{\sigma}_{\text{neg}} \rangle.$$

Also, we have an epimorphism $\langle \widetilde{\Delta_1} \rangle \times \cdots \times \langle \widetilde{\Delta_k} \rangle \rtimes \widetilde{\sigma}_{\text{neg}} \to \langle \Delta_1, \dots, \Delta_k; \sigma_{\text{neg}} \rangle^{\sim}$. We know that the group $\mathrm{GSp}(n)$ acts on $\widetilde{\mathrm{Sp}(n)}$, by [Mœglin et al. 1987, II.1(3)]. Moreover, by [ibid., page 92], this action extends to the action on irreducible representations, which is equivalent to taking contragredients. We choose an element $\eta' = (1, \eta) \in \mathrm{GSp}(n)$, where $\eta \in \mathrm{GSp}(n')$ is an element with similitude equal to -1, and 1 denotes the identity acting on the GL part. Thus, we obtain an epimorphism

$$\alpha \langle \widetilde{\Delta_1} \rangle \times \cdots \times \alpha \langle \widetilde{\Delta_k} \rangle \rtimes \widetilde{\sigma}_{\text{neg}}^{\eta} \to \langle \Delta_1, \dots, \Delta_k; \sigma_{\text{neg}} \rangle^{\sim \eta'}.$$

Since $\widetilde{\sigma}_{\text{neg}}^{\eta} \simeq \sigma_{\text{neg}}$, we have $\langle \Delta_1, \dots, \Delta_k; \sigma_{\text{neg}} \rangle^{\sim} = \langle \alpha \Delta_1, \dots, \alpha \Delta_k; \widetilde{\sigma}_{\text{neg}} \rangle$.

Remark. When we are dealing with the action of the group of similitudes on the symplectic groups, α does not appear in the situation similar to the one above. However, since the action of $\mathrm{GSp}(n)$ on the metaplectic group is not trivial on its center μ_2, one has to compare the action η on the metaplectic part of the Levi subgroup with the action of η' on the whole Levi subgroup. The calculation is not very complicated and resembles the calculations in [Hanzer and Muić 2010, Lemma 3.2].

First, we classify Hermitian irreducible genuine representations.

Proposition 4.1. *Let $\chi, \zeta, \zeta_1, \zeta_2 \in \widehat{F^{\times}}$ such that $\zeta^2 = \zeta_i^2 = 1_{F^\times}$ for $i = 1, 2$, with ζ_1 and ζ_2 not necessarily different. Let $s, s_1, s_2 > 0$. The following families of representations are Hermitian and exhaust all irreducible Hermitian genuine representations of $\widetilde{\mathrm{Sp}(2)}$ supported in the minimal parabolic subgroup $\widetilde{P_{(1,1)}}$:*

1. *irreducible tempered representations supported in $\widetilde{P_{(1,1)}}$,*
2. $\langle \chi_{V,\psi} \nu^s \chi, \chi_{V,\psi} \nu^s \chi^{-1}; \omega_0 \rangle$,
3. $\langle \chi_{V,\psi} \nu^{s_1} \zeta_1, \chi_{V,\psi} \nu^{s_2} \zeta_2; \omega_0 \rangle$,
4. $\langle \chi_{V,\psi} \nu^s \zeta \, 1_{\mathrm{GL}(2)}; \omega_0 \rangle$,
5. $\langle \chi_{V,\psi} \nu^s \zeta; \chi_{V,\psi} \chi \rtimes \omega_0 \rangle$,
6. $\langle \chi_{V,\psi} \nu^s \zeta; \omega_{\psi_a,1}^{+} \rangle$,
7. $\omega_{\psi_a,2}^{+}$.

Proof. Using the reasoning before this proposition, we see that a representation $\langle \Delta_1, \dots, \Delta_k; \sigma_{\text{neg}} \rangle$ is Hermitian, if and only if

$$\langle \Delta_1, \dots, \Delta_k; \sigma_{\text{neg}} \rangle = \langle \alpha \overline{\Delta}_1, \dots, \alpha \overline{\Delta}_k; \overline{\widetilde{\sigma}}_{\text{neg}} \rangle.$$

