TWISTED SYMMETRIC GROUP ACTIONS

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Let $K$ be any field, let $K(x_1, \ldots, x_n)$ be the rational function field of $n$ variables over $K$, and let $S_n$ and $A_n$ be the symmetric group and the alternating group of degree $n$, respectively. For any $a \in K \setminus \{0\}$, define an action of $S_n$ on $K(x_1, \ldots, x_n)$ by $\sigma \cdot x_i = x_{\sigma(i)}$ for $\sigma \in A_n$ and $\sigma \cdot x_i = a/x_{\sigma(i)}$ for $\sigma \in S_n \setminus A_n$. We prove that for any field $K$ and $n = 3, 4, 5$, the fixed field $K(x_1, \ldots, x_n)^{S_n}$ is rational (that is, purely transcendental) over $K$.

1. Introduction

Let $K$ be any field, let $K(x_1, \ldots, x_n)$ be the rational function field of $n$ variables over $K$, and let $S_n$ and $A_n$ be the symmetric group and the alternating group of degree $n$, respectively. For any $a \in K \setminus \{0\}$, define a twisted action of $S_n$ on $K(x_1, \ldots, x_n)$ by

$$
\sigma(x_i) := \begin{cases} 
  x_{\sigma(i)} & \text{if } \sigma \in A_n, \\
  a/x_{\sigma(i)} & \text{if } \sigma \in S_n \setminus A_n.
\end{cases}
$$

Consider the fixed subfield

$$
K(x_1, \ldots, x_n)^{S_n} = \{ \alpha \in K(x_1, \ldots, x_n) : \sigma(\alpha) = \alpha \text{ for any } \sigma \in S_n \}.
$$

If $n = 2$, then $K(x_1, x_2)^{S_2} = K(x_1 + (a/x_2), ax_1/x_2)$ is rational (that is, purely transcendental) over $K$. When $a = 1$ (equivalently when $a \in K^\times$), we have the following theorem.

**Theorem 1.1** [Hajja and Kang 1997, Theorem 3.5]. Let $K$ be any field and let $a \in K^\times$. Then $K(x_1, \ldots, x_n)^{S_n}$ is rational over $K$.

The case when $a \in K^\times \setminus K^\times^2$ and $n \geq 3$ had been intractable for many years; see [Hajja and Kang 1997, page 638; Hajja 2000, Example 5.12, page 147; Kang 2001, Question 3.8, page 215]. Even the case $n = 3$ was unsolved. The next theorem is our recent result for the cases $n = 3, 4, 5$.

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Theorem 1.2. Let $K$ be any field, let $a \in K \setminus \{0\}$, and let $S_n$ act on $K(x_1, \ldots, x_n)$ as defined in (1-1). If $n = 3, 4, 5$, then $K(x_1, \ldots, x_n)^{S_n}$ is rational over $K$.

We will prove Theorem 1.2 in Section 2. It is interesting that we use three different methods for the three cases of $n$; it seems that there is no unified proof for the three cases. One of the reasons is that the solutions to Noether’s problem for the alternating group $A_n$ are rather different when $n = 3$ and when $n = 5$; see Theorem 2.2 and Theorem 2.5. Since Noether’s problem for $A_n$ is still open in the case $n \geq 6$ (see [Maeda 1989] and [Hajja and Kang 1995, Section 4] for the statement of this problem), it is not so surprising that our question is solvable at present only for $n \leq 5$. It is still unknown whether the fixed field $K(x_1, \ldots, x_n)^{S_n}$ is rational when $n \geq 6$.

In Section 3 we propose another approach to the rationality of $K(x_1, \ldots, x_n)^{S_n}$. We show in Theorem 3.4 that it is isomorphic to the function field of a conic bundle over $\mathbb{P}^{n-1}$ of the form $x^2 - ay^2 = h(v_1, \ldots, v_{n-1})$ with affine coordinates $v_1, \ldots, v_{n-1}$. Although this approach is valid only when $\text{char } K \neq 2$, it does provide a new technique in studying rationality problems. The structure of a conic bundle together with its rationality problem is a central subject in algebraic geometry [Iskovskih 1991]. Fortunately, when $n = 3$ and $n = 4$, the conic bundle in our case contains singularities and the rationality problem can be solved by a suitable blowing-up process. In particular, we find another proof of Theorem 1.2 when $\text{char } K \neq 2$ and $n = 3, 4$. For other rationality problems of conic bundles, see [Kang 2007, Section 4].

Since the fixed field $K(x_1, \ldots, x_n)^{S_n}$ is the quotient field of the ring of invariants $K[x_1, \ldots, x_n]^{S_n}$, it seems plausible to study it through the structure of the latter. This strategy is carried out in Section 4, and we give another proof of Theorem 1.2 when $\text{char } K = 2$ and $n = 3, 4$.

2. Proof of Theorem 1.2

Theorem 2.1 [Kang 2004, Theorem 2.4]. Let $K$ be any field and let $K(x, y)$ be the rational function field of two variables over $K$. Let $\sigma$ be a $K$-automorphism on $K(x, y)$ defined by

$$\sigma : x \mapsto a/x, \quad y \mapsto b/y,$$

where $a \in K \setminus \{0\}$ and $b = c(x + (a/x)) + d$ such that $c, d \in K$ and at least one of $c$ and $d$ is nonzero. Then $K(x, y)^{\langle \sigma \rangle} = K(s, t)$, where

$$s = \frac{x - (a/x)}{xy - (ab/xy)}, \quad t = \frac{y - (b/y)}{xy - (ab/xy)}.$$

The next result is essentially due to Masuda [1955, page 62] when $\text{char } K \neq 3$ (with a misprint in the original expression). We thank Y. Rikuna who pointed out
that the same formula is still valid when \( \text{char } K = 3 \) if we compare this formula with the proof in [Kuniyoshi 1955]. For convenience, we provide a new proof.

**Theorem 2.2** [Masuda 1955, Theorem 3]. Let \( K \) be any field, \( K(x_1, x_2, x_3) \) be the rational function field of three variables over \( K \). Let \( \sigma \) be a \( K \)-automorphism on \( K(x_1, x_2, x_3) \) defined by

\[
\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1.
\]

Then \( K(x_1, x_2, x_3)^{(\sigma)} = K(s_1, u, v) = K(s_3, u, v) \), where \( s_i \) is the elementary symmetric function of degree \( i \) for \( 1 \leq i \leq 3 \), and \( u \) and \( v \) are defined by

\[
u := \frac{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1},
\]

\[
u := \frac{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1}.
\]

Moreover, we have the identities

\[
s_2 = s_1(u + v) - 3(u^2 - uv + v^2),
\]

\[
s_3 = s_1uv - (u^3 + v^3),
\]

\[
x_1^2x_2 + x_2^2x_3 + x_3^2x_1 = s_1^2u - 3s_1u^2 + 3(2u - v)(u^2 - uv + v^2),
\]

\[
x_1^2x_2 + x_2^2x_3 + x_3^2x_1 = s_1^2v - 3s_1v^2 - 3(u - 2v)(u^2 - uv + v^2).
\]

**Proof.** With the aid of computer packages, say Mathematica or Maple, it is easy to verify the theorem’s identities. We have \([K(x_1, x_2, x_3) : K(s_1, s_2, s_3)] = 6 \) and \([K(x_1, x_2, x_3)^{(\sigma)} : K(s_1, s_2, s_3)] = 2 \). Since \( x_1x_2^2 + x_2x_3^2 + x_3x_1^2 \not\in K(s_1, s_2, s_3) \), it follows that \( K(x_1, x_2, x_3)^{(\sigma)} = K(s_1, s_2, s_3) \subset K(s_1, u, v) \). Hence \( K(x_1, x_2, x_3)^{(\sigma)} = K(s_1, u, v) = K(s_3, u, v) \). \( \square \)

**Proof of Theorem 2.2 when \( n = 3 \)**. Let \( \sigma = (1, 2, 3) \), \( \tau = (1, 2) \in S_3 \).

By Theorem 2.2, we find that \( K(x_1, x_2, x_3)^{(\sigma)} = K(s_3, u, v) \).

