Pacific
Journal of
Mathematics

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Volume 248 No. 1 November 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 248, No. 1, 2010

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Let $P \subset \mathbb{R}^3$ be a polyhedron. It was conjectured that if P is weakly convex (that is, its vertices lie on the boundary of a strictly convex domain) and decomposable (that is, P can be triangulated without adding new vertices), then it is infinitesimally rigid. We prove this conjecture under a weak additional assumption of codecomposability.

The proof relies on a result of independent interest about the Hilbert–Einstein function of a triangulated convex polyhedron. We determine the signature of the Hessian of that function with respect to deformations of the interior edges. In particular, if there are no interior vertices, then the Hessian is negative definite.

1. Introduction

The rigidity of convex polyhedra. The rigidity of convex polyhedra is a classical result in geometry, first proved by Cauchy [1813] using ideas going back to Legendre [1794, note XII, pages 321–334].

Theorem 1.1 [Cauchy 1813; Legendre 1794]. *Let P, $Q \subset \mathbb{R}^3$ be two convex polyhedra with the same combinatorics whose corresponding faces are isometric. Then P and Q are congruent.*

This result had a profound influence on geometry over the last two centuries. It led for instance to the discovery of the rigidity of smooth convex surfaces in $\mathbb{R}^3$, to Alexandrov's rigidity, and to his results on the realization of positively curved cone-metrics on the boundary of polyhedra; see [Alexandrov 2005].

From a practical viewpoint, global rigidity is perhaps not as relevant as infinitesimal rigidity (see Definition 1.8). Although the infinitesimal rigidity of convex polyhedra can be proved using Cauchy's argument, the first proof was given much later, and is completely different from Cauchy's.

Theorem 1.2 [Dehn 1916]. *Any convex polyhedron is infinitesimally rigid.*

MSC2000: 52B10, 52C25.
Keywords: rigidity, polyhedra, nonconvex, Hilbert–Einstein functional.
Izmestiev was supported by the DFG Research Unit "Polyhedral Surfaces".

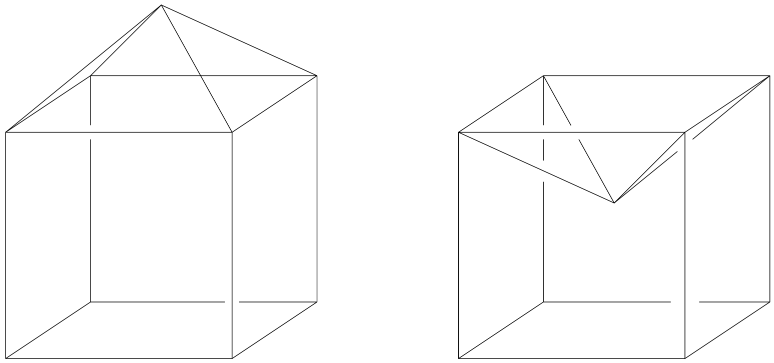

Figure 1. Cauchy's rigidity theorem fails for nonconvex polyhedra.

Neither Theorem 1.1 nor Theorem 1.2 extends to nonconvex polyhedra. It is easy to find a counterexample to the extension of Theorem 1.1 to nonconvex polyhedra; see Figure 1. Counterexamples to the extension of Theorem 1.2 are more complicated; see Figure 2.

In this paper we deal with a generalization of Theorem 1.2 to a vast class of nonconvex polyhedra. The main idea is that it is not necessary to consider convex polyhedra; what is important is that the vertices are in convex position. Additional assumptions are necessary, but are automatically satisfied for convex polyhedra.

Main result. By a polyhedron we mean a body in $\mathbb{R}^3$ bounded by a closed polyhedral surface.

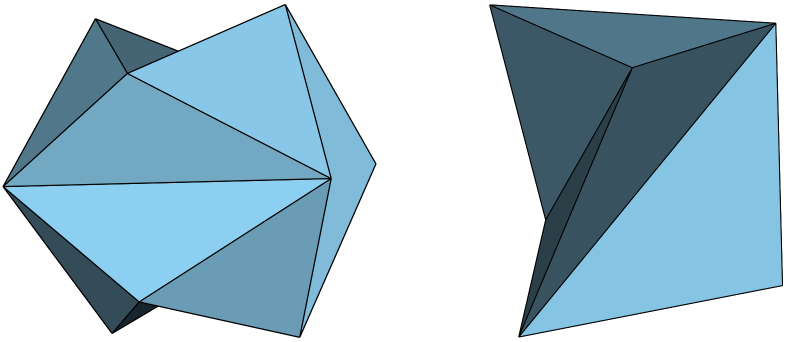

Figure 2. Examples of infinitesimally flexible polyhedra: Jessen's orthogonal icosahedron [Jessen 1967; Weisstein] and Schönhardt's polyhedron [Schönhardt 1928; Wunderlich 1965]. Both examples are weakly convex (Definition 1.3) but not decomposable (Definition 1.4).

Definition 1.3. A polyhedron $P \subset \mathbb{R}^3$ is called *weakly convex* if its vertices are in convex position in $\mathbb{R}^3$.

In other words, P is weakly convex if its vertices are the vertices of a strictly convex polyhedron.

Definition 1.4. A polyhedron P is called *decomposable* if it can be triangulated without adding new vertices.

In other words, every simplex of the triangulation must have vertices among those of P.

Our work was motivated by the following conjecture.

Conjecture 1.5. *Every weakly convex decomposable polyhedron is infinitesimally rigid.*

Any infinitesimally flexible polyhedron known to us fails to satisfy one of the assumptions of Conjecture 1.5. Thus, both polyhedra on Figure 2 are weakly convex but not decomposable. The infinitesimally flexible nonconvex octahedron pictured in [Gluck 1975] is decomposable but not weakly convex.

The main result of this paper is the proof of a weakening of Conjecture 1.5. We state it for polyhedra with triangular faces for simplicity, and explain in the next subsection how it extends to polyhedra with nontriangular faces. To state it, we need another, simple definition.

Definition 1.6. We call a polyhedron P *codecomposable* if its complement in $\operatorname{conv} P$ can be triangulated without adding new vertices. We call P *weakly codecomposable* if P is contained in a convex polyhedron Q, such that all vertices of P are vertices of Q and that the complement of P in Q can be triangulated without adding new vertices.

