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Let K be any field, let $K(x_1, \dots, x_n)$ be the rational function field of n variables over K, and let S_n and A_n be the symmetric group and the alternating group of degree n, respectively. For any $a \in K \setminus \{0\}$, define an action of S_n on $K(x_1, \dots, x_n)$ by $\sigma \cdot x_i = x_{\sigma(i)}$ for $\sigma \in A_n$ and $\sigma \cdot x_i = a/x_{\sigma(i)}$ for $\sigma \in S_n \setminus A_n$. We prove that for any field K and $n = 3, 4, 5$, the fixed field $K(x_1, \dots, x_n)^{S_n}$ is rational (that is, purely transcendental) over K.

1. Introduction

Let K be any field, let $K(x_1, \dots, x_n)$ be the rational function field of n variables over K, and let S_n and A_n be the symmetric group and the alternating group of degree n, respectively. For any $a \in K \setminus \{0\}$, define a twisted action of S_n on $K(x_1, \dots, x_n)$ by

$$\sigma(x_i) := \begin{cases} x_{\sigma(i)} & \text{if } \sigma \in A_n, \\ a/x_{\sigma(i)} & \text{if } \sigma \in S_n \setminus A_n. \end{cases} \tag{1-1}$$

Consider the fixed subfield

$$K(x_1, \dots, x_n)^{S_n} = \{\alpha \in K(x_1, \dots, x_n) : \sigma(\alpha) = \alpha \text{ for any } \sigma \in S_n\}.$$

If $n = 2$, then $K(x_1, x_2)^{S_2} = K(x_1 + (a/x_2), ax_1/x_2)$ is rational (that is, purely transcendental) over K. When $a = 1$ (equivalently when $a \in K^{\times 2}$), we have the following theorem.

Theorem 1.1 [Hajja and Kang 1997, Theorem 3.5]. *Let K be any field and let $a \in K^{\times 2}$. Then $K(x_1, \dots, x_n)^{S_n}$ is rational over K.*

The case when $a \in K^{\times} \setminus K^{\times 2}$ and $n \geq 3$ had been intractable for many years; see [Hajja and Kang 1997, page 638; Hajja 2000, Example 5.12, page 147; Kang 2001, Question 3.8, page 215]. Even the case $n = 3$ was unsolved. The next theorem is our recent result for the cases $n = 3, 4, 5$.

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Theorem 1.2. *Let K be any field, let $a \in K \setminus \{0\}$, and let S_n act on $K(x_1, \dots, x_n)$ as defined in* (1-1). *If $n = 3, 4, 5$, then $K(x_1, \dots, x_n)^{S_n}$ is rational over K.*

We will prove Theorem 1.2 in Section 2. It is interesting that we use three different methods for the three cases of n; it seems that there is no unified proof for the three cases. One of the reasons is that the solutions to Noether's problem for the alternating group A_n are rather different when $n = 3$ and when $n = 5$; see Theorem 2.2 and Theorem 2.5. Since Noether's problem for A_n is still open in the case $n \geq 6$ (see [Maeda 1989] and [Hajja and Kang 1995, Section 4] for the statement of this problem), it is not so surprising that our question is solvable at present only for $n \leq 5$. It is still unknown whether the fixed field $K(x_1, \dots, x_n)^{S_n}$ is rational when $n \geq 6$.

In Section 3 we propose another approach to the rationality of $K(x_1, \dots, x_n)^{S_n}$. We show in Theorem 3.4 that it is isomorphic to the function field of a conic bundle over $\mathbb{P}^{n-1}$ of the form $x^2 - ay^2 = h(v_1, \dots, v_{n-1})$ with affine coordinates $v_1, \dots, v_{n-1}$. Although this approach is valid only when $\operatorname{char} K \neq 2$, it does provide a new technique in studying rationality problems. The structure of a conic bundle together with its rationality problem is a central subject in algebraic geometry [Iskovskih 1991]. Fortunately, when $n = 3$ and $n = 4$, the conic bundle in our case contains singularities and the rationality problem can be solved by a suitable blowing-up process. In particular, we find another proof of Theorem 1.2 when $\operatorname{char} K \neq 2$ and $n = 3, 4$. For other rationality problems of conic bundles, see [Kang 2007, Section 4].

Since the fixed field $K(x_1, \dots, x_n)^{S_n}$ is the quotient field of the ring of invariants $K[x_1, \dots, x_n]^{S_n}$, it seems plausible to study it through the structure of the latter. This strategy is carried out in Section 4, and we give another proof of Theorem 1.2 when $\operatorname{char} K = 2$ and $n = 3, 4$.

2. Proof of Theorem 1.2

Theorem 2.1 [Kang 2004, Theorem 2.4]. *Let K be any field and let $K(x, y)$ be the rational function field of two variables over K. Let σ be a K-automorphism on $K(x, y)$ defined by*

$$\sigma : x \mapsto a/x, \quad y \mapsto b/y,$$

where $a \in K \setminus \{0\}$ and $b = c(x + (a/x)) + d$ such that $c, d \in K$ and at least one of c and d is nonzero. Then $K(x, y)^{\langle\sigma\rangle} = K(s, t)$, where

$$s = \frac{x - (a/x)}{xy - (ab/xy)}, \quad t = \frac{y - (b/y)}{xy - (ab/xy)}.$$

The next result is essentially due to Masuda [1955, page 62] when $\operatorname{char} K \neq 3$ (with a misprint in the original expression). We thank Y. Rikuna who pointed out that the same formula is still valid when char $K = 3$ if we compare this formula with the proof in [Kuniyoshi 1955]. For convenience, we provide a new proof.

Theorem 2.2 [Masuda 1955, Theorem 3]. *Let K be any field, $K(x_1, x_2, x_3)$ be the rational function field of three variables over K. Let σ be a K-automorphism on $K(x_1, x_2, x_3)$ defined by*

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1.$$

Then $K(x_1, x_2, x_3)^{\langle\sigma\rangle} = K(s_1, u, v) = K(s_3, u, v)$, where s_i is the elementary symmetric function of degree i for $1 \le i \le 3$, and u and v are defined by

$$u := \frac{x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1},$$
$$v := \frac{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1}.$$

Moreover, we have the identities

$$s_2 = s_1(u + v) - 3(u^2 - uv + v^2),$$
$$s_3 = s_1uv - (u^3 + v^3),$$
$$x_1x_2^2 + x_2x_3^2 + x_3x_1^2 = s_1^2u - 3s_1u^2 + 3(2u - v)(u^2 - uv + v^2),$$
$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 = s_1^2v - 3s_1v^2 - 3(u - 2v)(u^2 - uv + v^2).$$

Proof. With the aid of computer packages, say Mathematica or Maple, it is easy to verify the theorem's identities. We have $[K(x_1, x_2, x_3) : K(s_1, s_2, s_3)] = 6$ and $[K(x_1, x_2, x_3)^{\langle\sigma\rangle} : K(s_1, s_2, s_3)] = 2$. Since $x_1x_2^2 + x_2x_3^2 + x_3x_1^2 \notin K(s_1, s_2, s_3)$, it follows that $K(x_1, x_2, x_3)^{\langle\sigma\rangle} = K(s_1, s_2, s_3, x_1x_2^2 + x_2x_3^2 + x_3x_1^2) \subset K(s_1, u, v)$. Hence $K(x_1, x_2, x_3)^{\langle\sigma\rangle} = K(s_1, u, v) = K(s_3, u, v)$. □

Proof of Theorem 1.2 when $n = 3$. Let $\sigma = (1, 2, 3)$, $\tau = (1, 2) \in S_3$.

By Theorem 2.2, we find that $K(x_1, x_2, x_3)^{\langle\sigma\rangle} = K(s_3, u, v)$.

