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We study biharmonic hypersurfaces in a generic Riemannian manifold. We first derive an invariant equation for such hypersurfaces generalizing the biharmonic hypersurface equation in space forms studied by Jiang, Chen, Caddeo, Montaldo, and Oniciuc. We then apply the equation to show that the generalized Chen conjecture is true for totally umbilical biharmonic hypersurfaces in an Einstein space, and construct a 2-parameter family of conformally flat metrics and a 4-parameter family of multiply warped product metrics, each of which turns the foliation of an upper-half space of $\mathbb{R}^m$ by parallel hyperplanes into a foliation with each leaf a proper biharmonic hypersurface. We also study the biharmonicity of Hopf cylinders of a Riemannian submersion.

1. Biharmonic maps and submanifolds

All manifolds, maps, and tensor fields that appear in this paper are assumed to be smooth unless stated otherwise.

A biharmonic map is a map $\varphi:(M, g)\to(N, h)$ between Riemannian manifolds that is a critical point of the bienergy functional

$$E^2(\varphi,\Omega)=\frac{1}{2}\int_\Omega|\tau(\varphi)|^2\mathrm{d}x$$

for every compact subset Ω of M, where $\tau(\varphi)=\mathrm{Trace}_g\,\nabla\mathrm{d}\varphi$ is the tension field of φ. The Euler–Lagrange equation of this functional gives the biharmonic map equation [Jiang 1986b]

$$\tau^2(\varphi):=\mathrm{Trace}_g(\nabla^\varphi\nabla^\varphi-\nabla^\varphi_{\nabla^M})\tau(\varphi)-\mathrm{Trace}_g\,R^N(\mathrm{d}\varphi,\tau(\varphi))\mathrm{d}\varphi=0, \tag{1}$$

which states that φ is biharmonic if and only if its bitension field $\tau^2(\varphi)$ vanishes identically. In this equation we used R^N to denote the curvature operator of (N, h)

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defined by

$$R^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

Clearly, it follows from (1) that any harmonic map is biharmonic and we call the nonharmonic biharmonic maps *proper biharmonic maps*.

Let M^m be a submanifold of Euclidean space $\mathbb{R}^n$ with the mean curvature vector H viewed as a map $H : M \to \mathbb{R}^n$. B. Y. Chen [1991] called M^m a biharmonic submanifold if $\triangle H = (\Delta H^1, \dots, \Delta H^n) = 0$, where Δ is the Beltrami–Laplace operator of the induced metric on M^m. If we use $i : M \to \mathbb{R}^n$ to denote the inclusion map of the submanifold, then the tension field of the inclusion map i is given by $\tau(i) = \Delta i = mH$, and hence the submanifold $M^n \subset \mathbb{R}^n$ is biharmonic if and only if $\triangle H = \triangle(\frac{1}{m}\Delta i) = \frac{1}{m}\Delta^2 i = \frac{1}{m}\tau^2(i) = 0$, that is, the inclusion map is a biharmonic map. In general, a submanifold M of (N, h) is called a *biharmonic submanifold* if the inclusion map $i : (M, i^*h) \to (N, h)$ is a biharmonic isometric immersion. It is well known that an isometric immersion is minimal if and only if it is harmonic. So a minimal submanifold is trivially biharmonic, and we call a nonminimal biharmonic submanifold a *proper biharmonic submanifold*.

Here are some known facts about biharmonic submanifolds:

Biharmonic submanifolds in Euclidean spaces. Jiang [1987] and then Chen and Ishikawa [1998] proved that any biharmonic submanifold in $\mathbb{R}^3$ is minimal. In [1992], Dimitrić showed that any biharmonic curve in $\mathbb{R}^n$ is a part of a straight line, any biharmonic submanifold of finite type in $\mathbb{R}^n$ is minimal, any pseudoumbilical submanifolds $M^m \subset \mathbb{R}^n$ with $m \neq 4$ is minimal, and any biharmonic hypersurface in $\mathbb{R}^n$ with at most two distinct principal curvatures is minimal. Hasanis and Vlachos [1995] proved that any biharmonic hypersurface in $\mathbb{R}^4$ is minimal. Based on these results, B. Y. Chen [1991] made the still-open conjecture that any biharmonic submanifold of Euclidean space is minimal.

Biharmonic submanifolds in hyperbolic space forms. Caddeo, Montaldo and Oniciuc [2002] showed that any biharmonic submanifold in hyperbolic 3-space is minimal, and that any m-dimensional pseudoumbilical biharmonic submanifold of hyperbolic n-space is minimal if $m \neq 4$. It is shown in [Balmuş et al. 2008] that any biharmonic hypersurface of hyperbolic n-space with at most two distinct principal curvatures is minimal. Based on these, Caddeo, Montaldo and Oniciuc [2001] extended Chen's conjecture to the *generalized Chen conjecture*: any biharmonic submanifold in (N, h) is minimal if $\mathrm{Riem}^N \leq 0$.