Theorem 1.7. *Let P be a weakly convex, decomposable, and weakly codecomposable polyhedron with triangular faces. Then P is infinitesimally rigid.*

Note that P is not required to be homeomorphic to a ball. The hypothesis that P is weakly codecomposable, however, appears to be quite weak for polyhedra homeomorphic to a ball. In the appendix we describe a simple example of a polyhedron that is not weakly codecomposable, but however is not homeomorphic to a ball; it's quite possible that this example can be modified fairly simply to make it contractible.

It is easy to come up with many examples of polyhedra to which Theorem 1.7 applies. Consider a convex polyhedron Q, and select an edge e of Q adjacent to two triangular faces f and f'. Cut out from Q the simplex that has f and f' as two of its faces, and let Q_1 be the nonconvex polyhedron obtained. Combinatorially, Q_1 is the same as Q, except that the edge e has been removed and replaced by the other diagonal of the quadrilateral made of the two triangular faces adjacent to e. By construction, Q_1 is weakly convex and weakly codecomposable; it is easy to check that it is decomposable (actually, star-shaped with respect to at least 2 of its vertices). This operation of cutting out a simplex can then be repeated, to obtain polyhedra $Q_2, Q_3, \dots$, which are always weakly convex and weakly codecomposable. It is not guaranteed, however, that they remain decomposable, and indeed the Schönhardt polyhedron depicted above shows that they might cease to be decomposable.

On the other hand, examples of noncodecomposable weakly convex polyhedra homeomorphic to a ball are quite complicated [Aichholzer et al. 2002], so a counterexample to Conjecture 1.5 would be difficult to construct. On the other hand, the codecomposability assumption is used in our proof of Theorem 1.7 in a very essential way. Thus the question whether the codecomposability assumption may be omitted remains wide open. (However, this assumption does not appear in [Schlenker 2005].)

There is another, clearly equivalent way to state Theorem 1.7. Let Q be a convex polyhedron, with a triangulation T with no interior vertex (all vertices of the simplices of T are vertices of Q). Let Σ be a subcomplex of T, homeomorphic to a closed surface. Then Σ, considered as a polyhedral surface, is infinitesimally rigid. This statement is also, in an obvious way, an extension of the Cauchy–Dehn rigidity result, Theorem 1.2.

Polyhedra with nontriangular faces. It is well known that to prove the infinitesimal rigidity of a polyhedron with nontriangular faces, it suffices to prove it is so after triangulating the faces; see for example [Alexandrov 2005]. In Theorem 1.7 the fact that two triangular faces are coplanar makes no difference, so it basically extends as is to polyhedra with some nontriangular faces. There is however a slightly subtle point that should be mentioned.

If a polyhedron P with some nontriangular faces is decomposable, then there is a triangulation of the faces that is compatible with its triangulation. Similarly, if P is codecomposable, there is a triangulation of its faces that is compatible with a decomposition of the complement of P in its convex hull. However, it is conceivable that P could be decomposable and codecomposable, but such that there is no triangulation of its nontriangular faces that is compatible both with a triangulation of P and with a triangulation of the complement of P in its convex hull. In this case, Theorem 1.7 would not apply to P. We have no example of such a polyhedron, and do not know whether any such example exists.

From this point on, we only consider polyhedra with triangular faces.

Earlier results. Conjecture 1.5 originated as a question in [Schlenker 2005], where a related result was proved: If P is a decomposable polyhedron such that there exists an ellipsoid that intersects all edges of P but contains none of its vertices, then P is infinitesimally rigid. The proof relies on hyperbolic geometry, more precisely the properties of the volume of hyperideal hyperbolic polyhedra.

Connelly and Schlenker [2010] then proved two special cases of the conjecture: when P is a weakly convex suspension containing its north-south axis, and when P has only one concave edge, or two concave edges adjacent to a vertex. The proof for suspensions was based on stress arguments, while the proof of the other result used a refinement of Cauchy's argument.

More recently, Schlenker [2009] proved that the conjecture holds when P is star-shaped with respect to one of its vertices. This implies the two results in [Connelly and Schlenker 2010]. The proof was based on recent results of [Izmestiev 2008] concerning convex caps. Schlenker's result — and therefore the two results of Connelly and Schlenker — are consequences of Theorem 1.7, since it is not difficult to show that a polyhedron that is star-shaped with respect to one of its vertices is codecomposable (the proof actually appears as a step in [Schlenker 2009]).

Definitions. Every polyhedron has faces, edges, and vertices. As mentioned above we only consider polyhedra with triangular faces.

Definition 1.8. A polyhedron P is called *infinitesimally rigid* if every infinitesimal flex of its 1-skeleton is trivial.

Definition 1.9. By an *infinitesimal flex* of a graph in $\mathbb{R}^3$, we mean an assignment of vectors to the vertices of the graph such that the displacements of the vertices in the assigned directions induce a zero first-order change of the edge lengths:

$$(p_i - p_j) \cdot (q_i - q_j) = 0 \quad \text{for every edge } p_i p_j,$$

where q_i is the vector associated to the vertex p_i. An infinitesimal flex is called trivial if it is the restriction of an infinitesimal rigid motion of $\mathbb{R}^3$.

Polyhedra with flat vertices. We will need to deal with triangulations of the boundaries of polyhedra that contain additional vertices. Infinitesimal flexes and infinitesimal rigidity for such *triangulated spheres* are defined in the same way. Note that a triangulated sphere may be infinitesimally flexible even if it bounds a convex polyhedron, see Figure 3.

Definition 1.10. Let S be a triangulation of the boundary of a polyhedron P. A vertex p of S is called a *flat vertex* if it lies in the interior of a face of P.

The following statement is an easy generalization of Dehn's theorem.

Theorem 1.11. *Let S be a triangulation of the boundary of a convex polyhedron P. Then every infinitesimal flex of S is the sum of an infinitesimal rigid motion and of displacements of flat vertices in the directions orthogonal to their ambient faces.*

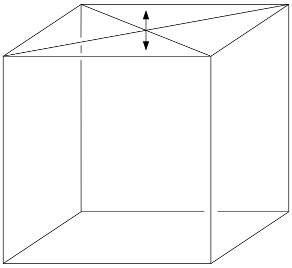

Figure 3. Convex triangulated sphere with a flat vertex. Moving this vertex in the direction orthogonal to the face produces a nontrivial infinitesimal flex.

The Hilbert–Einstein function. The proof of Theorem 1.7 is based on some striking properties of the discrete Hilbert–Einstein function, also known in the physics community as the Regge function [1961]. First we have to define a space of deformations of a triangulated polyhedron.