Now $\tau(x_1) = a/x_2$, $\tau(x_2) = a/x_3$, and $\tau(x_3) = a/x_3$. Note that

$$\tau(s_1) = as_2/s_3, \quad \tau(s_2) = a^2s_1/s_3, \quad \tau(s_3) = a^3/s_3,$$
$$\tau(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) = a^3(x_1x_2^2 + x_2x_3^2 + x_3x_1^2)/s_3^2,$$
$$\tau(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) = a^3(x_1^2x_2 + x_2^2x_3 + x_3^2x_1)/s_3^2.$$

With the aid of Theorem 2.2, it is not difficult to find that

$$\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad u \mapsto \frac{au}{u^2-uv+v^2}, \quad v \mapsto \frac{av}{u^2-uv+v^2}. \tag{2-1}$$

Define $w := u/v$. Then $K(s_3, u, v) = K(s_3, v, w)$ and

$$\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad v \mapsto \frac{a}{v(1-w+w^2)}, \quad w \mapsto w.$$

By Theorem 2.1, $K(s_3, v, w)^{\langle\tau\rangle}$ is rational over $K(w)$. Hence $K(x_1, x_2, x_3)^{S_3} = K(s_3, v, w)^{\langle\tau\rangle}$ is rational over K. □

Proof of Theorem 1.2 when $n = 4$. Define

$$\begin{aligned}
\sigma &:= (123) &&: x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1,\\
\tau &:= (12) &&: x_1 \mapsto a/x_2, \quad x_2 \mapsto a/x_1, \quad x_3 \mapsto a/x_3, \quad x_4 \mapsto a/x_4,\\
\rho_1 &:= (12)(34) &&: x_1 \mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto x_4, \quad x_4 \mapsto x_3,\\
\rho_2 &:= (13)(24) &&: x_1 \mapsto x_3, \quad x_3 \mapsto x_1, \quad x_2 \mapsto x_4, \quad x_4 \mapsto x_2.
\end{aligned}$$

Note that $\{1\} \lhd V_4 = \langle\rho_1, \rho_2\rangle \lhd A_4 = \langle\sigma, \rho_1, \rho_2\rangle \lhd S_4 = \langle\sigma, \tau, \rho_1, \rho_2\rangle$ is a normal series.

First we will show that $K(x_1, \dots, x_4)^{V_4}$ is rational over K. Define

$$s_1 := x_1 + x_2 + x_3 + x_4, \quad s_4 := x_1x_2x_3x_4,$$
$$S := \frac{x_1 + x_2 - x_3 - x_4}{x_1x_2 - x_3x_4}, \quad T := \frac{x_1 - x_2 - x_3 + x_4}{x_1x_4 - x_2x_3}, \quad U := \frac{x_1 - x_2 + x_3 - x_4}{x_1x_3 - x_2x_4}.$$

Then we have $K(s_1, s_4, S, T, U) \subset K(x_1, x_2, x_3, x_4)^{V_4}$ and

$$\sigma : s_1 \mapsto s_1, \quad s_4 \mapsto s_4, \quad S \mapsto T, \quad T \mapsto U, \quad U \mapsto S. \tag{2-2}$$

Lemma 2.3. (i) *$K(x_1, x_2, x_3, x_4)^{V_4} = K(s_1, S, T, U) = K(s_4, S, T, U)$.*

(ii) *$K(x_1, x_2, x_3, x_4)^{A_4} = K(s_4, f, g, h)$ where f, g, h are defined by*

$$f = S + T + U, \quad g = \frac{ST^2+TU^2+US^2-3STU}{S^2+T^2+U^2-ST-TU-US},$$
$$h = \frac{S^2T+T^2U+U^2S-3STU}{S^2+T^2+U^2-ST-TU-US}.$$

Proof. Define $u_1 := S+T+U$, $u_2 := ST+TU+SU$ and $u_3 := STU$. Then it can be checked that $K(x_1, x_2, x_3, x_4) = K(s_1, S, T, U)(x_4)$ directly from the equalities

$$x_1 = \frac{4 - s_1T + (-2u_1 + s_1T(S+U))x_4 + SU(1 - s_1T)x_4^2 + u_3x_4^3}{S - T + U - SUx_4},$$
$$x_2 = \frac{4 - s_1U + (-2u_1 + s_1U(T+S))x_4 + TS(1 - s_1U)x_4^2 + u_3x_4^3}{T - U + S - TSx_4},$$
$$x_3 = \frac{4 - s_1S + (-2u_1 + s_1S(U+T))x_4 + UT(1 - s_1S)x_4^2 + u_3x_4^3}{U - S + T - UTx_4}.$$

We see that $[K(s_1, S, T, U)(x_4) : K(s_1, S, T, U)] \le 4$ by the equality

$$u_1^2 - 4u_2 + s_1u_3 + (8 - s_1u_1)u_3x_4 - (2u_1 - s_1u_2)u_3x_4^2 - s_1u_3^2x_4^3 + u_3^2x_4^4 = 0.$$

Hence we get $K(x_1, x_2, x_3, x_4)^{V_4} = K(s_1, S, T, U)$. It follows from the equality $s_4 = (u_1^2 - 4u_2 + u_3s_1)/u_3^2$ that $K(s_1, S, T, U) = K(s_4, S, T, U)$.

As for the field $K(x_1, x_2, x_3, x_4)^{A_4}$, apply Theorem 2.2 to $K(s_4, S, T, U)^{\langle\sigma\rangle} = K(S, T, U)^{\langle\sigma\rangle}(s_4)$. □

We have $K(x_1, x_2, x_3, x_4)^{S_4} = (K(x_1, x_2, x_3, x_4)^{V_4})^{S_4/V_4} = K(s_4, S, T, U)^{\langle\sigma,\tau\rangle}$. The action of $\langle\sigma, \tau\rangle$ on $K(s_4, S, T, U)$ is given by

$$\sigma : s_4 \mapsto s_4, \quad S \mapsto T, \quad T \mapsto U, \quad U \mapsto S,$$
$$\tau : s_4 \mapsto \frac{a^4}{s_4}, \quad S \mapsto \frac{-S+T+U}{aTU}, \quad T \mapsto \frac{S+T-U}{aST}, \quad U \mapsto \frac{S-T+U}{aSU}.$$

Define

$$N := \begin{cases} \dfrac{s_4 + a^2}{s_4 - a^2} & \text{if char } K \ne 2, \\ \dfrac{s_4}{s_4 + a^2} & \text{if char } K = 2. \end{cases}$$

Then we get $K(s_4, S, T, U) = K(N, S, T, U)$, $\sigma(N) = N$ and

$$\tau(N) = \begin{cases} -N & \text{if char } K \ne 2, \\ N+1 & \text{if char } K = 2. \end{cases}$$

Applying [Hajja and Kang 1995, Theorem 1], we find that $K(x_1, x_2, x_3, x_4)^{S_4} = K(N, S, T, U)^{\langle\sigma,\tau\rangle}$ is rational over K, provided that $K(S, T, U)^{\langle\sigma,\tau\rangle}$ is rational over K. Explicitly, define P by

$$P := \begin{cases} N \cdot \left(S + T + U + \dfrac{S^2 + T^2 + U^2 - 2(ST + TU + US)}{aSTU}\right) & \text{if char } K \ne 2, \\ N + \dfrac{S+T+U}{S+T+U+aSTU} & \text{if char } K = 2. \end{cases}$$

Then we have that $K(N, S, T, U) = K(P, S, T, U)$ and $K(x_1, x_2, x_3, x_4)^{S_4} = K(P, S, T, U)^{\langle\sigma,\tau\rangle} = K(S, T, U)^{\langle\sigma,\tau\rangle}(P)$, where $\sigma(P) = \tau(P) = P$.

Thus it remains to prove this:

Theorem 2.4. *Let K be any field and let $K(S, T, U)$ be the rational function field of three variables S, T and U over K. Let σ and τ be K-automorphisms of $K(S, T, U)$ defined by*

$$\sigma : S \mapsto T, \quad T \mapsto U, \quad U \mapsto S,$$
$$\tau : S \mapsto \frac{-S+T+U}{aTU}, \quad T \mapsto \frac{S+T-U}{aST}, \quad U \mapsto \frac{S-T+U}{aSU},$$

where $a \in K \setminus \{0\}$. Then $\langle\sigma, \tau\rangle \cong S_3$ and $K(S, T, U)^{\langle\sigma,\tau\rangle}$ is rational over K.

Proof. By Theorem 2.2, we may choose a transcendence basis of $K(S,T,U)^{\langle\sigma\rangle}$ over K by $K(S,T,U)^{\langle\sigma\rangle}=K(f,g,h)$, where

$$f=S+T+U,\quad g=\frac{ST^2+TU^2+US^2-3STU}{S^2+T^2+U^2-ST-TU-US},$$
$$h=\frac{S^2T+T^2U+U^2S-3STU}{S^2+T^2+U^2-ST-TU-US}.$$

Thus we have $K(S,T,U)^{\langle\sigma,\tau\rangle}=(K(S,T,U)^{\langle\sigma\rangle})^{\langle\tau\rangle}=K(f,g,h)^{\langle\tau\rangle}$. The action of τ on $K(f,g,h)$ is given by

$$f\mapsto\frac{f^2-4f(g+h)+12X}{aY},$$
$$g\mapsto\frac{-f^2h(f-4h)+2f(f-2g-8h)X+24X^2-8gY}{a(f^2-2f(g+h)+4X)Y},$$
$$h\mapsto\frac{-f^2(fg+4h^2)+6f(f-2g)X+24X^2-4(f+2h)Y}{a(f^2-2f(g+h)+4X)Y},$$

where $X=g^2-gh+h^2$ and $Y=g^3-fgh+h^3$.