Biharmonic submanifolds in spheres. The first example of a proper biharmonic submanifold in S^{n+1} was found in [Jiang 1986a] to be the generalized Clifford torus $S^p(1/\sqrt{2}) \times S^q(1/\sqrt{2})$ with $p \neq q$ and $p + q = n$. Caddeo, Montaldo, and Oniciuc [2001] found a second type of proper biharmonic submanifolds in S^{n+1}

to be the hypersphere $S^n(1/\sqrt{2})$, and also gave a complete classification of biharmonic submanifolds in S^3. Balmuş, Montaldo, and Oniciuc [2008] proved that any pseudoumbilical biharmonic submanifold $M^m \subset S^{n+1}$ with $m \neq 4$ has constant mean curvature, and also showed that if a hypersurface $M^n \subset S^{n+1}$ with at most two distinct principal curvatures (which by [Nishikawa and Maeda 1974] is equivalent for $n > 3$ to saying that M is a quasiumbilical or conformally flat hypersurface in S^{n+1}) is biharmonic, then M is an open part of the hypersphere $S^n(1/\sqrt{2})$ or the generalized Clifford torus $S^p(1/\sqrt{2}) \times S^q(1/\sqrt{2})$ with $p \neq q$ and $p+q=n$. Zhang [2008] found some examples of proper biharmonic real hypersurfaces in $\mathbb{C}P^n$ and determined all proper biharmonic tori $T^{n+1} = S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{n+1})$ in S^{2n+1}. All known examples of biharmonic submanifolds in spheres are consistent with the conjecture of [Balmuş et al. 2008] that any biharmonic submanifold in sphere has constant mean curvature, and any proper biharmonic hypersurface in S^{n+1} is an open part of the hypersphere $S^n(1/\sqrt{2})$ or the generalized Clifford torus $S^p(1/\sqrt{2}) \times S^q(1/\sqrt{2})$ with $p \neq q$ and $p+q=n$.

Biharmonic submanifolds in other model spaces. The survey article [Montaldo and Oniciuc 2006] contains an account of the study of biharmonic curves in various models. See [Arslan et al. 2005; Inoguchi 2004; Fetcu and Oniciuc 2009a; 2009b; Sasahara 2005; 2008] for special biharmonic submanifolds in contact manifolds or Sasakian space forms.

Biharmonic submanifolds in other senses. Some authors, for example, Javaloyes and Meroño [2003], use the condition $\Delta H = 0$ to define a biharmonic submanifold of a Riemannian manifold; this definition agrees with ours only if the ambient space is flat. For conformal biharmonic submanifolds (that is, conformal biharmonic immersions), see [Ou 2009].

This paper studies biharmonic hypersurfaces in a generic Riemannian manifold. In Section 2, we derive an invariant equation for such hypersurfaces that involves the mean curvature function, the norm of the second fundamental form, the shape operator of the hypersurface, and the Ricci curvature of the ambient space. We prove that the generalized Chen conjecture holds for totally umbilical hypersurfaces in an Einstein space. Section 3 is devoted to constructing a family of conformally flat metrics and a family of multiply warped product metrics, each of which turns the foliation of an upper-half space of $\mathbb{R}^m$ by parallel hyperplanes into a foliation with each leaf a proper biharmonic hypersurface. We accomplish these by starting with hyperplanes in Euclidean space and then looking for a type of conformally flat or multiply warped product metric on the ambient space that will reduce the biharmonic hypersurface equation into ordinary differential equations whose solutions give the metrics that render the inclusion maps proper biharmonic isometric immersions. In Section 4, we study biharmonicity of Hopf cylinders given by a Riemannian submersion from a complete 3-manifold. Our method shows that there is no proper biharmonic Hopf cylinder in S^3, thus recovering [Inoguchi 2004, Proposition 3.1].

2. The equations of biharmonic hypersurfaces

Recall that if $\varphi : M \to (N, h)$ is the inclusion map of a submanifold, or more generally, an isometric immersion, then we have an orthogonal decomposition of the vector bundle $\varphi^{-1}TN = \tau M \oplus \nu M$ into the tangent and normal bundles. We use $\mathrm{d}\varphi$ to identify TM with its image τM in $\varphi^{-1}TN$. Then, for any $X, Y \in \Gamma(TM)$ we have $\nabla^{\varphi}_X(\mathrm{d}\varphi(Y)) = \nabla^N_X Y$, whereas $\mathrm{d}\varphi(\nabla^M_X Y)$ equals the tangential component of $\nabla^N_X Y$. It follows that

$$\nabla \mathrm{d}\varphi(X, Y) = \nabla^{\varphi}_X(\mathrm{d}\varphi(Y)) - \mathrm{d}\varphi(\nabla^M_X Y) = B(X, Y), \tag{2}$$

that is, the second fundamental form $\nabla \mathrm{d}\varphi(X, Y)$ of the isometric immersion φ agrees with the second fundamental form $B(X, Y)$ of the immersed submanifold $\varphi(M)$ in N; for details see [Kobayashi and Nomizu 1969, Chapter 7] and [Baird and Wood 2003, Example 3.2.3]. From (2) we see that the tension field $\tau(\varphi)$ of an isometric immersion and the mean curvature vector field η of the submanifold are related by

$$\tau(\varphi) = m\eta.$$

For a hypersurface, that is, a codimension one isometric immersion φ of M^m into N^{m+1}, we can choose a local unit vector field ξ normal to $\varphi(M) \subset N$. Then $\eta = H\xi$ for H the mean curvature function, and we can write $B(X, Y) = b(X, Y)\xi$, where $b : TM \times TM \to C^{\infty}(M)$ is the second fundamental form. The relationship between the shape operator A of the hypersurface with respect to the unit normal vector field ξ and the second fundamental form is given by

$$B(X, Y) = \langle \nabla^N_X Y, \xi\rangle\xi = -\langle Y, \nabla^N_X \xi\rangle\xi = \langle AX, Y\rangle\xi, \tag{3}$$

$$\langle AX, Y\rangle = \langle B(X, Y), \xi\rangle = \langle b(X, Y)\xi, \xi\rangle = b(X, Y). \tag{4}$$