Now \( \tau(x_1) = a/x_2 \), \( \tau(x_2) = a/x_3 \), and \( \tau(x_3) = a/x_1 \). Note that

\[
\tau(s_1) = as_2/s_3, \quad \tau(s_2) = a^2s_1/s_3, \quad \tau(s_3) = a^3/s_3,
\]

\[
\tau(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) = a^3(x_1x_2^2 + x_2x_3^2 + x_3x_1^2)/s_3^2,
\]

\[
\tau(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) = a^3(x_1^2x_2 + x_2^2x_3 + x_3^2x_1)/s_3^2.
\]

With the aid of Theorem 2.2, it is not difficult to find that

\[
(2-1) \quad \tau : s_3 \mapsto \frac{a^3}{s_3}, \quad u \mapsto \frac{au}{u^2-uv+v^2}, \quad v \mapsto \frac{av}{u^2-uv+v^2}.
\]
Define $w := u/v$. Then $K(s_3, u, v) = K(s_3, v, w)$ and
\[
\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad v \mapsto \frac{a}{v(1-w+w^2)}, \quad w \mapsto w.
\]

By Theorem 2.1, $K(s_3, v, w)^{(\tau)}$ is rational over $K(w)$. Hence $K(x_1, x_2, x_3)^{S_3} = K(s_3, v, w)^{(\tau)}$ is rational over $K$. □

**Proof of Theorem 1.2 when $n = 4$.** Define
\[
\sigma := (123) : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1,
\tau := (12) : x_1 \mapsto a/x_2, \quad x_2 \mapsto a/x_1, \quad x_3 \mapsto a/x_3, \quad x_4 \mapsto a/x_4,
\rho_1 := (12)(34) : x_1 \mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto x_4, \quad x_4 \mapsto x_3,
\rho_2 := (13)(24) : x_1 \mapsto x_3, \quad x_3 \mapsto x_1, \quad x_2 \mapsto x_4, \quad x_4 \mapsto x_2.
\]

Note that $\{1\} \triangleleft V_4 = \langle \rho_1, \rho_2 \rangle \triangleleft A_4 = \langle \sigma, \rho_1, \rho_2 \rangle \triangleleft S_4 = \langle \sigma, \tau, \rho_1, \rho_2 \rangle$ is a normal series.

First we will show that $K(x_1, \ldots, x_4)^{V_4}$ is rational over $K$. Define
\[
s_1 := x_1 + x_2 + x_3 + x_4, \quad s_4 := x_1x_2x_3x_4,
S := \frac{x_1 + x_2 - x_3 - x_4}{x_1x_2 - x_3x_4}, \quad T := \frac{x_1 - x_2 - x_3 + x_4}{x_1x_4 - x_2x_3}, \quad U := \frac{x_1 - x_2 + x_3 - x_4}{x_1x_3 - x_2x_4}.
\]
Then we have $K(s_1, s_4, S, T, U) \subset K(x_1, x_2, x_3, x_4)^{V_4}$ and
\[
(2-2) \quad \sigma : s_1 \mapsto s_1, \quad s_4 \mapsto s_4, \quad S \mapsto T, \quad T \mapsto U, \quad U \mapsto S.
\]

**Lemma 2.3.**

(i) $K(x_1, x_2, x_3, x_4)^{V_4} = K(s_1, S, T, U) = K(s_4, S, T, U)$.

(ii) $K(x_1, x_2, x_3, x_4)^{A_4} = K(s_4, f, g, h)$ where $f, g, h$ are defined by
\[
f = S + T + U, \quad g = \frac{ST^2 + TU^2 + US^2 - 3STU}{S^2 + T^2 + U^2 - ST - TU - US},
\]
\[
h = \frac{S^2 T^2 + T^2 U^2 + U^2 S - 3STU}{S^2 + T^2 + U^2 - ST - TU - US}.
\]

**Proof.** Define $u_1 := S + T + U, \ u_2 := ST + TU + SU$ and $u_3 := STU$. Then it can be checked that $K(x_1, x_2, x_3, x_4) = K(s_1, S, T, U)(x_4)$ directly from the equalities
\[
x_1 = \frac{4 - s_1 T + (-2u_1 + s_1 T(S + U))x_4 + SU(1 - s_1 T)x_4^2 + u_3x_4^3}{S - T + U - SUx_4},
\]
\[
x_2 = \frac{4 - s_1 U + (-2u_1 + s_1 U(T + S))x_4 + TS(1 - s_1 U)x_4^2 + u_3x_4^3}{T - U + S - TSx_4},
\]
\[
x_3 = \frac{4 - s_1 S + (-2u_1 + s_1 S(U + T))x_4 + UT(1 - s_1 S)x_4^2 + u_3x_4^3}{U - S + T - UTx_4}.
\]
We see that \([K(s_1, S, T, U)(x_4) : K(s_1, S, T, U)] \leq 4\) by the equality
\[u_1^2 - 4u_2 + s_1u_3 + (8 - s_1u_1)u_3x_4 - (2u_1 - s_1u_2)u_3x_4^2 - s_1u_3^2x_4^3 + u_3^2x_4 = 0.\]
Hence we get \(K(x_1, x_2, x_3, x_4)^{V_4} = K(s_1, S, T, U)\). It follows from the equality
\[s_4 = (u_1^2 - 4u_2 + u_3s_1)/u_3^2\] that \(K(s_1, S, T, U) = K(s_4, S, T, U)\).

As for the field \(K(x_1, x_2, x_3, x_4)^{A_4}\), apply Theorem 2.2 to \(K(s_4, S, T, U)^{(\sigma)} = K(S, T, U)^{(\sigma)}(s_4)\).

We have \(K(x_1, x_2, x_3, x_4)^{S_4} = (K(x_1, x_2, x_3, x_4)^{V_4})^{S_4/V_4} = K(s_4, S, T, U)^{(\sigma, \tau)}\).

The action of \(\langle \sigma, \tau \rangle\) on \(K(s_4, S, T, U)\) is given by
\[
\begin{align*}
\sigma &: s_4 \mapsto s_4, \quad S \mapsto T, \quad T \mapsto U, \quad U \mapsto S, \\
\tau &: s_4 \mapsto \frac{a^4}{s_4}, \quad S \mapsto \frac{-S+T+U}{aTU}, \quad T \mapsto \frac{S+T-U}{aST}, \quad U \mapsto \frac{S-T+U}{aSU}.
\end{align*}
\]
Define
\[
N := \begin{cases} 
\frac{s_4 + a^2}{s_4 - a^2} & \text{if } \text{char } K \neq 2, \\
\frac{s_4}{s_4 + a^2} & \text{if } \text{char } K = 2.
\end{cases}
\]
Then we get \(K(s_4, S, T, U) = K(N, S, T, U), \quad \sigma(N) = N\) and
\[
\tau(N) = \begin{cases} 
-N & \text{if } \text{char } K \neq 2, \\
N + 1 & \text{if } \text{char } K = 2.
\end{cases}
\]
Applying [Hajja and Kang 1995, Theorem 1], we find that \(K(x_1, x_2, x_3, x_4)^{S_4} = K(N, S, T, U)^{(\sigma, \tau)}\) is rational over \(K\), provided that \(K(S, T, U)^{(\sigma, \tau)}\) is rational over \(K\). Explicitly, define \(P\) by
\[
P := \begin{cases} 
N \cdot \left( S + T + U + \frac{S^2 + T^2 + U^2 - 2(ST + TU + US)}{aSTU} \right) & \text{if } \text{char } K \neq 2, \\
N + \frac{S+T+U}{S+T+U+aSTU} & \text{if } \text{char } K = 2.
\end{cases}
\]
Then we have that \(K(N, S, T, U) = K(P, S, T, U)\) and \(K(x_1, x_2, x_3, x_4)^{S_4} = K(P, S, T, U)^{(\sigma, \tau)} = K(S, T, U)^{(\sigma, \tau)}(P)\), where \(\sigma(P) = \tau(P) = P\).

Thus it remains to prove this:

**Theorem 2.4.** Let \(K\) be any field and let \(K(S, T, U)\) be the rational function field of three variables \(S, T\) and \(U\) over \(K\). Let \(\sigma\) and \(\tau\) be \(K\)-automorphisms of \(K(S, T, U)\) defined by
\[
\begin{align*}
\sigma &: S \mapsto T, \quad T \mapsto U, \quad U \mapsto S, \\
\tau &: S \mapsto \frac{-S+T+U}{aTU}, \quad T \mapsto \frac{S+T-U}{aST}, \quad U \mapsto \frac{S-T+U}{aSU},
\end{align*}
\] where \(a \in K \setminus \{0\}\). Then \(\langle \sigma, \tau \rangle \cong S_3\) and \(K(S, T, U)^{(\sigma, \tau)}\) is rational over \(K\).
Proof. By Theorem 2.2, we may choose a transcendence basis of \( K(S, T, U)^{(\sigma)} \) over \( K \) by \( K(S, T, U)^{(\sigma)} = K(f, g, h) \), where

\[
    f = S + T + U, \quad g = \frac{ST^2 + TU^2 + US^2 - 3STU}{S^2 + T^2 + U^2 - ST - TU - US},
\]

\[
    h = \frac{ST + T^2U + US^2 - 3STU}{S^2 + T^2 + U^2 - ST - TU - US}.
\]

Thus we have \( K(S, T, U)^{(\sigma, \tau)} = (K(S, T, U)^{(\sigma)})^{(\tau)} = K(f, g, h)^{(\tau)} \). The action of \( \tau \) on \( K(f, g, h) \) is given by

\[
    f \mapsto \frac{f^2 - 4f(g + h) + 12X}{aY},
\]

\[
    g \mapsto \frac{-f^2h(f - 4h) + 2f(f - 2g - 8h)X + 24X^2 - 8gY}{a(f^2 - 2f(g + h) + 4X)Y},
\]

\[
    h \mapsto \frac{-f^2(fg + 4h^2) + 6f(f - 2g)X + 24X^2 - 4(f + 2h)Y}{a(f^2 - 2f(g + h) + 4X)Y},
\]

where \( X = g^2 - gh + h^2 \) and \( Y = g^3 - fgh + h^3 \).