Definition 1.12. Let T be a triangulation of a polyhedron P, and let $e_1, \dots, e_n$ be the interior edges of T. We denote by $\mathfrak{D}_{P,T}$ the space of n-tuples $(l_1, \dots, l_n) \in \mathbb{R}^n_{>0}$ such that for every simplex σ of T, replacing the lengths of the edges of σ that are interior edges of T by the corresponding l_j produces a nondegenerate simplex.

For every element $l \in \mathfrak{D}_{P,T}$, there is an associated metric structure on P obtained by gluing the simplices with changed edge lengths. The resulting metric space is locally Euclidean except that it has cone singularities along the interior edges of T. For every $i \in \{1, \dots, n\}$, denote by ω_i the total angle around e_i and by $\kappa_i := 2\pi - \omega_i$ the singular curvature along e_i. Let $e'_1, \dots, e'_r$ be the boundary edges of P; for every $j \in \{1, \dots, r\}$ denote by α_j the dihedral angle of P at e'_j, and by l'_j the length of e'_j.

Definition 1.13. The *Hilbert–Einstein function* on $\mathfrak{D}_{P,T}$ is given by the formula

$$\mathcal{S}(l) := \sum_{i=1}^{n} l_i \kappa_i + \sum_{j=1}^{r} l'_j (\pi - \alpha_j) \ .$$

The Schläfli formula. A key tool in polyhedral geometry, this formula has several generalizations. The 3-dimensional Euclidean version states simply that, under a first-order deformation of any Euclidean polyhedron,

$$\sum_e l_e d\alpha_e = 0 \ , \tag{1}$$

where the sum is taken over all edges e, with l_e denoting the length of the edge e, and α_e the dihedral angle at e. This equality is also known as the Regge formula.

It follows directly from the Schläfli formula that, under any first-order variation of the lengths of the interior edges of a triangulation T of the polyhedron P — that is, for any vector tangent to $\mathcal{D}_{P,T}$ — the first-order variation of $\mathcal{S}$ is simply

$$d\mathcal{S} = \sum_{i=1}^{n} \kappa_i dl_i \ . \tag{2}$$

Therefore, the Hessian of $\mathcal{S}$ equals the Jacobian of the map $(l_i)_{i=1}^n \mapsto (\kappa_i)_{i=1}^n$.

Definition 1.14. Let T be a triangulation of a polyhedron P with n interior edges. Define the $n \times n$ matrix M_T as

$$M_T = \left(\frac{\partial \omega_i}{\partial l_j}\right) = -\left(\frac{\partial^2 \mathcal{S}}{\partial l_i \partial l_j}\right).$$

The derivatives are taken at the point $l \in \mathcal{D}_{P,T}$ that corresponds to the actual edge lengths in T.

The arguments in this paper use only M_T, and not directly the Hilbert–Einstein function $\mathcal{S}$. The fact that M_T is minus the Hessian of $\mathcal{S}$ does imply, however, that M_T is symmetric.

The matrix M_T is directly related to the infinitesimal rigidity of P, an idea that, in the smooth rather than the polyhedral context, goes back to Blaschke and Herglotz.[1]

Lemma 1.15. *Let T be a triangulation of a polyhedron P without interior vertices. Then P is infinitesimally rigid if and only if M_T is nondegenerate.*

The proof can be found in [Bobenko and Izmestiev 2008; Schlenker 2009] and is based on the observation that an isometric deformation of P induces a first-order variation of the interior edge lengths but a zero variation of the angles around them.

The second-order behavior of $\mathcal{S}$. The following is the key technical statement of the paper.

Theorem 1.16. *Let P be a convex polyhedron, and let T be a triangulation of P with* $\mathrm{Vert}(T) = \mathrm{Vert}(P)$. *Then M_T is positive definite.*

Theorem 1.16 is actually a special case of the following theorem that describes the signature of M_T for T *any* triangulation of P.

Theorem 1.17. *Let P be a convex polyhedron, and let T be a triangulation of P with m interior and k flat vertices. Then the dimension of the kernel of M_T is $3m + k$, and M_T has m negative eigenvalues.*

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[1] Blaschke and Herglotz suggested that the critical points of the Hilbert–Einstein function on a manifold with boundary (in the smooth case), with fixed boundary metric, correspond to Einstein metrics, that is, to constant curvature metrics in dimension 3. The analog of M_T in this context is the Hessian of the Hilbert–Einstein function.

From Theorem 1.16 to Theorem 1.7. Let P be a polyhedron satisfies the assumptions of Theorem 1.7. Since P is decomposable and weakly codecomposable, there exists a convex polyhedron Q such that all vertices of P are vertices of Q, and a triangulation $\overline{T}$ of Q that contains a triangulation T of P and whose vertices are only the vertices of Q. It is easy to see that the matrix M_T is then a principal minor of the matrix $M_{\overline{T}}$. By Theorem 1.16, $M_{\overline{T}}$ is positive definite; thus so is M_T. In particular, M_T is nondegenerate. Lemma 1.15 implies that the polyhedron P is infinitesimally rigid.

Since Theorem 1.16 is a special case of Theorem 1.17, the rest of this paper deals with proving Theorem 1.17.

Plan 1.18 (for proving Theorem 1.17). The proof is based on a standard procedure. To show that the matrix M_T has the desired property for every triangulation T, we prove three points:

- any two triangulations can be connected by a sequence of moves;
- the moves don't affect the desired property;
- the property holds for a special triangulation.

These points are dealt with in the given order in the next three sections.

2. Connectedness of the set of triangulations

Moves on simplicial complexes are well studied; see [Lickorish 1999] for an overview. Several theorems state that any two triangulations of a given manifold can be connected by certain kinds of simplicial moves. However, we are in a different situation here, since we deal with triangulations of a fixed geometric object. Taking a closer look, one sees that a simplicial move is defined as a geometric move preceded and followed by a simplicial isomorphism. Performing an isomorphism is the possibility that is missing in our case.

To emphasize the difference between the combinatorial and the geometric situation, let us cite a negative result concerning geometric moves. Santos [2005] exhibited two triangulations with the same set of vertices in $\mathbb{R}^5$ that cannot be connected via $2 \leftrightarrow 5$ and $3 \leftrightarrow 4$ bistellar moves. For an overview on geometric bistellar moves, see [Santos 2006].