Case 1: char $K\neq 2$.

Define

$$F:=g+h,\quad G:=g-h,\quad H:=f-(g+h).$$

Then $K(S,T,U)^{\langle\sigma\rangle}=K(f,g,h)=K(F,G,H)$ and τ acts on $K(F,G,H)$ by

$$F\mapsto\frac{4(27G^4-7FG^2H+5G^2H^2-FH^3)}{a(4FG^2-F^2H+G^2H)(3G^2+H^2)},$$
$$G\mapsto\frac{4G(FG^2+7G^2H-FH^2+H^3)}{a(4FG^2-F^2H+G^2H)(3G^2+H^2)},$$
$$H\mapsto\frac{4H(FG^2+7G^2H-FH^2+H^3)}{a(4FG^2-F^2H+G^2H)(3G^2+H^2)}.$$

Note that $\tau(G/H)=G/H$. Define

$$A:=F/G,\quad B:=G,\quad C:=G/H.$$

Then $K(S,T,U)^{\langle\sigma\rangle}=K(F,G,H)=K(A,B,C)$ and τ acts on $K(A,B,C)$ by

$$A\mapsto\frac{-A+5C-7AC^2+27C^3}{1-AC+7C^2+AC^3},$$
$$B\mapsto\frac{4(1-AC+7C^2+AC^3)}{aB(1-A^2+4AC)(1+3C^2)},\quad C\mapsto C.$$

Define

$$D:=1-AC+7C^2+AC^3,\quad E:=2C(C^2-1)/B.$$

Then $K(A, B, C) = K(C, D, E)$ and the action of τ on $K(C, D, E)$ is given by

$$\begin{aligned} &C \mapsto C, \quad D \mapsto (1+3C^2)^3/D, \\ &E \mapsto -a(1+3C^2)\big(D+(1+3C^2)^3/D-2(1+5C^2+2C^4)\big)/E. \end{aligned}$$

Hence the assertion follows from Theorem 2.1.

Case 2: char $K = 2$.

The action of τ on $K(f, g, h)$ is given by

$$\tau : f \mapsto \frac{f^2}{aY}, \quad g \mapsto \frac{fh}{aY}, \quad h \mapsto \frac{fg}{aY},$$

where $Y = g^3 + fgh + h^3$. Define

$$A := f/(g+h), \quad B := g/h, \quad C := 1/h.$$

Then $K(f, g, h) = K(A, B, C)$ and τ acts on $K(A, B, C)$ by

$$A \mapsto A, \quad B \mapsto \frac{1}{B}, \quad C \mapsto \frac{a}{A}\Big(B + \frac{1}{B} + A + 1\Big)\Big/C.$$

Hence the assertion follows from Theorem 2.1. We will give another proof when $n = 4$ and char $K = 2$ in Section 4. □

This concludes the proof of Theorem 1.2 when $n = 4$. □

Proof of Theorem 1.2 when $n = 5$.

We recall Maeda's theorem for the A_5 action.

Theorem 2.5 [Maeda 1989]. *Let K be any field, $K(x_1, \dots, x_5)$ be the rational function field of five variables over K. Then $K(x_1, \dots, x_5)^{A_5}$ is rational over K. Moreover a transcendental basis $F_1, \dots, F_5$ of $K(x_1, \dots, x_5)^{A_5}$ over K may be given explicitly as follows*:

(i) *When* char $K \neq 2$,

$$F_1 = \frac{\sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4 x_1)}{\sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4)},$$

$$F_2 = \frac{\sum_{\sigma \in S_5} \sigma([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10})}{\prod_{i<j}[ij]^2 \cdot \sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4)},$$

$$F_3 = \frac{\sum_{\sigma \in S_5} \sigma([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10} x_1)}{\prod_{i<j}[ij]^2 \cdot \sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4)},$$

$$F_4 = \frac{\sum_{\mu \in R_1} \mu([12]^2[13]^2[23]^2[45]^4)}{\prod_{i<j}[ij]},$$

$$F_5 = \frac{\sum_{\mu \in R_1} \mu([12]^2[13]^2[23]^2[14]^4[24]^4[34]^4[15]^4[25]^4[35]^4)}{\prod_{i<j}[ij]^3},$$

where $[ij]=x_i-x_j$ *and* $R_1=\{1, (34), (354), (234), (2354), (24)(35), (1234), (12354), (124)(35), (13524)\}$.

(ii) *When* char $K=2$,

$$F_1=\frac{\sum_{i<j<k}x_ix_jx_k}{\sum_{i<j}x_ix_j},\qquad F_4=\frac{\sum_{\nu\in R_3}\nu([12]^2[34]^2[13][24][15][25][35][45])}{\prod_{i<j}[ij]},$$

$$F_2=\frac{\sum_{i=1}^5([12][13][14][15]\cdot I^2)^{(1i)}}{\prod_{i<j}[ij]\cdot\sum_{i<j}x_ix_j},\qquad F_5=\textit{the same } F_5 \textit{ as in (i)},$$

$$F_3=\frac{\sum_{i=1}^5([12][13][14][15]\cdot I^2\cdot x_1)^{(1i)}}{\prod_{i<j}[ij]\cdot\sum_{i<j}x_ix_j},$$

where $[ij]=x_i-x_j$, $I=\sum_{\tau\in R_2}\tau(x_2x_3(x_2x_3+x_4^2+x_5^2))$, $R_2=\{1, (34), (354), (234), (2354), (24)(35)\}$ *and* $R_3=\{1, (234), (243), (152), (15234), (15243), (125), (12345), (12435), (15432), (154), (15423), (15342), (15324), (153)\}$.

In the theorem, note that R_1, R_2 and R_3 are coset representatives with respect to various subgroups:

$$S_5=\bigcup_{\mu\in R_1}H_1\mu,\quad H=\bigcup_{\tau\in R_2}H_2\tau,\quad A_5=\bigcup_{\nu\in R_3}H_3\nu,$$

where

$$H=\langle(23),(24),(25)\rangle\cong S_4,\quad H_1=\langle(12),(13),(45)\rangle\cong D_6,$$
$$H_2=\langle(23),(45)\rangle\cong V_4,\qquad H_3=\langle(12)(34),(13)(24)\rangle\cong V_4,$$

and D_6 is the dihedral group of order 12.

Now we start to prove Theorem 1.2 when $n=5$. Let $\tau=(12)\in S_5$. By Theorem 2.5, we see that $K(x_1,\dots,x_5)^{A_5}=K(F_1,\dots,F_5)$.

With the aid of a computer, we can evaluate the action of τ on $K(F_1,\dots,F_5)$ as follows:

$$\tau: F_1\mapsto a/F_1,\quad F_2\mapsto F_3/F_1,\quad F_3\mapsto aF_2/F_1,$$
$$F_4\mapsto -F_4,\quad F_5\mapsto -F_5\qquad \text{when char } K\neq 2;$$
$$\tau: F_1\mapsto a/F_1,\quad F_2\mapsto F_3/F_1,\quad F_3\mapsto aF_2/F_1,$$
$$F_4\mapsto F_4+1,\quad F_5\mapsto F_5\qquad \text{when char } K=2.$$

Case 1: char $K\neq 2$.

Define

$$G_1:=F_1,\qquad G_2:=F_4+1/F_4-1,\quad G_3:=F_4(F_2-F_3/F_1),$$
$$G_4:=F_2+F_3/F_1,\quad G_5:=F_4F_5.$$

Then we have $K(x_1, \dots, x_5)^{A_5} = K(F_1, \dots, F_5) = K(G_1, \dots, G_5)$ and

$$\tau : G_1 \mapsto a/G_1, \quad G_2 \mapsto 1/G_2, \quad G_3 \mapsto G_3, \quad G_4 \mapsto G_4, \quad G_5 \mapsto G_5.$$

So it follows from Theorem 2.1 that $K(x_1, \dots, x_5)^{S_5} = K(G_3, G_4, G_5)(G_1, G_2)^{\langle\tau\rangle}$ is rational over K.

Case 2: char $K = 2$.

Define

$$G_1 := F_1, \quad G_2 := F_2, \quad G_3 := \frac{F_2 F_3}{F_1}, \quad G_4 := F_4 + \frac{F_3}{F_1 F_2 + F_3}, \quad G_5 := F_5.$$

Then we have $K(x_1, \dots, x_5)^{A_5} = K(F_1, \dots, F_5) = K(G_1, \dots, G_5)$ and

$$\tau : G_1 \mapsto a/G_1, \quad G_2 \mapsto G_3/G_2, \quad G_3 \mapsto G_3, \quad G_4 \mapsto G_4, \quad G_5 \mapsto G_5.$$

We use Theorem 2.1 and find that $K(x_1, \dots, x_5)^{S_5} = K(G_3, G_4, G_5)(G_1, G_2)^{\langle\tau\rangle}$ is rational over K. □

3. Conic bundles: Another approach when char $K \neq 2$

Throughout this section we assume that char $K \neq 2$.