Case 1: \( \text{char} \ K \neq 2 \).

Define

\[ F := g + h, \quad G := g - h, \quad H := f - (g + h). \]

Then \( K(S, T, U)^{(\sigma)} = K(f, g, h) = K(F, G, H) \) and \( \tau \) acts on \( K(F, G, H) \) by

\[
    F \mapsto \frac{4(27G^4 - 7FG^2H + 5G^2H^2 - FH^3)}{a(4FG^2 - F^2H + G^2H)(3G^2 + H^2)},
\]

\[
    G \mapsto \frac{4G(FG^2 + 7G^2H - FH^2 + H^3)}{a(4FG^2 - F^2H + G^2H)(3G^2 + H^2)},
\]

\[
    H \mapsto \frac{4H(FG^2 + 7G^2H - FH^2 + H^3)}{a(4FG^2 - F^2H + G^2H)(3G^2 + H^2)}.
\]

Note that \( \tau(G/H) = G/H \). Define

\[ A := F/G, \quad B := G, \quad C := G/H. \]

Then \( K(S, T, U)^{(\sigma)} = K(F, G, H) = K(A, B, C) \) and \( \tau \) acts on \( K(A, B, C) \) by

\[
    A \mapsto \frac{-A + 5C - 7AC^2 + 27C^3}{1 - AC + 7C^2 + AC^3},
\]

\[
    B \mapsto \frac{4(1 - AC + 7C^2 + AC^3)}{aB(1 - A^2 + 4AC)(1 + 3C^2)}, \quad C \mapsto C.
\]

Define

\[ D := 1 - AC + 7C^2 + AC^3, \quad E := 2C(C^2 - 1)/B. \]
Then $K(A, B, C) = K(C, D, E)$ and the action of $\tau$ on $K(C, D, E)$ is given by
\[ C \mapsto C, \quad D \mapsto (1 + 3C^2)^3/D, \]
\[ E \mapsto -a(1 + 3C^2)(D + (1 + 3C^2)^3/D - 2(1 + 5C^2 + 2C^4))/E. \]
Hence the assertion follows from Theorem 2.1.

**Case 2:** $\text{char } K = 2$.

The action of $\tau$ on $K(f, g, h)$ is given by
\[ \tau: f \mapsto \frac{f^2}{aY}, \quad g \mapsto \frac{fh}{aY}, \quad h \mapsto \frac{fg}{aY}, \]
where $Y = g^3 + fgh + h^3$. Define
\[ A := f/(g + h), \quad B := g/h, \quad C := 1/h. \]
Then $K(f, g, h) = K(A, B, C)$ and $\tau$ acts on $K(A, B, C)$ by
\[ A \mapsto A, \quad B \mapsto \frac{1}{B}, \quad C \mapsto \frac{a}{A}(B + \frac{1}{B} + A + 1)/C. \]
Hence the assertion follows from Theorem 2.1. We will give another proof when $n = 4$ and $\text{char } K = 2$ in Section 4.

This concludes the proof of Theorem 1.2 when $n = 4$. \qed

**Proof of Theorem 1.2 when $n = 5$.**

We recall Maeda’s theorem for the $A_5$ action.

**Theorem 2.5** [Maeda 1989]. Let $K$ be any field, $K(x_1, \ldots, x_5)$ be the rational function field of five variables over $K$. Then $K(x_1, \ldots, x_5)^{A_5}$ is rational over $K$. Moreover a transcendental basis $F_1, \ldots, F_5$ of $K(x_1, \ldots, x_5)^{A_5}$ over $K$ may be given explicitly as follows:

(i) When $\text{char } K \neq 2$,
\[
F_1 = \frac{\sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4x_1)}{\sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4)}, \quad F_2 = \frac{\sum_{\sigma \in S_5} \sigma([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10})}{\prod_{i < j} [ij]^2 \cdot \sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4)}, \quad F_3 = \frac{\sum_{\sigma \in S_5} \sigma([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10}x_1)}{\prod_{i < j} [ij]^2 \cdot \sum_{\sigma \in S_5} \sigma([12][13][14][15][23]^4[45]^4)}, \quad F_4 = \frac{\sum_{\mu \in R_1} \mu([12]^2[13]^2[23]^2[45]^4)}{\prod_{i < j} [ij]}, \quad F_5 = \frac{\sum_{\mu \in R_1} \mu([12]^2[13]^2[23]^2[45]^4[24]^4[34]^4[15]^4[25]^4[35]^4)}{\prod_{i < j} [ij]^3},
\]
where \([ij] = x_i - x_j\) and \(R_1 = \{1, (34), (354), (234), (2354), (24)(35), (1234), (12354), (124)(35), (13524)\}.

(ii) When \(\text{char } K = 2\),
\[
F_1 = \frac{\sum_{i<j<k} x_i x_j x_k}{\sum_{i<j} x_i x_j}, \quad F_4 = \frac{\sum_{v \in R_4} v([12][4][13][2][24][15][25][35][45])}{\prod_{i<j}[ij]},
\]
\[
F_2 = \frac{\sum_{i=1}^{5} ([12][13][14][15] \cdot I^2)^{(1i)}}{\prod_{i<j}[ij] \cdot \sum_{i<j} x_i x_j}, \quad F_5 = \text{the same } F_5 \text{ as in (i)},
\]
\[
F_3 = \frac{\sum_{i=1}^{5} ([12][13][14][15] \cdot I^2 \cdot x_1)^{(1i)}}{\prod_{i<j}[ij] \cdot \sum_{i<j} x_i x_j},
\]
\[
\text{where } [ij] = x_i - x_j, \quad I = \sum_{\tau \in R_2} \tau(x_2 x_3 (x_2 x_3 + x_1^2 + x_5^2)), \quad R_2 = \{1, (34), (354), (234), (2354), (24)(35)\} \text{ and } R_3 = \{1, (234), (243), (152), (15234), (15243), (125), (12345), (12435), (15432), (154), (15423), (15342), (15324), (153)\}.
\]

In the theorem, note that \(R_1, R_2\) and \(R_3\) are coset representatives with respect to various subgroups:
\[
S_5 = \bigcup_{\mu \in R_1} H_1 \mu, \quad H = \bigcup_{\tau \in R_2} H_2 \tau, \quad A_5 = \bigcup_{v \in R_3} H_3 v,
\]
where
\[
H = \langle (23), (24), (25) \rangle \cong S_4, \quad H_1 = \langle (12), (13), (45) \rangle \cong D_6,
\]
\[
H_2 = \langle (23), (45) \rangle \cong V_4, \quad H_3 = \langle (12)(34), (13)(24) \rangle \cong V_4,
\]
and \(D_6\) is the dihedral group of order 12.

Now we start to prove Theorem 1.2 when \(n = 5\). Let \(\tau = (12) \in S_5\). By Theorem 2.5, we see that \(K(x_1, \ldots, x_5)^{A_5} = K(F_1, \ldots, F_5)\).

With the aid of a computer, we can evaluate the action of \(\tau\) on \(K(F_1, \ldots, F_5)\) as follows:
\[
\tau : F_1 \mapsto a/F_1, \quad F_2 \mapsto F_3/F_1, \quad F_3 \mapsto aF_2/F_1, \quad F_4 \mapsto -F_4, \quad F_5 \mapsto -F_5 \quad \text{when char } K \neq 2;
\]
\[
\tau : F_1 \mapsto a/F_1, \quad F_2 \mapsto F_3/F_1, \quad F_3 \mapsto aF_2/F_1, \quad F_4 \mapsto F_4 + 1, \quad F_5 \mapsto F_5 \quad \text{when char } K = 2.
\]

Case 1: char \(K \neq 2\).

