PREALTERNATIVE ALGEBRAS
AND PREALTERNATIVE BIALGEBRAS

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We introduce a notion of prealternative algebra, which may be viewed as an
alternative algebra whose product can be decomposed into two compatible
pieces. It is also an alternative algebra analogue of a dendriform dialgebra
or a pre-Lie algebra. The left and right multiplication operators of a pre-
alternative algebra give a bimodule structure of the associated alternative
algebra. There exists a (coboundary) bialgebra theory for prealternative
algebras, namely, prealternative bialgebras, which exhibits all the familiar
properties of the Lie bialgebra theory. In particular, a prealternative bial-
gebra is equivalent to a phase space of an alternative algebra, and our study
leads to what we call the PA equations in a prealternative algebra, which
are analogues of the classical Yang–Baxter equation.

1. Introduction

A dendriform dialgebra is a vector space $D$ together with two bilinear operations
$<, > : D \otimes D \to D$ such that for any $x, y, z \in D$
\begin{align}
(x < y) < z &= x < (y \circ z), \\
(x > y) < z &= x > (y < z), \\
(x \circ y) > z &= x > (y > z),
\end{align}
where $x \circ y = x < y + x > y$. Dendriform dialgebras were introduced by J.-L.
Loday in 1995 as the (Koszul) dual of the associative dialgebra, which is related to
periodicity phenomena in algebraic K-theory [Loday 2001]. It was further studied
in connection with several areas in mathematics and physics, including operads
[Loday 2004], homology [Frabetti 1997; 1998], Hopf algebras [Chapoton 2002;
Holtkamp 2004; Ronco 2002], Lie and Leibniz algebras [Frabetti 1998], combi-
natorics [Aguiar and Sottile 2005; 2006], arithmetic [Loday 2002] and quantum

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For any dendriform dialgebra \((D, <, >)\), the bilinear operation \(\circ\) defines an associative algebra. Thus, a dendriform dialgebra may be seen as an associative algebra whose multiplication can be decomposed into two coherent operations.

We may reexamine the identity (1-1) as follows. Let \(A\) be a vector space together with two operations \(<, >: A \otimes A \to A\). The right associator \((r\text{-associator})\), middle associator \((m\text{-associator})\) and left associator \((l\text{-associator})\) are defined for all \(x, y, z \in A\) by

\[
(x, y, z)_r = (x \prec y) \prec z - x \prec (y \circ z),
\]

\[
(x, y, z)_m = (x \succ y) \prec z - x \prec (y \prec z),
\]

\[
(x, y, z)_l = (x \circ y) \succ z - x \succ (y \succ z),
\]

respectively, where \(x \circ y = x \prec y + x \succ y\). So \((D, <, >)\) is a dendriform dialgebra if and only if all the above three associators are zero.

On the other hand, alternative algebras are a class of important nonassociative algebras [Kuz’min and Shestakov 1995; Schafer 1952; 1954]. Alternative algebras are closely related to Lie algebras [Schafer 1954], Jordan algebras [Jacobson 1968] and Malcev algebras [Kuz’min and Shestakov 1995]. Due to the relationships between associative algebras and alternative algebras (see Section 2), it is natural to consider the algebraic structure on an alternative algebra as an analogue of a dendriform dialgebra on an associative algebra. So we introduce a notion of prealternative algebra, one of the main objects in this paper. Just as an alternative algebra is a generalization of an associative algebra that weakens the condition of associativity, a prealternative algebra is a generalization of a dendriform dialgebra that weakens the conditions of \(l\text{-associativity}, m\text{-associativity and } r\text{-associativity.}\)

There has already been a Lie algebraic version of the relationship between associative algebras and dendriform dialgebras. A class of nonassociative algebras, the pre-Lie algebras (also called left-symmetric algebras, Vinberg algebras and so on — see a survey article [Burde 2006] and the references therein) play a role similar to dendriform dialgebras. Therefore, in this sense, prealternative algebras are just alternative algebra analogues of pre-Lie algebras or dendriform dialgebras.

Goncharov [2007] constructed alternative D-bialgebras, a bialgebra theory for alternative algebras. In this paper, we show that prealternative bialgebras serve as a (coboundary) bialgebra theory for prealternative algebras, and exhibit all the familiar properties of the Lie bialgebra theory of Drinfeld [1983]. Just as an alternative D-bialgebra is equivalent to an alternative analogue of Manin triple [Goncharov 2007; Chari and Pressley 1994], a prealternative bialgebra is equivalent to a phase space of an alternative algebra [Kupershmidt 1994; Bai 2006]. In particular, there
exists an unexpected Drinfeld double construction for a prealternative bialgebra. Also, there is a clear analogy between alternative D-bialgebras and prealternative bialgebras. On the other hand, we emphasize that the representation theories of alternative and prealternative algebras play an essential role in establishing the bialgebra theories. We also point out that both alternative D-bialgebras and prealternative bialgebras can be fit into the general framework of generalized bialgebras introduced in [Loday 2008]. So it would be interesting to find the relationship to Loday’s question, that is, to find, as he put it, “good triples of operads”.

The paper is organized as follows. In Section 2, we study bimodules of alternative algebras and introduce various methods to construct prealternative algebras. In Section 3, we recall the properties of alternative D-bialgebras of Goncharov and prove some new results. In Section 4, we generalize the notion of phase space in mathematical physics [Kupershmidt 1994] to the realm of alternative algebras, and show that prealternative algebras are the natural underlying structures. In Section 5, we define and study bimodules and matched pairs of prealternative algebras. In Section 6, we introduce the notion of prealternative bialgebra, which is equivalent to a phase space of an alternative algebra. In Section 7, we show that there is a reasonable coboundary (prealternative) bialgebra theory; what we study leads to what we call PA equations. Section 8 discusses the properties of the PA equations. We compare alternative D-bialgebras and prealternative algebras in Section 9. In the appendix, we prove the main results in [Goncharov 2007] by a somewhat different approach; we point out a Drinfeld double construction for an alternative D-bialgebra that was not given there.

Throughout this paper, all the algebras are finite-dimensional over a fixed base field $k$ of characteristic not 2. We give some notations as follows.

Let $V$ be a vector space. Let $\mathcal{B} : V \otimes V \to F$ be a symmetric or skew-symmetric bilinear form on $V$. If $W$ is a subspace of $V$, then we define

\begin{equation}
W^\perp = \{ x \in V \mid \mathcal{B}(x, y) = 0 \text{ for all } y \in W \}.
\end{equation}

We say $W$ is isotropic if $W \subset W^\perp$ and Lagrangian if $W = W^\perp$.

Let $(A, \diamond)$ be a vector space with a binary operation $\diamond : A \otimes A \to A$. Let $l_\diamond(x)$ and $r_\diamond(x)$ denote the left and right multiplication operators, that is, $l_\diamond(x) y = r_\diamond(y) x = x \diamond y$ for any $x, y \in A$. We may sometimes write instead $l(x)$ or $r(x)$ when no confusion would result. Let $l_\diamond, r_\diamond : A \to \mathfrak{gl}(A)$ be two linear maps with $x \mapsto l_\diamond(x)$ and $x \mapsto r_\diamond(x)$, respectively.

Let $V$ be a vector space and $r = \sum_i a_i \otimes b_i \in V \otimes V$. Set

\begin{equation}
r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i,
\end{equation}

where $i$ ranges over the index set.
where 1 is a symbol playing a role similar to the unit. If in addition there exists a binary operation \( \diamond : V \otimes V \to V \) on \( V \), then the operation between two of the \( r \) is done in the obvious way. For example,

\[
\begin{align*}
 r_{12} \diamond r_{13} &= \sum_{i,j} a_i \diamond a_j \otimes b_i \otimes b_j, \\
 r_{13} \diamond r_{23} &= \sum_{i,j} a_i \otimes a_j \otimes b_i \odot b_j, \\
 r_{23} \diamond r_{12} &= \sum_{i,j} a_j \otimes a_i \otimes b_j \odot b_i.
\end{align*}
\]

(1-5)

Let \( V \) be a vector space. Let \( \sigma : V \otimes V \to V \otimes V \) be the flip defined by

\[
\sigma(x \otimes y) = y \otimes x \quad \text{for all } x, y \in V.
\]

(1-6)

We call \( r \in V \otimes V \) symmetric if \( r = \sigma(r) \) and skew-symmetric if \( r = -\sigma(r) \). On the other hand, any \( r \in V \otimes V \) can be identified as a linear map \( T_r : V^* \to V \) via

\[
\langle u^* \odot v^*, r \rangle = \langle u^*, T_r(v^*) \rangle \quad \text{for all } u^*, v^* \in V^*,
\]

(1-7)

where \( \langle \cdot, \cdot \rangle \) is the canonical pairing between \( V \) and \( V^* \). We call \( r \in V \otimes V \) nondegenerate if \( T_r \) is invertible. Any invertible linear map \( T : V^* \to V \) induces a nondegenerate bilinear form \( \mathcal{B} \) on \( V \) by

\[
\mathcal{B}(u, v) = \langle T^{-1}u, v \rangle \quad \text{for all } u, v \in V.
\]

(1-8)

We call \( T \) symmetric (respectively skew-symmetric) if the induced bilinear form \( \mathcal{B} \) is symmetric (respectively skew-symmetric). Obviously, the symmetry or skew-symmetry of both \( T \) and the corresponding \( r \in V \otimes V \) coincide.

Let \( V_1, V_2 \) be two vector spaces and \( T : V_1 \to V_2 \) be a linear map. Denote the dual (linear) map by \( T^* : V_2^* \to V_1^* \) defined by

\[
\langle v_1, T^*(v_2^*) \rangle = \langle T(v_1), v_2^* \rangle \quad \text{for all } v_1 \in V_1, v_2^* \in V_2^*.
\]

(1-9)

On the other hand, \( T \) can be identified as an element \( r_T \in V_2 \otimes V_1^* \) by

\[
\langle r_T, v_2^* \otimes v_1 \rangle = \langle T(v_1), v_2^* \rangle \quad \text{for all } v_1 \in V_1, v_2^* \in V_2^*.
\]

(1-10)

Note that (1-7) is exactly the case that \( V_1 = V_2^* \). In the above sense, any linear map \( T : V_1 \to V_2 \) is obviously an element in \((V_2 \oplus V_1^*) \oslash (V_2 \oplus V_1^*)\).

Let \( A \) be an algebra and \( V \) be a vector space. For any linear map \( \rho : A \to \text{gl}(V) \), define a linear map \( \rho^* : A \to \text{gl}(V^*) \) by

\[
\langle \rho^*(x)v^*, u \rangle = \langle v^*, \rho(x)u \rangle \quad \text{for all } x \in A, u \in V, v^* \in V^*.
\]

(1-11)

Note that in this case \( \rho^* \) not the map dual to \( \rho \) in the sense of (1-9).

For vector spaces \( V_1 \) and \( V_2 \), we denote the elements of \( V_1 \oplus V_2 \) by \( u + v \) or \( (u, v) \) for \( u \in V_1 \) and \( v \in V_2 \).

We may use 1 to denote the identity transformation of a vector space \( V \).
2. Representation theory of alternative algebras and prealternative algebras

Definition 2.1. An alternative algebra \((A, \circ)\) is a vector space \(A\) equipped with a bilinear operation \((x, y) \mapsto x \circ y\) satisfying

\[(2-1) \quad (x, x, y) = (y, x, x) = 0 \quad \text{for all } x, y, z \in A,\]

where \((x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)\) is the associator.

Remark 2.2. If the characteristic of the field is not 2, then an alternative algebra \((A, \circ)\) also satisfies for all \(x_1, x_2, y \in A\) the stronger axioms

\[(x_1, x_2, y) + (x_2, x_1, y) = 0 \quad \text{and} \quad (y, x_1, x_2) + (y, x_2, x_1) = 0.\]

Definition 2.3 [Schafer 1952]. Let \((A, \circ)\) be an alternative algebra and \(V\) be a vector space. Let \(L, R : A \to \text{gl}(V)\) be two linear maps. We call \(V\) (or the pair \((L, R)\), or \((V, L, R)\)) a representation or a bimodule of \(A\) if for any \(x, y \in A\)

\[(2-2) \quad L(x^2) = L(x)L(x), \quad R(x^2) = R(x)R(x)\]

and

\[(2-3) \quad R(y)L(x) - L(x)R(y) = R(x \circ y) - R(y)R(x), \]

\[L(y \circ x) - L(y)L(x) = L(y)R(x) - R(x)L(y).\]

By [Schafer 1995], \((V, L, R)\) is a bimodule of an alternative algebra \((A, \circ)\) if and only if the direct sum \(A \oplus V\) of vector spaces is turned into an alternative algebra (the semidirect sum) by defining multiplication in \(A \oplus V\) by

\[(2-4) \quad (x_1 + v_1) \ast (x_2 + v_2) = x_1 \circ x_2 + (L(x_1)v_2 + R(x_2)v_1)\]

for all \(x_1, x_2 \in A\) and \(v_1, v_2 \in V\).