In this section, we will give another proof of Theorem 1.2 when $n = 3, 4$ (and char $K \neq 2$) by presenting $K(x_1, \dots, x_n)^{S_n}$ as the function field of a conic bundle over $\mathbb{P}^{n-1}$.

Consider the action of S_n on $K(x_1, \dots, x_n)$ defined by Equation (1-1). Because of Theorem 1.1, we may assume that $a \in K^\times \setminus K^{\times 2}$ without loss of generality.

Define $\alpha := \sqrt{a}$ and $\mathrm{Gal}(K(\alpha)/K) = \langle\rho\rangle$, where $\rho(\alpha) = -\alpha$. Extend the actions of S_n and ρ to $K(\alpha)(x_1, \dots, x_n) = K(\alpha) \otimes_K K(x_1, \dots, x_n)$ by requiring that S_n acts trivially on $K(\alpha)$ and τ acts trivially on $K(x_1, \dots, x_n)$.

Define $z_i := (\alpha - x_i)/(\alpha + x_i)$ for $1 \le i \le n$. We find that $K(\alpha)(x_1, \dots, x_n) = K(\alpha)(z_1, \dots, z_n)$ and

$$\sigma : z_i \mapsto -z_{\sigma(i)}$$

for any $\sigma \in S_n \setminus A_n$, and

$$\rho : \alpha \mapsto -\alpha, \quad z_i \mapsto 1/z_i.$$

Define $z_0 := z_1 + \cdots + z_n$, $y_i := z_i/z_0$ for $1 \le i \le n$. Hence $y_1 + \cdots + y_n = 1$.

Let $t_1, \dots, t_n$ be the elementary symmetric functions of $y_1, \dots, y_n$. In particular, $t_1 = 1$. Define $\Delta := \prod_{1 \le i < j \le n} (y_i - y_j) \in K(y_1, \dots, y_n)$ and $u := z_0 \cdot \Delta$. Note that Δ^2 can be written as a polynomial in $t_1, \dots, t_n$, and thus in $t_2, \dots, t_n$.

Lemma 3.1. *$K(x_1, \dots, x_n)^{S_n} = K(\alpha)(t_2, \dots, t_n, u)^{\langle\rho\rangle}$ and*

$$\rho : \alpha \mapsto -\alpha, \quad t_i \mapsto t_{n-i}(t_n/t_{n-1})^i t_n^{-1}, \quad u \mapsto f(t_2, \dots, t_n) \cdot u^{-1},$$

where $f(t_2,\dots,t_n)\in K(t_2,\dots,t_n)$ *is given by*

$$f(t_2,\dots,t_n):=(-1)^{n(n-1)/2}t_n^{-(n-1)}(t_n/t_{n-1})^{(n+1)(n-2)/2}\Delta^2 \tag{3-1}$$

and we adopt the convention that $t_0=t_1=1$.

Proof. Note that $K(\alpha)(y_1,\dots,y_n,z_0)=K(\alpha)(y_1,\dots,y_n,u)$. Since u is fixed by the action of S_n, it follows that $K(\alpha)(y_1,\dots,y_n,z_0)^{S_n}=K(\alpha)(y_1,\dots,y_n)^{S_n}(u)=K(\alpha)(t_2,\dots,t_n,u)$; the last equality follows, for example, from the proof of [Hajja and Kang 1995, Lemma 1] because $\sigma(y_i)=y_{\sigma(i)}$ for any $\sigma\in S_n$ and i in $1\le i\le n$.

Thus $K(x_1,\dots,x_n)^{S_n}=(K(\alpha)^{\langle\rho\rangle}(x_1,\dots,x_n))^{S_n}=K(\alpha)(x_1,\dots,x_n)^{\langle S_n,\rho\rangle}=(K(\alpha)(x_1,\dots,x_n)^{S_n})^{\langle\rho\rangle}=K(\alpha)(t_2,\dots,t_n,u)^{\langle\rho\rangle}$.

It is easy to verify that the action of ρ on $K(\alpha)(t_2,\dots,t_n,u)$ is as stated. □

We write $n=2m+1$ if n is odd, and $n=2m$ otherwise. Define

$$u_i:=t_{i+1},\quad u_{n-i}:=\rho(t_{i+1})=t_{n-(i+1)}t_n^i/t_{n-1}^{i+1}\quad\text{for } i=1,\dots,m-1 \tag{3-2}$$

and

$$\begin{cases} u_m:=t_{m+1},\quad u_{m+1}:=\rho(t_{m+1})=t_mt_n^m/t_{n-1}^{m+1} & \text{if } n \text{ is odd,}\\ u_m:=t_n/t_{n-1}, & \text{if } n \text{ is even.}\end{cases} \tag{3-3}$$

Lemma 3.2. $K(x_1,\dots,x_n)^{S_n}=K(\alpha)(u_1,\dots,u_{n-1},u)^{\langle\rho\rangle}$ *and*

$$\begin{aligned}\rho:\alpha\mapsto -\alpha,\quad u_i\mapsto u_{n-i}\quad \textit{for } i=1,\dots,n-1,\\ u\mapsto g(u_1,\dots,u_{n-1})\cdot u^{-1},\end{aligned}$$

where $g(u_1,\dots,u_{n-1})=f(t_2,\dots,t_n)$ *and* $f(t_2,\dots,t_n)$ *is given as in* (3-1).

Proof. The assertion follows from $K(\alpha)(t_2,\dots,t_n,u)=K(\alpha)(u_1,\dots,u_{n-1},u)$ and Lemma 3.1. Indeed we may show $K(t_2,\dots,t_n)\subset K(u_1,\dots,u_{n-1})$ as follows.

Case 1: $n=2m+1$ is odd.

The fact that $t_2,\dots,t_{m+1}\in K(u_1,\dots,u_{n-1})$ follows from (3-2) and (3-3). We have $t_n\in K(u_1,\dots,u_{n-1})$ because

$$\left(\frac{u_m^{m+1}}{u_{m-1}^m}\right)u_{m+1}^m\left(\frac{1}{u_{m+2}}\right)^{m+1}=\left(\frac{t_{m+1}^{m+1}}{t_m^m}\right)\left(\frac{t_mt_n^m}{t_{n-1}^{m+1}}\right)^m\left(\frac{t_{n-1}^m}{t_{m+1}t_n^{m-1}}\right)^{m+1}=t_n.$$

and $t_{n-1}\in K(u_1,\dots,u_{n-1})$ because

$$t_n\left(\frac{u_{m-1}}{u_m}\right)u_{m+2}\left(\frac{1}{u_{m+1}}\right)=t_n\left(\frac{t_m}{t_{m+1}}\right)\left(\frac{t_{m+1}t_n^{m-1}}{t_{n-1}^m}\right)\left(\frac{t_{n-1}^{m+1}}{t_mt_n^m}\right)=t_{n-1}.$$

From (3-2) we find that $t_{n-(i+1)}=u_{n-i}t_{n-1}^{i+1}/t_n^i$ for $1\le i\le m-2$. Thus $t_{m+2},\dots,t_{n-2}\in K(u_1,\dots,u_{n-1})$.

Case 2: $n=2m$ is even. That $t_2,\dots,t_m\in K(u_1,\dots,u_{n-1})$ follows from (3-2).

From (3-2) and (3-3), we get

$$\frac{u_{k+1}}{u_{k+2}} = \frac{t_k}{t_{k+1}} \cdot \frac{t_n}{t_{n-1}} = \frac{t_k}{t_{k+1}} \cdot u_m,$$

where $k = m, \dots, 2m-3$. We find that $t_{k+1} = t_k u_m u_{k+2}/u_{k+1} \in K(u_1, \dots, u_{n-1})$ for $m \le k \le 2m-3$. From (3-2), we have $u_{n-1} = t_{n-2}t_n/t_{n-1}^2 = t_{n-2}u_m/t_{n-1}$. Hence $t_{n-1} = t_{n-2}u_m/u_{n-1} \in K(u_1, \dots, u_{n-1})$.