The representation σ_{neg} also has to be Hermitian. Now we just check this requirement on the set of all irreducible representations of $\widetilde{\mathrm{Sp}(2)}$ with the support in the minimal parabolic subgroup; we have classified them in the previous section. For example, if we analyze the representation $\Pi = \langle \chi_{V,\psi}\nu^{s_1}\chi_1, \chi_{V,\psi}\nu^{s_2}\chi_2; \omega_0\rangle$ with $s_2 \geq s_1 > 0$, we have

$$\bar{\tilde{\Pi}} = \langle \alpha\overline{\chi_{V,\psi}\nu^{s_1}\chi_1}, \alpha\overline{\chi_{V,\psi}\nu^{s_2}\chi_2}; \omega_0\rangle.$$

Now we see that this representation is isomorphic to Π if and only if $\chi_1^2 = 1 = \chi_2^2$ or $s_1 = s_2$ and $\chi_1^{-1} = \chi_2$. This gives us the second and the third case from the proposition. All other cases are dealt with analogously. □

Theorem 4.2. *Let $\chi, \zeta, \zeta_1, \zeta_2 \in (\widehat{F^\times})$ such that $\zeta^2 = \zeta_i^2 = 1_{F^\times}$ for $i = 1, 2$, with ζ_1 and ζ_2 not necessarily different. The following families of representations are unitary and exhaust all irreducible unitary genuine representations of $\widetilde{\mathrm{Sp}(2)}$ that are supported in the minimal parabolic subgroup $\widetilde{P_{(1,1)}}$:*

(1) *irreducible tempered representations supported in $\widetilde{P_{(1,1)}}$.*

(2) $\langle \chi_{V,\psi}\nu^{s}\chi, \chi_{V,\psi}\nu^{s}\chi^{-1}; \omega_0\rangle$ *for* $0 < s \leq 1/2$,

(3) $\langle \chi_{V,\psi}\nu^{s_1}\zeta_1, \chi_{V,\psi}\nu^{s_2}\zeta_2; \omega_0\rangle$ *for* $s_2 \leq s_1$ *and* $0 < s_1 \leq 1/2$,

(4) $\langle \chi_{V,\psi}\nu^{s}\zeta; \chi_{V,\psi}\chi \rtimes \omega_0\rangle$ *for* $0 < s \leq 1/2$,

(5) $\langle \chi_{V,\psi}\nu^{s}\zeta; \omega^{+}_{\psi_a,1}\rangle$ *for* $s \leq 1/2$,

(6) $\omega^{+}_{\psi_a,2}$.

Proof. We first review some basic facts of representation theory of reductive groups, which directly carry over to the case of metaplectic groups.

Unitarizability of the complementary series. As explained in detail in [Tadić 1993, Section 3], it is enough to have a continuous family of $\widetilde{\mathrm{Sp}(2)}$-invariant hermitian forms (and the representations should be realized on one space — a compact picture). Then, a linear algebra argument (involving finite-dimensional representations of a compact subgroup) ensures that if this family of hermitian forms is positive definite at one point, it has to be positive definite everywhere, and this finishes the argument. So, we first have to show that, the restriction of an irreducible admissible (hermitian) representation π of $\widetilde{\mathrm{Sp}(2)}$ to the inverse image $\widetilde{K}$ in $\widetilde{\mathrm{Sp}(2)}$ of a maximal, good compact subgroup of Sp(2) (for example, $K = \mathrm{Sp}(2, O_F)$, where O_F is the ring of integers of F) decomposes into a direct sum of irreducible representations of $\widetilde{K}$ with finite multiplicities. But this follows directly from the admissibility of the representation π. Second, we have to have a way to form a continuous families of hermitian forms. This is obtained using intertwining operators, in the same way as for the algebraic groups. To define them, we note that the unipotent radicals of the standard parabolic subgroups of Sp(2) are split in $\widetilde{\mathrm{Sp}(2)}$.

Then we can define standard intertwining operators for the complex argument deep enough in the Weyl chamber in the same way as in [Shahidi 1981]. These operators can be meromorphically continued, using the results on filtration via Bruhat cells; see [Casselman 1995, Section 6 and 7] or [Muić 2008]. The arguments in the last reference carry over to the metaplectic groups without change, through the splitting of the unipotent radicals and Frobenius reciprocity, also valid for the metaplectic group. This passage in the construction of the intertwining operators from the linear case to the case of metaplectic groups is explained in detail also in [Zorn 2007; 2010]. We now illustrate how the hermitian form is defined. For example, suppose that $\chi_1, \chi_2 \in \hat{F^{\times}}$ such that $\chi_1^2 = \chi_2^2 = 1$, so that, for the longest element w_0 of the Weyl group, we have a map $A(s_1, s_2, \chi_1, \chi_2, w_0)$ from

$$\chi_{V,\psi}\chi_1\nu^{s_1} \times \chi_{V,\psi}\chi_2\nu^{s_2} \rtimes \omega_0 \quad \text{to} \quad \chi_{V,\psi}\nu^{-s_1}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2 \rtimes \omega_0.$$