We denote it by \(A \ltimes_{L,R} V\) or simply \(A \ltimes V\).

Proposition 2.4. If \((V, L, R)\) is a bimodule of an alternative algebra \((A, \circ)\), then \((V^*, R^*, L^*)\) is a bimodule of \((A, \circ)\).

Proof. By (2-2) and (2-3), we have

\[L(x \circ y) - L(x)L(y) = -L(y \circ x) + L(y)L(x) = R(x)L(y) - L(y)R(x)\]

for all \(x, y \in A\). So for any \(u^* \in V^*, v \in V\), we have

\[
\langle (L^*(y)R^*(x) - R^*(x)L^*(y))u^*, v \rangle = \langle u^*, (R(x)L(y) - L(y)R(x))v \rangle \\
= \langle u^*, (L(x \circ y) - L(x)L(y))v \rangle \\
= \langle (L^*(x \circ y) - L^*(y)L^*(x))u^*, v \rangle.
\]

So \(L^*(y)R^*(x) - R^*(x)L^*(y) = L^*(x \circ y) - L^*(y)L^*(x)\). Similarly, \((V^*, R^*, L^*)\) also satisfies the other axioms defining a bimodule of \((A, \circ)\). \(\square\)
Definition 2.5. A prealternative algebra \((A, \prec, \succ)\) is a vector space \(A\) with two bilinear operations denoted by \(\prec, \succ: A \otimes A \to A\) satisfying
\[
(x, y, z)_m + (y, x, z)_r = 0, \\
(x, y, z)_m + (x, z, y)_l = 0, \\
(y, x, x)_r = (x, x, y)_l = 0
\]
for all \(x, y, z \in A\), where \((x, y, z)_r, (x, y, z)_l, (x, y, z)_m\) are defined by (1-2) and \(x \circ y = x \succ y + x \prec y\).

Remark 2.6. If the characteristic of the field is not 2, then a prealternative algebra \((A, \prec, \succ)\) satisfies for any \(x, y, z \in A\) the strong axioms
\[
(2-5) \quad (x, y, z)_m + (y, x, z)_r = 0, \quad (x, y, z)_m + (x, z, y)_l = 0, \\
(2-6) \quad (x, y, z)_l + (y, x, z)_l = 0, \quad (x, y, z)_r + (x, z, y)_r = 0.
\]

It would be interesting to describe free prealternative algebras; see [Loday 2001].

Proposition 2.7. Let \((A, \prec, \succ)\) be a prealternative algebra. Then the operation
\[
x \circ y = x \succ y + x \prec y \quad \text{for all } x, y \in A,
\]
defines an alternative algebra, which is called the associated alternative algebra of \(A\) and denoted by \(\text{Alt}(A)\). We call \((A, \prec, \succ)\) a compatible prealternative algebra structure on the alternative algebra \(\text{Alt}(A)\).

Proof. In fact, for any \(x, y \in A\), we have
\[
(x, x, y) = (x \circ x) \circ y - x \circ (x \circ y)
\]
\[
= (x \circ x) \succ y + (x \succ x) \prec y + (x \prec x) \prec y - x \succ (x \prec y) - x \prec (x \circ y)
\]
\[
= (x, x, y)_l + (x, x, y)_m + (x, x, y)_r = 0.
\]

Similarly, we show that \((y, x, x) = 0\). \qed

Remark 2.8. Thus a prealternative algebra can be viewed as an alternative algebra whose operation decomposes into two compatible pieces. On the other hand, it is obvious that an associative algebra is an alternative algebra and a dendriform dialgebra is a prealternative algebra.

If \((A, \circ)\) is an alternative algebra, then \((A, l_\circ, r_\circ)\) is a bimodule of \(A\).

Proposition 2.9. Let \((A, \prec, \succ)\) be a prealternative algebra. Then \((A, l_\prec, r_\prec)\) is a bimodule of the associated alternative algebra \((\text{Alt}(A), \circ)\).

Proof. For any \(x, y, z \in A\), we have
\[
(r_\prec(y)l_\prec(x) - l_\prec(x)r_\prec(y))z = (x \succ z) \prec y - x \succ (z \prec y) = z \prec (x \circ y) - (z \prec x) \prec y
\]
\[
= (r_\prec(x \circ y) - r_\prec(y)r_\prec(x))z.
\]
Proposition 2.11. Let $T$ be an example. The proof of the others is similar. For any $u$ and $v$, we only prove one identity, with

\begin{equation}
T(u) \circ T(v) = T(L(T(u))v + R(T(v))u) \quad \text{for all } u, v \in V.
\end{equation}

Proposition 2.11. Let $T : V \to A$ be an $\mathcal{O}$-operator of an alternative algebra $(A, \circ)$ associated to a bimodule $(V, L, R)$. Then there exists a prealternative algebra structure on $V$ given by

\begin{equation}
u < v = R(T(v))u \quad \text{and} \quad u > v = L(T(u))v \quad \text{for all } u, v \in V.
\end{equation}

Therefore $V$ is an alternative algebra as the associated alternative algebra of this prealternative algebra and $T$ is a homomorphism of alternative algebras. Furthermore, $T(V) = \{T(v) \mid v \in V\} \subset A$ is an alternative subalgebra of $(A, \circ)$ and there is induced prealternative algebra structure on $T(V)$ given by

\begin{equation}
T(u) \prec T(v) = T(u \prec v) \quad \text{and} \quad T(u) \succ T(v) = T(u \succ v) \quad \text{for all } u, v \in V.
\end{equation}

Moreover, the associated alternative algebra structure is just the alternative subalgebra structure of $(A, \circ)$ and $T$ is a homomorphism of prealternative algebras.

Proof. We only prove one identity, with $(V, \prec, \succ)$ being a prealternative algebra as an example. The proof of the others is similar. For any $u, v, w \in V$,

\[ (u \succ v) \prec w + (v \prec u) \prec w = R(T(w))L(T(u))v + R(T(w))R(T(u))v, \]

\[ u \succ (v \prec u) + v \prec (u \circ v) = L(T(u))R(T(w))v + R(T(u \circ w))v. \]

By (2-3), (2-7) and (2-8), we show that

\[ (u, v, w)_m + (u, v, w)_r = (u \succ v) \prec w + (v \prec u) \prec w - u \succ (v \prec w) - v \prec (u \circ v) = 0. \]

The remaining parts of the conclusion are obvious. □

Definition 2.12 [Schafer 1952]. Let $(A, \circ)$ be an alternative algebra and $(V, L, R)$ be a bimodule. A 1-cocycle of $A$ into $V$ is a linear map $D : A \to V$ satisfying

\begin{equation}
D(x \circ y) = L(x)D(y) + R(y)D(x) \quad \text{for all } x, y \in A.
\end{equation}

Proposition 2.13. For $(A, \circ)$ an alternative algebra, the following conditions are equivalent.
Thus, for any $x$ given by $\prec$, Proposition 2.13, there is a compatible prealternative algebra structure $\prec$ on $A$, $\circ$. Let $\rho$.

Example 2.14. Let $(A, \circ)$ be an alternative algebra graded by positive integers, that is, $A = \bigoplus_{i \in \mathbb{N}} A_i$ and $A_i \circ A_j \subset A_{i+j}$. Then there is a bijective 1-cocycle associated to the bimodule $(A, l_\circ, r_\circ)$ defined by $D(x_i) = ix_i$ for $x_i \in A_i$. Therefore there exists a compatible prealternative algebra structure on $(A, \circ)$ given by $x_i \circ x_j = \frac{j}{i+j} x_i \circ x_j$ and $x_i \circ x_j = \frac{i}{i+j} x_i \circ x_j$ for all $x_i \in A_i, x_j \in A_j$.

Definition 2.15. Let $(A, \circ)$ be an arbitrary algebra (not necessarily associative) and $\omega$ be a skew-symmetric bilinear form on $A$. The bilinear form $\omega$ is said to be closed if $\omega$ satisfies

$$\omega(a \circ b, c) + \omega(b \circ c, a) + \omega(c \circ a, b) = 0 \quad \text{for all } a, b, c \in A.$$

If $\omega$ is also nondegenerate, then $\omega$ is said to be symplectic. An alternative algebra $A$ equipped with a symplectic form is called a symplectic alternative algebra.

Proposition 2.16. Let $(A, \circ, \omega)$ be an alternative algebra with symplectic form $\omega$. Then $A$ has a compatible prealternative algebra structure $\prec, \succ$ given by

$$\omega(x \prec y, z) = \omega(x, y \circ z) \quad \text{and} \quad \omega(x \succ y, z) = \omega(y, z \circ x) \quad \text{for all } x, y, z \in A.$$

Proof. Define a linear map $T : A \to A^*$ by $(T(x), y) = \omega(x, y)$ for all $x, y \in A$. Then $T$ is invertible and $T$ is a 1-cocycle of $A$ into the bimodule $(A^*, l_\circ^*, r_\circ^*)$. So by Proposition 2.13, there is a compatible prealternative algebra structure $\prec, \succ$ on $A$ given by

$$x \prec y = T^{-1}(l_\circ^*(y)T(x)) \quad \text{and} \quad x \succ y = T^{-1}(l_\circ^*(x)T(y)) \quad \text{for all } x, y \in A.$$

Thus, for any $x, y \in A$,

$$\omega(x \prec y, z) = \langle T(x \prec y), z \rangle = \langle l_\circ^*(y)T(x), z \rangle = \omega(x, y \circ z),$$

$$\omega(x \succ y, z) = \langle T(x \succ y), z \rangle = \langle r_\circ^*(x)T(y), z \rangle = \omega(y, z \circ x).$$
3. Alternative D-bialgebras and an alternative analogue of the classical Yang–Baxter equation

**Definition 3.1** [Goncharov 2007; Zhelyabin 1997]. Let $M$ be an arbitrary variety of $k$-algebras and $(A, \circ)$ be an algebra in $M$ with comultiplication $\Delta$. Then $(A, \circ, \Delta)$ is called an $M$-bialgebra in the sense of Drinfeld if $D(A)$ belongs to $M$, where $D(A) = A \oplus A^*$ is equipped with the multiplication

\[(a + f) \star (b + g) = (a \circ b + f \cdot b + a \cdot g) + (f \ast g + f \bullet b + a \bullet g)\]

for all $a, b \in A$ and $f, g \in A^*$, where

\[f \cdot a = \sum a_{(1)} \langle f, a_{(2)} \rangle, \quad (f \cdot a, b) = \langle fa \circ b \rangle,\]
\[a \cdot f = \sum a_{(1)} \langle f, a_{(2)} \rangle a_{(2)}, \quad (a \cdot f, b) = \langle f, b \circ a \rangle,\]
\[\Delta(a) = \sum a_{(1)} \otimes a_{(2)},\]

and the multiplication $\star$ on $A^*$ is induced by $\Delta$. In this case, $D(A) = A \oplus A^*$ is called the Drinfeld double of $A$. In particular, when $M$ is a variety of alternative algebras, $(A, \circ, \Delta)$ is called an alternative $D$-bialgebra.

**Remark 3.2.** Goncharov [2007] notes that an alternative $D$-bialgebra $(A, \circ, \Delta)$ is equivalent to an alternative analogue of Manin triple [Chari and Pressley 1994]: There is an alternative algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of $A$ and $A^*$ such that both $A$ and $A^*$ are subalgebras and the symmetric bilinear form on $A \oplus A^*$ given by

\[(B + a^*, y + b^*) = \langle a^*, y \rangle + \langle x, b^* \rangle \quad \text{for all } x, y \in A \text{ and } a^*, b^* \in A^*\]

is invariant. Recall that a bilinear form $B$ on an alternative algebra $(A, \circ)$ is called invariant if

\[B(x \circ y, z) = B(x, y \circ z) \quad \text{for all } x, y, z \in A.\]

It is easy to show that $(A, \circ, \Delta)$ being an alternative $D$-bialgebra is equivalent to $(A, A^*, \circ^*, l^*_o, r^*_o, l^*_r, r^*_r)$ being a matched pair of alternative algebras in the sense of Proposition 4.7.