Since $t_n = u_m t_{n-1}$, it follows that $t_n \in K(u_1, \dots, u_{n-1})$. □

We will change the variables $u_1, \dots, u_{n-1}$ to $v_1, \dots, v_{n-1}$ as follows. When $n = 2m+1$ is odd, define

$$v_i := \tfrac{1}{2}(u_i + u_{n-i}), \quad v_{n-i} := \tfrac{1}{2}(\alpha(u_i - u_{n-i})) \quad \text{for } i = 1, \dots, m.$$

When $n = 2m$ is even, define

$$v_m := u_m, \quad v_i := \tfrac{1}{2}(u_i + u_{n-i}), \quad v_{n-i} := \tfrac{1}{2}(\alpha(u_i - u_{n-i})) \quad \text{for } i = 1, \dots, m-1.$$

Thus $K(\alpha)(u_1, \dots, u_{n-1}, u) = K(\alpha)(v_1, \dots, v_{n-1}, u)$.

In these variables, Lemma 3.2 reads as follows:

Lemma 3.3. *$K(x_1, \dots, x_n)^{S_n} = K(\alpha)(v_1, \dots, v_{n-1}, u)^{\langle\rho\rangle}$ and*

$$\rho : \alpha \mapsto -\alpha, \quad v_i \mapsto v_i \quad \text{for } i = 1, \dots, n-1, \quad u \mapsto h(v_1, \dots, v_{n-1}) \cdot u^{-1},$$

where $h(v_1, \dots, v_{n-1}) = f(t_2, \dots, t_n)$ and $f(t_2, \dots, t_n)$ is given as in (3-1).

Hence we get the following theorem, which asserts that $K(x_1, \dots, x_n)^{S_n}$ is the function field of a conic bundle over $\mathbb{P}^{n-1}$ of the form $x^2 - ay^2 = h(v_1, \dots, v_{n-1})$ with affine coordinates $v_1, \dots, v_{n-1}$; see for example [Shafarevich 1974, page 73] for conic bundles over $\mathbb{P}^1$.

Theorem 3.4. *$K(x_1, \dots, x_n)^{S_n} = K(x, y, v_1, \dots, v_{n-1})$ and the generators x, y, $v_1, \dots, v_{n-1}$ satisfy the relation*

$$x^2 - ay^2 = h(v_1, \dots, v_{n-1}),$$

where $h(v_1, \dots, v_{n-1}) = f(t_2, \dots, t_n)$ and $f(t_2, \dots, t_n)$ is given as in (3-1).

Proof. Define

$$x := \frac{1}{2}\Big(u + \frac{h(v_1, \dots, v_{n-1})}{u}\Big), \quad y := \frac{1}{2\alpha}\Big(u - \frac{h(v_1, \dots, v_{n-1})}{u}\Big).$$

Then we get $K(x, y, v_1, \dots, v_{n-1}) \subset K(x_1, \dots, x_n)^{S_n} = K(\alpha)(v_1, \dots, v_{n-1}, u)$. Thus $K(x, y, v_1, \dots, v_{n-1}) = K(x_1, \dots, x_n)^{S_n}$, since $K(x, y, v_1, \dots, v_n)(u) = K(\alpha)(v_1, \dots, v_{n-1}, u)$ and $[K(x, y, v_1, \dots, v_n)(u) : K(x, y, v_1, \dots, v_n)] = 2$. We also have $x^2 - ay^2 = h(v_1, \dots, v_{n-1})$ by definition. □

Proof of Theorem 1.2 when $n = 3$ and char $K \neq 2$.
Step 1. By Lemma 3.1 we find that $K(x_1, x_2, x_3)^{S_3} = K(\alpha)(t_2, t_3, u)^{\langle\rho\rangle}$, where

$$\rho : \alpha \mapsto -\alpha, \quad t_2 \mapsto t_2^{-2} t_3, \quad t_3 \mapsto t_2^{-3} t_3^2, \quad u \mapsto -t_2^{-2}\Delta^2 \cdot u^{-1}.$$

Note that $\Delta^2 = \prod_{1\le i<j\le 3}(y_i - y_j)^2 = t_2^2 - 4t_2^3 - 4t_3 + 18t_2t_3 - 27t_3^2$ because $t_1 = 1$.

Define $u_1 := t_2$, $u_2 := \rho(t_2) = t_2^{-2}t_3$. Then $K(\alpha)(t_2, t_3, u) = K(\alpha)(u_1, u_2, u)$ and

$$\rho : u_1 \mapsto u_2 \mapsto u_1, \quad u \mapsto g(u_1, u_2) \cdot u^{-1},$$

where $g(u_1, u_2) = -1 + 4u_1 + 4u_2 - 18u_1u_2 + 27u_1^2u_2^2$.

Define $v_1 := (u_1 + u_2)/2$ and $v_2 := \alpha(u_1 - u_2)/2$. Then $\rho : v_1 \mapsto v_1, v_2 \mapsto v_2$ and $g(u_1, u_2) = h(v_1, v_2)$, where

$$h(v_1, v_2) = -1 + 8v_1 - 18v_1^2 + 27v_1^4 + (18/a)v_2^2 - (54/a)v_1^2v_2^2 + (27/a^2)v_2^4.$$

Hence $K(x_1, x_2, x_3)^{S_3} = K(\alpha)(v_1, v_2, u)^{\langle\rho\rangle} = K(x, y, v_1, v_2)$, where

$$x = \frac{1}{2}\Big(u + \frac{h(v_1, v_2)}{u}\Big), \quad y = \frac{1}{2\alpha}\Big(u - \frac{h(v_1, v_2)}{u}\Big).$$

Note that x and y satisfy the relation

$$\begin{aligned} x^2 - ay^2 &= h(v_1, v_2) \\ &= (1 + v_1)(-1 + 3v_1)^3 - (18/a)v_2^2(-1 + 3v_1^2) + (27/a^2)v_2^4. \end{aligned} \tag{3-4}$$

Step 2. Suppose that char $K = 3$. Then (3-4) becomes $x^2 - ay^2 = -1 - v_1$. Hence $K(x_1, x_2, x_3)^{S_3} = K(x, y, v_1, v_2) = K(x, y, v_2)$ is rational over K.

Step 3. From now on, we assume that char $K \neq 2, 3$.

We normalize the generators v_1 and v_2 by defining $T_1 := 3v_1$ and $T_2 := 3v_2/a$. We get $K(x_1, x_2, x_3)^{S_3} = K(x, y, T_1, T_2)$ with a relation

$$3x^2 - 3ay^2 = -3 + 8T_1 - 6T_1^2 + T_1^4 + 6aT_2^2 - 2aT_1^2T_2^2 + a^2T_2^4. \tag{3-5}$$

Step 4. We find the singularities of (3-5). We get $x = y = -1 + T_1 = T_2 = 0$. Define $T_3 := -1 + T_1$. The relation (3-5) becomes

$$3x^2 - 3ay^2 = 4aT_2^2 + a^2T_2^4 - 4aT_2^2T_3 - 2aT_2^2T_3^2 + 4T_3^3 + T_3^4. \tag{3-6}$$

Step 5. We blow-up Equation (3-6), that is, define $X_2 := x/T_3$, $Y_2 := y/T_3$ and $T_4 := T_2/T_3$. Then $K(x, y, T_1, T_2) = K(x, y, T_2, T_3) = K(X_2, Y_2, T_3, T_4)$ and the relation (3-6) becomes

$$\begin{aligned} 3X_2^2 - 3aY_2^2 &= 4T_3 + T_3^2 + 4aT_4^2 - 4aT_3T_4^2 - 2aT_3^2T_4^2 + a^2T_3^2T_4^4 \\ &= (T_3 - aT_3T_4^2)^2 + 4(T_3 - aT_3T_4^2) + 4aT_4^2 \\ &= (T_3 - aT_3T_4^2)(4 + T_3 - aT_3T_4^2) + 4aT_4^2. \end{aligned} \tag{3-7}$$

Define

$$X_3 := \frac{X_2}{T_3 - aT_3T_4^2}, \qquad Y_3 := \frac{Y_2}{T_3 - aT_3T_4^2},$$
$$S_1 := \frac{4 + T_3 - aT_3T_4^2}{T_3 - aT_3T_4^2}, \quad S_2 := \frac{T_4}{T_3 - aT_3T_4^2}.$$

Note that $K(X_2, Y_2, T_3, T_4) = K(X_3, Y_3, S_1, S_2)$. For $S_1 \in K(X_3, Y_3, S_1, S_2)$, S_1 is a fractional linear transformation of $T_3 - aT_3T_4^2$. Hence $T_3 - aT_3T_4^2 \in K(X_3, Y_3, S_1, S_2)$. Thus $T_4 = S_2 \cdot (T_3 - aT_3T_4^2) \in K(X_3, Y_3, S_1, S_2)$ also. Now S_1 is a fractional linear transformation of T_3 with coefficients in $K(T_4)$. Hence $T_3 \in K(X_3, Y_3, S_1, S_2)$. It follows that $X_2, Y_2 \in K(X_3, Y_3, S_1, S_2)$ also.