Theorem 2.1. *Let $\varphi : M^m \to N^{m+1}$ be an isometric immersion of codimension one with mean curvature vector $\eta = H\xi$. Then φ is biharmonic if and only if*

$$\begin{cases} \Delta H - H|A|^2 + H\,\mathrm{Ric}^N(\xi, \xi) = 0, \\ 2A(\mathrm{grad}\, H) + \frac{1}{2}m\, \mathrm{grad}\, H^2 - 2H(\mathrm{Ric}^N(\xi))^{\top} = 0, \end{cases} \tag{5}$$

where $\mathrm{Ric}^N : T_qN \to T_qN$ denotes the Ricci operator of the ambient space, defined by $\langle \mathrm{Ric}^N(Z), W\rangle = \mathrm{Ric}^N(Z, W)$, and A is the shape operator of the hypersurface with respect to the unit normal vector ξ.

Proof. First choose a local orthonormal frame $\{e_i\}_{i=1,\dots,m}$ on M such that the orthonormal frame $\{\mathrm{d}\varphi(e_1), \dots, \mathrm{d}\varphi(e_m), \xi\}$ is adapted to the ambient space defined on the hypersurface. Identifying $\mathrm{d}\varphi(X) = X$ and $\nabla^\varphi_X W = \nabla^N_X W$ and noting that the tension field of φ is $\tau(\varphi) = mH\xi$, we can compute the bitension field of φ as

$$\begin{aligned}
\tau^2(\varphi) &= \sum_{i=1}^m \{\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(mH\xi) - \nabla^\varphi_{\nabla_{e_i}e_i}(mH\xi) - R^N(\mathrm{d}\varphi(e_i), mH\xi)d\varphi(e_i)\}\\
&= m\sum_{i=1}^m (e_ie_i(H)\xi + 2e_i(H)\nabla^N_{e_i}\xi + H\nabla^N_{e_i}\nabla^N_{e_i}\xi - (\nabla_{e_i}e_i)(H)\xi - H\nabla^N_{\nabla_{e_i}e_i}\xi)\\
&\qquad\qquad - mH\sum_{i=1}^m R^N(\mathrm{d}\varphi(e_i), \xi)d\varphi(e_i)\\
&= m(\Delta H)\xi - 2mA(\operatorname{grad} H) - mH\Delta^\varphi\xi - mH\sum_{i=1}^m R^N(\mathrm{d}\varphi(e_i), \xi)d\varphi(e_i).
\end{aligned}$$

To find the tangential and normal parts of the bitension field, we first compute the tangential and normal components of the curvature term, getting

$$\begin{aligned}
&\sum_{i,k=1}^m \langle R^N(\mathrm{d}\varphi(e_i), \xi)d\varphi(e_i), e_k\rangle e_k = -(\mathrm{Ric}^N(\xi, e_k))e_k = -(\mathrm{Ric}(\xi))^\top,\\
&\sum_{i=1}^m \langle R^N(\mathrm{d}\varphi(e_i), \tau(\varphi))d\varphi(e_i), \xi\rangle = -mH\,\mathrm{Ric}^N(\xi, \xi).
\end{aligned}$$

To find the normal part of $\Delta^\varphi\xi$, we compute

$$\langle \Delta^\varphi\xi, \xi\rangle = \sum_{i=1}^m \langle -\nabla^N_{e_i}\nabla^N_{e_i}\xi + \nabla^N_{\nabla_{e_i}e_i}\xi, \xi\rangle = \sum_{i=1}^m \langle \nabla^N_{e_i}\xi, \nabla^N_{e_i}\xi\rangle. \tag{6}$$

On the other hand, using (3) and (4), we have

$$\begin{aligned}
|A|^2 &= \sum_{i,j=1}^m \langle Ae_i, e_j\rangle^2 = \sum_{i,j=1}^m \langle \nabla^N_{e_i}\xi, e_j\rangle^2 = \sum_{i=1}^m \langle \nabla^N_{e_i}\xi, \sum_{j=1}^m \langle \nabla^N_{e_i}\xi, e_j\rangle e_j\rangle\\
&= \sum_{i=1}^m \langle \nabla^N_{e_i}\xi, \nabla^N_{e_i}\xi\rangle,
\end{aligned}$$

which, together with (6), implies that

$$(\Delta^\varphi\xi)^\perp = \langle \Delta^\varphi\xi, \xi\rangle\xi = \sum_{i=1}^m \langle \nabla^N_{e_i}\xi, \nabla^N_{e_i}\xi\rangle\xi = |A|^2\xi.$$

A straightforward computation gives the tangential part of $\Delta^{\varphi}\xi$ as

$$(7)\qquad \begin{aligned}(\Delta^{\varphi}\xi)^{\top} &= \sum_{i,k=1}^{m}\langle -\nabla^N_{e_i}\nabla^N_{e_i}\xi + \nabla^N_{\nabla_{e_i}e_i}\xi, e_k\rangle e_k\\ &= \sum_{i,k=1}^{m}\langle \nabla^N_{e_i}Ae_i - A(\nabla_{e_i}e_i),\ e_k\rangle e_k = \sum_{i,k=1}^{m}((\nabla_{e_i}b)(e_k, e_i))e_k.\end{aligned}$$