**Definition 3.3.** Let $(A, \circ)$ be an alternative algebra and \(r = \sum_i a_i \otimes b_i \in A \otimes A\). Then the pair $(A, r)$ is called a coboundary alternative $D$-bialgebra if $(A, \circ, \Delta_r)$, where

\[\Delta_r(x) = \sum_i a_i \circ x \otimes b_i - \sum_i a_i \otimes x \circ b_i \quad \text{for all } x \in A\]

is an alternative $D$-bialgebra.
Theorem 3.4 [Goncharov 2007]. Let \((A, \circ)\) be an alternative algebra and let \(r \in A \otimes A\). Assume that \(r\) is skew-symmetric and
\[
C_A(r) = r_{23} \circ r_{12} - r_{12} \circ r_{13} - r_{13} \circ r_{23} = 0.
\]
Then \((A, \circ, \Delta_r)\) is an alternative \(D\)-bialgebra.

Definition 3.5. Let \((A, \circ)\) be an alternative algebra and let \(r \in A \otimes A\). Goncharov [2007] calls Equation (3-5) an alternative analogue of the classical Yang–Baxter equation. We also call it the alternative Yang–Baxter equation in \((A, \circ)\).

Proposition 3.6. Let \((A, \circ)\) be an alternative algebra and \(r \in A \otimes A\). Then \(r\) is a solution of the alternative Yang–Baxter equation in \((A, \circ)\) if and only if \(T_r\) is an \(\mathcal{O}\)-operator associated to the bimodule \((A^*, T^*_r, l^*_r)\), that is, \(T_r\) satisfies the equation
\[
T_r(a^*) \circ T_r(b^*) = T_r(T^*_r(a^*)b^*) + l^*_r(T_r(b^*)a^*) \quad \text{for all } a^*, b^* \in A^*.
\]
So there is a prealternative algebra structure on \(A^*\) given by
\[
a^* < b^* = l^*_r(T_r(b^*))a^* \quad \text{and} \quad a^* > b^* = T^*_r(a^*)b^* \quad \text{for all } a^*, b^* \in A^*.
\]
Moreover, the associated alternative algebra structure is exactly the alternative algebra structure on \(A^*\) as a subalgebra of \(D(A) = A \oplus A^*\) that is induced from the comultiplication defined by (3-4). We denote this alternative algebra structure on \(A^*\) by \(A^*(r)\).

Proof. Let \(\{e_1, \ldots, e_n\}\) be a basis of \(A\) and \(\{e_1^*, \ldots, e_n^*\}\) its dual. Suppose that \(e_i \circ e_j = \sum_k c^k_{ij} e_k\) and \(r = \sum_{i,j} a_{ij} e_i \otimes e_j\). Hence \(a_{ij} = -a_{ji}\) and \(T_r(e_j^*) = \sum_k a_{kl} e_k\). Then \(r\) is a solution of the alternative Yang–Baxter equation in \((A, \circ)\) if and only if for any \(i, k, t\)
\[
\sum_{js} a_{st} a_{ij} c^k_{sj} - a_{jk} a_{st} c^i_{js} - a_{ij} a_{ks} c^t_{js} = 0.
\]
The left hand side of this equation is precisely the coefficient of \(e_i\) in
\[
-T_r(e_k^*) \circ T_r(e_j^*) + T_r(T^*_r(e_k^*)e_j^*) + l^*_r(T_r(e_j^*))e_k^*).
\]
Thus the first half part of the conclusion holds. It is easy to get the other results. \(\square\)

Corollary 3.7. Let \((A, \circ)\) be an alternative algebra and \(r \in A \otimes A\). Assume \(r\) is skew-symmetric and there exists a nondegenerate symmetric invariant bilinear form \(\mathcal{B}\) on \((A, \circ)\). Define a linear map \(\varphi : A \rightarrow A^*\) by \(\langle \varphi(x), y \rangle = \mathcal{B}(x, y)\) for any \(x, y \in A\). Then \(r\) is a solution of the alternative Yang–Baxter equation in \((A, \circ)\) if and only if \(\tilde{T}_r = T_r \varphi : A \rightarrow A\) is an \(\mathcal{O}\)-operator associated to the bimodule \((A, l_0, T_0)\), that is, \(\tilde{T}_r\) satisfies the equation
\[
\tilde{T}_r(x) \circ \tilde{T}_r(y) = \tilde{T}_r(\tilde{T}_r(x) \circ y + x \circ \tilde{T}_r(y)) \quad \text{for all } x, y \in A.
\]
Hence there is a prealternative algebra structure on $A$ given by
\begin{equation}
(3-9) \quad x < y = x \circ \tilde{T}_r(y) \quad \text{and} \quad x > y = \tilde{T}_r(x) \circ y \quad \text{for all} \ x, y \in A.
\end{equation}

**Proof.** For all $x, y, z \in A$, we have
\[ \langle \varphi(l_o(x)y), z \rangle = \mathcal{B}(x \circ y, z) = \mathcal{B}(z, x \circ y) = \mathcal{B}(y, z \circ x) = \{r_o^*(x)\varphi(y), z\}. \]

Hence $\varphi(l_o(x)y) = r_o^*(x)\varphi(y)$ and similarly $\varphi(r_o(x)y) = l_o^*(x)\varphi(y)$ for any $x, y \in A$.

Let $a^* = \varphi(x)$, $b^* = \varphi(y)$. Then by Proposition 3.6, $r$ is a solution of the alternative Yang–Baxter equation in $A$ if and only if
\[ T_r\varphi(x) \circ T_r\varphi(y) = T_r(a^*) \circ T_r(b^*) = T_r(r_o^*(T_r(a^*))b^* + l_o^*(T_r(b^*))a^*) \]
\[ = T_r\varphi(T_r(x) \circ y + x \circ T_r\varphi(y)). \quad \square \]

**Remark 3.8.** Equation (3-8) is exactly the Rota–Baxter relation of weight zero for an alternative algebra; see [Baxter 1960; Rota 1969].

**Proposition 3.9.** Let $(A, \circ)$ be an alternative algebra, $(V, L, R)$ a bimodule of $A$, and $(V^*, R^*, L^*)$ the dual bimodule. Let $T : V \to A$ be a linear map that can be identified as an element in $A \rtimes_{R^*, L^*} V^* \otimes A \rtimes_{R^*, L^*} V^*$. Then $T$ is an $\mathcal{O}$-operator of $A$ associated to $(V, L, R)$ if and only if $r = T - \sigma(T)$ is a skew-symmetric solution of the alternative Yang–Baxter equation in $A \rtimes_{R^*, L^*} V^*$.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a basis of $A$. Let $\{v_1, \ldots, v_m\}$ be a basis of $V$ and $\{v_1^*, \ldots, v_m^*\}$ be its dual. Set $T(v_i) = \sum_{k=1}^n a_{ik}e_k$ for $i = 1, \ldots, m$. Then
\[ T = \sum_{i=1}^m T(v_i) \otimes v_i^* = \sum_{i=1}^m \sum_{k=1}^n a_{ik}e_k \otimes v_i^* \in A \otimes V^* \subset (A \rtimes_{R^*, L^*} V^*) \otimes (A \rtimes_{R^*, L^*} V^*). \]

Therefore we have
\[ r_{12} \circ r_{13} = \sum_{i,k=1}^m (T(v_i) \circ T(v_k) \otimes v_k^* \otimes v_k^* - R^*(T(v_i))v_k^* \otimes v_i \otimes T(v_k) \]
\[ - L^*(T(v_k))v_i^* \otimes T(v_i) \otimes v_k^*), \]
\[ r_{23} \circ r_{12} = \sum_{i,j=1}^m (T(v_k) \otimes R^*(T(v_j))v_k^* \otimes v_i^* - v_k^* \otimes T(v_i) \otimes v_i^* + v_k^* \otimes L^*(T(v_k))v_i^* \otimes T(v_i), \]
\[ r_{13} \circ r_{23} = \sum_{i,k=1}^m (v_i^* \otimes v_k^* \otimes T(v_i) \circ T(v_k) - T(v_i) \otimes v_k^* \otimes L^*(T(v_k))v_i^* \]
\[ - v_i^* \otimes T(v_k) \otimes R^*(T(v_i))v_i^*). \]

By the definition of a dual bimodule, we know
\[ L^*(T(v_k))v_i^* = \sum_{j=1}^m [v_i^*, L(T(v_k))v_j^*], \quad R^*(T(v_k))v_i^* = \sum_{j=1}^m [v_i^*, R(T(v_k))v_j^*. \]
Then
\[
\sum_{i,k=1}^{m} T(v_i) \otimes v_k^* \otimes L^*(T(v_k))v_i^* = \sum_{i,k=1}^{m} \sum_{j=1}^{m} (v_j^*, L(T(v_k))v_j) T(v_i) \otimes v_k^* \otimes v_i^*
\]
\[
= \sum_{i,k=1}^{m} T((v_j^*, L(T(v_k))v_j)^*) \otimes v_k^* \otimes v_i^*
\]
\[
= \sum_{i,k=1}^{m} T(L(T(v_k))v_i) \otimes v_k^* \otimes v_i^*.
\]

Hence, we get
\[
r_{12} \circ r_{13} + r_{13} \circ r_{23} - r_{23} \circ r_{12}
\]
\[
= \sum_{i,k=1}^{m} ((T(v_i) \circ T(v_k) - T(L(T(v_i))v_k) - T(R(T(v_k))v_i)) \otimes v_i^* \otimes v_k^*
\]
\[
+ v_k^* \otimes (T(v_i) \circ T(v_k) - T(L(T(v_i))v_k) - T(R(T(v_k))v_i)) \otimes v_i^*
\]
\[
+ v_i^* \otimes v_k^* \otimes (T(v_i) \circ T(v_k) - T(L(T(v_i))v_k) - T(R(T(v_k))v_i)).
\]

So \( r \) is a solution of the alternative Yang–Baxter equation in \( A \ltimes_{R^*, L^*} V^* \) if and only if \( T \) is an \( \mathcal{O} \)-operator of \( A \) associated to \( (V, L, R) \). \qed

Proposition 3.10 [Goncharov 2007]. Let \( (A, \circ) \) be an alternative algebra and let \( r \in A \otimes A \). Suppose that \( r \) is skew-symmetric and nondegenerate. Then \( r \) is a solution of the alternative Yang–Baxter equation in \( A \) if and only if the bilinear form \( \omega \) induced by \( r \) through (1-8) is a symplectic form.

Corollary 3.11. Let \( (A, \prec, \succ) \) be a prealternative algebra. Let \( \{e_1, \ldots, e_n\} \) be a basis of \( A \) and let \( \{e_1^*, \ldots, e_n^*\} \) be its dual. Then \( r = \sum_i (e_i \otimes e_i^* - e_i^* \otimes e_i) \) is a non-degenerate solution of the alternative Yang–Baxter equation in \( \text{Alt}(A) \ltimes_{\tau^*, \tau^*} A^* \).

The symplectic form \( \omega_p \) induced by \( r \) through (1-8) is given by
\[
(3-10) \quad \omega_p(x + a^*, y + b^*) = (a^*, y) - (x, b^*) \quad \text{for all } x, y \in A \text{ and } a^*, b^* \in A^*.
\]

Proof. It follows from the fact that \( T = \text{id} \) is an \( \mathcal{O} \)-operator of \( \text{Alt}(A) \) associated to the bimodule \((A, l_-, r_-)\). \( \Box \)

Proposition 3.12. Let \( (A, \circ, \omega) \) be an alternative algebra with symplectic form \( \omega \). Suppose that there is a compatible prealternative algebra structure \( <, > \) on \( A \) given by Proposition 2.16 and a prealternative algebra structure \( <_s, >_s \) on \( A^* \) given by (3-7), where the solution \( r \) of the alternative Yang–Baxter equation in \( (A, \circ) \) is induced by \( \omega \) through (1-8). Let \( a^* \star b^* = a^* \prec_b^* b^* + a^* \succ_b^* b^* \) for any \( a^*, b^* \in A^* \). Then there is a prealternative algebra structure \( <_0, >_0 \) on \( A \oplus A^* \) given for any \( x, y \in A \) and \( a^*, b^* \in A^* \) by
\[
(x, a^*) < (y, b^*) = (x < y + l_0^*(b^*)x, a^* \prec b^* + l_0^*(y)a^*),
\]
\[
(x, a^*) > (y, b^*) = (x > y + r_0^*(a^*)y, a^* \succ b^* + r_0^*(x)b^*).
\]
Moreover, the associated alternative algebra is just the Drinfeld double $D(A)$ for the coboundary alternative $D$-bialgebra $(A, \circ, \Delta_r)$.