Define
\[
G_1 := F_1, \quad G_2 := F_4 + 1/F_4 - 1, \quad G_3 := F_4(F_2 - F_3/F_1),
\]
\[
G_4 := F_2 + F_3/F_1, \quad G_5 := F_4 F_5.
\]
Then we have $K(x_1, \ldots, x_5)^{A_5} = K(F_1, \ldots, F_5) = K(G_1, \ldots, G_5)$ and

$$\tau : G_1 \mapsto a/G_1, \quad G_2 \mapsto 1/G_2, \quad G_3 \mapsto G_3, \quad G_4 \mapsto G_4, \quad G_5 \mapsto G_5.$$ 

So it follows from Theorem 2.1 that $K(x_1, \ldots, x_5)^{S_5} = K(G_3, G_4, G_5)(G_1, G_2)^{(\tau)}$ is rational over $K$.

**Case 2:** $\text{char } K = 2$.

Define

$$G_1 := F_1, \quad G_2 := F_2, \quad G_3 := \frac{F_2F_3}{F_1}, \quad G_4 := F_4 + \frac{F_3}{F_1F_2 + F_3}, \quad G_5 := F_5.$$ 

Then we have $K(x_1, \ldots, x_5)^{A_5} = K(F_1, \ldots, F_5) = K(G_1, \ldots, G_5)$ and

$$\tau : G_1 \mapsto a/G_1, \quad G_2 \mapsto G_3/G_2, \quad G_3 \mapsto G_3, \quad G_4 \mapsto G_4, \quad G_5 \mapsto G_5.$$ 

We use Theorem 2.1 and find that $K(x_1, \ldots, x_5)^{S_5} = K(G_3, G_4, G_5)(G_1, G_2)^{(\tau)}$ is rational over $K$. \hfill \square

## 3. Conic bundles: Another approach when $\text{char } K \neq 2$

Throughout this section we assume that $\text{char } K \neq 2$.

In this section, we will give another proof of Theorem 1.2 when $n = 3, 4$ (and $\text{char } K \neq 2$) by presenting $K(x_1, \ldots, x_n)^{S_n}$ as the function field of a conic bundle over $\mathbb{P}^{n-1}$.

Consider the action of $S_n$ on $K(x_1, \ldots, x_n)$ defined by Equation (1-1). Because of Theorem 1.1, we may assume that $a \in K^\times \setminus K^\times 2$ without loss of generality.

Define $\alpha := \sqrt{a}$ and $\text{Gal}(K(\alpha)/K) = \langle \rho \rangle$, where $\rho(\alpha) = -\alpha$. Extend the actions of $S_n$ and $\rho$ to $K(\alpha)(x_1, \ldots, x_n) = K(\alpha) \otimes_K K(x_1, \ldots, x_n)$ by requiring that $S_n$ acts trivially on $K(\alpha)$ and $\tau$ acts trivially on $K(x_1, \ldots, x_n)$.

Define $z_i := (\alpha - x_i)/\alpha + x_i$ for $1 \leq i \leq n$. We find that $K(\alpha)(x_1, \ldots, x_n) = K(\alpha)(x_1, \ldots, z_n)$ and

$$\sigma : z_i \mapsto -z_{\sigma(i)}$$

for any $\sigma \in S_n \setminus A_n$, and

$$\rho : \alpha \mapsto -\alpha, \quad z_i \mapsto 1/z_i.$$

Define $z_0 := z_1 + \cdots + z_n, y_i := z_i/z_0$ for $1 \leq i \leq n$. Hence $y_1 + \cdots + y_n = 1$.

Let $t_1, \ldots, t_n$ be the elementary symmetric functions of $y_1, \ldots, y_n$. In particular, $t_1 = 1$. Define $\Delta := \prod_{1 \leq i < j \leq n}(y_i - y_j) \in K(y_1, \ldots, y_n)$ and $u := z_0 \cdot \Delta$. Note that $\Delta^2$ can be written as a polynomial in $t_1, \ldots, t_n$, and thus in $t_2, \ldots, t_n$.

**Lemma 3.1.** $K(x_1, \ldots, x_n)^{S_n} = K(\alpha)(t_2, \ldots, t_n, u)^{(\rho)}$ and

$$\rho : \alpha \mapsto -\alpha, \quad t_i \mapsto t_{n-i}(t_n/t_{n-1})^iy_{n-1}^{-1}, \quad u \mapsto f(t_2, \ldots, t_n) \cdot u^{-1},$$
where \( f(t_2, \ldots, t_n) \in K(t_2, \ldots, t_n) \) is given by

\[
(3-1) \quad f(t_2, \ldots, t_n) := (-1)^{n(n-1)/2} t_n^{-(n-1)}(t_n/t_{n-1})^{(n+1)(n-2)/2} \Delta^2
\]

and we adopt the convention that \( t_0 = 1 \).

**Proof.** Note that \( K(\alpha)(y_1, \ldots, y_n, z_0) = K(\alpha)(y_1, \ldots, y_n, u) \). Since \( u \) is fixed by the action of \( S_n \), it follows that \( K(\alpha)(y_1, \ldots, y_n, z_0)^{S_n} = K(\alpha)(y_1, \ldots, y_n)^{S_n}(u) = K(\alpha)(t_2, \ldots, t_n, u) \); the last equality follows, for example, from the proof of [Hajja and Kang 1995, Lemma 1] because \( \sigma(y_i) = y_{\sigma(i)} \) for any \( \sigma \in S_n \) and \( i \) in \( 1 \leq i \leq n \).

Thus \( K(x_1, \ldots, x_n)^{S_n} = (K(\alpha)(\rho)(x_1, \ldots, x_n))^{S_n} = K(\alpha)(x_1, \ldots, x_n)^{(S_n, \rho)} = (K(\alpha)(x_1, \ldots, x_n)^{(\rho)} = K(\alpha)(t_2, \ldots, t_n, u)^{(\rho)}.

It is easy to verify that the action of \( \rho \) on \( K(\alpha)(t_2, \ldots, t_n, u) \) is as stated. □

We write \( n = 2m + 1 \) if \( n \) is odd, and \( n = 2m \) otherwise. Define

\[
(3-2) \quad u_i := t_{i+1}, \quad u_{n-i} := \rho(t_{i+1}) = t_{n-(i+1)}t_n^{i+1}/t_{n-1} \quad \text{for} \quad i = 1, \ldots, m - 1
\]

and

\[
(3-3) \quad \begin{cases} u_m := t_{m+1}, & u_{m+1} := \rho(t_{m+1}) = t_m t_n^{m+1}/t_{n-1} & \text{if} \quad n \text{ is odd}, \\
 u_m := t_n/t_{n-1}, & \end{cases} \quad \text{if} \quad n \text{ is even.}
\]

**Lemma 3.2.** \( K(x_1, \ldots, x_n)^{S_n} = K(\alpha)(u_1, \ldots, u_{n-1}, u)^{(\rho)} \) and

\[
\rho : \alpha \mapsto -\alpha, \quad u_1 \mapsto u_{n-1} \quad \text{for} \quad i = 1, \ldots, n - 1,
\]

\[
u \mapsto g(u_1, \ldots, u_{n-1}) \cdot u^{-1},
\]

where \( g(u_1, \ldots, u_{n-1}) = f(t_2, \ldots, t_n) \) and \( f(t_2, \ldots, t_n) \) is given as in \( (3-1) \).

**Proof.** The assertion follows from \( K(\alpha)(t_2, \ldots, t_n, u) = K(\alpha)(u_1, \ldots, u_{n-1}, u) \) and Lemma 3.1. Indeed we may show \( K(t_2, \ldots, t_n) \subset K(u_1, \ldots, u_{n-1}) \) as follows.

**Case 1:** \( n = 2m + 1 \) is odd.

The fact that \( t_2, \ldots, t_{m+1} \in K(u_1, \ldots, u_{n-1}) \) follows from \( (3-2) \) and \( (3-3) \). We have \( t_n \in K(u_1, \ldots, u_{n-1}) \) because

\[
(u_{m+1}^{m+1}/u_m^{m+2})^{m+1} = (\frac{1}{u_{m+2}})^{m+1}(\frac{t_{m+1}^{m+1}t_n}{t_{m-1}^{m+1}})^m \cdot (\frac{t_{n-1}}{t_{m+1}^{m+1}t_{n-1}})^{m+1} = t_n.
\]

and \( t_{n-1} \in K(u_1, \ldots, u_{n-1}) \) because

\[
t_n (u_{n-1}^{m+1}/u_{m+2}) u_{m+2} \cdot (\frac{1}{u_{m+1}}) = t_n \cdot (\frac{t_{m+1}^{m+1}t_n}{t_{m-1}^{m+1}}) \cdot (\frac{t_{n-1}^{m+1}}{t_{m+1}^{m+1}t_{n-1}}) = t_{n-1}.
\]

From \( (3-2) \) we find that \( t_{n-(i+1)} = u_{n-i}^{i+1}/t_n^i \) for \( 1 \leq i \leq m - 2 \). Thus \( t_{m+2}, \ldots, t_{n-2} \in K(u_1, \ldots, u_{n-1}) \).