Substituting the Codazzi–Mainardi equation for a hypersurface, namely,

$$(\nabla_{e_i}b)(e_k, e_i) - (\nabla_{e_k}b)(e_i, e_i) = (R^N(e_i, e_k)e_i)^{\perp} = \langle R^N(e_i, e_k)e_i, \xi\rangle,$$

into (7) and using normal coordinates at a point, we have

$$\begin{aligned}(\Delta^{\varphi}\xi)^{\top} &= \sum_{i,k=1}^{m}((\nabla_{e_i}b)(e_k e_i))e_k\\ &= \sum_{k=1}^{m}\Big(\sum_{i=1}^{m}(\nabla_{e_k}b)(e_i, e_i) - \mathrm{Ric}(\xi, e_k)\Big)e_k = m\,\mathrm{grad}(H) - (\mathrm{Ric}(\xi, e_k))e_k.\end{aligned}$$

Therefore, by collecting all the tangent and normal parts of the bitension field separately, we finally have

$$\begin{aligned}(\tau^2(\varphi))^{\perp} &= \langle \tau^2(\varphi), \xi\rangle\xi = m(\Delta H - H|A|^2 + H\,\mathrm{Ric}^N(\xi, \xi))\xi,\\ (\tau^2(\varphi))^{\top} &= \sum_{k=1}^{m}\langle \tau^2(\varphi), e_k\rangle e_k\\ &= -m\big(2A(\mathrm{grad}\,H) + \tfrac{1}{2}m(\mathrm{grad}\,H^2) - 2H(\mathrm{Ric}(\xi))^{\top}\big).\end{aligned}$$ □

As an immediate consequence of Theorem 2.1 is this:

Corollary 2.2. *A constant mean curvature hypersurface in a Riemannian manifold is biharmonic if and only if it is minimal or* $\mathrm{Ric}^N(\xi, \xi) = |A|^2$ *and* $(\mathrm{Ric}^N(\xi))^{\top} = 0$. *In particular*, *we recover* [Oniciuc 2002, Proposition 2.4], *which states that a constant mean curvature hypersurface in a Riemannian manifold* (N^{m+1}, h) *with nonpositive Ricci curvature is biharmonic if and only if it is minimal.*

Corollary 2.3. *A hypersurface in an Einstein space* (N^{m+1}, h) *is biharmonic if and only if its mean curvature function* H *is a solution of the PDEs*

$$(8)\qquad \begin{aligned}\Delta H - H|A|^2 + \frac{rH}{m+1} &= 0,\\ 2A(\mathrm{grad}\,H) + \tfrac{1}{2}m\,\mathrm{grad}\,H^2 &= 0,\end{aligned}$$

where r *is the scalar curvature of the ambient space. In particular*, *a hypersurface* $\varphi : (M^m, g) \to (N^{m+1}(C), h)$ *in a space of constant sectional curvature* C *is* *biharmonic if and only if its mean curvature function H is a solution of the PDEs* [Jiang 1987; Chen 1991; Caddeo et al. 2002]

$$\begin{aligned} \Delta H - H|A|^2 + mCH &= 0, \\ 2A(\operatorname{grad} H) + \tfrac{1}{2}m \operatorname{grad} H^2 &= 0. \end{aligned} \tag{9}$$

Proof. It is well known that if (N^{m+1}, h) is an Einstein manifold, then

$$\mathrm{Ric}^N(Z, W) = \frac{r}{m+1} h(Z, W) \quad \text{for any } Z, W \in TN$$

and hence $(\mathrm{Ric}^N(\xi))^\top = 0$ and $\mathrm{Ric}^N(\xi, \xi) = r/(m+1)$. From these and (5) we obtain (8). When $(N^{m+1}(C), h)$ is a space of constant sectional curvature C, it is an Einstein space with scalar curvature $r = m(m+1)C$. Substituting this into (8) we obtain (9). □

Theorem 2.4. *A totally umbilical hypersurface in an Einstein space with non-positive scalar curvature is biharmonic if and only if it is minimal.*

Proof. Take an orthonormal frame $\{e_1, \dots, e_m, \xi\}$ of (N^{m+1}, h) adapted to the hypersurface M so that $Ae_i = \lambda_i e_i$, where A is the Weingarten map of the hypersurface and λ_i is the principal curvature in the direction e_i. Since M is assumed to be totally umbilical, all principal normal curvatures at any point $p \in M$ are equal to the same number $\lambda(p)$. It follows that

$$H = \frac{1}{m} \sum_{i=1}^{m} \langle Ae_i, e_i \rangle = \lambda, \quad |A|^2 = m\lambda^2,$$

$$A(\operatorname{grad} H) = A\Big(\sum_{i=1}^{m} (e_i \lambda) e_i\Big) = \tfrac{1}{2} \operatorname{grad} \lambda^2,$$

The biharmonic hypersurface equations (8) become

$$\Delta\lambda - m\lambda^3 + \frac{r\lambda}{m+1} = 0 \quad \text{and} \quad (2+m) \operatorname{grad} \lambda^2 = 0.$$

Solving these, we have either $\lambda = 0$ and hence $H = 0$, or $\lambda = \pm\sqrt{r/(m(m+1))}$ is a constant. The latter happens only if the scalar curvature is nonnegative, from which we obtain the theorem. □

Remark 2.5. Theorem 2.4 generalizes the results of [Balmuş et al. 2008; Caddeo et al. 2002; Dimitrić 1992] about totally umbilical biharmonic hypersurfaces in a space form. It also implies that the generalized Chen conjecture is true for totally umbilical hypersurfaces in an Einstein space with nonpositive scalar curvature. Note that nonpositive scalar curvature is a much weaker condition than nonpositive sectional curvature.