Geometric stellar moves: the Morelli–Włodarczyk theorem. Morelli [1996] and Włodarczyk [1997] obtained a positive result on geometric simplicial moves. As a crucial step in the proof of the weak Oda conjecture, they showed that any two triangulations of a convex polyhedron can be connected by a sequence of *geometric stellar moves*.

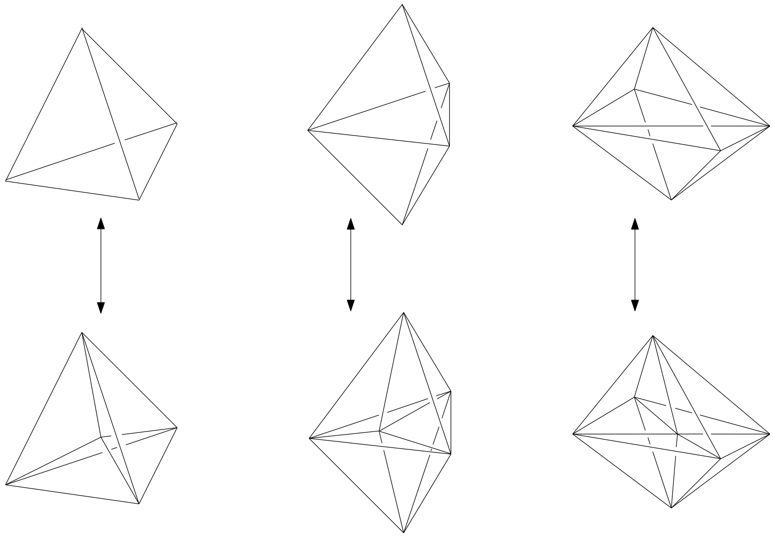

Figure 4. Interior stellar moves in dimension 3.

Definition 2.1. Let p be an interior point of a simplex $\sigma \subset \mathbb{R}^n$. The *starring* of σ at p is an operation that replaces σ by the cone with the apex p over the boundary of σ.

Let T be a triangulation of a subset of $\mathbb{R}^n$, let σ be a simplex of T, and let p be a point in the relative interior of σ. The operation of *starring* of T at p consists of replacing the star $\operatorname{st}\sigma$ of σ by the cone with apex p over the boundary of $\operatorname{st}\sigma$. The operation inverse to starring is called *welding*.

Starrings and weldings are called *stellar moves*.

See Figures 4 and 5 for stellar moves in dimension 3. Figure 4 depicts starrings and weldings at interior points of T, while Figure 5 shows starrings and weldings at boundary points. In the case of a boundary point our definition is not completely correct: A starring replaces $\operatorname{st}\sigma$ by the cone over $\partial \operatorname{st}\sigma \setminus \partial T$.

Theorem 2.2 [Morelli 1996; Włodarczyk 1997]. *Any two triangulations of a convex polyhedron $P \subset \mathbb{R}^n$ can be connected by a sequence of geometric stellar moves.*

We outline of Morelli's proof using more elementary language and tools.

Outline of the proof. Let T and T' be two triangulations of P. A triangulation Σ of $P \times [0, 1]$ with $\Sigma_{P\times\{1\}} = T$ and $\Sigma|_{P\times\{0\}} = T'$ is called a *simplicial cobordism* between T and T'.

Definition 2.3. Let pr denote the orthogonal projection $P \times [0, 1] \to P$. A simplex $\sigma \in \Sigma$ is called a *circuit* if $\dim \operatorname{pr}(\sigma) < \dim \sigma$ and σ is inclusion minimal with this property.

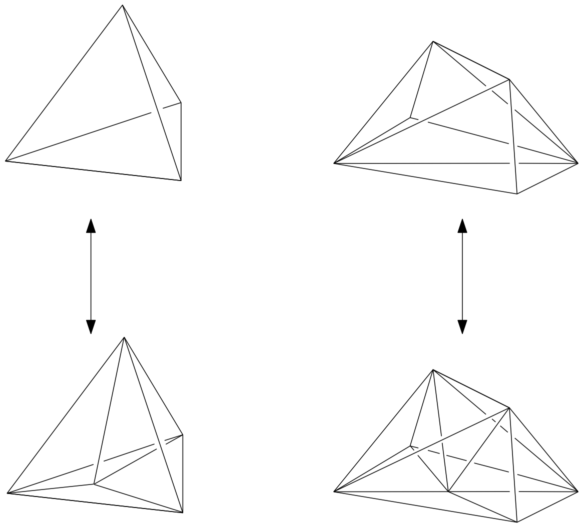

Figure 5. Boundary stellar moves in dimension 3.

Clearly, the stars of the circuits are simplicial balls with no vertical faces and $\Sigma = \bigcup_{\text{circuits } \sigma} \operatorname{st} \sigma$ with disjoint interiors.

Definition 2.4. We call a simplicial cobordism Σ *collapsible* if there is a sequence of triangulations $\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_N = \Sigma_{P\times\{0\}}$ such that

- $\Sigma_{i+1} = \Sigma_i \setminus \operatorname{st} \sigma_i$ for a circuit σ_i;
- the upper boundary of Σ_i projects one-to-one on P for every i.

In other words, Σ is collapsible if it can be "dismantled with a crane".

Lemma 2.5. *The triangulation* $\operatorname{pr}(\partial^+\Sigma_{i+1})$ *can be obtained from* $\operatorname{pr}(\partial^+\Sigma_i)$ *by a starring with a subsequent welding. Here* ∂^+ *denotes the upper boundary.*

Proof. For every circuit σ, the transformation $\operatorname{pr}(\partial^+\sigma) \rightsquigarrow \operatorname{pr}(\partial^-\sigma)$ is a bistellar move and can be realized by a starring and a welding. These extend to a starring and a welding in $\operatorname{st}\sigma$. ☐

Thus, a collapsible simplicial cobordism between two triangulations gives rise to a sequence of stellar moves joining the triangulations.

Definition 2.6. We call a triangulation Σ *coherent* if there is a function $h : |\Sigma| \to \mathbb{R}$ that is piecewise linear with respect to Σ and strictly convex across every facet of Σ. (Here $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$ is the support of Σ.)

The barycentric subdivision of any convex polytope Q is coherent — one can choose values of h at the barycenters of faces of Q consecutively, along with increasing dimension. Each time the value must be sufficiently large.