Let f_{s_1,s_2}, g_{s_1,s_2} be sections from the compact picture of the induced representation $\chi_{V,\psi}\chi_1\nu^{s_1} \times \chi_{V,\psi}\chi_2\nu^{s_2} \rtimes \omega_0$. Then, a hermitian form indexed by (s_1, s_2) is defined by

$$(f, g)_{(s_1,s_2)} = \int_{\widetilde{K}} A(s_1, s_2, \chi_1, \chi_2, w_0) f_{s_1,s_2}(\widetilde{k})\overline{g_{s_1,s_2}(\widetilde{k})}\, \mathrm{d}\widetilde{k}.$$

The $\widetilde{\mathrm{Sp}(2)}$-invariance of this form follows from [Casselman 1981, Theorem 2.4.2] (in the context of totally disconnected groups) and then from Proposition 3.1.3 therein, after normalizing the measure on $\widetilde{P}$ so that $\widetilde{P} \cap \widetilde{K}$ is of volume one (since $\widetilde{P}\widetilde{K} = \widetilde{G}$).

Unitarizability of the ends of the complementary series. For the reductive algebraic groups, the unitarizability of the ends of the complementary series is proved by Miličić [1973] using C^*-algebra arguments. To avoid that (although this argument may also apply in the case of metaplectic groups), we use a similar result, that is, [Tadić 1986, Theorem 2.5]. The proof of this result relies on calculations of the limits of the operators acting on the finite-dimensional complex vector spaces, and the only requirements are admissibility of the irreducible smooth representations in question (our Lemma 2.1) and a result of Bernstein about uniform admissibility. But, since we do not require the generality in which that theorem is posed, we actually do not need Bernstein's argument, since we are dealing with the family of representations in the complementary series — all of them have the same restriction to the compact open subgroup K_1 (which splits in $\widetilde{\mathrm{Sp}}$), and the requirement labeled $(*)$ there is automatically fulfilled. Hecke algebra $H(\widetilde{\mathrm{Sp}(2)}, K_1)$ is defined in the same way as in the case of reductive groups.

The asymptotics of the matrix coefficients of the representations of the metaplectic group. These can also be estimated in terms of Jacquet modules of the representations [Casselman 1981, Section 4]. Indeed, the arguments there rely on the calculation of the spaces of the coinvariants for the unipotent subgroups (which split in $\widetilde{\mathrm{Sp}(2)}$) and spaces of vectors fixed by some small compact subgroups of $\widetilde{\mathrm{Sp}(2)}$. These subgroups can always be taken to belong to the maximal compact subgroup of $\mathrm{Sp}(2)$ that splits in $\widetilde{\mathrm{Sp}(2)}$ (if the residual characteristic is odd) or to some smaller open compact subgroup that splits, so we actually take the fixed vectors by these splittings of compact subgroups.

On the other hand, the reducibility points of the principal series for $\mathrm{SO}(5)$ are analogous to those for $\widetilde{\mathrm{Sp}(2)}$, so the unbounded areas of [Matić 2010, Figure 1 of Theorem 3.5], through the asymptotics explained above, give rise to the representations with unbounded matrix coefficients. Thus none of these representations are unitarizable (because of the continuity of the hermitian forms on these unbounded parts).

The arguments above (plus the irreducibility of the unitary principal series) were the main tools in the proof of [Matić 2010, Theorem 3.5]; there was only the problem of how to deal with certain isolated representations.