Corollary 2.6. *Any totally umbilical biharmonic hypersurface in a Ricci flat manifold is minimal.*

Proof. This follows from Theorem 2.4 and the fact that a Ricci flat manifold is an Einstein space with zero scalar curvature. □

3. Proper biharmonic foliations of codimension one

In general, proper biharmonic maps as local solutions of a system of fourth order PDEs are extremely difficult to unearth. Even in the case of biharmonic submanifolds (viewed as biharmonic maps with geometric constraints), few examples have been found. In this section, we construct families of metrics that turn some foliations of hypersurfaces into proper biharmonic foliations, thus providing infinitely many proper biharmonic hypersurfaces.

Theorem 3.1. *For any constant C, let $N = \{(x_1, \dots, x_m, z) \in \mathbb{R}^{m+1} \mid z > -C\}$ denote the upper half space. Then, the conformally flat space*

$$\Big(N, h = f^{-2}(z)\Big(\sum_{i=1}^{m} \mathrm{d}{x_i}^2 + \mathrm{d}z^2\Big)\Big)$$

is foliated by proper biharmonic hyperplanes $z = k$, where $k \in \mathbb{R}$ and $k > -C$, if and only if $f(z) = D/(z + E)$, where $E \geq C$ and $D \in \mathbb{R} \setminus \{0\}$.

Proof. Consider the isometric immersion

$$\varphi : (\mathbb{R}^m, g) \to \Big(\mathbb{R}^{m+1}, h = f^{-2}(z)\Big(\sum_{i=1}^{m} \mathrm{d}{x_i}^2 + \mathrm{d}z^2\Big)\Big)$$

with $\varphi(x_1, \dots, x_m) = (x_1, \dots, x_m, k)$ and k being a constant, where the induced metric g with respect to the natural frame $\partial_i = \partial/\partial x_i$ for $i = 1, 2, \dots, m$ and $\partial_{m+1} = \partial/\partial z$ has components

$$g_{ij} = g(\partial_i, \partial_j) = h(d\varphi(\partial_i), d\varphi(\partial_j)) \circ \varphi = \begin{cases} f^{-2}(k) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

One can check that $e_A = f(z)\partial_A$, where $A = 1, 2, \dots, m, m+1$, constitutes a local orthonormal frame on $\mathbb{R}^{m+1}$ adapted to the hypersurface $z = k$ with $\xi = e_{m+1}$ being the unit normal vector field. A straightforward computation using Koszul's formula gives the coefficients of the Levi-Civita connection of the ambient space:

$$(\nabla_{e_A} e_B) = \begin{pmatrix} f'e_{m+1} & 0 & \cdots & 0 & -f'e_1 \\ 0 & f'e_{m+1} & \cdots & 0 & -f'e_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f'e_{m+1} & -f'e_m \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{(m+1)\times(m+1)}$$

Noting that $\xi = e_{m+1}$ is the unit normal vector field, we can easily compute the components of the second fundamental form as

$$h(e_i, e_j) = \langle \nabla_{e_i} e_j, e_{m+1} \rangle = \begin{cases} f' & \text{if } i = j = 1, 2, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

from which we conclude that each of the hyperplanes $z = k$ is a totally umbilical hypersurface in the conformally flat space.

We compute the mean curvature of the hypersurface and the norm of the second fundamental form to be

$$H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i) = f' \quad \text{and} \quad |A|^2 = \sum_{i=1}^{m} |h(e_i, e_i)|^2 = m f'^2.$$

Since H depends only on z, we have $\operatorname{grad}_g H = \sum_{i=1}^{m} e_i(H) e_i = 0$ and hence $\Delta_g H = \operatorname{div}(\operatorname{grad}_g H) = 0$. Therefore, by Theorem 2.1, the biharmonic equation of the isometric immersion reduces to the system

$$-|A|^2 + \operatorname{Ric}^N(\xi, \xi) = 0 \quad \text{and} \quad \sum_{i=1}^{m} (\operatorname{Ric}^N(\xi, e_i)) e_i = 0. \tag{10}$$

We can compute the Ricci curvature of the ambient space:

$$\operatorname{Ric}(e_i, \xi) = \operatorname{Ric}(e_i, e_{m+1}) = \sum_{j=1}^{m} \langle R(e_{m+1}, e_j) e_j, e_i \rangle = 0 \quad \text{for all } i = 1, 2, \dots, m,$$

$$\operatorname{Ric}(\xi, \xi) = \operatorname{Ric}(e_{m+1}, e_{m+1}) = \sum_{j=1}^{m} \langle R(e_{m+1}, e_j) e_j, e_{m+1} \rangle = m f f'' - m f'^2.$$

Substituting these into the system (10), we conclude that all isometric immersions $\varphi : \mathbb{R}^m \to (\mathbb{R}^{m+1}, h = f^{-2}(z)(\sum_{i=1}^{m} \mathrm{d}x_i{}^2 + \mathrm{d}z^2))$ with $\varphi(x_1, \dots, x_m) = (x_1, \dots, x_m, k)$ are biharmonic if and only if $f f'' - 2 f'^2 = 0$. This equation can be written as $(f'/f)' - (f'/f)^2 = 0$. This ordinary differential equation has solution $f(z) = D/(z + C)$, where C and D are constants. Since the mean curvature of the hypersurface $H = f'(k)$ is never zero, we conclude that each of the hyperplanes $z = k$ for $k \neq -C$ is a proper biharmonic hypersurface in the conformally flat space $(N, h = ((z + C)/D)^2 (\sum_{i=1}^{m} \mathrm{d}x_i{}^2 + \mathrm{d}z^2))$. □