Lemma 2.7. *A coherent simplicial cobordism is collapsible.*

Proof. Let σ and σ' be two circuits of Σ such that some point of $\operatorname{st}\sigma$ lies directly above a point of $\operatorname{st}\sigma'$. It follows that $\partial h/\partial x_{n+1}|_\sigma > \partial h/\partial x_{n+1}|_{\sigma'}$, where $\partial/\partial x_{n+1}$ denotes the derivative in the vertical direction. Thus the stars of the circuits can be lifted up in nondecreasing order of the vertical derivative of h on the circuits. □

To prove the theorem, we construct a coherent cobordism between stellar subdivisions of T and T'. (By a stellar subdivision, we mean the result of a sequence of starrings.)

Lemma 2.8. *Let Σ and Σ' be two triangulations with the same support. Then Σ can be stellarly subdivided to a triangulation Σ'' that refines Σ'.*

The reader can find a proof of this classical statement in [Glaser 1970].

Lemma 2.9. *Let Σ be an arbitrary triangulation of a convex polytope. Then Σ can be stellarly subdivided to a coherent triangulation.*

Proof. By Lemma 2.8, the barycentric triangulation of the convex polytope $|\Sigma|$ can be stellarly subdivided to a triangulation Σ' that refines Σ. But the barycentric subdivision of any polytope is coherent. Since starring a coherent triangulation produces a coherent triangulation, Σ' is coherent.

Again by Lemma 2.8, the triangulation Σ can be stellarly subdivided to a triangulation Σ'' that refines Σ'. We claim that Σ'' is coherent.

Let us show that there exists a function $s'' : |\Sigma| \to \mathbb{R}$ that is piecewise linear with respect to Σ'' and strictly convex across all facets of Σ'' that are not contained in the codimension 1 skeleton of Σ. We construct s'' by induction on the number of stellar subdivisions that transform Σ to Σ''. As the induction base, we take the zero function. When a stellar subdivision is done, we redefine the function at the center of the subdivision by increasing it a little. Then we get strict convexity across appearing facets and don't destroy convexity where it already takes place. Note that s'' can be concave across facets of Σ'' that are contained in facets of Σ.

Now, since Σ' is coherent, there exists a function h' piecewise linear and strictly convex with respect to Σ'. Since Σ' refines Σ, the function h' is strictly convex across the codimension 1 skeleton of Σ. Therefore the function $h'' = h' + \epsilon s''$ is strictly convex on Σ'' for a sufficiently small positive ϵ. □

Outline of the proof of Theorem 2.2. Applying Lemma 2.9 to an arbitrary simplicial cobordism Σ between T and T', we get a coherent simplicial cobordism Σ''. Since $\Sigma''|_{P\times 1}$ and $\Sigma''|_{P\times 0}$ are stellar subdivisions of T and T' respectively, this yields a sequence of stellar moves connecting T and T'.

An alternative way to derive Theorem 2.2 from Lemma 2.9 was suggested to us by Francisco Santos and is as follows. By Lemma 2.9, the triangulations T and T' can be stellarly subdivided to coherent triangulations S and S', respectively. Let

$h : P \times \{1\} \to \mathbb{R}$ and $h' : P \times \{0\} \to \mathbb{R}$ be corresponding convex piecewise linear functions. Then their lower envelope $\widetilde{h} : P \times [0, 1]$ is a convex function whose linearity domains determine a polyhedral subdivision of $P \times [0, 1]$. If h and h' are in general position, then this subdivision is a coherent triangulation, and thus a collapsible simplicial cobordism between T and T'. ☐

Realizing interior stellar moves by bistellar moves. To simplify our task in the next section, we show that instead of interior stellar moves one can use *bistellar* or *Pachner moves* and continuous displacements of the vertices of the triangulation.

Definition 2.10. Let T be a triangulation of a subset of $\mathbb{R}^3$.

- Let σ be a 3-dimensional simplex of T. A $1 \to 4$ Pachner move replaces σ by four smaller simplices sharing a vertex that is an interior point of σ.
- Let σ and τ be two 3-simplices of T such that the union $\sigma \cup \tau$ is a strictly convex bipyramid. A $2 \to 3$ Pachner move replaces σ and τ by three simplices sharing the edge that joins the opposite vertices of σ and τ.
- A $3 \to 2$ Pachner move is the inverse of a $2 \to 3$ Pachner move.
- A $4 \to 1$ Pachner move is the inverse of a $1 \to 4$ Pachner move.

The Pachner moves are depicted on Figure 6.

Lemma 2.11. *Any two triangulations of a convex polyhedron P can be connected by a sequence of Pachner moves, boundary stellar moves and continuous displacements of the interior vertices.*

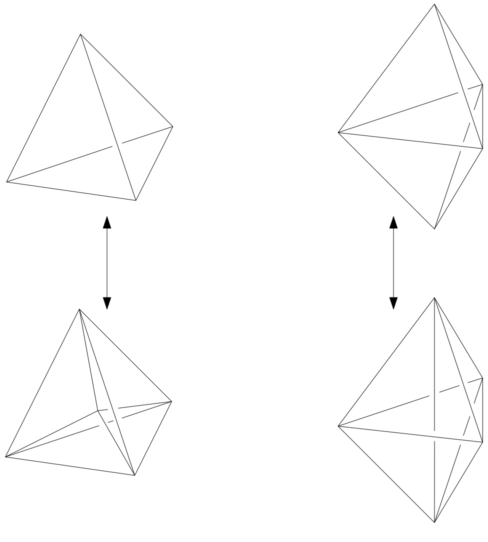

Figure 6. The $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ Pachner moves.

Proof. By Theorem 2.2, it suffices to show that every interior stellar move can be realized as a sequence of Pachner moves and vertex displacements. Since Pachner moves are invertible, we realize only interior starrings.

The starring in a 3-simplex is a $1 \to 4$ Pachner move.

Consider the starring in a triangle (middle of Figure 4). Denote the vertices of the triangle to be starred by $1, 2, 3$, and the two remaining vertices by a and b. Perform a $1 \to 4$ move on the tetrahedron $a123$ and denote the new vertex by p. Then perform a $2 \to 3$ move on the tetrahedra $p123$ and $b123$. Finally move the vertex p so that it lies in the triangle 123.