**Case 2:** \( n = 2m \) is even. That \( t_2, \ldots, t_m \in K(u_1, \ldots, u_{n-1}) \) follows from \( (3-2) \).
From (3-2) and (3-3), we get
\[
\frac{u_{k+1}}{u_{k+2}} = \frac{t_k}{t_{k+1}} \cdot \frac{t_n}{t_{n-1}} = \frac{t_k}{t_{k+1}} \cdot u_m,
\]
where \(k = m, \ldots, 2m - 3\). We find that \(t_{k+1} = t_k u_mu_{k+2}/u_{k+1} \in K(u_1, \ldots, u_{n-1})\) for \(m \leq k \leq 2m - 3\). From (3-2), we have \(u_{n-1} = t_{n-2}t_n/t_{n-1}^2 = t_{n-2}u_m/t_{n-1}\). Hence \(t_{n-1} = t_{n-2}u_m/u_{n-1} \in K(u_1, \ldots, u_{n-1})\).

Since \(t_n = u_m t_{n-1}\), it follows that \(t_n \in K(u_1, \ldots, u_{n-1})\). \(\square\)

We will change the variables \(u_1, \ldots, u_{n-1}\) to \(v_1, \ldots, v_{n-1}\) as follows. When \(n = 2m + 1\) is odd, define
\[
v_i := \frac{1}{2}(u_i + u_{n-i}), \quad v_{n-i} := \frac{1}{2}(\alpha(u_i - u_{n-i})) \quad \text{for } i = 1, \ldots, m.
\]
When \(n = 2m\) is even, define
\[
v_m := u_m, \quad v_i := \frac{1}{2}(u_i + u_{n-i}), \quad v_{n-i} := \frac{1}{2}(\alpha(u_i - u_{n-i})) \quad \text{for } i = 1, \ldots, m - 1.
\]
Thus \(K(\alpha)(u_1, \ldots, u_{n-1}, u) = K(\alpha)(v_1, \ldots, v_{n-1}, u)\).

In these variables, Lemma 3.2 reads as follows:

**Lemma 3.3.** \(K(x_1, \ldots, x_n)^{S_n} = K(\alpha)(v_1, \ldots, v_{n-1}, u)^{(\rho)}\) and
\[
\rho : \alpha \mapsto -\alpha, \quad v_i \mapsto v_i \quad \text{for } i = 1, \ldots, n-1, \quad u \mapsto h(v_1, \ldots, v_{n-1}) \cdot u^{-1},
\]
where \(h(v_1, \ldots, v_{n-1}) = f(t_2, \ldots, t_n)\) and \(f(t_2, \ldots, t_n)\) is given as in (3-1).

Hence we get the following theorem, which asserts that \(K(x_1, \ldots, x_n)^{S_n}\) is the function field of a conic bundle over \(\mathbb{P}^{n-1}\) of the form \(x^2 - ay^2 = h(v_1, \ldots, v_{n-1})\) with affine coordinates \(v_1, \ldots, v_{n-1}\); see for example [Shafarevich 1974, page 73] for conic bundles over \(\mathbb{P}^1\).

**Theorem 3.4.** \(K(x_1, \ldots, x_n)^{S_n} = K(x, y, v_1, \ldots, v_{n-1})\) and the generators \(x, y, v_1, \ldots, v_{n-1}\) satisfy the relation
\[
x^2 - ay^2 = h(v_1, \ldots, v_{n-1}),
\]
where \(h(v_1, \ldots, v_{n-1}) = f(t_2, \ldots, t_n)\) and \(f(t_2, \ldots, t_n)\) is given as in (3-1).

**Proof.** Define
\[
x := \frac{1}{2}(u + \frac{h(v_1, \ldots, v_{n-1})}{u}), \quad y := \frac{1}{2\alpha}(u - \frac{h(v_1, \ldots, v_{n-1})}{u}).
\]
Then we get \(K(x, y, v_1, \ldots, v_{n-1}) \subset K(x_1, \ldots, x_n)^{S_n} = K(\alpha)(v_1, \ldots, v_{n-1}, u)\). Thus \(K(x, y, v_1, \ldots, v_{n-1}) = K(x_1, \ldots, x_n)^{S_n}\), since \(K(x, y, v_1, \ldots, v_{n})(u) = K(\alpha)(v_1, \ldots, v_{n-1}, u)\) and \([K(x, y, v_1, \ldots, v_{n})(u) : K(x, y, v_1, \ldots, v_{n})] = 2\). We also have \(x^2 - ay^2 = h(v_1, \ldots, v_{n-1})\) by definition. \(\square\)
Proof of Theorem 1.2 when \( n = 3 \) and \( \text{char } K \neq 2 \).

Step 1. By Lemma 3.1 we find that \( K(x_1, x_2, x_3)^{S_3} = K(\alpha)(t_2, t_3, u)^{\langle \rho \rangle} \), where

\[
\rho : \alpha \mapsto -\alpha, \quad t_2 \mapsto t_2^{-2}t_3, \quad t_3 \mapsto t_2^{-3}t_3^2, \quad u \mapsto -t_2^{-2}u^2 \cdot u^{-1}.
\]

Note that \( \Delta^2 = \prod_{1 \leq i < j \leq 3} (y_i - y_j)^2 = t_2^2 - 4t_2^3 - 4t_3 + 18t_2t_3 - 27t_3^2 \) because \( t_1 = 1 \).

Define \( u_1 := t_2, \ u_2 := \rho(t_2) = t_2^{-2}t_3 \). Then \( K(\alpha)(t_2, t_3, u) = K(\alpha)(u_1, u_2, u) \) and

\[
\rho : u_1 \mapsto u_2 \mapsto u_1, \quad u \mapsto g(u_1, u_2) \cdot u^{-1},
\]

where \( g(u_1, u_2) = -1 + 4u_1 + 4u_2 - 18u_1u_2 + 27u_1^2u_2^2 \).

Define \( v_1 := (u_1 + u_2)/2 \) and \( v_2 := \alpha(u_1 - u_2)/2 \). Then \( \rho : v_1 \mapsto v_1, v_2 \mapsto v_2 \) and \( g(u_1, u_2) = h(v_1, v_2) \), where

\[
h(v_1, v_2) = -1 + 8v_1 - 18v_1^2 + 27v_1^4 + (18/a)v_2^2 - (54/a)v_1^2v_2^2 + (27/a^2)v_2^4.
\]

Hence \( K(x_1, x_2, x_3)^{S_3} = K(\alpha)(v_1, v_2, u)^{\langle \rho \rangle} = K(x, y, v_1, v_2) \), where

\[
x = \frac{1}{2}(u + \frac{h(v_1, v_2)}{u}), \quad y = \frac{1}{2\alpha}(u - \frac{h(v_1, v_2)}{u}).
\]

Note that \( x \) and \( y \) satisfy the relation

\[
x^2 - ay^2 = h(v_1, v_2) \tag{3-4}
\]

\[
= (1 + v_1)(-1 + 3v_1^3) - (18/a)v_2^2(-1 + 3v_1^2) + (27/a^2)v_2^4.
\]

Step 2. Suppose that char \( K = 3 \). Then (3-4) becomes \( x^2 - ay^2 = -1 - v_1 \). Hence \( K(x_1, x_2, x_3)^{S_3} = K(x, y, v_1, v_2) = K(x, y, v_2) \) is rational over \( K \).

Step 3. From now on, we assume that char \( K \neq 2, 3 \).