Theorem 3.2. *The isometric immersion*

$$\varphi : \mathbb{R}^2 \to (\mathbb{R}^3, h = e^{2p(z)} \mathrm{d}x^2 + e^{2q(z)} \mathrm{d}y^2 + \mathrm{d}z^2)$$

with $\varphi(x, y) = (x, y, c)$ is biharmonic if and only if

$$p'' + 2p'^2 + q'' + 2q'^2 = 0. \tag{11}$$

In particular, for any positive constants A, B, C, D, the upper half space $\mathbb{R}^3_+ = \{(x, y, z) \mid z > 0\}$ with metric $h = (Az + B)\mathrm{d}x^2 + (Cz + D)\mathrm{d}y^2 + \mathrm{d}z^2$ is foliated by proper biharmonic planes $z =$ constant.

Proof. Let φ be as stated, with c being a positive constant. Using the notation $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$ and $\partial_3 = \partial/\partial z$ we can easily check that the induced metric is given by

$$\begin{aligned} g_{11} &= g(\partial_1, \partial_1) = h(d\varphi(\partial_1), d\varphi(\partial_1)) \circ \varphi = e^{2p(c)}, \\ g_{12} &= g(\partial_1, \partial_2) = h(d\varphi(\partial_1), d\varphi(\partial_2)) \circ \varphi = 0, \\ g_{22} &= g(\partial_2, \partial_2) = h(d\varphi(\partial_2), d\varphi(\partial_2)) \circ \varphi = e^{2q(c)}. \end{aligned}$$

One can also check that $e_1 = e^{-p(z)}\partial_1$, $e_2 = e^{-q(z)}\partial_2$ and $e_3 = \partial_3$ constitute an orthonormal frame on $\mathbb{R}^3_+$ adapted to the surface $z = c$, with $\xi = e_3$ being the unit normal vector field. A further computation gives the Lie brackets

$$[e_1, e_2] = 0, \quad [e_1, e_3] = p'e_1, \quad [e_2, e_3] = q'e_2, \tag{12}$$

and the coefficients of the Levi-Civita connection:

$$\begin{aligned} &\nabla_{e_1} e_1 = -p'e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = p'e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = -q'e_3, && \nabla_{e_2} e_3 = q'e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned} \tag{13}$$

Since that $\xi = e_3$ is the unit normal vector field, the components of the second fundamental form are

$$\begin{aligned} h(e_1, e_1) &= \langle \nabla_{e_1} e_1, e_3 \rangle = -p', \\ h(e_1, e_2) &= \langle \nabla_{e_1} e_2, e_3 \rangle = 0, \\ h(e_2, e_2) &= \langle \nabla_{e_2} e_2, e_3 \rangle = -q'. \end{aligned}$$

From these, the mean curvature of the isometric immersion is

$$H = \tfrac{1}{2}(h(e_1, e_1) + h(e_2, e_2)) = -(p' + q')/2, \tag{14}$$

and the norm of the second fundamental form is

$$|A|^2 = \sum_{i=1}^{2} |h(e_i, e_i)|^2 = p'^2 + q'^2.$$

Since H depends only on z we have $\mathrm{grad}_g\, H = e_1(H)e_1 + e_2(H)e_2 = 0$ and hence $\Delta_g H = \mathrm{div}(\mathrm{grad}_g\, H) = 0$. Therefore, by Theorem 2.1, the biharmonic equation of the isometric immersion reduces to (10) with $m = 2$. To compute the Ricci curvature of the ambient space we can use (12) and (13) to get

$$\begin{aligned}\mathrm{Ric}(e_1,\xi)&=\mathrm{Ric}(e_1,e_3)=\langle R(e_3,e_2)e_2,e_1\rangle=0,\\ \mathrm{Ric}(e_2,\xi)&=\mathrm{Ric}(e_2,e_3)=\langle R(e_3,e_1)e_1,e_3\rangle=0,\\ \mathrm{Ric}(\xi,\xi)&=\mathrm{Ric}(e_3,e_3)=\langle R(e_3,e_1)e_1,e_3\rangle+\langle R(e_3,e_2)e_2,e_3\rangle\\ &=-p''-p'^2-q''-q'^2.\end{aligned}$$

Substituting these into (10) with $m=2$, we conclude that φ is biharmonic if and only if (11) holds, which gives the theorem's first statement. The second is obtained by looking for the solutions of (11) satisfying $p''+2p'^2=0$ and $q''+2q'^2=0$. In fact, we have special solutions $p(z)=\frac{1}{2}\ln(Az+B)$ and $q(z)=\frac{1}{2}\ln(Cz+D)$ with positive constants A, B, C, D. By (14) and the choice of these constants, we see that the mean curvature of the surface $z=c$ is

$$H=-\frac{2ACz+AD+BC}{2(Az+B)(Cz+D)}\neq 0,$$

and hence each such surface is a nonminimal biharmonic surface. □

Remark 3.3. Theorem 3.2 has a generalization to a higher dimensional space $\mathbb{R}^m_+$ for $m>3$.

Example 3.4. Let $\lambda(t)=\sqrt{At+B}$, where A and B are positive constants. Then the warped product space $N=(S^2\times\mathbb{R}^+, h=\lambda^2(t)g^{S^2}+\mathrm{d}t^2)$ is foliated by the spheres $(S^2\times\{t\}, \lambda^2(t)g^{S^2})$, each of which is a totally umbilical proper biharmonic surface.