Recall that in R we have $\chi_{V,\psi}\nu^{3/2}\zeta \times \chi_{V,\psi}\nu^{1/2}\zeta \rtimes \omega_0$ is equal to

$$\langle \chi_{V,\psi}\nu^{3/2}\zeta, \chi_{V,\psi}\nu^{1/2}\zeta; \omega_0\rangle + \omega^+_{\psi_a,2} + \langle \chi_{V,\psi}\nu^{3/2}\zeta; \omega^+_{\psi_a,1}\rangle + \langle \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle,$$

where $\langle \chi_{V,\psi}\nu^{3/2}\zeta, \chi_{V,\psi}\nu^{1/2}\zeta; \omega_0\rangle$ and $\omega^+_{\psi_a,2}$ are unitarizable. Observe that the representation $\langle \chi_{V,\psi}\nu^{3/2}\zeta; \omega^+_{\psi_a,1}\rangle$ (respectively, $\langle \chi_{V,\psi}\nu\zeta\, 1_{\mathrm{GL}(2)}; \omega_0\rangle$) has Jacquet modules analogous to those of the representation $L(\delta([\nu^{1/2}, \nu^{3/2}]), 1)$ (respectively, $L(\nu^{3/2}, \mathrm{St}_{\mathrm{SO}(3)})$) of the group $\mathrm{SO}(5)$. Hence, nonunitarizability of these two representations can be proved analagously to the nonunitarizability of the representations $L(\delta([\nu^{1/2}, \nu^{3/2}]), 1)$ and $L(\nu^{3/2}, \mathrm{St}_{\mathrm{SO}(3)})$, which is a special case of [Hanzer and Tadić 2010, Propositions 4.1 and 4.6]. The arguments used there rely on the Jacquet modules method, which also applies to group $\widetilde{\mathrm{Sp}(2)}$, and the simple fact that every unitary representation is also semisimple. □

5. Unitary dual supported in maximal parabolic subgroups

5.1. ***The Siegel case.*** Using [Hanzer and Muić 2009], [Matić 2010, Proposition 4.1] and previously discussed issues of complementary series and nonunitarizability of the representations indexed by the (geometrically) unbounded pieces of the plane, we directly get the following:

Proposition 5.1. *Let ρ be an irreducible cuspidal representation of* $\mathrm{GL}(2, F)$. *There is at most one $s \geq 0$ such that $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ reduces. One of the following holds*:

(1) *If ρ is not self-dual, then $\chi_{V,\psi}\rho \rtimes \omega_0$ is irreducible and unitarizable. Also, the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ are irreducible and nonunitarizable for $s > 0$.*

(2) *If ρ is self-dual and $\omega_\rho = 1$, where ω_ρ denotes the central character of ρ, then the representation $\chi_{V,\psi}\rho \rtimes \omega_0$ reduces, while all of the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ are nonunitarizable for $s > 0$.*

(3) *If ρ is self-dual and $\omega_\rho \neq 1$, then the unique $s \geq 0$ such that $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ reduces is equal to $1/2$. For $0 \leq s \leq 1/2$, the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ are all unitarizable; for $s > 1/2$, the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ are all nonunitarizable. All irreducible subquotients of $\chi_{V,\psi}\nu^{1/2}\rho \rtimes \omega_0$ are unitarizable.*

5.2. *The non-Siegel case.* Hanzer and Muić [2009, Section 5.2] determine the reducibility points of the representations $\chi_{V,\psi}\nu^s\zeta \rtimes \pi$, where $s \in \mathbb{R}$, $\zeta \in \widehat{F^\times}$ and π is an irreducible cuspidal representation of $\widetilde{\mathrm{Sp}(1)}$. After determining the reducibility points, the unitarizability of the induced representations and irreducible subquotients follow in the same way as in Proposition 5.1. For the convenience of the reader, we write down all the results.

To the fixed quadratic character χ_V we attach, as in [Kudla 1996, Chapter V], two odd-orthogonal towers, the $+$-tower and the $-$-tower. We denote by $\Theta^\pm(\pi)$ the first appearance of the representation $\Theta(\pi)$ in the respective $\pm$-tower. Analogously, for $r \geq 0$, we denote by $\Theta^\pm(\pi, r)$ the lift of the representation π to the r-th level of the respective $\pm$-tower.

Since the representation $\chi_{V,\psi}\nu^s\zeta \rtimes \pi$ is irreducible for $\zeta^2 \neq 1$, we suppose $\zeta^2 = 1$ and consider two cases:

(a) $\zeta \neq 1$. Applying [Hanzer and Muić 2009, Theorem 3.5] we see that $\chi_{V,\psi}\nu^s\zeta \rtimes \pi$ reduces if and only if $\zeta\nu^s \rtimes \Theta^+(\pi)$ reduces (in the $+$-tower) if and only if $\zeta\nu^s \rtimes \Theta^-(\pi)$ reduces (in the $-$-tower).