We normalize the generators \( v_1 \) and \( v_2 \) by defining \( T_1 := 3v_1 \) and \( T_2 := 3v_2/a \). We get \( K(x_1, x_2, x_3)^{S_3} = K(x, y, T_1, T_2) \) with a relation

\[
(3-5) \quad 3x^2 - 3ay^2 = -3 + 8T_1 - 6T_1^2 + T_1^4 + 6aT_2^2 - 2aT_1^2T_2^2 + a^2T_2^4.
\]

Step 4. We find the singularities of (3-5). We get \( x = y = -1 + T_1 = T_2 = 0 \). Define \( T_3 := -1 + T_1 \). The relation (3-5) becomes

\[
(3-6) \quad 3x^2 - 3ay^2 = 4aT_2^2 + a^2T_2^4 - 4aT_2^2T_3 - 2aT_2^2T_3^2 + 4T_3^3 + T_3^4.
\]

Step 5. We blow-up Equation (3-6), that is, define \( X_2 := x/T_3, \ Y_2 := y/T_3 \) and \( T_4 := T_2/T_3 \). Then \( K(x, y, T_1, T_2) = K(x, y, T_2, T_3) = K(X_2, Y_2, T_3, T_4) \) and the
relation (3-6) becomes

\[3X_2^2 - 3aY_2^2 = 4T_3 + T_3^2 + 4aT_4^2 - 4aT_3T_4^2 - 2aT_3^2T_4^2 + a^2T_3^2T_4^4\]

(3-7) \[= (T_3 - aT_3T_4^2)^2 + 4(T_3 - aT_3T_4^2) + 4aT_4^2\]

\[= (T_3 - aT_3T_4^2)(4 + T_3 - aT_3T_4^2) + 4aT_4^2.\]

Define

\[X_3 := \frac{X_2}{T_3 - aT_3T_4^2}, \quad Y_3 := \frac{Y_2}{T_3 - aT_3T_4^2},\]

\[S_1 := \frac{4 + T_3 - aT_3T_4^2}{T_3 - aT_3T_4^2}, \quad S_2 := \frac{T_4}{T_3 - aT_3T_4^2}.\]

Note that \(K(X_2, Y_2, T_3, T_4) = K(X_3, Y_3, S_1, S_2)\). For \(S_1 \in K(X_3, Y_3, S_1, S_2)\), \(S_1\) is a fractional linear transformation of \(T_3 - aT_3T_4^2\). Hence \(T_3 - aT_3T_4^2 \in K(X_3, Y_3, S_1, S_2)\). Thus \(T_4 = S_2 \cdot (T_3 - aT_3T_4^2) \in K(X_3, Y_3, S_1, S_2)\) also. Now \(S_1\) is a fractional linear transformation of \(T_3\) with coefficients in \(K(T_4)\). Hence \(T_3 \in K(X_3, Y_3, S_1, S_2)\). It follows that \(X_2, Y_2 \in K(X_3, Y_3, S_1, S_2)\) also.

The relation (3-7) becomes \(3X_3^2 - 3aY_3^2 = S_1 + 4aS_2^2\), which is linear in \(S_1\). Hence \(K(x_1, x_2, x_3)^{S_3} = K(X_3, Y_3, S_1, S_2) = K(X_3, Y_3, S_2)\) is rational over \(K\).

**Step 6.** Here is another proof. Instead of the method in Step 5, we may proceed as follows:

Define \(X_4 := x/T_3^2, \quad Y_4 := y/T_3^2, \quad T_4 := T_2/T_3, \) and \(T_5 := 1/T_3\). Then 
\(K(x, y, T_2, T_3) = K(X_4, Y_4, T_4, T_5)\) and (3-6) becomes

\[3X_4^2 - 3aY_4^2 = 1 - 2aT_4^2 + a^2T_4^4 + 4T_5 - 4aT_4^2T_5 + 4aT_4^2T_5^2.\]

The singularities here are \(X_4 = Y_4 = T_4 \pm (1/\sqrt{a}) = T_5 = 0\). If we blow-up with respect to \(1 - aT_4^2\), that is, define

\[X_5 := X_4/(1 - aT_4^2), \quad Y_5 := Y_4/(1 - aT_4^2), \quad T_6 := T_5/(1 - aT_4^2),\]

then \(K(X_4, Y_4, T_4, T_5) = K(X_5, Y_5, T_4, T_6)\) and the relation becomes

(3-8) \[3X_5^2 - 3aY_5^2 = 1 + 4T_6 + 4aT_4^2T_6^2.\]

Thus we get \(K(x_1, x_2, x_3)^{S_3} = K(X_5, Y_5, T_4T_6, T_6) = K(X_5, Y_5, T_4T_6)\) is rational over \(K\) because (3-8) becomes linear in \(T_6\). \(\square\)

**Proof of Theorem 1.2 when \(n = 4\) and \(\text{char } K \neq 2\).**

**Step 1.** By Lemma 3.1 we find that \(K(x_1, x_2, x_3, x_4)^{S_4} = K(\alpha)(t_2, t_3, t_4, u)^{(\rho)}\), where

\[\rho : \alpha \mapsto -\alpha, \quad t_2 \mapsto t_2t_3^{-2}t_4, \quad t_3 \mapsto t_3^{-3}t_4^2, \quad t_4 \mapsto t_3^{-4}t_4^3, \quad u \mapsto t_3^{-5}t_4^2\Delta^2 \cdot u^{-1}.\]
\[ \Delta^2 = \prod_{1 \leq i < j \leq 4} (y_i - y_j)^2 = t_2^2 t_3^2 - 4t_2^3 t_3^2 - 4t_3^3 + 18t_2 t_3^3 - 27t_3^4 - 4t_2^3 t_4 + 16t_2^4 t_4 + 18t_2 t_3 t_4 - 80t_2^2 t_3 t_4 - 6t_2^4 t_4 + 144t_2^2 t_3 t_4 - 27t_4^2 + 144t_2^2 t_3^2 - 128t_2^2 t_4^2 - 192t_3 t_4^2 + 256t_4^3. \]

Define \( u_1 := t_2, \ u_2 := t_4/t_3 \) and \( u_3 := \rho(t_2) = t_2 t_4/t_3^2 \). Then \( K(\alpha)(t_2, t_3, t_4, u) = K(\alpha)(u_1, u_2, u_3, u) \) and
\[ \rho : \alpha \mapsto -\alpha, \quad u_1 \mapsto u_3 \mapsto u_1, \quad u_2 \mapsto u_2, \quad u \mapsto g(u_1, u_2, u_3) \cdot u^{-1}, \]
where
\[ g(u_1, u_2, u_3) = \frac{u_2}{u_1 u_3} (-27u_1^2 u_2^2 - 4u_1 u_2 u_3 + 18u_1^2 u_2 u_3 - 6u_1^2 u_3^2 + 144u_1^2 u_2^2 u_3 - 192u_1 u_2^3 u_3 + 256u_1 u_2^4 u_3 + u_1 u_2^2 u_3 - 4u_1^2 u_3^2 + 18u_1 u_2 u_3^2 - 80u_1 u_2^2 u_3^2 - 27u_2^2 u_3^2 + 144u_1 u_2^2 u_3^2 - 128u_1^2 u_2^2 u_3^2 - 4u_1^2 u_3^3 + 16u_1^3 u_3^3). \]

Define \( v_1 := (u_1 + u_3)/2, \ v_2 := u_2 \) and \( v_3 = \alpha(u_1 - u_3)/2 \). Then we obtain \( K(\alpha)(u_1, u_2, u_3, u) = K(\alpha)(v_1, v_2, v_3, u) \) and
\[ \rho : \alpha \mapsto -\alpha, \quad v_1 \mapsto v_1, \quad v_2 \mapsto v_2, \quad v_3 \mapsto v_3, \quad u \mapsto h(v_1, v_2, v_3) \cdot u^{-1}, \]
where \( h(v_1, v_2, v_3) = g(u_1, u_2, u_3) \in K(v_1, v_2, v_3) \) is given as
\[ h(v_1, v_2, v_3) = \frac{v_2}{av_1^2 - v_3^2} (av_1^2 v_2 (v_1^2 + v_2^2 + v_3^2) - 2v_2 v_3 (v_1^2 - 8v_3^2 + 24v_1 v_2 - 80v_1 v_2^2 + 18v_1 v_2 - 80v_2^2 + 144v_1 v_2 v_3 - 128v_1 v_2^2 - 96v_1 v_3^2 + 128v_3^2) - (1/\alpha)v_2 v_3^4 (-1 + 8v_1 - 48v_1^2 + 80v_2 + 128v_2^2) - (16/\alpha^2)v_2 v_3^6). \]

**Step 2.** Because \( h(v_1, v_2, v_3) \) is still complicated, we define \( p, q \) and \( r \) as
\[ p := \frac{1}{2} \left( \frac{1}{u_1} + \frac{1}{u_3} \right) u_2, \quad q := \frac{\alpha}{2} \left( \frac{1}{u_1} - \frac{1}{u_3} \right) u_2, \quad r := 4u_2. \]
Then \( K(\alpha)(v_1, v_2, v_3, u) = K(\alpha)(p, q, r, u) \). Indeed we have
\[ p = \frac{av_1 v_2}{av_1^2 - v_3^2}, \quad q = -\frac{av_2 v_3}{av_1^2 - v_3^2}, \quad r = 4v_2, \]
\[ v_1 = -\frac{apr}{4(ap^2 - q^2)}, \quad v_2 = r/4, \quad v_3 = -\frac{apr}{4(ap^2 - q^2)}. \]