The relation (3-7) becomes $3X_3^2 - 3aY_3^2 = S_1 + 4aS_2^2$, which is linear in S_1. Hence $K(x_1, x_2, x_3)^{S_3} = K(X_3, Y_3, S_1, S_2) = K(X_3, Y_3, S_2)$ is rational over K.

Step 6. Here is another proof. Instead of the method in Step 5, we may proceed as follows:

Define $X_4 := x/T_3^2$, $Y_4 := y/T_3^2$, $T_4 := T_2/T_3$, and $T_5 := 1/T_3$. Then $K(x, y, T_2, T_3) = K(X_4, Y_4, T_4, T_5)$ and (3-6) becomes

$$3X_4^2 - 3aY_4^2 = 1 - 2aT_4^2 + a^2T_4^4 + 4T_5 - 4aT_4^2T_5 + 4aT_4^2T_5^2.$$

The singularities here are $X_4 = Y_4 = T_4 \pm (1/\sqrt{a}) = T_5 = 0$. If we blow-up with respect to $1 - aT_4^2$, that is, define

$$X_5 := X_4/(1 - aT_4^2), \quad Y_5 := Y_4/(1 - aT_4^2), \quad T_6 := T_5/(1 - aT_4^2),$$

then $K(X_4, Y_4, T_4, T_5) = K(X_5, Y_5, T_4, T_6)$ and the relation becomes

$$3X_5^2 - 3aY_5^2 = 1 + 4T_6 + 4aT_4^2T_6^2. \tag{3-8}$$

Thus we get $K(x_1, x_2, x_3)^{S_3} = K(X_5, Y_5, T_4T_6, T_6) = K(X_5, Y_5, T_4T_6)$ is rational over K because (3-8) becomes linear in T_6. □

Proof of Theorem 1.2 when $n = 4$ *and* char $K \neq 2$.

Step 1. By Lemma 3.1 we find that $K(x_1, x_2, x_3, x_4)^{S_4} = K(\alpha)(t_2, t_3, t_4, u)^{\langle \rho \rangle}$, where

$$\rho : \alpha \mapsto -\alpha, \quad t_2 \mapsto t_2t_3^{-2}t_4, \quad t_3 \mapsto t_3^{-3}t_4^2, \quad t_4 \mapsto t_3^{-4}t_4^3, \quad u \mapsto t_3^{-5}t_4^2\Delta^2 \cdot u^{-1},$$

where

$$\begin{aligned}\Delta^2 &= \prod_{1\le i<j\le 4}(y_i-y_j)^2\\ &= t_2^2t_3^2-4t_2^3t_3^2-4t_3^3+18t_2t_3^3-27t_3^4-4t_2^3t_4+16t_2^4t_4+18t_2t_3t_4-80t_2^2t_3t_4\\ &\qquad -6t_3^2t_4+144t_2t_3^2t_4-27t_4^2+144t_2t_4^2-128t_2^2t_4^2-192t_3t_4^2+256t_4^3.\end{aligned}$$

Define $u_1 := t_2,\ u_2 := t_4/t_3$ and $u_3 := \rho(t_2) = t_2t_4/t_3^2$. Then $K(\alpha)(t_2, t_3, t_4, u) = K(\alpha)(u_1, u_2, u_3, u)$ and

$$\rho : \alpha \mapsto -\alpha, \quad u_1 \mapsto u_3 \mapsto u_1, \quad u_2 \mapsto u_2, \quad u \mapsto g(u_1, u_2, u_3)\cdot u^{-1},$$

where

$$\begin{aligned}g(u_1,u_2,u_3) = \frac{u_2}{u_1u_3}\big(&-27u_1^2u_2^2-4u_1u_2u_3+18u_1^2u_2u_3-6u_1u_2^2u_3+144u_1^2u_2^2u_3\\ &-192u_1u_2^3u_3+256u_1u_2^4u_3+u_1^2u_3^2-4u_1^3u_3^2+18u_1u_2u_3^2\\ &-80u_1^2u_2u_3^2-27u_2^2u_3^2+144u_1u_2^2u_3^2-128u_1^2u_2^2u_3^2-4u_1^2u_3^3+16u_1^3u_3^3\big).\end{aligned}$$

Define $v_1 := (u_1+u_3)/2$, $v_2 := u_2$ and $v_3 = \alpha(u_1-u_3)/2$. Then we obtain $K(\alpha)(u_1, u_2, u_3, u) = K(\alpha)(v_1, v_2, v_3, u)$ and

$$\rho : \alpha \mapsto -\alpha, \quad v_1 \mapsto v_1, \quad v_2 \mapsto v_2, \quad v_3 \mapsto v_3, \quad u \mapsto h(v_1, v_2, v_3)\cdot u^{-1},$$

where $h(v_1, v_2, v_3) = g(u_1, u_2, u_3) \in K(v_1, v_2, v_3)$ is given as

$$\begin{aligned}h(v_1,v_2,v_3) = \frac{v_2}{av_1^2-v_3^2}\big(&av_1^2v_2(-1+4v_1-8v_2)^2(v_1^2-4v_2+4v_1v_2+4v_2^2)\\ &-2v_2v_3^2\big(v_1^2-8v_1^3+24v_1^4-2v_2+18v_1v_2-80v_1^2v_2\\ &\qquad+24v_2^2+144v_1v_2^2-128v_1^2v_2^2-96v_2^3+128v_2^4\big)\\ &-(1/a)v_2v_3^4(-1+8v_1-48v_1^2+80v_2+128v_2^2)-(16/a^2)v_2v_3^6\big).\end{aligned}$$

Step 2. Because $h(v_1, v_2, v_3)$ is still complicated, we define p, q and r as

$$p := \frac{1}{2}\Big(\frac{1}{u_1}+\frac{1}{u_3}\Big)u_2, \quad q := \frac{\alpha}{2}\Big(\frac{1}{u_1}-\frac{1}{u_3}\Big)u_2, \quad r := 4u_2.$$

Then $K(\alpha)(v_1, v_2, v_3, u) = K(\alpha)(p, q, r, u)$. Indeed we have

$$\begin{aligned}p &= \frac{av_1v_2}{av_1^2-v_3^2}, & q &= -\frac{av_2v_3}{av_1^2-v_3^2}, & r &= 4v_2,\\ v_1 &= \frac{apr}{4(ap^2-q^2)}, & v_2 &= r/4, & v_3 &= -\frac{apr}{4(ap^2-q^2)}.\end{aligned}$$

Hence we obtain $K(x_1, x_2, x_3, x_4)^{S_4} = K(\alpha)(p, q, r, u)^{\langle\rho\rangle}$ and

$$\rho : \alpha \mapsto -\alpha, \quad p \mapsto p, \quad q \mapsto q, \quad r \mapsto r, u \mapsto \frac{r^2}{64(ap^2-q^2)^2}\cdot\frac{H(p,q,r)}{u},$$

where

$$H(p,q,r)=a^2(p-r+2pr)^2(-16p^2+r+4pr+4p^2r) \\ -a\big(-32p^2+r+36pr-12p^2r-20r^2+72pr^2 \\ -96p^2r^2-8r^3+32p^2r^3\big)q^2+16(-1+r)^3q^4. \tag{3-9}$$

Define $U:=u\cdot r/(8(ap^2-q^2))$. Then $K(\alpha)(p,q,r,u)=K(\alpha)(p,q,r,U)$, and ρ acts on $K(\alpha)(p,q,r,U)$ by

$$\rho:\alpha\mapsto-\alpha,\quad p\mapsto p,\quad q\mapsto q,\quad r\mapsto r,\quad U\mapsto H(p,q,r)/U.$$

Hence $K(x_1,\dots,x_4)^{S_4}=K(\alpha)(p,q,r,U)^{\langle\rho\rangle}=K(X,Y,p,q,r)$ where

$$X=\frac{1}{2}\Big(U+\frac{g(p,q,r)}{U}\Big),\quad Y=\frac{1}{2\alpha}\Big(U-\frac{g(p,q,r)}{U}\Big).$$

Note that X and Y satisfy the relation

$$X^2-aY^2=H(p,q,r). \tag{3-10}$$

Step 3. Because $H(p,q,r)$ in (3-9) is a biquadratic equation with respect to q and its constant term has the square factor $(p-r+2pr)^2$, we define $p_2:=p-r+2pr$. Then $p=(p_2+r)/(1+2r)$. We also define $X_2:=X(1+2r)$ and $Y_2:=Y(1+2r)$. Then $K(x_1,x_2,x_3,x_4)^{S_4}=K(X_2,Y_2,p_2,q,r)$ and (3-10) becomes

$$X_2^2-aY_2^2=a^2p_2^2(-16p_2^2+r-28p_2r+4p_2^2r-8r^2+16p_2r^2+16r^3) \\ -a\big(-32p_2^2+r-28p_2r-12p_2^2r-12r^2+120p_2r^2 \\ -96p_2^2r^2+48r^3-48p_2r^3+32p_2^2r^3-64r^4+64p_2r^4\big)q^2 \\ +16(-1+r)^3(1+2r)^2q^4.$$