To see what is claimed in Example 3.4, we parametrize the unit sphere S^2 by spherical polar coordinates:

$$\mathbb{R}\times\mathbb{R}\ni(\rho,\theta)\to(\cos\rho, \sin\rho\cos\theta, \sin\rho\sin\theta))\in\mathbb{R}^3.$$

Then, the standard metric can be written as $g^{S^2}=\mathrm{d}\rho^2+\sin^2\rho\,\mathrm{d}\theta^2$, and the warped product metric on N takes the form $h=\lambda^2(t)\mathrm{d}\rho^2+\lambda^2(t)\sin^2\rho\,\mathrm{d}\theta^2+\mathrm{d}t^2$. Consider the isometric immersion $\varphi: S^2\to(\mathbb{R}^+\times S^2, \mathrm{d}t^2+\lambda^2(t)g^{S^2})$ with $\varphi(\rho,\theta)=(\rho,\theta,c)$ and c being a positive constant. Using the notation $\partial_1=\partial/\partial\rho$, $\partial_2=\partial/\partial\theta$ and $\partial_3=\partial/\partial t$, we can easily check that the induced metric is given by

$$\begin{aligned}g_{11}&=g(\partial_1,\partial_1)=h(d\varphi(\partial_1), d\varphi(\partial_1))\circ\varphi=\lambda^2(c),\\ g_{12}&=g(\partial_1,\partial_2)=h(d\varphi(\partial_1), d\varphi(\partial_2))\circ\varphi=0,\\ g_{22}&=g(\partial_2,\partial_2)=h(d\varphi(\partial_2), d\varphi(\partial_2))\circ\varphi=\lambda^2(c)\sin^2\rho.\end{aligned}$$

Using the orthonormal frame $e_1=\lambda^{-1}(t)\partial_1$, $e_2=(\lambda(t)\sin\rho)^{-1}\partial_2$ and $e_3=\partial_3$, we have the Lie brackets

$$[e_1,e_2]=-(\cot\rho/\lambda)e_2,\quad [e_1,e_3]=fe_1,\quad [e_2,e_3]=fe_2,$$

where here and in the sequel we use the notation $f = (\ln\lambda)' = \lambda'/\lambda$. Clearly, e_1, e_2 and $\xi = e_3 = \partial_3$ constitute a local orthonormal frame of N adapted to the surface with ξ being the unit vector field normal to the surface. We can use the Kozsul formula to compute the components of the second fundamental form as

$$\begin{aligned}
h(e_1, e_1) &= \langle \nabla_{e_1} e_1, \xi \rangle = \langle \nabla_{e_1} e_1, e_3 \rangle \\
&= \tfrac{1}{2}(-\langle e_1, [e_1, e_3]\rangle - \langle e_1, [e_1, e_3]\rangle + \langle e_3, [e_1, e_1]\rangle) = -f, \\
h(e_1, e_2) &= \langle \nabla_{e_1} e_2, \xi \rangle = \langle \nabla_{e_1} e_2, e_3 \rangle = 0, \\
h(e_2, e_2) &= \langle \nabla_{e_2} e_2, \xi \rangle = \langle \nabla_{e_2} e_2, e_3 \rangle = -f,
\end{aligned}$$

from which we conclude that each such sphere is a totally umbilical surface in N.

The mean curvature of the isometric immersion and the norm of the second fundamental form are

$$H = \tfrac{1}{2}(h(e_1, e_1) + h(e_2, e_2)) = -f \quad \text{and} \quad |A|^2 = \sum_{i=1}^{2} |h(e_i, e_i)|^2 = 2f^2,$$

which depend only on t. It follows that $\operatorname{grad}_g H = 0$ and $\Delta_g H = 0$. Therefore, by Theorem 2.1, the proper biharmonic equation of φ reduces to (10) with $m = 2$.

On the other hand, using the Ricci curvature formula (for example [Besse 2008]) of the warped product $M = B \times_\lambda F$, we have

$$\begin{aligned}
\operatorname{Ric}(e_1, \xi) &= \operatorname{Ric}(e_1, e_3) = 0, \qquad \operatorname{Ric}(e_2, \xi) = \operatorname{Ric}(e_2, e_3) = 0, \\
\operatorname{Ric}(\xi, \xi) &= \operatorname{Ric}(e_3, e_3) = \operatorname{Ric}^{\mathbb{R}}(e_3, e_3) - (2/\lambda)\operatorname{Hess}_\lambda(e_3, e_3) \\
&= -(2/\lambda)(e_3(e_3\lambda) - d\lambda(\nabla_{e_3} e_3)) = -2\lambda''/\lambda.
\end{aligned}$$

Substituting these into (10) with $m = 2$ we conclude that the isometric immersion $\varphi : S^2 \to (S^2 \times \mathbb{R}^+, \lambda^2(t) g^{S^2} + dt^2)$ with $\varphi(\rho, \theta) = (\rho, \theta, c)$ is biharmonic if and only if $-2(\lambda'/\lambda)^2 - 2\lambda''/\lambda = 0$. Solving this final equation, we have $\lambda(t) = \sqrt{At + B}$, proving the claim in Example 3.4.

Remark 3.5. The referee points out that the biharmonicity of the inclusion maps in Example 3.4 is in fact a special case of [Balmuş et al. 2007, Corollary 3.4], which was proved by a different method.