To realize a starring of an edge, we also first perform a sequence of Pachner moves to obtain a triangulation combinatorially equivalent to the starring, and then move the new vertex. Denote by a and b the vertices of the edge to be starred, and denote the vertices in the link of the edge ab by $1, 2, \dots, n$ in the cyclic order. Perform a $1 \to 4$ move on the tetrahedron $ab1n$. The new vertex p should be chosen so that the plane of the triangle abp does not pass through any other vertex. Let $(k, k+1)$ be the edge intersected by this plane. Perform a $2 \to 3$ move on the tetrahedra $ab1p$ and $ab12$, then a $2 \to 3$ move on the tetrahedra $ab2p$ and $ab23$, and so on. This sequence finishes with a $2 \to 3$ move on $ab(k-1)p$ and $ab(k-1)k$. After that apply a similar sequence of $2 \to 3$ moves on the other side starting with the tetrahedra $abnp$ and $abn(n-1)$ and finishing with $ab(k+2)p$ and $ab(k+2)(k+1)$. Finally perform a $3 \to 2$ move over the tetrahedra $abpk$, $abp(k+1)$ and $abk(k+1)$. It remains to move the vertex p so that it lies on the edge ab. ☐

3. The effect of the elementary moves on the signature of M_T

In this section we realize the second point of Plan 1.18. Namely, we show that if Theorem 1.17 holds for some triangulation T, then it holds for a triangulation T' that is obtained from T by an elementary move. An elementary move is either a Pachner move or a boundary stellar move or a continuous displacement of the interior vertices of T.

The rank of the matrix M_T. Here we prove a part of Theorem 1.17:

Lemma 3.1. *The corank of the matrix M_T equals $3m + k$, with m the number of interior vertices and k the number of flat boundary vertices in the triangulation T:*

$$\dim \ker M_T = 3m + k.$$

Proof. If $m > 0$ or $k > 0$, then it is easy to find a whole bunch of vectors in the kernel of M_T. Any continuous displacement of the interior vertices of T changes the lengths of the interior edges, but doesn't change the angles around them, which stay equal to 2π. Similarly, moving a flat boundary vertex in the direction orthogonal to its ambient face doesn't change any of the angles ω_i. It does change the lengths of the boundary edges incident to this vertex, but only in the second order. It follows that the variations of interior edge lengths induced by the orthogonal displacement of a flat boundary vertex belong to the kernel of M_T.

Being formal, let $Q : \mathcal{V}(T) \to \mathbb{R}^3$ be an assignment to every vertex p_i of T of a vector q_i such that

(1) $q_i = 0$ if p_i is a nonflat boundary vertex of T;

(2) $q_i \perp F_i$ if p_i is a flat boundary vertex lying in the face F_i of P.

For every edge ij of T put

$$\ell_{ij}^Q = \frac{p_i - p_j}{\|p_i - p_j\|} \cdot (q_i - q_j).$$

It is easy to see that this formula gives the infinitesimal change of ℓ_{ij} that results from the infinitesimal displacements of the vertices p_i, p_j by the vectors q_i, q_j. By the previous paragraph, $\ell_{ij}^Q \in \ker M_T$.

Let us show that the span of the vectors ℓ^Q has dimension $3m+k$. The correspondence between Q and ℓ^Q is linear, and the space of assignments Q with properties (1) and (2) has dimension $3m+k$, so it suffices to show that $\ell^Q = 0$ implies $Q = 0$. Indeed, $\ell^Q = 0$ means that Q is an infinitesimal flex of the 1-skeleton of T; see Definition 1.9. But T is infinitesimally rigid, since every simplex is. Thus $\ell^Q = 0$ implies that Q is trivial. Since $q_i = 0$ on the vertices of P, we have $Q = 0$.

It remains to show that any vector $\dot{\ell} \in \ker M_T$ has the form ℓ^Q for some Q. Let $p_1p_2p_3$ be a triangle of T. Choose q_1, q_2, and q_3 arbitrarily. Let p_4 be a vertex such that there is a simplex $p_1p_2p_3p_4$ in T. The values of $\dot{\ell}_{i4}$ for $i = 1, 2, 3$ determine uniquely a vector q_4 such that $\dot{\ell}_{i4} = \ell_{i4}^Q$ for $i = 1, 2, 3$. If ij is a boundary edge of T, we put $\dot{\ell}_{ij} = 0$. Similarly, we define q_5 for the vertex p_5 of a simplex that shares a face with $p_1p_2p_3p_4$. Proceeding in this manner, we can assign a vector q_i to every vertex p_i if we show that this is well-defined (we extend our assignment along paths in the dual graph of T, and it needs to be shown that the extension does not depend on the choice of a path). It is not hard to see that this is ensured by the property $M_T\dot{\ell} = 0$. Thus we have constructed an assignment $Q : \mathcal{V}(T) \to \mathbb{R}^3$ such that $\dot{\ell} = \ell^Q$. Since $\dot{\ell}_{ij} = 0$ for every boundary edge ij of T, the vectors $(q_i)|_{p_i \in \partial P}$ define an infinitesimal flex of the boundary of P. Due to Theorem 1.11, Q satisfies properties (1) and (2) above, after subtracting an infinitesimal motion. Thus the kernel of M_T consists of the vectors of the form ℓ^Q. □

Corollary 3.2. *Let T be a triangulation of a convex polyhedron P. Consider a continuous displacement of the vertices of T such that no simplex of the triangulation degenerates, the underlying space of T remains a convex polyhedron, all flat* *boundary vertices remain flat, and nonflat remain nonflat. Then the signature of the matrix M_T does not change during this deformation.*

Proof. Due to Lemma 3.1, the rank of M_T does not change during the deformation. Hence no eigenvalue changes its sign. ☐

The effect of the Pachner moves.

Lemma 3.3. *Let P be a convex polyhedron, and let T and T' be two triangulations of P such that T' is obtained from T by a $2 \to 3$ Pachner move. Then the statement of Theorem 1.17 applies to T if and only if it applies to T'.*

Proof. Since triangulations T and T' have the same number of interior and flat boundary vertices, the matrices M_T and $M_{T'}$ have the same corank by Lemma 3.1. It remains to show that M_T and $M_{T'}$ have the same number of negative eigenvalues.

Matrices M_T and $M_{T'}$ define symmetric bilinear forms (that are denoted by the same letters) on the spaces $\mathbb{R}^{\mathscr{E}_{\mathrm{int}}(T)}$ and $\mathbb{R}^{\mathscr{E}_{\mathrm{int}}(T')}$, respectively. Here $\mathscr{E}_{\mathrm{int}}(T)$ denotes the set of interior edges of the triangulation T. Note that $\mathscr{E}_{\mathrm{int}}(T') = \mathscr{E}_{\mathrm{int}}(T) \cup \{e_0\}$, where e_0 is the vertical edge on the lower right of Figure 6. Extend M_T to a symmetric bilinear form on $\mathbb{R}^{\mathscr{E}_{\mathrm{int}}(T')}$ by augmenting the matrix M_T with a zero row and a zero column, and put $\Phi = M_{T'} - M_T$. By Definition 1.14, we have

$$\Phi = \left(\frac{\partial(\omega'_i - \omega_i)}{\partial \ell_j} \right)_{i,j \in \mathscr{E}_{\mathrm{int}}(T')},$$

where we put $\partial\omega_0/\partial\ell_j = 0$ for all j.