The right hand side is biquadratic in q with constant term on the first line. Hence we define $p_3:=p_2/q$, $X_3:=X_2/q$ and $Y_3:=Y_2/q$, and the equation becomes quadratic in q:

$$X_3^2-aY_3^2=ar(-1+4r)^2(-1+ap_3^2+4r) \\ +4ap_3r(7-7ap_3^2-30r+4ap_3^2r+12r^2-16r^3)q \\ +4(-1+ap_3^2-4r-4r^2)(4-4ap_3^2-12r+ap_3^2r+12r^2-4r^3)q^2.$$

Define $q_2:=1/q$, $r_2:=4r$, $X_4:=4X_3/q$, $Y_4:=4Y_3/q$. Then

$$X_4^2-aY_4^2=4ar_2(-1+r_2)^2(-1+ap_3^2+r_2)q_2^2 \\ +4ap_3r_2(28-28ap_3^2-30r_2+4ap_3^2r_2+3r_2^2-r_2^3)q_2 \\ +(-4+4ap_3^2-4r_2-r_2^2)(64-64ap_3^2-48r_2+4ap_3^2r_2+12r_2^2-r_2^3). \tag{3-11}$$

Because (3-11) is quadratic in q_2, we may eliminate a linear term of q_2 in the usual manner by putting

$$q_3 := 2q_2 + \frac{p_3(28 - 28ap_3^2 - 30r_2 + 4ap_3^2 r_2 + 3r_2^2 - r_2^3)}{(-1+r_2)^2(-1+ap_3^2+r_2)}.$$

Define

$$X_5 := X_4(-1+r_2)(-1+ap_3^2+r_2), \quad Y_5 := Y_4(-1+r_2)(-1+ap_3^2+r_2).$$

Then (3-11) becomes

$$X_5^2 - aY_5^2 = (2+r_2)^2(-1+ap_3^2+r_2)(4-4ap_3^2-5r_2+r_2^2)^3 + ar_2(-1+r_2)^4(-1+ap_3^2+r_2)^3 q_3^2.$$

Defining

$$q_4 := \frac{q_3(-1+r_2)^2(-1+ap_3^2+r_2)}{(2+r_2)(4-4ap_3^2-5r_2+r_2^2)}$$

and

$$X_6 := \frac{X_5}{(2+r_2)(4-4ap_3^2-5r_2+r_2^2)}, \quad Y_6 := \frac{Y_5}{(2+r_2)(4-4ap_3^2-5r_2+r_2^2)},$$

we get $K(x_1,\dots,x_4)^{S_4} = K(X_6, Y_6, p_3, q_4, r_2)$ and the equation becomes

$$X_6^2 - aY_6^2 = (-1+ap_3^2+r_2)((4-4ap_3^2-5r_2+r_2^2) + ar_2q_4^2). \tag{3-12}$$

Step 4. We find the singularities of (3-12). We get $p_3 \pm (1/\sqrt{a}) = r_2 = X_6 = Y_6 = 0$. Blow-up with respect to $-1+ap_3^2$, that is, define

$$r_3 := r_2/(-1+ap_3^2), \quad X_7 := X_6/(-1+ap_3^2), \quad Y_7 := Y_6/(-1+ap_3^2).$$

Then $K(p_3, q_4, r_2, X_6, Y_6) = K(p_3, q_4, r_3, X_7, Y_7)$ and (3-12) becomes

$$X_7^2 - aY_7^2 = (1+r_3)(-4-5r_3+aq_4^2r_3 - r_3^2 + ap_3^2r_3^2).$$

Define $p_4 := p_3r_3$. Then

$$X_7^2 - aY_7^2 = (1+r_3)(-4-5r_3+aq_4^2r_3 - r_3^2 + ap_4^2). \tag{3-13}$$

Step 5. Equation (3-13) still has a singular locus $p_4 \pm q_4 = r_3 + 1 = X_7 = Y_7 = 0$. If we define $p_5 := p_4 + q_4$ and $r_4 := r_3 + 1$, it becomes

$$X_7^2 - aY_7^2 = r_4(ap_5^2 - 2ap_5q_4 - 3r_4 + aq_4^2r_4 - r_4^2) \tag{3-14}$$

with singular locus $S = (p_5 = r_4 = X_7 = Y_7 = 0)$. Blowing this up along S by defining $r_5 := r_4/p_5$, $X_8 := X_7/p_5$, and $Y_8 := Y_7/p_5$, we get

$$X_8^2 - aY_8^2 = r_5(ap_5 - 2aq_4 - 3r_5 + aq_4^2r_5 - p_5r_5^2).$$

Note that this is linear in p_5. Hence we conclude that the fixed field $K(x_1, \dots, x_4)^{S_4} = K(X_8, Y_8, q_4, r_5)$ is rational over K. □

4. Using the structures of rings of invariants

In this section, we give an another proof of Theorem 1.2 in the case of $n = 3, 4$ and char $K = 2$ by using the structure of $K(x_1, \dots, x_n)^{A_n}$. Throughout, we assume that char $K = 2$.

For $1 \le i \le n$, let s_i be the elementary symmetric function in $x_1, \dots, x_n$ of degree i.

By Revoy's theorem [1982], the invariant ring $K[x_1, \dots, x_n]^{A_n}$ is a free module of rank 2 over the subring $K[x_1, \dots, x_n]^{S_n} = K[s_1, \dots, s_n]$. Revoy's theorem is valid for all characteristics. We will find explicitly a free basis of $K[x_1, \dots, x_n]^{A_n}$ over $K[x_1, \dots, x_n]^{S_n}$ for the case $n = 3, 4$. For $n = 3$ and $n = 4$, it suffices by [Neusel and Smith 2002, Example 1, page 75] to find a polynomial of degree 3 and 6, respectively, that is in $K[x_1, \dots, x_n]^{A_n}$ but not in $K[x_1, \dots, x_n]^{S_n}$.

Define

$$b_3 := \sum_{\sigma \in A_3} \sigma(x_1 x_2^2) = x_1x_2^2 + x_2x_3^2 + x_3x_1^2,$$

$$\begin{aligned} b_4 := \sum_{\sigma \in A_4} \sigma(x_1 x_2^2 x_3^3) = {} & x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1x_2^2x_3^3 + x_1^3x_2^2x_4 + x_2^3x_3^2x_4 + x_1^2x_3^3x_4 \\ & + x_1x_2^3x_4^2 + x_1^3x_3x_4^2 + x_2x_3^3x_4^2 + x_1^2x_2x_4^3 + x_2^2x_3x_4^3 + x_1x_3^2x_4^3. \end{aligned}$$

For $n = 3, 4$, it follows that $\{1, b_n\}$ is a free basis of $K[x_1, \dots, x_n]^{A_n}$, that is,

$$K[x_1, x_2, x_3]^{A_3} = K[s_1, s_2, s_3] \oplus b_3 K[s_1, s_2, s_3],$$
$$K[x_1, x_2, x_3, x_4]^{A_4} = K[s_1, s_2, s_3, s_4] \oplus b_4 K[s_1, s_2, s_3, s_4].$$

We have proved this:

Lemma 4.1. *Let K be a field of* char $K = 2$. *Then the fields $K(x_1, x_2, x_3)^{A_3}$ and $K(x_1, x_2, x_3, x_4)^{A_4}$ of invariants are given as follows.*

(i) $K(x_1, x_2, x_3)^{A_3} = K(s_1, s_2, s_3, b_3)$ *with the relation*

$$b_3^2 + b_3s_1s_2 + s_2^3 + b_3s_3 + s_1^3s_3 + s_3^2 = 0.$$

(ii) $K(x_1, x_2, x_3, x_4)^{A_4} = K(s_1, s_2, s_3, s_4, b_4)$ *with the relation*

$$b_4^2 + b_4s_1s_2s_3 + b_4s_3^2 + s_2^3s_3^2 + s_1^3s_3^3 + s_3^4 + b_4s_1^2s_4 + s_1^2s_2^3s_4 + s_1^4s_4^2 = 0.$$

Proof of Theorem 1.2 when $n = 3$ and char $K = 2$. First, τ acts on $K(x_1, x_2, x_3)^{A_3} = K(s_1, s_2, s_3, b_3)$ as

$$s_1 \mapsto as_2/s_3, \quad s_2 \mapsto a^2s_1/s_3, \quad s_3 \mapsto a^3/s_3, \quad b_3 \mapsto a^3b_3/s_3^2.$$

Apply Theorem 2.2. We find $K(x_1, x_2, x_3)^{A_3} = K(s_3, u, v)$, where u and v are the same as in Theorem 2.2. It is not difficult to check that

$$u = \frac{b_3 + s_3}{s_1^2 + s_2} \quad \text{and} \quad v = \frac{b_3 + s_1 s_2}{s_1^2 + s_2}.$$

Moreover, the action of τ is given by

$$\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad u \mapsto \frac{au}{u^2 - uv + v^2}, \quad v \mapsto \frac{av}{u^2 - uv + v^2}.$$

Define $w := u/v$. Then $K(x_1, x_2, x_3)^{A_3} = K(s_3, v, w)$ and

$$\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad v \mapsto \frac{a}{v(1 - w + w^2)}, \quad w \mapsto w.$$

By Theorem 2.1, $K(x_1, x_2, x_3)^{S_3} = K(s_3, v, w)^{\langle \tau \rangle}$ is rational over K. □

Proof of Theorem 1.2 when $n = 4$ *and* $\operatorname{char} K = 2$.