**Proof.** In fact, since $r$ is invertible, it is easy to show for any $x, y \in A$ and $a^*, b^* \in A^*$ that
\[
\begin{align*}
\Gamma^*_o(t^*_r(b^*)) &= x < T_r(b^*), & \Gamma^*_o(t^*_r(a^*)) &= y < T_r(a^*), & \Gamma^*_o(t^*_r(y)) &= a < T_r^{-1}(y), & \Gamma^*_o(t^*_r(x)) &= b^* > T_r^{-1}(x) \Rightarrow b^*.
\end{align*}
\]
So for any $z \in A$ and $c^* \in A^*$,
\[
((x, a^*) >_0 (y, b^*)) <_0 (z, c^*)
\]
\[
= ((x + T_r(a^*)) > y, (T_r^{-1}(x) + a^*) >_r b^*) <_0 (z, c^*)
\]
\[
= ((x > y) < z + (x > y) < T_r(c^*) + (T_r(a^*) > y) < z + (T_r(a^*) > y) < T_r(c^*) + (T_r^{-1}(x) >_r b^*) <_r T_r^{-1}(z) + (T_r^{-1}(x) >_r b^*) <_r c^* + (a^* > b^*) <_r T_r^{-1}(z) + (a^* > b^*) <_r c^*).
\]

Similarly,
\[
((y, b^*) <_0 (x, a^*)) <_0 (z, c^*)
\]
\[
= ((y < x) < z + (y < x) < T_r(c^*) + (y < T_r(a^*)) < z + (y < T_r(a^*)) < T_r(c^*) + (b^* <_r T_r^{-1}(x)) <_r T_r^{-1}(z) + (b^* <_r T_r^{-1}(x)) <_r c^* + (b^* <_r a^*) <_r c^*),
\]
\[
(x, a^*) >_0 ((y, b^*) <_0 (z, c^*))
\]
\[
= (x > (y < T_r(c^*)) + x > (y < z + T_r(a^*) > (y < z) + T_r(a^*) > (y < T_r(c^*)) + (T_r^{-1}(x) >_r b^*) <_r T_r^{-1}(x) >_r (b^* <_r T_r^{-1}(z)) + a^* >_r (b^* <_r c^*) + a^* >_r (b^* <_r T_r^{-1}(z))),
\]
\[
(y, b^*) <_r ((x, a^*) \bullet (z, c^*))
\]
\[
= (y < (T_r(a^*) \circ T_r(c^*)) + y < (T_r(a^*) < z) + y < (x > T_r(c^*)) + y < (T_r(a^*) < z), b^* <_r (a^* \bullet c^*) + b^* <_r (T_r^{-1}(z)) + b^* <_r (T_r^{-1}(x) >_r c^*) + b^* <_r (T_r^{-1}(x) >_r T_r^{-1}(x)) + b^* <_r (T_r^{-1}(x) >_r c^*) + b^* <_r (a^* >_r T_r^{-1}(z))),
\]

where $\bullet = <_0 \circ >_0$. Hence
\[
((x, a^*) >_0 (y, b^*)) <_0 (z, c^*) + ((y, b^*) <_0 (x, a^*)) <_0 (z, c^*)
\]
\[
= (x, a^*) >_0 ((y, b^*) <_0 (z, c^*)) + (y, b^*) <_r ((x, a^*) \bullet (z, c^*)).
\]

Using a similar argument, we can prove that $(<_0, >_0)$ also satisfies (2-6) and the second equation in (2-5).
Proposition 3.13. Let \((A, \circ)\) be an alternative algebra.

(1) For any skew-symmetric solution \(r\) of the alternative Yang–Baxter equation, the Drinfeld double \(D(A)\) of the coboundary alternative \(D\)-bialgebra \((A, \circ, \Delta_r)\) is isomorphic to \(A \ltimes \mathbb{C}_r, \iota\) as alternative algebras.

(2) The skew-symmetric solutions of the alternative Yang–Baxter equation in \((A, \circ)\) are in one-to-one correspondence with the linear maps \(T_r : A^* \to A\) whose graphs

\[
\text{graph}(T_r) = \{(T_r(a^*), a^*) \in A \ltimes \mathbb{C}_r, \iota\ A^* \mid a^* \in A^*\}
\]

are Lagrangian subalgebras of \(A \ltimes \mathbb{C}_r, \iota\ A^*\) with respect to the bilinear form given by \((3-2)\). Consequently every alternative subalgebra that is also a Lagrangian graph \((T_r)\) of \(A \ltimes \mathbb{C}_r, \iota\ A^*\) carries a prealternative algebra structure defined for any \(a^*, b^* \in A^*\) by

\[
\begin{align*}
(T_r(a^*), a^*) &< (T_r(b^*), b^*) = (T_r(I_0^*(T_r(b^*))a^*), I_0^*(T_r(b^*))a^*), \\
(T_r(a^*), a^*) &> (T_r(b^*), b^*) = (T_r(T_r(a^*))b^*, T_r(a^*))b^*).
\end{align*}
\]

Proof. (1) Let \(r\) be a skew-symmetric solution of the alternative Yang–Baxter equation in \((A, \circ)\). Let the operation in \(A^*(r)\) be \(*\). Then by Proposition 3.6, we know \(a^* * b^* = r_o^*(T_r(a^*))b^* + l_o^*(T_r(b^*))a^*\) for any \(a^*, b^* \in A^*.\) We claim that for any \(x, y \in A\) and \(a^*, b^* \in A^*\), we have

\[
(3-11) \quad r^*_*(a^*)y + l^*_o(b^*)x = T_r(a^*) \circ y + x \circ T_r(b^*) - T_r(r_o^*(x)b^* + l_o^*(y)a^*).
\]

In fact, it follows from the computation (for any \(c^* \in A^*\))

\[
\begin{align*}
& \langle r^*_o(a^*)y + l^*_o(b^*)x, c^* \rangle = \langle y, c^* \circ a^* \rangle + \langle x, b^* \circ c^* \rangle \\
& = \langle y, r^*_o(T_r(c^*)a^*) + l^*_o(T_r(a^*)c^*) \rangle + \langle x, r^*_o(T_r(b^*)c^*) + l^*_o(T_r(c^*))b^* \rangle \\
& = \langle y \circ T_r(c^*), a^* \rangle + \langle T_r(a^*) \circ y, c^* \rangle + \langle x \circ T_r(b^*), c^* \rangle + \langle T_r(c^*) \circ x, b^* \rangle \\
& = \langle T_r(a^*) \circ y + x \circ T_r(b^*) - T_r(r_o^*(x)b^* + l_o^*(y)a^*), c^* \rangle,
\end{align*}
\]

where we use that \(T_r(a^*, b^*) = -(a^*, T_r(b^*))\), which follows from the fact that \(r\) is skew-symmetric. Define a linear map \(\lambda : (D(A) = A \oplus A^*, \circ) \to (A \ltimes \mathbb{C}_r, \iota, A^*, \circ)\) by \(\lambda((x, a^*)) = (T_r(a^* + x), a^*)\) for all \(x \in A, a^* \in A^*\). Then we have

\[
\lambda((x, a^*)) \circ \lambda((y, b^*)) = \lambda(((T_r(a^* + x) \circ (T_r(b^*) + y), r_o^*(T_r(a^*) + x)b^* + l_o^*(T_r(b^*) + x)a^*) \\
= \lambda(T_r(a^* \circ b^* + r_o^*(x)b^* + l_o^*(y)a^*) + x \circ y + r_o^*(a^*)y + l_o^*(b^*)x, a^* \circ b^*) \\
= \lambda((x, a^*) \circ (y, b^*)).
\]
where we used (3-6) and (3-11). Furthermore, it is easy to show that λ is bijective. Therefore λ is an isomorphism of alternative algebras.

(2) First, λ(A*(r)) = \text{graph}(T_r). So \text{graph}(T_r) is a subalgebra of \( A \times \mathbb{C} \mathfrak{e} \). Since \( r \) is skew-symmetric, \text{graph}(T_r) is isotropic with respect to the bilinear form defined by (3-2). Moreover, it has a complementary isotropic algebra \( \lambda(A) = A \). So it is a Lagrangian subalgebra of \( A \times \mathbb{C} \mathfrak{e} \). Conversely, let \( T : A^* \rightarrow A \) be a linear map whose \text{graph}(T) is a Lagrangian subalgebra of \( A \times \mathbb{C} \mathfrak{e} \). So \( T \) is skew-symmetric, that is, \( \langle T(a^*), b^* \rangle = -\langle T(b^*), a^* \rangle \) for any \( a^*, b^* \in A^* \). Since graph(T) is a subalgebra, we have

\[
(T(a^*), a^*) \star (T(b^*), b^*) = (T(a^*) \circ T(b^*), \tau^s_0(T_r(a^*))b^* + \tau^s_0(T_r(b^*))a^*) \\
= (T_r(\tau^s_0(T_r(a^*)))b^* + \tau^s_0(T_r(b^*))a^*, \tau^s_0(T_r(a^*))b^* + \tau^s_0(T_r(b^*))a^*).
\]

Thus \( T(a^*) \circ T(b^*) = T(\tau^s_0(T_r(a^*)))b^* + \tau^s_0(T_r(b^*))a^* \). By Proposition 3.6, \( T \) corresponds to a skew-symmetric solution of the alternative Yang–Baxter equation in \((A, \circ)\). The last statement is obtained by transferring (by the isomorphism \( \lambda \)) the prealternative algebra structure of \( A^*(r) \) to \text{graph}(T_r).

\[\square\]

4. Phase spaces of alternative algebras

and matched pairs of alternative algebras

**Definition 4.1.** Let \((A, \circ, \omega)\) be a symplectic alternative algebra. We call \( A \) an \textit{L-symplectic alternative algebra} if it is a direct sum of the underlying vector spaces of two Lagrangian subalgebras \( A^+ \) and \( A^- \), we denote it by \((A, \circ, A^+, A^-, \omega)\). Two \textit{L-symplectic alternative algebras} \((A_1, \circ, A_1^+, A_1^-, \omega_1)\) and \((A_2, \circ, A_2^+, A_2^-, \omega_2)\) are \textit{isomorphic} if there exists an isomorphism \( \varphi : A_1 \rightarrow A_2 \) of alternative algebras such that, for all \( a, b \in A_1 \),

\[
(4-1) \quad \varphi(A_1^+) = A_2^+, \quad \varphi(A_1^-) = A_2^-, \quad \omega_1(a, b) = \varphi^* \omega_2(\varphi(a), \varphi(b))
\]

It is straightforward to show that a symplectic alternative algebra \((A, \circ, \omega)\) is an \textit{L-symplectic alternative algebra} if and only if \( A \) is a direct sum of the underlying vector space of two isotropic subalgebras.

**Proposition 4.2.** Let \((A, \circ, A^+, A^-, \omega)\) be an \textit{L-symplectic alternative algebra}. Then there exists a prealternative algebra structure on \( A \) given by Proposition 2.16 such that \( A^+ \) and \( A^- \) are prealternative subalgebras. Two \textit{L-symplectic alternative algebras} \((A_i, \circ, A_i^+, A_i^-, \omega_i)\) for \( i = 1, 2 \) are isomorphic if and only if there exists an isomorphism of prealternative algebras satisfying (4-1) in which the compatible prealternative algebras are given by Proposition 2.16.

**Proof.** If \( a, b, c \in A^+ \), then \( \omega(a \prec b, c) = \omega(a, b \circ c) = 0 \). Since \( A^+ \) is a Lagrangian subalgebra of \( A \), we have \( a \prec b \in A^+ \) for all \( a, b \in A^+ \). Similar arguments apply to \( \succ \) and \( A^- \). So the first conclusion holds. It is easy to get the second. \[\square\]
**Definition 4.3.** Let \((A, \circ)\) be an alternative algebra. If there exists an alternative algebra structure on the direct sum of the underlying vector space of \(A\) and \(A^*\) such that \(A\) and \(A^*\) are alternative subalgebras and the natural skew-symmetric bilinear form \(\omega_p\) on \(A \oplus A^*\) given by (3-10) is a symplectic form, then it is called a *phase space* of the alternative algebra \(A\).