Denote those edges on the upper right of Figure 6 that are interior edges of T by $e_1, \dots, e_s$. Note that $\omega_i = \omega'_i$ as functions of the edge lengths for all $i \notin \{0, \dots, s\}$. Thus, the matrix Φ reduces to an $(s+1) \times (s+1)$ matrix with rows corresponding to the edges $e_0, \dots, e_s$.

We claim that the matrix Φ is positive semidefinite of rank 1. To construct a vector in the kernel of Φ, note that during any continuous deformation of the bipyramid on Figure 6 we have $\omega_i = \omega'_i$ as functions of edge lengths for $i = 1, \dots, s$, while ω'_0 is identically 2π. Thus if we choose $\dot{\ell}_1, \dots, \dot{\ell}_s$ arbitrarily and define $\dot{\ell}_0$ as the infinitesimal change of the length of e_0 under the corresponding infinitesimal deformation of the bipyramid, then we have $\Phi\dot{\ell} = 0$. Therefore $\operatorname{rank}\Phi \le 1$. The infinitesimal rigidity of the bipyramid implies $\partial\omega'_0/\partial\ell_0 \neq 0$; thus $\operatorname{rank}\Phi = 1$. Since the space of convex bipyramids is connected, it suffices to prove the positive semidefiniteness of Φ in some special case. In the case when all edges of the bipyramid have equal length, one can easily see that $\partial\omega'_0/\partial\ell_0 > 0$, which implies the positivity of the unique eigenvalue of Φ.

The equation

$$\operatorname{rank} M_{T'} = \operatorname{rank} M_T + 1 = \operatorname{rank} M_T + \operatorname{rank}\Phi$$

implies that $\ker M_T$ and $\ker \Phi$ intersect transversally and $\ker M_{T'} = \ker M_T \cap \ker \Phi$. Therefore

$$\operatorname{rank}(M_T + t\Phi) = \operatorname{rank} M_T + 1 \quad \text{for all } t \neq 0.$$

The Courant minimax principle [Courant and Hilbert 1953, Chapter I, Section 4] implies that the eigenvalues of $M_T + \epsilon\Phi$ are larger than or equal to the corresponding eigenvalues of M_T. It follows that when M_T is deformed into $M_{T'}$ via $\{M_T + t\Phi\}_{t\in[0,1]}$, exactly one of the zero eigenvalues of M_T becomes positive, and all of the nonzero eigenvalues preserve their sign. Thus $M_{T'}$ has the same number of negative eigenvalues as M_T and the lemma is proved. □

Lemma 3.4. *Let P be a convex polyhedron, and let T and T' be two triangulations of P such that T' is obtained from T by a $1 \to 4$ Pachner move. Then the statement of Theorem 1.17 applies to T if and only if it applies to T'.*

Proof. The same arguments as in the proof of Lemma 3.3 work. The triangulation T' has one interior vertex more that the triangulation T and four interior edges more than T. By Lemma 3.1, we have $\operatorname{rank} M_{T'} = \operatorname{rank} M_T + 1$, and we have to prove that $M_{T'}$ has the same number of positive eigenvalues as M_T and one negative eigenvalue more. For this it suffices to show that the quadratic form $\Phi = M_{T'} - M_T$ is negative semidefinite of rank 1. In the same way as in the proof of Lemma 3.3, one shows that $\operatorname{rank} \Phi \leq 1$. After that, it suffices to show that the restriction of Φ to the space spanned by the variations of lengths of the four interior edges on the lower left of Figure 6 is nontrivial and negative semidefinite. The nontriviality follows from the infinitesimal rigidity of the simplex, and it suffices to check the negative semidefiniteness in some convenient special case. □

The effect of the boundary stellar moves.

Lemma 3.5. *Let P be a convex polyhedron, and let T and T' be two triangulations of P such that T' is obtained from T by the starring of a boundary 2-simplex. Then the statement of Theorem 1.17 applies to T if and only if it applies to T'.*

Proof. We have $\operatorname{rank} M_{T'} = \operatorname{rank} M_T$ and need to show that $M_{T'}$ has the same signature as M_T. This is true because in fact $M_{T'} = M_T$ — more precisely, $M_{T'}$ is obtained from M_T by adding a column and a row, each with all elements equal to zero. This can be shown using the explicit formulas for $\partial\omega_i/\partial\ell_j$ and $\partial\omega'_i/\partial\ell_j$ from [Bobenko and Izmestiev 2008, Section 3.1] and [Korepanov 2000]. □

Lemma 3.6. *Let P be a convex polyhedron, and let T and T' be two triangulations of P such that T' is obtained from T by the starring of a boundary 1-simplex. Then the statement of Theorem 1.17 applies to T if and only if it applies to T'.*

Proof. The strategy is the same as in the proofs of Lemmas 3.3 and 3.4. Put $\Phi = M_{T'} - M_T$ and note that by Lemma 3.1

$$\operatorname{rank} M_{T'} = \operatorname{rank} M_T + i, \tag{3}$$

where i is one less than the number of simplices incident to the starred edge (for example, $i = 2$ in the right column of Figure 5). As in the proof of Lemma 3.3, one shows that $\operatorname{rank} \Phi \leq i$. Then (3) implies $\operatorname{rank} \Phi = i$. Since we aim to show that $M_{T'}$ has the same number of negative eigenvalues as M_T, it suffices to show that Φ is positively semidefinite.