In this case, τ acts on $K(x_1, x_2, x_3, x_4)^{A_4} = K(s_1, s_2, s_3, s_4, b_4)$ as

$$s_1 \mapsto as_3/s_4, \quad s_2 \mapsto a^2 s_2/s_4, \quad s_3 \mapsto a^3 s_1/s_4, \quad s_4 \mapsto a^4/s_4,$$
$$b_4 \mapsto a^6(b_4 + s_1 s_2 s_3 + s_3^2 + s_1^2 s_4)/s_4^3.$$

Define

$$t_1 := \frac{s_1 s_3}{s_2}, \quad t_2 := s_2, \quad t_3 := s_3, \quad t_4 := \frac{s_1 s_2 s_3 + s_3^2 + s_1^2 s_4}{s_2^2}, \quad t_5 := \frac{b_4 + s_2^3}{s_2}.$$

It follows that $K(s_1, s_2, s_3, s_4, b_4) = K(t_1, t_2, t_3, t_4, t_5)$. It is easy to check that the relation among the generators $t_1, \dots, t_5$ is given by

$$t_1^3 + t_1^2 t_2 + t_1 t_2^2 + t_2^3 + t_2 t_4^2 + t_2 t_4 t_5 + t_2 t_5^2 = 0.$$

Define

$$u_1 := t_1, \quad u_2 := \frac{t_2}{t_1}, \quad u_3 := t_3, \quad u_4 := \frac{t_4}{(t_1 + t_2)}, \quad u_5 := \frac{t_5}{(t_1 + t_2)}.$$

Then we get $K(t_1, \dots, t_5) = K(u_1, \dots, u_5)$ with the relation

$$u_2(u_4^2 + u_4 u_5 + u_5^2 + 1) + 1 = 0.$$

Because this relation is linear in u_2, we obtain the following lemma.

Lemma 4.2. $K(x_1, \dots, x_4)^{A_4} = K(u_1, u_3, u_4, u_5)$, *where*

$$u_1 = \frac{s_1 s_3}{s_2}, \quad u_3 = s_3, \quad u_4 = \frac{s_1 s_2 s_3 + s_3^2 + s_1^2 s_4}{s_2(s_2^2 + s_1 s_3)}, \quad u_5 = \frac{b_4 + s_2^3}{s_2(s_2^2 + s_1 s_3)}.$$

Now we will prove Theorem 1.2 when $n = 4$ and char $K = 2$.

Write $p = u_1$, $q = u_3$, $r = u_4$, $s = u_5$ and $\tau = (12) \in S_4 \setminus A_4$. Note that $K(x_1, \dots, x_4)^{S_4} = K(p, q, r, s)^{\langle\tau\rangle}$ and the action of τ on $K(p, q, r, s)$ is given by

$$\begin{aligned} &p \mapsto \frac{r^2+rs+s^2+1}{ap}, \\ &q \mapsto \frac{a^3p^6q}{(r^2+rs+s^2+1)^3 + p^3q((r+1)(r^2+rs+s^2+1)+1)}, \\ &r \mapsto r, \quad s \mapsto s+r. \end{aligned}$$

Define

$$t := \frac{(r^2+rs+s^2+1)^3}{p^3q((r+1)(r^2+rs+s^2+1)+1)}.$$

Then $K(x_1, x_2, x_3, x_4)^{S_4} = K(p, q, r, s)^{\langle\tau\rangle} = K(p, t, r, s)^{\langle\tau\rangle}$ and the action of τ on $K(p, t, r, s)$ is given by

$$\tau : p \mapsto (r^2+rs+s^2+1)/(ap), \quad t \mapsto t+1, \quad r \mapsto r, \quad s \mapsto s+r.$$

Define

$$A := r+s+rt, \quad B := (r+s)/s, \quad C := pr/s.$$

It follows that $K(p, q, r, s) = K(r, A, B, C)$. Thus we have $K(x_1, x_2, x_3, x_4)^{S_4} = K(r)(A, B, C)^{\langle\tau\rangle}$ and

$$\tau : r \mapsto r, \quad A \mapsto A, \quad B \mapsto \frac{1}{B}, \quad C \mapsto \frac{1}{a}\Big((r^2+1)\Big(\frac{1}{B}+B\Big)+r^2\Big)\Big/C.$$

Apply Theorem 2.1. We find that $K(x_1, x_2, x_3, x_4)^{S_4}$ is rational over K. □

References

[Hajja 2000] M. Hajja, "Linear and monomial automorphisms of fields of rational functions: Some elementary issues", pp. 137–148 in *Algebra and number theory*, edited by M. Boulagouaz and J. P. Tignol, Lecture Notes in Pure and Appl. Math. **208**, Dekker, New York, 2000. MR 2001b:12009 Zbl 0958.12003

[Hajja and Kang 1995] M. Hajja and M. C. Kang, "Some actions of symmetric groups", *J. Algebra* **177**:2 (1995), 511–535. MR 96i:20013 Zbl 0837.20054

[Hajja and Kang 1997] M. Hajja and M.-C. Kang, "Twisted actions of symmetric groups", *J. Algebra* **188**:2 (1997), 626–647. MR 98b:13003 Zbl 0988.13007

[Iskovskih 1991] V. A. Iskovskih, "Towards the problem of rationality of conic bundles", pp. 50–56 in *Algebraic geometry* (Chicago, IL, 1989), edited by S. Bloch et al., Lecture Notes in Math. **1479**, Springer, Berlin, 1991. MR 93i:14033

[Kang 2001] M.-c. Kang, "The rationality problem of finite group actions", pp. 211–218 in *First International Congress of Chinese Mathematicians* (Beijing, 1998), AMS/IP Stud. Adv. Math. **20**, Amer. Math. Soc., Providence, RI, 2001. MR 2002b:12009

[Kang 2004] M.-C. Kang, "Rationality problem of GL_4 group actions", *Adv. Math.* **181**:2 (2004), 321–352. MR 2004k:12008 Zbl 1084.14508

[Kang 2007] M.-c. Kang, “Some rationality problems revisited”, pp. 211–218 in *International Congress of Chinese Mathematicians* (Hangzhou, 2007), Higher Education Press, Beijing, 2007.

[Kuniyoshi 1955] H. Kuniyoshi, “On a problem of Chevalley”, *Nagoya Math. J.* **8** (1955), 65–67. MR 16,993d Zbl 0065.02602

[Maeda 1989] T. Maeda, “Noether’s problem for A_5”, *J. Algebra* **125**:2 (1989), 418–430. MR 91c:12004 Zbl 0697.12018

[Masuda 1955] K. Masuda, “On a problem of Chevalley”, *Nagoya Math. J.* **8** (1955), 59–63. MR 16,993c Zbl 0065.02601

[Neusel and Smith 2002] M. D. Neusel and L. Smith, *Invariant theory of finite groups*, Mathematical Surveys and Monographs **94**, American Mathematical Society, Providence, RI, 2002. MR 2002k:13012 Zbl 0999.13002

[Revoy 1982] P. Revoy, “Anneau des invariants du groupe alterné”, *Bull. Sci. Math.* (2) **106**:4 (1982), 427–431. MR 84m:13007 Zbl 0507.13009

[Shafarevich 1974] I. R. Shafarevich, *Basic algebraic geometry*, Die Grundlehren der mathematischen Wissenschaften **213**, Springer, New York, 1974. MR 51 #3163 Zbl 0284.14001

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