**Remark 4.4.** The notion of phase space is borrowed from mathematical physics [Kupershmidt 1994; Bai 2006].

**Proposition 4.5.** Every \(L\)-symplectic alternative algebra \((A, \circ, A^+, A^-, \omega)\) is isomorphic to a phase space of \(A^+\).

**Proof.** Since \(A^-\) and \((A^+)^*\) are identified by the symplectic form, we can transfer the alternative algebra structure on \(A^-\) to \((A^+)^*\). Hence the alternative algebra structure on \(A^+ \oplus A^-\) can be transferred to \(A^+ \oplus (A^+)^*\). \(\square\)

**Remark 4.6.** By symmetry of \(A^+\) and \(A^-\), every \(L\)-symplectic alternative algebra \((A, \circ, A^+, A^-, \omega)\) is isomorphic to a phase space of \(A^-\).

**Proposition 4.7.** Let \((A, \circ)\) and \((B, \ast)\) be two alternative algebras. Suppose that there are linear maps \(L_A, R_A : A \to \text{gl}(B)\) and \(L_B, R_B : B \to \text{gl}(A)\) such that \((L_A, R_A)\) is a bimodule of \(A\) and \((L_B, R_B)\) is a bimodule of \(B\) and they satisfy the conditions

\[
\begin{align*}
\text{(4-2)} & \quad L_B(\text{Ass}_A(x)a)y + (\text{Ass}_B(a)x) \circ y = L_B(a)(x \circ y) + R_B(R_A(y)a)x + x \circ (L_B(a)y), \\
\text{(4-3)} & \quad R_B(a)(x \circ y + y \circ x) = R_B(L_A(y)a)x + x \circ (R_B(a)y) + R_B(L_A(a)x)y + y \circ (R_B(a)x), \\
\text{(4-4)} & \quad R_B(a)(x \circ y) + L_B(L_A(x)a)y + (R_B(a)x) \circ y = R_B(\text{Ass}_A(y)a)x + x \circ (\text{Ass}_B(a)y), \\
\text{(4-5)} & \quad L_B(a)(x \circ y + y \circ x) = (L_B(a)x) \circ y + L_B(R_A(x)a)y + (L_B(a)y) \circ x + L_B(R_A(y)a)x, \\
\text{(4-6)} & \quad L_A(\text{Ass}_B(a)x)b + (\text{Ass}_A(x)a) \ast b = L_A(x)(a \ast b) + R_A(R_B(b)x)a + a \ast (L_A(x)b), \\
\text{(4-7)} & \quad R_A(x)(a \ast b + b \ast a) = R_A(L_B(b)x)a + a \ast (R_A(x)b) + R_A(L_B(a)x)b + b \ast (R_A(x)a), \\
\text{(4-8)} & \quad R_A(x)(a \ast b + L_A(L_B(a)x)b + (R_A(x)a) \ast b = R_A(\text{Ass}_B(b)x)a + a \ast (\text{Ass}_A(x)b), \\
\text{(4-9)} & \quad L_A(x)(a \ast b + b \ast a) = (L_A(x)a) \ast b + L_A(R_B(a)x)b + (L_A(x)b) \ast a + L_A(R_B(b)x)a,
\end{align*}
\]
where \( x, y \in A, \ a, b \in B \) and \( \text{Ass}_i = L_i + R_i \) for \( i = A, B \). Then there is an alternative algebra structure on the vector space \( A \oplus B \) given for all \( x, y \in A \) and \( a, b \in B \) by

\[
(x + a) \star (y + b) = (x \circ y + L_B(a)y + R_B(b)x) + (a \ast b + L_A(x)b + R_A(y)a)
\]

We denote this alternative algebra by \( A \triangleright\triangleleft_{L_A, R_A} B \) or simply \( A \triangleright\triangleleft B \). We call any \((A, B, L_A, R_A, L_B, R_B)\) satisfying the conditions above a matched pair of alternative algebras. Every alternative algebra that is a direct sum of the underlying vector spaces of two subalgebras can be obtained this way.

**Proof.** Straightforward. \( \square \)

**Proposition 4.8.** Let \((A, \prec, \succ)\) be a prealternative algebra and \((\text{Alt}(A), \circ)\) be the associated alternative algebra. Suppose there exists a prealternative algebra structure \(\prec, \succ\) on the dual space \(A^*\), with \((\text{Alt}(A^*), \star)\) the associated alternative algebra. Then there exists an \(L\)-symplectic alternative algebra structure on \(A \oplus A^*\) such that \((\text{Alt}(A), \circ)\) and \((\text{Alt}(A^*), \star)\) are Lagrangian subalgebras associated to the symplectic form \((3-10)\) if and only if \((\text{Alt}(A), \text{Alt}(A^*), \tau^*_<, \tau^*_>, \tau^*_l, \tau^*_r, \tau^*_\triangleright, \tau^*_\triangleleft)\) is a matched pair of alternative algebras. Every \(L\)-symplectic alternative algebra can be obtained in this way.

**Proof.** If \((\text{Alt}(A), \text{Alt}(A^*), \tau^*_<, \tau^*_>, \tau^*_l, \tau^*_r, \tau^*_\triangleright, \tau^*_\triangleleft)\) is a matched pair of alternative algebras, then it is straightforward to show that the bilinear form \((3-10)\) is a symplectic form of the alternative algebra \(A_{\triangleright\triangleleft} := \text{Alt}(A) \triangleright\triangleleft_{\tau^*_<, \tau^*_>, \tau^*_l, \tau^*_r} \text{Alt}(A^*)\). Conversely, set \(x \ast a^* = L_\circ(x)a^* + R_\triangleright(a^*)x, \ a^* \star x = L_\triangleright(a^*)x + R_\circ(x)a^*\) for all \(x \in A, a^* \in A^*\), where \(\ast\) is the alternative algebra structure of \(A_{\triangleright\triangleleft}\). Then \((A, A^*, L_\circ, R_\triangleright, L_\triangleright, R_\circ)\) is a matched pair of alternative algebras. Note that

\[
\langle R_\circ(x)a^*, y \rangle = \langle a^* \star x, y \rangle = -\omega_p(y, a^* \ast x) = -\omega_p(x \succ_1 y, a^*) = \langle \tau^*_< \triangleright_1 (x)a^*, y \rangle
\]

\[
\langle L_\triangleright(a^*)x, b^* \rangle = \langle a^* \star x, b^* \rangle = \omega_p(b^*, a^* \ast x) = \omega_p(b^* \prec_2 a^*, x) = \langle \tau^*_< (a^*)x, b^* \rangle
\]

where \(x, y \in A\) and \(a^*, b^* \in A^*\). Hence, \(R_\circ = \tau^*_< \triangleright_1\) and \(L_\triangleright = \tau^*_< \prec_2\). Similarly, \(L_\circ = \tau^*_> \prec_1\) and \(R_\triangleright = \tau^*_> \triangleright_2\). \( \square \)

5. Bimodules and matched pairs of prealternative algebras

**Definition 5.1.** Let \((A, \prec, \succ)\) be a prealternative algebra and \(V\) be a vector space. Let \(L_\prec, R_\prec, L_\succ, R_\succ : A \rightarrow \text{gl}(V)\) be linear maps. We call \(V\) (or \((L_\prec, R_\prec, L_\succ, R_\succ)\) or \((V, L_\prec, R_\prec, L_\succ, R_\succ)\)) a representation or a bimodule of \(A\) if (for any \(x, y \in A\))

\[
L_\succ(x \circ y + y \circ x) = L_\succ(x)L_\succ(y) + L_\succ(y)L_\succ(x)
\]

\[
R_\succ(y)(L_\circ(x) + R_\circ(x)) = L_\succ(x)R_\succ(y) + R_\succ(x \succ y)
\]

\]
Proof. We only prove (3) as an example. The others are straightforward. Since $v \in \text{Alt}(A)$ and $A$ is a bimodule of $\text{Alt}(A)$, we have

$$R_v(x) = R_{\circ}(x) \circ y = R_{\circ}(x) \circ y,$$

$$(x + a) < (y + b) = x < y + L_{\times}(x)b + R_{\times}(y)a,$$

$$(x + a) > (y + b) = x > y + L_{\times}(x)b + R_{\times}(y)a.$$

We denote it by $A \ltimes_{L_{\times}, R_{\circ}, L_{\times}, R_{\circ}} V$ or simply $A \ltimes V$.

**Proposition 5.2.** Suppose $(V, L_{\times}, R_{\circ}, L_{\times}, R_{\circ})$ is a bimodule of a prealternative algebra $(A, \ltimes, \succ)$. Let $(\text{Alt}(A), \circ)$ be the associated alternative algebra.

1. Both $(V, L_{\times}, R_{\circ})$ and $(V, L_{\circ} = L_{\times} + L_{\times}, R_{\circ} = R_{\times} + R_{\circ})$ are bimodules of $(\text{Alt}(A), \circ)$.

2. If $(V, L, R)$ is a bimodule of $(\text{Alt}(A), \circ)$, then $(V, 0, R, L, 0)$ is a bimodule of $(A, \ltimes, \succ)$.

3. $(V^*, -R_{\circ}^*, L_{\circ}^*, R_{\circ}^*, -L_{\times}^*)$ is a bimodule of $(A, \ltimes, \succ)$.

**Proof.** We only prove (3) as an example. The others are straightforward. Since $(V, L_{\circ}, R_{\circ})$ is a bimodule of $\text{Alt}(A)$, we have $R_{\circ}(x^2) = R_{\circ}(x) \circ R_{\circ}(x)$ for any $x \in A$. Hence $R_{\circ}(x \circ y + y \circ x) = R_{\circ}(x) \circ R_{\circ}(y) + R_{\circ}(y) \circ R_{\circ}(x)$ for all $x, y \in A$. So for any $v \in V$ and $u^* \in V^*$,

$$\langle R_{\circ}^*(x \circ y + y \circ x)u^*, v \rangle = \langle u^*, R_{\circ}(x \circ y + y \circ x)v \rangle = \langle u^*, (R_{\circ}(x) \circ R_{\circ}(y) + R_{\circ}(y) \circ R_{\circ}(x))v \rangle$$

$$= \langle (R_{\circ}^*(x))R_{\circ}^*(y) + R_{\circ}^*(y)R_{\circ}^*(x))u^*, v \rangle.$$

Therefore $R_{\circ}^*(x \circ y + y \circ x) = R_{\circ}^*(x)R_{\circ}^*(y) + R_{\circ}^*(y)R_{\circ}^*(x)$. Similarly, we can prove that $(-R_{\circ}^*, L_{\circ}^*, R_{\circ}^*, -L_{\times}^*)$ also satisfies the remaining requirements (5-2)–(5-10) of a bimodule. \qed
Example 5.3. Let $(A, \langle, \rangle)$ be a prealternative algebra. Then $(I_{\prec}, r_{\prec}, I_{\succ}, r_{\succ}), (0, r_{\prec}, I_{\prec}, 0), (0, r_{\succ}, I_{\succ}, 0)$ and $(-r_{\prec}, I_{\prec}, r_{\succ}, -I_{\succ})$ are bimodules of $(A, \langle, \rangle)$.

Definition 5.4. Let $(A, \langle A, \rangle_A)$ and $(B, \langle B, \rangle_B)$ be two prealternative algebras. Suppose that there are linear maps

\[ L_{\langle A}, R_{\langle A}, L_{\rangle A}, R_{\rangle A} : A \rightarrow \text{gl}(B) \quad \text{and} \quad L_{\langle B}, R_{\langle B}, L_{\rangle B}, R_{\rangle B} : B \rightarrow \text{gl}(A) \]

such that the products

\[(x+a) \langle y+b \rangle = x \langle A y \rangle + L_{\langle B}(a) y + R_{\langle B}(b) x + a \langle B b \rangle + L_{\langle A}(x) b + R_{\langle A}(y) a, \]

\[(x+a) \rangle (y+b) = x \rangle A y + L_{\rangle B}(a) y + R_{\rangle B}(b) x + a \rangle B b + L_{\rangle A}(x) b + R_{\rangle A}(y) a, \]

on the vector space $A \oplus B$ (for any $x, y \in A$ and $a, b \in B$) define a prealternative algebra structure. Then we call $(A, B, L_{\langle A}, R_{\langle A}, L_{\rangle A}, R_{\rangle A}, L_{\langle B}, R_{\langle B}, L_{\rangle B}, R_{\rangle B})$ a matched pair of prealternative algebras, and we denote this pair by $A \triangleright L_{\langle B}, R_{\langle B}, L_{\rangle B}, R_{\rangle B} B$ or simply $A \triangleright B$.