Now $\Theta^+(\pi)$ is an irreducible cuspidal representation of some of the groups $O(1)$, $O(3)$ or $O(5)$. Let r denote the first occurrence of a nonzero lift of π in the odd orthogonal $+$-tower. We have several cases depending on r:

- If $r = 0$, that is, if π equals $\omega_{\psi_a^-,1}$, which is an odd part of the Weil representation attached to additive character ψ, then $\Theta^+(\pi, 0) = \mathrm{sgn}_{O(1)}$, so the representation $\zeta\nu^s \rtimes \mathrm{sgn}_{O(1)}$ reduces if and only if $\zeta\nu^s \rtimes 1$ reduces in SO(3). It is well known that this representation reduces when $s = 1/2$.
- If $r = 1$, the representation $\zeta\nu^s \rtimes \pi$ reduces if and only if the representation $\zeta\nu^s \rtimes \Theta^+(\pi, 1)|_{\mathrm{SO}(3)}$ reduces. As in [Matić 2010], we obtain that the unique s such that $\zeta\nu^s \rtimes \pi$ reduces is equal to $1/2$.
- If $r = 2$, the representation $\zeta\nu^s \rtimes \pi$ reduces if and only if $\zeta\nu^s \rtimes \Theta^+(\pi, 2)$ reduces, and that is if and only if $\zeta\nu^s \rtimes \Theta^+(\pi, 2)|_{\mathrm{SO}(5)}$ reduces. We do

not know if the representation $\Theta^+(\pi, 2)$ is generic, so we turn our attention to the representation $\zeta\nu^s \rtimes \Theta^-(\pi, 0)$, because we know that $\Theta^-(\pi, 0)$ is a nonzero representation of $O(1)$ (since π is cuspidal, the dichotomy conjecture holds). Recall that $\zeta\nu^s \rtimes \Theta^-(\pi, 0)$ reduces for $s = 0$ if and only if $\mu(s, \zeta \otimes \Theta^-(\pi, 0)) \neq 0$ for $s = 0$ and that $\zeta\nu^s \rtimes \Theta^-(\pi, 0)$ reduces for $s_0 > 0$ if and only if $\mu(s, \zeta \otimes \Theta^-(\pi, 0))$ has a pole for $s = s_0$. In the same way as in [Hanzer and Muić 2009, Section 5.2, case 3], we obtain $\mu(s, \zeta \otimes \Theta^-(\pi, 0)) \cong \mu(s, \zeta \otimes JL(\Theta^-(\pi, 0)))$, where $JL(\Theta^-(\pi, 0)))$ denotes the Jacquet–Langlands lift of $\Theta^-(\pi, 0))$. Now we consider two possibilities:

(I) $\Theta^-(\pi, 0)$ is not one-dimensional. In this case, $JL(\Theta^-(\pi, 0))$ is a cuspidal generic representation of $\mathrm{SO}(3, F)$ and the reducibility point is $s = 1/2$.

(II) $\Theta^-(\pi, 0) = \zeta_1 \circ \nu_D$, where ζ_1 is a quadratic character of $F^\times$, while ν_D is a reduced norm on $D^\times$ (here D is a nonsplit quaternion algebra over F). We have $JL(\Theta^-(\pi, 0)) = \zeta_1 \mathrm{St}_{\mathrm{GL}(2,F)}$. If $\zeta_1 = \zeta$, then the reducibility point is $s = 1/2$, otherwise the reducibility point is $s = 3/2$.

(b) $\zeta = 1$. This case can be completely solved using [Hanzer and Muić 2009, Theorem 4.2]. We again denote by r the first occurrence of nonzero lift of representation π in the odd orthogonal $+$-tower and consider all the possible cases:

- If $r = 0$, then π equals $\omega_{\psi_a^-,1}$ and the representation $\chi_{V,\psi}\nu^s \rtimes \omega_{\psi_a^-,1}$ reduces for $s = \pm 3/2$.
- If $r = 1$, the representation $\chi_{V,\psi}\nu^s \rtimes \pi$ reduces for $s = 1/2$.
- If $r = 2$, the representation $\chi_{V,\psi}\nu^s \rtimes \pi$ reduces for $s = 1/2$.

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