Hence we obtain \( K(x_1, x_2, x_3, x_4)^{S_4} = K(\alpha)(p, q, r, u)^{\rho} \) and
\[ \rho : \alpha \mapsto -\alpha, \quad p \mapsto p, \quad q \mapsto q, \quad r \mapsto r, \quad u \mapsto \frac{r^2}{64(ap^2 - q^2)^2} \cdot \frac{H(p, q, r)}{u}, \]
(3-9) \[ H(p, q, r) = a^2(p - r + 2pr)^2(-16p^2 + r + 4pr + 4p^2r) \]
\[-a(-32p^2 + r + 36pr - 12p^2r - 20r^2 + 72p^2) \]
\[-96p^2r^2 - 8r^3 + 32p^2r^3)q^2 + 16(-1 + r)^3q^4. \]

Define \( U := u \cdot r/(8(ap^2 - q^2)) \). Then \( K(\alpha)(p, q, r, u) = K(\alpha)(p, q, r, U) \), and \( \rho \) acts on \( K(\alpha)(p, q, r, U) \) by
\[ \rho : \alpha \mapsto -\alpha, \quad p \mapsto p, \quad q \mapsto q, \quad r \mapsto r, \quad U \mapsto H(p, q, r)/U. \]

Hence \( K(x_1, \ldots, x_4)^{S_4} = K(\alpha)(p, q, r, U)^{(\rho)} = K(X, Y, p, q, r) \) where
\[ X = \frac{1}{2}(U + g(p, q, r)/U), \quad Y = \frac{1}{2\alpha}(U - g(p, q, r)/U). \]

Note that \( X \) and \( Y \) satisfy the relation
\[ (3-10) \quad X^2 - aY^2 = H(p, q, r). \]

**Step 3.** Because \( H(p, q, r) \) in (3-9) is a biquadratic equation with respect to \( q \) and its constant term has the square factor \((p - r + 2pr)^2\), we define \( p_2 := p - r + 2pr \).

Then \( p = (p_2 + r)/(1 + 2r) \). We also define \( X_2 := X(1 + 2r) \) and \( Y_2 := Y(1 + 2r) \).

Then \( K(x_1, x_2, x_3, x_4)^{S_4} = K(X_2, Y_2, p_2, q, r) \) and (3-10) becomes
\[ X_2^2 - aY_2^2 = a^2 p_2^2(-16p_2^2 + r - 28p_2r + 4p_2^2r - 8r^2 + 16p_2r^2 + 16r^3) \]
\[-a(-32p_2^2 + r - 28p_2r - 12p_2^2r - 12r^2 + 120p_2r^2 \]
\[-96p_2^2r^2 + 48r^3 - 48p_2r^3 + 32p_2^2r^3 - 64r^4 + 64p_2r^4)q^2 \]
\[+16(-1 + r)^3(1 + 2r)^2q^4. \]

The right hand side is biquadratic in \( q \) with constant term on the first line. Hence we define \( p_3 := p_2/q, \ X_3 := X_2/q \) and \( Y_3 := Y_2/q \), and the equation becomes quadratic in \( q \):
\[ X_3^2 - aY_3^2 = ar(-1 + 4r)^2(-1 + ap_3^2 + 4r) \]
\[+4ap_3r(7 - 7ap_3^2 - 30r + 4ap_3^2r + 12r^2 - 16r^3)q \]
\[+4(-1) + ap_3^2 - 4r - 4r^2)(4 - 4ap_3^2 - 12r + ap_3^2r + 12r^2 - 4r^3)q^2. \]

Define \( q_2 := 1/q, \ r_2 := 4r, \ X_4 := 4X_3/q, \ Y_4 := 4Y_3/q. \) Then
\[ (3-11) \quad X_4^2 - aY_4^2 = 4ar_2(-1 + r_2)^2(-1 + ap_3^2 + r_2)q_2^2 \]
\[+4ap_3r_2(28 - 28ap_3^2 - 30r_2 + 4ap_3^2r_2 + 3r_2 - r_2^3)q_2 \]
\[+(-4 + 4ap_3^2 - 4r_2 - r_2^3)(64 - 64ap_3^2 - 48r_2 + 4ap_3^2r_2 + 12r_2^2 - r_2^3). \]
Because (3-11) is quadratic in $q_2$, we may eliminate a linear term of $q_2$ in the usual manner by putting
\[
q_3 := 2q_2 + \frac{p_3(28 - 28ap_3^2 - 30r_2 + 4ap_3^2r_2 + 3r_2^2 - r_3^2)}{(-1 + r_2)^2(-1 + ap_3^2 + r_2)}.
\]
Define
\[
X_5 := X_4(-1 + r_2)(-1 + ap_3^2 + r_2), \quad Y_5 := Y_4(-1 + r_2)(-1 + ap_3^2 + r_2).
\]
Then (3-11) becomes
\[
X_5^2 - aY_5^2 = (2 + r_2)^2(-1 + ap_3^2 + r_2)(4 - 4ap_3^2 - 5r_2 + r_2^2)^3 + ar_2(-1 + r_2)^4(-1 + ap_3^2 + r_2)^3q_3^2.
\]
Defining
\[
q_4 := \frac{q_3(-1 + r_2)^2(-1 + ap_3^2 + r_2)}{(2 + r_2)(4 - 4ap_3^2 - 5r_2 + r_2^2)}
\]
and
\[
X_6 := \frac{X_5}{(2 + r_2)(4 - 4ap_3^2 - 5r_2 + r_2^2)}, \quad Y_6 := \frac{Y_5}{(2 + r_2)(4 - 4ap_3^2 - 5r_2 + r_2^2)},
\]
we get $K(x_1, \ldots, x_4)^{S_1} = K(X_6, Y_6, p_3, q_4, r_2)$ and the equation becomes
\[
(3-12) \quad X_6^2 - aY_6^2 = (-1 + ap_3^2 + r_2)((4 - 4ap_3^2 - 5r_2 + r_2^2) + ar_2q_4^2).
\]

**Step 4.** We find the singularities of (3-12). We get $p_3 \pm (1/\sqrt{a}) = r_2 = X_6 = Y_6 = 0$. Blow-up with respect to $-1 + ap_3^2$, that is, define
\[
r_3 := r_2/(-1 + ap_3^2), \quad X_7 := X_6/(-1 + ap_3^2), \quad Y_7 := Y_6/(-1 + ap_3^2).
\]
Then $K(p_3, q_4, r_2, X_6, Y_6) = K(p_3, q_4, r_3, X_7, Y_7)$ and (3-12) becomes
\[
X_7^2 - aY_7^2 = (1 + r_3)(-4 - 5r_3 + aq_4^2r_3 - r_3^2 + ap_3^2r_3^2).
\]
Define $p_4 := p_3r_3$. Then
\[
(3-13) \quad X_7^2 - aY_7^2 = (1 + r_3)(-4 - 5r_3 + aq_4^2r_3 - r_3^2 + ap_3^2r_3^2).
\]

**Step 5.** Equation (3-13) still has a singular locus $p_4 \pm q_4 = r_3 + 1 = X_7 = Y_7 = 0$. If we define $p_5 := p_4 + q_4$ and $r_4 := r_3 + 1$, it becomes
\[
(3-14) \quad X_7^2 - aY_7^2 = r_4(ap_3^2 - 2ap_5q_4 - 3r_4 + aq_4^2r_4 - r_4^2)
\]
with singular locus $S = (p_5 = r_4 = X_7 = Y_7 = 0)$. Blowing this up along $S$ by defining $r_5 := r_4/p_5$, $X_8 := X_7/p_5$, and $Y_8 := Y_7/p_5$, we get
\[
X_8^2 - aY_8^2 = r_5(ap_5 - 2aq_4 - 3r_5 + aq_4^2r_5 - p_5r_5^2).
\]
Note that this is linear in $p_5$. Hence we conclude that the fixed field $K(x_1, \ldots, x_4)^{S_4} = K(X_8, Y_8, q_4, r_5)$ is rational over $K$. \hfill \qedsymbol

4. Using the structures of rings of invariants

In this section, we give an another proof of **Theorem 1.2** in the case of $n = 3, 4$ and $\text{char } K = 2$ by using the structure of $K(x_1, \ldots, x_n)^{A_n}$. Throughout, we assume that $\text{char } K = 2$.

For $1 \leq i \leq n$, let $s_i$ be the elementary symmetric function in $x_1, \ldots, x_n$ of degree $i$.