Remark 5.5. The analogue of Proposition 4.7 for a matched pair of prealternative algebras contains 20 equations. We omit them. Note that $(B, L_{\langle A}, R_{\langle A}, L_{\rangle A}, R_{\rangle A})$ and $(A, L_{\langle B}, R_{\langle B}, L_{\rangle B}, R_{\rangle B})$ must be bimodules of $A$ and $B$, respectively.

Corollary 5.6. Suppose $(A, B, L_{\langle A}, R_{\langle A}, L_{\rangle A}, R_{\rangle A}, L_{\langle B}, R_{\langle B}, L_{\rangle B}, R_{\rangle B})$ is a matched pair of prealternative algebras. Then

\[(\text{Alt}(A), \text{Alt}(B), L_{\langle A} + L_{\rangle A}, R_{\langle A} + R_{\rangle A}, L_{\langle B} + L_{\rangle B}, R_{\langle B} + R_{\rangle B})\]

is a matched pair of alternative algebras.

Proof. It follows from the relationship between a prealternative algebra and the associated alternative algebra. 

\[\square\]

Proposition 5.7. Let $(A, \langle 1, \rangle 1)$ be a prealternative algebra and $(\text{Alt}(A), \circ_1)$ be the associated alternative algebra. Suppose there is a prealternative algebra structure $\langle 2, \rangle 2$ on the dual space $A^*$ and $(\text{Alt}(A^*), \circ_2)$ is the associated alternative algebra. Then $(\text{Alt}(A), \text{Alt}(A^*), r^e_{\langle 1}, I^e_{\langle 1}, r^e_{\langle 2}, I^e_{\langle 2})$ is a matched pair of alternative algebras if and only if $(A, A^*, -r^e_{\langle 1}, I^e_{\langle 1}, r^e_{\langle 2}, I^e_{\langle 2})$ is also.

Proof. By Corollary 5.6, we only need to prove the “only if” part of the conclusion. If $(\text{Alt}(A), \text{Alt}(A^*), r^e_{\langle 1}, I^e_{\langle 1}, r^e_{\langle 2}, I^e_{\langle 2})$ is a matched pair of alternative algebras, then by Proposition 4.8,

\[\mathcal{A} := \text{Alt}(A) \triangleright L_{\langle B}, R_{\langle B}, L_{\rangle B}, R_{\rangle B} \text{Alt}(B)\]

is an $L$-symplectic alternative algebra with symplectic form given by $(3-10)$. Hence Proposition 2.16 gives a compatible prealternative algebra structure on $\mathcal{A}$. Then
for any \( x, y \in A \) and \( a^*, b^* \in A^* \) we have

\[
\langle a^* < x, y \rangle = \omega_p(a^* < x, y) = \omega_p(a^*, x \circ_1 y) = \langle l_{o_1}^*(x)a^*, y \rangle,
\]

\[
\langle a^* < x, b^* \rangle = -\omega_p(a^* < x, b^*) = -\langle a^*, l_{x_2}^*(b^*)x \rangle = -\langle b^* >_2 a^*, x \rangle = \langle -l_{x_2}^*(a^*)x, b^* \rangle.
\]

So \( a^* < x = -l_{x_2}^*(a^*)x + l_{o_1}(x)a^* \). Similarly,

\[
x < a^* = -l_{x_1}^*(x)a^* + l_{o_2}(a^*)x,
\]

\[
x > a^* = r_{o_1}^*(x)a^* - l_{x_2}^*(a^*)x,
\]

\[
a^* > x = r_{o_2}(a^*)x - l_{x_1}^*(x)a^*.
\]

Therefore \((A, A^*, -l_{x_1}^*, l_{o_1}^*, r_{o_1}^*, -l_{x_2}^*, l_{o_2}^*, r_{o_2}^*, -l_{x_2}^*)\) is a matched pair of prealternative algebras. \(\Box\)

### 6. Prealternative bialgebras

**Theorem 6.1.** Let \((A, <, >, \alpha, \beta)\) be a prealternative algebra \((A, <, >)\) equipped with two comultiplications \(\alpha, \beta : A \rightarrow A \otimes A\) and let \((\text{Alt}(A), \circ)\) be the associated alternative algebra. Suppose \(\alpha^*, \beta^* : A^* \otimes A^* \rightarrow A^*\) induce a prealternative algebra structure \(<_*, >_*\) on the dual space \(A^*\). Then \((\text{Alt}(A), \text{Alt}(A^*), \tau_{x_*}, l_{x_*}^*, r_{x_*}^*, l_{x_*}, r_{x_*})\) is a matched pair of alternative algebras if and only if \(\alpha, \beta\) satisfy the following eight equations for any \(x, y \in A\):

\[
(6-1) \quad \alpha(x \circ y + y \circ x) = (r_{o}(y) \otimes 1 + 1 \otimes l_{x}(y))\alpha(x) + (r_{o}(x) \otimes 1 + 1 \otimes l_{x}(x))\alpha(y).
\]

\[
(6-2) \quad \beta(x \circ y + y \circ x) = (r_{x}(y) \otimes 1 + 1 \otimes l_{o}(y))\beta(x) + (r_{x}(x) \otimes 1 + 1 \otimes l_{o}(x))\beta(y),
\]

\[
(6-3) \quad \alpha(x \circ y) = (1 \otimes r_{x}(x) + 1 \otimes l_{x}(x) - l_{o}(x) \otimes 1)\alpha(y) + (r_{o}(y) \otimes 1)\alpha(x)
\]

\[
\quad + (r_{o}(y) \otimes 1 - 1 \otimes l_{x}(y))\sigma\beta(x),
\]

\[
(6-4) \quad \beta(x \circ y) = (l_{x}(y) \otimes 1 + r_{x}(y) \otimes 1 - 1 \otimes r_{o}(y))\beta(x) + (1 \otimes l_{o}(x))\beta(y)
\]

\[
\quad + (1 \otimes l_{o}(x) - r_{x}(x) \otimes 1)\sigma\alpha(y),
\]

\[
(6-5) \quad (\alpha + \beta)(x < y) = (1 \otimes l_{x}(x))(\sigma\alpha + \beta)(y)
\]

\[
\quad + (r_{x}(y) \otimes 1 + l_{x}(y) \otimes 1 - 1 \otimes r_{x}(y))((\alpha + \beta)(x)
\]

\[
\quad -(r_{x}(x) \otimes 1)\sigma\beta(y),
\]
\[(6-6) \ (\alpha + \beta)(x > y)\]
\[= (\tau_>(y) \otimes 1)(\alpha + \sigma \beta)(x)\]
\[+ (1 \otimes I_>(x) + 1 \otimes \tau_<(x) - I_>(x) \otimes 1)(\alpha + \beta)(y)\]
\[\quad - (1 \otimes I_>(y))\sigma \alpha(x),\]
\[(6-7) \ (\alpha + \beta + \sigma \alpha + \sigma \beta)(x > y)\]
\[= (\tau_>(y) \otimes 1)\alpha(x) + (1 \otimes I_>(x))(\alpha + \beta)(y) + (1 \otimes \tau_<(y))\sigma \alpha(x)\]
\[\quad + (I_>(x) \otimes 1)(\sigma \alpha + \sigma \beta)(y),\]
\[(6-8) \ (\alpha + \beta + \sigma \alpha + \sigma \beta)(x < y)\]
\[= (1 \otimes I_>(x))\beta(y) + (\tau_<(y) \otimes 1)(\alpha + \beta)(x) + (I_>(x) \otimes 1)\sigma \beta(y)\]
\[\quad + (1 \otimes \tau_<(y))(\sigma \alpha + \sigma \beta)(y).\]

**Proof.** By Proposition 4.7, we need to prove (6-1)–(6-8) equivalent to (4-3)–(4-9) if we replace \((A, B, L_A, R_A, L_B, R_B)\) by \((\text{Alt}(A), \text{Alt}(A^*), \tau^*_{\alpha}, I^*_{\alpha}, \tau^*_{\beta}, I^*_{\beta}).\)
As an example, we give an explicit proof of the equivalence between (4-3) and (6-3). The proofs of the others are similar. In this case, (4-3) becomes
\[
\tau^*_{\alpha}(\tau^*_{\alpha}(x)a^* + I^*_{\alpha}(x)a^*)y + (\tau^*_{\alpha}(a^*)x + I^*_{\alpha}(a^*)x) \circ y
\]
\[= \tau^*_{\alpha}(a^*)(x \circ y) + I^*_{\alpha}(I^*_{\alpha}(y)a^*)x + x \circ (\tau^*_{\alpha}(a^*)y),\]
where \(x, y \in A\) and \(a^* \in A^*\). Let both the left and the right side of this equation act on \(b^* \in A^*\). Then
\[
\langle \text{l.h.s., } b^* \rangle = \langle \tau^*_{\alpha}(\tau^*_{\alpha}(x)a^* + I^*_{\alpha}(x)a^*)y + (\tau^*_{\alpha}(a^*)x + I^*_{\alpha}(a^*)x) \circ y, b^* \rangle
\]
\[= \langle y, b^* < (\tau^*_{\alpha}(x)a^* + I^*_{\alpha}(x)a^*)) + (\tau^*_{\alpha}(a^*)x + I^*_{\alpha}(a^*)x, \tau^*_{\alpha}(y)b^*) \rangle
\]
\[= \langle \alpha(y), b^* \otimes \tau^*_{\alpha}(x)a^* + b^* \otimes I^*_{\alpha}(x)a^* \rangle + \langle \alpha(x), \tau^*_{\alpha}(y)b^* \otimes a^* \rangle
\]
\[\quad + \langle \beta(x), a^* \otimes \tau^*_{\alpha}(y)b^* \rangle
\]
\[= \langle (1 \otimes \tau_<(x) + 1 \otimes I_>(x))\alpha(y) + (\tau_{\circ}(y) \otimes 1)(\alpha + \sigma \beta)(x), b^* \otimes a^* \rangle,
\]
\[
\langle \text{r.h.s., } b^* \rangle = \langle x \circ y, b^* < a^* \rangle + \langle x, (I^*_{\alpha}(y)a^*) > b^* \rangle + \langle \tau^*_{\alpha}(a^*)y, I^*_{\alpha}(x)b^* \rangle
\]
\[= \langle \alpha(x \circ y), b^* \otimes a^* \rangle + (\beta(x), I^*_{\alpha}(y)a^* \otimes b^*) + (\alpha(y), I^*_{\alpha}(x)b^* \otimes a^*)
\]
\[= \langle \alpha(x \circ y) + (I_>(x) \otimes 1)\sigma \beta(x) + (I_<(x) \otimes 1)\alpha(y), b^* \otimes a^* \rangle.
\]
So (4-3) holds if and only if (6-3) holds.

**Definition 6.2.** (1) Let \((A, \alpha, \beta)\) be a vector space with two comultiplications \(\alpha, \beta : A \to A \otimes A\). If \((A, \alpha^*, \beta^*)\) is a prealternative algebra, then we call the triple \((A, \alpha, \beta)\) a prealternative coalgebra.

(2) If \((A, <, >, \alpha, \beta)\) is a prealternative algebra \((A, <, >)\) with two comultiplications \(\alpha, \beta : A \to A \otimes A\) such that \((A, \alpha, \beta)\) is a prealternative coalgebra
and \( \alpha \) and \( \beta \) satisfy (6-1)–(6-8), then we call \((A, \langle , \rangle, \alpha, \beta)\) a prealternative bialgebra.