4. Biharmonic cylinders of a Riemannian submersion

Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannian submersion with totally geodesic fibers from a complete manifold. Let $\alpha : I \to (N^2, h)$ be an immersed regular curve parametrized by arclength. Then $\Sigma = \bigcup_{t \in I} \pi^{-1}(\alpha(t))$ is a surface in M that can be viewed as a disjoint union of all horizontal lifts of the curve α. Let $\{\overline{X} = \alpha', \bar{\xi}\}$ be a Frenet frame along α, and let $\bar{\kappa}$ be the geodesic curvature of the curve. Then the Frenet formula for α is given by

$$\begin{aligned}\tilde{\nabla}_{\bar{X}}\bar{X} &= \bar{\kappa}\bar{\xi},\\ \tilde{\nabla}_{\bar{X}}\bar{\xi} &= -\bar{\kappa}\bar{X},\end{aligned}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection of (N, h). Let $\beta : I \to (M^3, g)$ be a horizontal lift of α. Let X and ξ be the horizontal lifts of $\bar{X}$ and $\bar{\xi}$, respectively. Let V be the unit vector field tangent to the fibers of the submersion π. Then $\{X, \xi, V\}$ is an orthonormal frame of M adapted to the surface, with ξ the unit normal vector of the surface. The restriction of this frame to the curve β is the Frenet frame along β. Therefore, the Frenet formula along β is given by

$$\begin{aligned}\nabla_X X &= \kappa\xi,\\ \nabla_X \xi &= -\kappa X + \tau V,\\ \nabla_X V &= -\tau\xi,\end{aligned} \tag{15}$$

where ∇ denotes the Levi-Civita connection of (M, g). Since a Riemannian submersion preserves the inner product of horizontal vector fields, we can check that $\kappa = \bar{\kappa} \circ \pi$ and $\tau = \langle \nabla_X \xi, V\rangle = \langle A_X \xi, V\rangle$ is the torsion of the horizontal lift that vanishes if the Riemannian submersion has integrable horizontal distribution; here A is the A-tensor of the submersion [O'Neill 1966]. In what follows, we will use the frame $\{X, \xi, V\}$ to compute the mean curvature, second fundamental form, and other terms that appear in the biharmonic equation of the surface Σ.

Using (15) we have

$$\begin{aligned}
A(X) &= -\langle \nabla_X \xi, X\rangle X - \langle \nabla_X \xi, V\rangle V = \kappa X - \tau V,\\
A(V) &= -\langle \nabla_V \xi, X\rangle X - \langle \nabla_V \xi, V\rangle V = -\tau X,\\
b(X, X) &= \langle A(X), X\rangle = \kappa, \qquad b(X, V) = \langle A(X), V\rangle = -\tau,\\
b(V, X) &= \langle A(V), X\rangle = -\tau, \qquad b(V, V) = \langle A(V), V\rangle = 0;\\
H &= \tfrac{1}{2}(b(X, X) + b(V, V)) = \tfrac{1}{2}\kappa,\\
A(\operatorname{grad} H) &= A(X(\tfrac{1}{2}\kappa)X + V(\tfrac{1}{2}\kappa)V) = X(\tfrac{1}{2}\kappa)A(X) = \tfrac{1}{2}\kappa'(\kappa X - \tau V);\\
\Delta H &= XX(H) - (\nabla_X X)H + VV(H) - (\nabla_V V)H = \tfrac{1}{2}\kappa'';\\
|A|^2 &= (b(X, X))^2 + (b(X, V))^2 + (b(V, X))^2 + (b(V, V))^2 = \kappa^2 + 2\tau^2.
\end{aligned}$$

Substituting these into the biharmonic hypersurface equation (5), we conclude that the surface Σ is biharmonic in (M^3, g) if and only if

$$\begin{aligned}
\tfrac{1}{2}\kappa'' - \tfrac{1}{2}\kappa(\kappa^2 + 2\tau^2) + \tfrac{1}{2}\kappa \operatorname{Ric}^M(\xi, \xi) &= 0,\\
\kappa'(\kappa X - \tau V) + \tfrac{1}{2}\kappa\kappa' X - \kappa \operatorname{Ric}^M(\xi, X)X - \kappa \operatorname{Ric}^M(\xi, V)V &= 0.
\end{aligned}$$

These are equivalent to

$$\begin{aligned} \kappa'' - \kappa(\kappa^2 + 2\tau^2) + \kappa\,\mathrm{Ric}^M(\xi,\xi) &= 0,\\ 3\kappa'\kappa - 2\kappa\,\mathrm{Ric}^M(\xi, X) &= 0, \qquad (16)\\ \kappa'\tau + \kappa\,\mathrm{Ric}^M(\xi, V) &= 0. \end{aligned}$$

Applying (16) to Hopf fibration $\pi : S^3 \to S^2$ we have the following corollary, which recovers [Inoguchi 2004, Proposition 3.1].

Corollary 4.1. *There is no proper biharmonic Hopf cylinder in S^3.*

Finally, applying (16) to the submersions $\pi : S^2 \times \mathbb{R} \to S^2$ and $\pi : H^2 \times \mathbb{R} \to H^2$ yields another corollary:

Corollary 4.2. (1) *The Hopf cylinder $\Sigma = \bigcup_{t\in I} \pi^{-1}(\alpha(t))$ is a proper biharmonic surface in $S^2 \times \mathbb{R}$ if and only if the directrix $\alpha : I \to (S^2, h)$ is a part of a circle in S^2 with radius $\sqrt{2}/2$;*

(2) *The Hopf cylinder $\Sigma = \bigcup_{t\in I} \pi^{-1}(\alpha(t))$ is biharmonic in $H^2 \times \mathbb{R}$ if and only if it is minimal.*

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