By Revoy’s theorem [1982], the invariant ring $K[x_1, \ldots, x_n]^{A_n}$ is a free module of rank 2 over the subring $K[x_1, \ldots, x_n]^{S_n} = K[s_1, \ldots, s_n]$. Revoy’s theorem is valid for all characteristics. We will find explicitly a free basis of $K[x_1, \ldots, x_n]^{A_n}$ over $K[x_1, \ldots, x_n]^{S_n}$ for the case $n = 3, 4$. For $n = 3$ and $n = 4$, it suffices by [Neusel and Smith 2002, Example 1, page 75] to find a polynomial of degree 3 and 6, respectively, that is in $K[x_1, \ldots, x_n]^{A_n}$ but not in $K[x_1, \ldots, x_n]^{S_n}$.

Define
\[
b_3 := \sum_{\sigma \in A_3} \sigma(x_1 x_2^2) = x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2,
\]
\[
b_4 := \sum_{\sigma \in A_4} \sigma(x_1 x_2 x_3 x_4) = x_1^2 x_2 x_3 x_4 + x_1 x_2^2 x_3^2 + x_1 x_2 x_3^2 x_4 + x_1^2 x_2 x_3 x_4^2 + x_1 x_2 x_3 x_4^3 + x_1^2 x_2 x_3 x_4^3.
\]

For $n = 3, 4$, it follows that $\{1, b_n\}$ is a free basis of $K[x_1, \ldots, x_n]^{A_n}$, that is,
\[
K[x_1, x_2, x_3]^{A_3} = K[s_1, s_2, s_3] \oplus b_3 K[s_1, s_2, s_3],
\]
\[
K[x_1, x_2, x_3, x_4]^{A_4} = K[s_1, s_2, s_3, s_4] \oplus b_4 K[s_1, s_2, s_3, s_4].
\]

We have proved this:

**Lemma 4.1.** Let $K$ be a field of char $K = 2$. Then the fields $K(x_1, x_2, x_3)^{A_3}$ and $K(x_1, x_2, x_3, x_4)^{A_4}$ of invariants are given as follows.

(i) $K(x_1, x_2, x_3)^{A_3} = K(s_1, s_2, s_3, b_3)$ with the relation
\[
b_3^2 + b_3 s_1 s_2 + s_2^3 + b_3 s_3 + s_1^3 s_3 + s_3^2 = 0.
\]

(ii) $K(x_1, x_2, x_3, x_4)^{A_4} = K(s_1, s_2, s_3, s_4, b_4)$ with the relation
\[
b_4^2 + b_4 s_1 s_2 s_3 + b_4 s_2^2 + s_2^2 s_3^2 + s_1^3 s_3^2 + s_3^4 + b_4 s_1^2 s_4 + s_1^3 s_2^3 s_4 + s_1^4 s_4^2 = 0.
\]

**Proof of Theorem 1.2** when $n = 3$ and char $K = 2$. First, $\tau$ acts on $K(x_1, x_2, x_3)^{A_3} = K(s_1, s_2, s_3, b_3)$ as
\[
s_1 \mapsto a s_2 / s_3, \quad s_2 \mapsto a^2 s_1 / s_3, \quad s_3 \mapsto a^3 / s_3, \quad b_3 \mapsto a^3 b_3 / s_3^2.
\]
Apply Theorem 2.2. We find $K(x_1,x_2,x_3)^{A_3} = K(s_3,u,v)$, where $u$ and $v$ are the same as in Theorem 2.2. It is not difficult to check that

$$u = \frac{b_3 + s_3}{s_1^2 + s_2^2} \quad \text{and} \quad v = \frac{b_3 + s_1s_2}{s_1^2 + s_2^2}.$$ 

Moreover, the action of $\tau$ is given by

$$\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad u \mapsto \frac{au}{u^2-uv+v^2}, \quad v \mapsto \frac{av}{u^2-uv+v^2}.$$ 

Define $w := u/v$. Then $K(x_1,x_2,x_3)^{A_3} = K(s_3,v,w)$ and

$$\tau : s_3 \mapsto \frac{a^3}{s_3}, \quad v \mapsto \frac{a}{v(1-w+w^2)}, \quad w \mapsto w.$$ 

By Theorem 2.1, $K(x_1,x_2,x_3)^{S_3} = K(s_3,v,w)^{(\tau)}$ is rational over $K$. \hfill $\Box$

Proof of Theorem 1.2 when $n = 4$ and char $K = 2$.

In this case, $\tau$ acts on $K(x_1,x_2,x_3,x_4)^{A_4} = K(s_1,s_2,s_3,s_4,b_4)$ as

$$s_1 \mapsto as_3/s_4, \quad s_2 \mapsto a^2s_2/s_4, \quad s_3 \mapsto a^3s_1/s_4, \quad s_4 \mapsto a^4/s_4, \quad b_4 \mapsto a^6(b_4 + s_1s_2s_3 + s_3^2 + s_1^2s_4)/s_4^3.$$ 

Define

$$t_1 := \frac{s_1s_3}{s_2}, \quad t_2 := s_2, \quad t_3 := s_3, \quad t_4 := \frac{s_1s_2s_3 + s_3^2 + s_1^2s_4}{s_2^2}, \quad t_5 := \frac{b_4 + s_2^3}{s_2}.$$ 

It follows that $K(s_1,s_2,s_3,s_4,b_4) = K(t_1,t_2,t_3,t_4,t_5)$. It is easy to check that the relation among the generators $t_1,\ldots,t_5$ is given by

$$t_1^3 + t_1^2t_2 + t_1t_2^2 + t_2^3 + t_2t_4^2 + t_2t_4t_5 + t_2t_5^2 = 0.$$ 

Define

$$u_1 := t_1, \quad u_2 := \frac{t_2}{t_1}, \quad u_3 := t_3, \quad u_4 := \frac{t_4}{(t_1 + t_2)}, \quad u_5 := \frac{t_5}{(t_1 + t_2)}.$$ 

Then we get $K(t_1,\ldots,t_5) = K(u_1,\ldots,u_5)$ with the relation

$$u_2(u_4^2 + u_4u_5 + u_5^2 + 1) + 1 = 0.$$ 

Because this relation is linear in $u_2$, we obtain the following lemma.

Lemma 4.2. $K(x_1,\ldots,x_4)^{A_4} = K(u_1,u_3,u_4,u_5)$, where

$$u_1 = \frac{s_1s_3}{s_2}, \quad u_3 = s_3, \quad u_4 = \frac{s_1s_2s_3 + s_3^2 + s_1^2s_4}{s_2(s_2^2 + s_1s_3)}, \quad u_5 = \frac{b_4 + s_2^3}{s_2(s_2^2 + s_1s_3)}.$$
Now we will prove Theorem 1.2 when $n = 4$ and char $K = 2$.

Write $p = u_1$, $q = u_3$, $r = u_4$, $s = u_5$ and $\tau = (12) \in S_4 \setminus A_4$. Note that $K(x_1, \ldots, x_4)^{S_4} = K(p, q, r, s)^{(\tau)}$ and the action of $\tau$ on $K(p, q, r, s)$ is given by

$$p \mapsto \frac{r^2 + rs + s^2 + 1}{ap},$$
$$q \mapsto \frac{a^3 p^6 q}{(r^2 + rs + s^2 + 1)^3 + p^3 q ((r + 1)(r^2 + rs + s^2 + 1) + 1)},$$
$$r \mapsto r, \quad s \mapsto s + r.$$

Define

$$t := \frac{(r^2 + rs + s^2 + 1)^3}{p^3 q ((r + 1)(r^2 + rs + s^2 + 1) + 1)}.$$

Then $K(x_1, x_2, x_3, x_4)^{S_4} = K(p, q, r, s)^{(\tau)} = K(p, t, r, s)^{(\tau)}$ and the action of $\tau$ on $K(p, t, r, s)$ is given by

$$\tau : p \mapsto (r^2 + rs + s^2 + 1)/(ap), \quad t \mapsto t + 1, \quad r \mapsto r, \quad s \mapsto s + r.$$

Define

$$A := r + s + rt, \quad B := (r + s)/s, \quad C := pr/s.$$

It follows that $K(p, q, r, s) = K(r, A, B, C)$. Thus we have $K(x_1, x_2, x_3, x_4)^{S_4} = K(r)(A, B, C)^{(\tau)}$ and

$$\tau : r \mapsto r, \quad A \mapsto A, \quad B \mapsto \frac{1}{B}, \quad C \mapsto \frac{1}{a} \left( (r^2 + 1) \left( \frac{1}{B} + B \right) + r^2 \right)/C.$$

Apply Theorem 2.1. We find that $K(x_1, x_2, x_3, x_4)^{S_4}$ is rational over $K$. \hfill \Box

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