Combining Propositions 4.8, 5.7 and Theorem 6.1, we have this:

**Corollary 6.3.** Let \((A, \langle 1, \rangle 1)\) be a prealternative algebra and \((\text{Alt}(A), \circ 1)\) be the associated alternative algebra. Let \(\alpha, \beta : A \to A \otimes A\) be two linear maps such that \(\alpha^*, \beta^* : A^* \otimes A^* \subset (A \otimes A)^* \to A^*\) induce a prealternative algebra structure \(\langle 2, \rangle 2\) on \(A^*\), that is, \((A, \alpha, \beta)\) is a prealternative coalgebra. Let \((\text{Alt}(A^*), \circ 2)\) be the associated alternative algebra of \((A^*, \langle 2, \rangle 2)\). Then the following conditions are equivalent:

1. \((\text{Alt}(A) \bowtie \text{Alt}(A^*), \text{Alt}(A), \text{Alt}(A^*), \omega_p)\) is an \(L\)-symplectic alternative algebra (or a phase space of \(\text{Alt}(A)\)), where \(\omega_p\) is given by (3-10).
2. \((\text{Alt}(A), \text{Alt}(A^*), r^*_1, l^*_1, l^*_2, r^*_2)\) is a matched pair of alternative algebras.
3. \((A, A^*, -r^*_1, l^*_1, r^*_2, l^*_2, -r^*_2, l^*_2)\) is a matched pair of prealternative algebras.
4. \((A, \langle 1, \rangle 1, \alpha, \beta)\) is a prealternative bialgebra.

**Definition 6.4.** Let \((A, \langle A, \rangle A, \alpha_A, \beta_A)\) and \((B, \langle B, \rangle B, \alpha_B, \beta_B)\) be two prealternative bialgebras. A homomorphism of prealternative bialgebras \(\varphi : A \to B\) is a homomorphism of prealternative algebras such that

\[
(6-9) \quad (\varphi \otimes \varphi)\alpha_A(x) = \alpha_B(\varphi(x)) \quad \text{and} \quad (\varphi \otimes \varphi)\beta_A(x) = \beta_B(\varphi(x)) \quad \text{for all } x \in A.
\]

**Proposition 6.5.** Two \(L\)-symplectic (hence phase spaces of) alternative algebras are isomorphic if and only if their corresponding prealternative bialgebras are isomorphic.

**Proof.** Let \((\text{Alt}(C) \bowtie \text{Alt}(C^*), \text{Alt}(C), \text{Alt}(C^*), \omega_p)\) for \(C = A, B\) be two \(L\)-symplectic alternative algebras, with \(\varphi : \text{Alt}(A) \bowtie \text{Alt}(A^*) \to \text{Alt}(B) \bowtie \text{Alt}(B^*)\) the isomorphism. Then \(\varphi|_A : A \to B\) and \(\varphi|_{A^*} : A^* \to B^*\) are isomorphisms of prealternative algebras by Proposition 4.2. Moreover, \(\varphi|_{A^*} = (\varphi|_A)^{*_{-1}}\) since

\[
\langle \varphi|_{A^*}(a^*), \varphi(x) \rangle = \omega_p(\varphi|_{A^*}(a^*), \varphi(x)) = \omega_p(a^*, x) = \langle a^*, x \rangle \\
= (\varphi^*(\varphi|_A)^{*_{-1}}(a^*), x) \\
= ((\varphi|_A)^{*_{-1}}(a^*), \varphi(x)) \quad \text{for all } x \in A \text{ and } a^* \in A^*.
\]

So \((\varphi|_A)^* : B^* \to A^*\) is a homomorphism of prealternative algebras, and then \((A, \langle A, \rangle A, \alpha_A, \beta_A)\) and \((B, \langle B, \rangle B, \alpha_B, \beta_B)\) are isomorphic as prealternative bialgebras. Conversely, suppose these two are isomorphic prealternative bialgebras, and let \(\varphi' : A \to B\) be the isomorphism. Let \(\varphi : A \oplus A^* \to B \oplus B^*\) be a linear map defined by

\[
\varphi(x) = \varphi'(x) \quad \text{and} \quad \varphi(a^*) = (\varphi'|_A)^{-1}(a^*) \quad \text{for all } x \in A \text{ and } a^* \in A^*.
\]
Then it is easy to show that \( \varphi \) is an isomorphism of the two \( L \)-symplectic alternative algebras in the statement.

**Example 6.6.** Let \((A, <, >, \alpha, \beta)\) be a prealternative bialgebra. Then the dual \((A, <_*, >_*, \gamma, \delta)\) is also a prealternative bialgebra, where the prealternative algebra structure \( <, > \) on \( A \) is defined by the linear maps \( \gamma^*, \delta^* : A \otimes A \to A \), and \( \alpha^*, \beta^* : A^* \otimes A^* \to A^* \) induce a prealternative algebra structure \( <_*, >_* \) on \( A^* \).

**Example 6.7.** Let \((A, <, >)\) be a prealternative algebra. Then \((A, <, >, 0, 0)\) is a prealternative bialgebra, and the corresponding prealternative algebra structure on \( A \oplus A^* \) is the semidirect sum \( A \ltimes \mathbb{C} \), \( \mathbb{C} \) being the \( \mathbb{C} \)-valued linear functionals on \( A \). The corresponding associated alternative algebra is the semidirect sum \( \text{Alt}(A) \ltimes \mathbb{C} \), with symplectic form \( \omega_p \) given by (3-10).

## 7. Coboundary prealternative bialgebras

**Definition 7.1.** A prealternative bialgebra \((A, <, >, \alpha, \beta)\) is called **coboundary** if the linear maps \( \alpha, \beta : A \to A \otimes A \) are given by

\[
\alpha(x) = (r_<(x) \otimes 1 - 1 \otimes r_>(x))r_>,
\]

\[
\beta(x) = (1 \otimes r_>(x) - r_<(x) \otimes 1)r_>,
\]

where \( x \circ y = x < y + x > y \), \( x, y \in A \) and \( r_>, r_\geq A \otimes A \).

**Remark 7.2.** The expression of (7-1) and (6-1)–(6-2) looks like certain kind of 1-coboundary and 1-cocycle.

**Theorem 7.3.** Let \((A, <, >)\) be a prealternative algebra with two linear maps \( \alpha, \beta : A \to A \otimes A \) defined by (7-1). If \( r_<= r_\geq = r \in A \otimes A \) and \( r \) is symmetric, then \( \alpha, \beta \) satisfy (6-1)–(6-8).

**Proof.** It is obvious that \( \alpha, \beta \) automatically satisfy (6-1) and (6-2). For (6-3)–(6-8), we give as an example an explicit proof of the fact that \( \alpha, \beta \) satisfy (6-5); the proof of the other cases is similar. Assume \( r = \sum_i u_i \otimes v_i \in A \otimes A \). After rearranging the terms suitably, we have, noting that \( r \) is symmetric,

\[
(\alpha + \beta)(x < y) - (1 \otimes r_<(x))(\sigma \alpha + \beta)(y) = \sum_i (u_i \circ (x < y) \otimes v_i - u_i < (x < y) \otimes v_i - (u_i \circ x) < y \otimes v_i + (u_i \circ x) < x \otimes v_i - y \otimes v_i \sim y \otimes v_i) + u_i \otimes (x < y) \otimes v_i - u_i \circ x \sim y \otimes v_i) - u_i \otimes x \sim (y \otimes v_i) - u_i \otimes (x > v_i) \sim x \otimes y \) + u_i \otimes (x \circ v_i) \sim y \otimes v_i - u_i \otimes x \circ v_i = \alpha + \beta(x) + (r_<(x) \otimes 1)\alpha \beta(y).
\]
The sum of the first seven terms is zero since it is equal to
\[ \sum_i (u_i > (x < y) - (u_i > x) < y - y > (u_i > x) + (y \circ u_i) > x) \otimes v_i = 0. \]

The sum of the 8th through the 13th term is zero since it is equal to
\[ \sum_i u_i \otimes ((x < y) < v_i - x < (y \circ v_i) - x < (v_i \circ y) + (x < v_i) < y) = 0. \]

The sum of the 14th through 16th term, the sum of 17th through 19th term, and the sum of the last three terms are all zero obviously. □

**Lemma 7.4.** Let \( A \) be a vector space and \( \alpha, \beta : A \to A \otimes A \) be two linear maps. Then \((A, \alpha, \beta)\) is a prealternative coalgebra if and only if the linear maps \( S^i_{\alpha, \beta} : A \to A \otimes A \otimes A \) for \( i = 1, 2, 3, 4 \) given by the following equations are all zero for any \( x \in A \):

\[
\begin{align*}
S^1_{\alpha, \beta}(x) &= ((\alpha + \beta) \otimes 1)\beta(x) + (\sigma \otimes 1)((\alpha + \beta) \otimes 1)\beta(x) \\
&\quad - (1 \otimes \beta)\beta(x) - (\sigma \otimes 1)(1 \otimes \beta)\beta(x), \\
S^2_{\alpha, \beta}(x) &= (\beta \otimes 1)\alpha(x) + (\sigma \otimes 1)(\alpha \otimes 1)\alpha(x) \\
&\quad - (1 \otimes \alpha)\beta(x) - (\sigma \otimes 1)(1 \otimes (\alpha + \beta))\alpha(x), \\
S^3_{\alpha, \beta}(x) &= ((\alpha + \beta) \otimes 1)\beta(x) + (1 \otimes \sigma)(\beta \otimes 1)\alpha(x) \\
&\quad - (1 \otimes \beta)\beta(x) - (1 \otimes \sigma)(1 \otimes \alpha)\beta(x), \\
S^4_{\alpha, \beta}(x) &= (\alpha \otimes 1)\alpha(x) + (1 \otimes \sigma)(\alpha \otimes 1)\alpha(x) \\
&\quad - (1 \otimes (\alpha + \beta))\alpha(x) - (1 \otimes \sigma)(1 \otimes (\alpha + \beta))\alpha(x).
\end{align*}
\]

**Proof.** It follows immediately from the definition 2.6 of a prealternative algebra. □

**Definition 7.5.** Let \((A, <, >)\) be a prealternative algebra and \((\text{Alt}(A), \circ)\) be the associated alternative algebra. Let \( r \in A \otimes A \). The following equations are called \( \text{PA}_j \) **equations** for \( i = 1, 2 \) and \( j = 1, 2, 3 \):

\[
\begin{align*}
\text{PA}_1^1 &= r_{12} \circ r_{13} - r_{23} > r_{12} - r_{13} < r_{23} = 0, \\
\text{PA}_1^2 &= r_{13} \circ r_{12} - r_{23} < r_{23} - r_{23} > r_{13} = 0, \\
\text{PA}_2^1 &= r_{12} \circ r_{23} - r_{23} < r_{13} - r_{13} > r_{12} = 0, \\
\text{PA}_2^2 &= r_{23} \circ r_{12} - r_{13} > r_{23} - r_{12} < r_{13} = 0, \\
\text{PA}_3^1 &= r_{13} \circ r_{23} - r_{12} > r_{13} - r_{23} < r_{12} = 0, \\
\text{PA}_3^2 &= r_{23} \circ r_{13} - r_{13} < r_{12} - r_{12} > r_{23} = 0.
\end{align*}
\]

We set \( \text{PA}_j = \text{PA}_j^1 + \text{PA}_j^2 \), where \( j = 1, 2, 3 \). Collectively the \( \text{PA}_j \) equations are called the **PA equations**.
**Proposition 7.6.** Let \((A, \prec, \succ)\) be a prealternative algebra and \((\text{Alt}(A), \circ)\) be the associated alternative algebra. Let \(r \in A \otimes A\) be symmetric. Let \(\alpha, \beta : A \rightarrow A \otimes A\) be two linear maps given by (7-1), where \(r_\prec = r_\succ = r\). Then \((A, \alpha, \beta)\) becomes a prealternative coalgebra if and only if for any \(x \in A\)

\[
\begin{align*}
-(1 \otimes 1 \otimes I_\circ(x)) PA_3 + (1 \otimes r_\prec(x) \otimes 1) PA_3^2 + (r_\prec(x) \otimes 1 \otimes 1) PA_3^1 &= 0, \\
-(1 \otimes 1 \otimes I_\succ(x)) PA_2 + (1 \otimes r_\circ(x) \otimes 1) PA_2^1 + (r_\circ(x) \otimes 1 \otimes 1) PA_2^0 &= 0, \\
(r_\prec(x) \otimes 1 \otimes 1) PA_3 - (1 \otimes 1 \otimes I_\circ(x)) PA_3^1 - (1 \otimes I_\prec(x) \otimes 1) PA_3^2 &= 0, \\
(r_\circ(x) \otimes 1 \otimes 1) PA_1 - (1 \otimes I_\succ(x) \otimes 1) PA_1 - (1 \otimes 1 \otimes I_\succ(x)) PA_1^0 &= 0.
\end{align*}
\]