Let Ψ be the $i \times i$ principal minor of Φ formed by the rows and columns that correspond to the interior edges of the triangulation on the lower right of Figure 5. We claim that Ψ is positively definite, which implies the nonnegativity of Φ. The proof is by continuity argument as in Lemma 3.3. To prove the nondegeneracy of Ψ, it suffices to show that the framework of the boundary edges on the lower right of Figure 5 is infinitesimally rigid. The framework on the upper right of Figure 5 is infinitesimally rigid, since it is formed by skeleta of 3-simplices that are rigid. This implies the infinitesimal rigidity of the boundary framework on the lower right (an easy exercise in applying the definition of an infinitesimal flex). Now consider a deformation of the triangulation on the upper right that makes the underlying polyhedron convex. This deformation can be extended to a deformation of the triangulation on the lower right. Since the matrix Ψ remains nondegenerate during the deformation, its signature is preserved. After the polyhedron is made convex, push the starring vertex off the starred edge so that the vertices of the triangulation are in the convex position. This also preserves the signature of Ψ. In the final position, Ψ is positive due to Theorem 4.1. □

4. Investigating M_T for a special triangulation T.

Let P be a convex polyhedron. Let S be a triangulation of ∂P such that $\operatorname{Vert}(S) = \operatorname{Vert}(P)$, and let p be a vertex of P. Consider the triangulation T consisting of simplices with a common vertex p and opposite faces the triangles of S disjoint from p.

Theorem 4.1. *The matrix M_T is positive definite.*

Proof. Formally, this is a special case of [Schlenker 2009, Theorem 1.5] that claims that M_T is positive if P is weakly convex and star-shaped with respect to the vertex p. The proof uses the positivity of the corresponding matrix for convex caps [Izmestiev 2008, Lemma 6 and Theorem 5] and the projective invariance of infinitesimal rigidity [Schlenker 2009, Section 5]. □

Theorem 4.1 accomplishes the plan outlined in Plan 1.18. Theorem 1.17 is proved, and therewith Theorem 1.7.

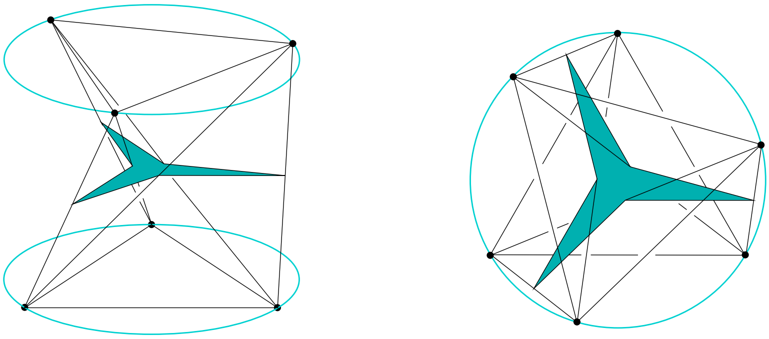

Figure 7. The twisted octahedron.

Appendix: A polyhedron that is not weakly codecomposable

Definition A.1. Let $\theta \in (-2\pi/3, 2\pi/3)$. The twisted octahedron Oct_θ of Figure 7 is the polyhedron with vertices A, B, C, A', B', C' of coordinates respectively

$$\begin{aligned} &(1, 0, 1), && (\cos(2\pi/3), \sin(2\pi/3), 1),\\ &(\cos(4\pi/3), \sin(4\pi/3), 1), && (\cos(-\pi+\theta), \sin(-\pi+\theta), -1),\\ &(\cos(-\pi/3+\theta), \sin(-\pi/3+\theta), -1), && (\cos(\pi/3+\theta), \sin(\pi/3+\theta), -1). \end{aligned}$$

The edges are the segments joining A to B' and C', B to A' and C', C to A' and B', and the faces are the triangles (ABC), $(A'B'C')$, $(AB'C')$, $(A'BC')$, $(A'B'C)$, (ABC'), $(AB'C)$, $(A'BC)$.

Note that $\mathrm{Oct}_{\pm\pi/2}$ is a Schönhardt polyhedron; see Figure 2, right.

Proposition A.2. Oct_θ *is embedded for all* $\theta \in (-2\pi/3, 2\pi/3)$.

For $\theta \in (-2\pi/3, 2\pi/3)$, we call $A_t(\theta)$ the area of the intersection of Oct_θ with the horizontal plane $\{z = t\}$.

Proposition A.3. $\lim_{\theta\to 2\pi/3} A_0(\theta) = 0$.

Let K be a large enough convex polygon in the plane Oxy (it suffices to require that the interior of K contains the disk $x^2 + y^2 \leq 1$). Consider the polyhedron $P_\theta = \mathrm{conv}(A, B, C, A', B', C', K) \setminus \mathrm{Oct}_\theta$ homeomorphic to a solid torus.

Lemma A.4. *For* θ *close enough to* $2\pi/3$, P_θ *is not weakly codecomposable.*

Proof. Suppose that P_θ is weakly codecomposable. Then there exists a convex polyhedron $Q_\theta \supset P_\theta$ such that $Q_\theta \setminus P_\theta$ can be triangulated without an interior vertex. Let $S_1, \dots, S_n$ be the simplices in this triangulation that intersect $\mathrm{Oct}_\theta \cap (Oxy)$. For each $i \in \{1, \dots, n\}$, let $a_i(t)$ be the area of the intersection of S_i with the horizontal plane $\{z = t\}$.

Each of the S_i can have either:

- Two vertices with $z \geq 1$ and two vertices with $z \leq -1$. Then the restriction of a_i to $(-1, 1)$ is a concave quadratic function, so that $2a_i(0) \geq a_i(-1) + a_i(1)$.
- One vertex with $z \geq 1$ and three vertices with $z \leq -1$. Then a_i is of the form $a_i(t) = c_i(t + b_i)^2$ with $b_i \geq 1$. It easily implies that $4a_i(0) \geq a_i(-1) + a_i(1)$.
- One vertex with $z \leq -1$ and three vertices with $z \geq 1$. The same argument then shows the same result.

So $4a_i(0) \geq a_i(-1) + a_i(1)$ for all i and the union of the S_i contains Oct_θ. It follows that $4A_0(\theta) \geq A_{-1}(\theta) + A_1(\theta)$. But $A_1(\theta)$ and $A_{-1}(\theta)$ are equal to the area of an equilateral triangle of fixed side length, while $A_0(\theta)$ goes to 0 as $\theta \to 2\pi/3$. This is a contradiction, and the claim follows. □

Acknowledgements

We are grateful to Bob Connelly, Günter Rote and Igor Pak for useful conversations and remarks concerning the subject considered here. We thank Francisco Santos for pointing out some mistakes in the proof of Lemma 2.9 and helping to correct them.

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Received August 22, 2009.

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at the University of California, Berkeley 94720-3840
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PACIFIC JOURNAL OF MATHEMATICS

Volume 248 No. 1 November 2010

0030-8730(201011)248:1;1-C