**(7-4)**

**Proof.** We give an explicit proof of the fact that the first of (7-4) is equivalent to \(S^1_{\alpha, \beta} = 0\) as an example. Using a similar argument, we can show that the rest are respectively equivalent to \(S^i_{\alpha, \beta} = 0\) for \(i = 2, 3, 4\). Set \(r = \sum_i u_i \otimes v_i\). Substituting \(\alpha(x) = \sum_i u_i \circ x \otimes v_i - u_i \otimes x > v_i\) and \(\beta(x) = \sum_i u_i \otimes x \circ v_i - u_i < x \otimes v_i\) for all \(x \in A\) into the first of (7-2) and after rearranging the terms suitably, we divide \(S^1_{\alpha, \beta}\) as

\[S^1_{\alpha, \beta} = (S1) + (S2) + (S3),\]

where

\[(S1) = \sum_{i, j} (u_i \circ u_j \otimes v_i \otimes x \circ v_j - u_i \otimes u_j \circ v_i \otimes x \circ v_j - u_i \otimes u_j \circ v_i - u_i \otimes u_j \circ v_i \otimes x \circ v_j)

+ u_j \circ v_i \otimes u_i \otimes x \circ v_j - v_i \otimes u_i \otimes x \circ v_j - u_j \circ u_i \circ (x \circ v_j) \circ v_i

- u_i \otimes u_j \circ (x \circ v_j) \circ v_i).

\[(S2) = \sum_{i, j} (u_i \otimes (u_j < x) > v_i \otimes v_j - u_i \otimes (u_j < x) \circ v_i \otimes v_j

- v_i \otimes u_i \circ (u_j < x) \otimes v_j + v_i \otimes u_i \otimes (u_j < x) \otimes v_j

+ u_j \otimes u_i \otimes (x \circ v_j) \otimes v_i + u_i \otimes u_j \otimes x \otimes v_j \circ v_i - u_i \otimes v_j \otimes u_j \otimes x \otimes v_i),

(S3) = \sum_{i, j} (-u_i \circ (u_j < x) \otimes v_i \otimes v_j + u_i \otimes (u_j < x) \otimes v_i \otimes v_j

+ (u_j < x) \circ v_i \otimes u_i \otimes v_j - (u_j < x) \otimes v_i \otimes u_i \otimes v_j + u_j \otimes u_i \otimes v_j \circ v_i

- u_j \otimes x \otimes u_i \otimes v_j + u_i \circ (x \circ v_j) \otimes u_j \otimes v_i)

+ (u_j < x) \circ v_i \otimes u_i \otimes v_j - (u_j < x) \otimes v_i \otimes u_i \otimes v_j + u_j \otimes u_i \otimes v_j \circ v_i

- u_j \otimes x \otimes u_i \otimes v_j + u_i \circ (x \circ v_j) \otimes u_j \otimes v_i).

Since \(r\) is symmetric and by Remark 2.6, we have

\[(S1) = -(1 \otimes 1 \otimes I_\circ(x)) PA_3, \quad (S2) = (1 \otimes r_\prec(x) \otimes 1) PA_3^2,

(S3) = (r_\prec(x) \otimes 1 \otimes 1) PA_3^1.\]
Theorem 7.7. Let \((A, \prec, \succ)\) be a prealternative algebra and \(r \in A \otimes A\) be symmetric. Let \(\alpha, \beta : A \rightarrow A \otimes A\) be linear maps given by (7-1), where \(r_\prec = r_\succ = r\). Then \((A, \prec, \succ, \alpha, \beta)\) is a prealternative bialgebra if and only if the equations of (7-4) are satisfied.

**Proof.** It follows from Theorem 7.3 and Proposition 7.6. \(\square\)

Next we give a Drinfeld double construction [Chari and Pressley 1994] for a prealternative bialgebra.

Theorem 7.8. Let \((A, \prec, \succ, \alpha, \beta)\) be a prealternative bialgebra. Then there is a canonical prealternative bialgebra structure on \(A \oplus A^*\) such that the inclusions \(i_1 : A \rightarrow A \oplus A^*\) and \(i_2 : A^* \rightarrow A \oplus A^*\) into the two summands are homomorphisms of prealternative bialgebras, where the prealternative bialgebra structure on \(A^*\) is given in Example 6.6.

**Proof.** Denote the prealternative algebra structures on \(A\) and \(A^*\) respectively, and the associated alternative algebra structure by \(\prec\) and \(\succ\) respectively, and the associated alternative algebra structure by \(\prec\). Let \(r \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)\) correspond to the identity map \(\text{id} : A \rightarrow A\). Then the prealternative algebra structure \(\prec, \succ\) on \(A \oplus A^*\) is given by

\[
\mathcal{P\mathcal{A\mathcal{D}}}(A) = A \prec -l_\prec, l_\prec, -l_\prec, A^*,
\]

that is, for all \(x, y \in A\) and \(a^*, b^* \in A^*\),

\[
\begin{align*}
x \prec \bullet y &= x \prec y, & x \succ \bullet y &= x \succ y, \\
ad^* \prec \bullet b^* &= a^* \prec \bullet b^*, & ad^* \succ \bullet b^* &= a^* \succ \bullet b^*, \\
x \prec \bullet a^* &= -l^*_\prec(x) b^* + l^*_\prec(a^*) x, & x \succ \bullet a^* &= v^*_\prec(x) a^* - l^*_\prec(a^*) x, \\
ad^* \prec \bullet x &= -v^*_\prec(a^*) x + l^*_\prec(x) a^*, & ad^* \succ \bullet x &= v^*_\prec(a^*) x - l^*_\prec(x) a^*,
\end{align*}
\]

We denote its associated alternative algebra structure by \(\bullet\). Let \(\{e_1, \ldots, e_n\}\) be a basis of \(A\) and \(\{e_1^*, \ldots, e_n^*\}\) be the dual basis. Then \(r = \sum_i e_i \otimes e_i^*\). Next we prove that

\[
\alpha_{\mathcal{P\mathcal{A\mathcal{D}}}}(u) = (\alpha_\circ(u) \otimes 1 - 1 \otimes l_\circ(u)) r \quad \text{and} \quad \beta_{\mathcal{P\mathcal{A\mathcal{D}}}}(u) = (1 \otimes l_\circ(u) - r_\circ(u) \otimes 1) r
\]

induce a (coboundary) prealternative bialgebra structure on \(A \oplus A^*\). Since \(r\) is not symmetric we cannot apply Theorem 7.7 and \(\beta_{\mathcal{P\mathcal{A\mathcal{D}}}}\) satisfies (6-1)–(6-8) and the conditions of Lemma 7.4. For the former, we prove that \(\alpha_{\mathcal{P\mathcal{A\mathcal{D}}}}\) and \(\beta_{\mathcal{P\mathcal{A\mathcal{D}}}}\) satisfy (6-3) as an example; the proof of the others is similar. In fact, we only need to prove

\[
(r_\circ(y) \otimes 1 - 1 \otimes l_\circ(y))(l_\circ(x) \otimes 1 - 1 \otimes r_\circ(x))(r - \sigma r) = 0 \quad \text{for all} \ x \in A.
\]

We can prove this equation in the following cases: \(x, y \in A; \ x, y \in A^*; \ x \in A\) and \(y \in A^*; \ x \in A^*\) and \(y \in A\). We prove the first case; the proof of the others
is similar. Let \( x = e_i \) and \( y = e_j \); then the equation becomes

\[
(7-5) \quad \sum_k ((e_i \cdot e_k^*) \cdot e_j \otimes e_k - e_k^* \cdot e_j \otimes e_k < \cdot e_i - e_i \cdot e_k^* \otimes e_j > \cdot e_k + e_k^* \otimes e_j > \cdot (e_k < \cdot e_i))
\]

\[
= \sum_k ((e_i \cdot e_k) \cdot e_j \otimes e_k - e_k \cdot e_j \otimes e_k < \cdot e_i - e_i \cdot e_k \otimes e_j > \cdot e_k^* + e_k \otimes e_j > \cdot (e_k < \cdot e_i)).
\]

The coefficient of \( e_m \otimes e_n \) on the left side of (7-5) is

\[
\sum_k ((e_i \cdot e_k^*) \cdot e_j \otimes e_m - (e_i \cdot e_k^*) \cdot e_m \otimes e_j) - (e_i \cdot e_k \otimes e_m) \cdot e_j^* + (e_j > \cdot e_k^* \cdot e_m) \cdot e_i^*)
\]

\[
= \sum_k ((e_i \cdot e_k) \cdot e_j \otimes e_m - (e_i \cdot e_k \otimes e_m) \cdot e_j^* + (e_j > \cdot e_k \cdot e_m) \cdot e_i^*)
\]

while on the right side that coefficient is the same:

\[
\sum_k ((e_i \cdot e_k^*) \cdot e_j \otimes e_m - (e_i \cdot e_k^*) \cdot e_m \otimes e_j) - (e_i \cdot e_k \otimes e_m) \cdot e_j^* + (e_j > \cdot e_k^* \cdot e_m) \cdot e_i^*)
\]

\[
= \sum_k ((e_i \cdot e_k) \cdot e_j \otimes e_m - (e_i \cdot e_k \otimes e_m) \cdot e_j^* + (e_j > \cdot e_k \cdot e_m) \cdot e_i^*)
\]

Similarly, the coefficients of \( e_m^* \otimes e_n, e_m \otimes e_n^* \) and \( e_m^* \otimes e_n \) on both sides of (7-5) are the same.

On the other hand, we prove that \( S_\rho^{i} \) for \( i = 1, 2, 3, 4 \). We prove it explicitly for \( i = 0 \). The coefficient of \( e_m \otimes e_n \otimes e_p \) in \( S_\rho^{i} \) is

\[
-\langle e_j > \cdot e_m^* \cdot e_n^* \rangle (e_k \cdot e_j^* \cdot e_p^*) + (e_j \cdot e_m \cdot e_n) \cdot e_k \cdot e_j^* \cdot e_p^*) - (e_j > \cdot e_m^* \cdot e_n^* \cdot e_p^*)
\]

\[
= \langle e_j, e_m^* \cdot e_n^* \rangle (e_k \cdot e_j^* > e_p^*) + (e_j, e_m \cdot e_n^* \cdot e_p^*) \cdot (e_k, e_j^* > e_p^*)
\]

\[
= \langle e_j, e_m^* \cdot e_n^* \cdot e_p^* \rangle > e_p^* - (e_m^* \cdot e_n^* \cdot e_p^*) > e_p^* = 0.
\]

Similarly, the remaining coefficients, of \( e_m^* \otimes e_n \otimes e_p, e_m \otimes e_n^* \otimes e_p, e_m^* \otimes e_n^* \otimes e_p, e_m \otimes e_n \otimes e_p, e_m^* \otimes e_n \otimes e_p, e_m \otimes e_n \otimes e_p \) and \( e_m^* \otimes e_n^* \otimes e_p \), are all zero. A similar study shows that \( S_\rho^{i} \) for \( i = 0 \) is a prealternative bialgebra. For \( e_i \in A \), we have

\[
\alpha_{\rho}(e_i) = \sum_j e_j \circ e_i \otimes e_j^* - e_j \otimes e_i > \cdot e_j^*
\]

\[
= \sum_{j,m} e_j \circ e_i \otimes e_j^* - e_j \otimes e_m \langle e_j^*, e_m \circ e_i \rangle + e_j \otimes e_m \langle e_i, e_j^* > e_m^* \rangle
\]

\[
= \sum_{j,m} \langle e_i, e_j^* > e_m^* \rangle e_j \otimes e_m = \alpha(e_i).
\]
Similarly we have $\beta_{\mathfrak{P}\mathfrak{A}\mathfrak{D}}(e_i) = \beta(e_i)$, so the inclusion $i_1: A \to A \oplus A^*$ is a homomorphism of prealternative bialgebras. Similarly, the inclusion $i_2: A^* \to A \oplus A^*$ is also a homomorphism of prealternative bialgebras.

**Definition 7.9.** Let $(A, \prec, \succ, \alpha, \beta)$ be a prealternative bialgebra. With the prealternative bialgebra structure given in Theorem 7.8, we call $A \oplus A^*$ a Drinfeld symplectic double of $A$ and denote it by $\mathfrak{P}\mathfrak{A}\mathfrak{D}(A)$.

**Proposition 7.10.** Let $(A, \prec, \succ, \alpha, \beta)$ be a prealternative bialgebra with $\alpha$, $\beta$ defined by (7-1), where $r_\prec = r_\succ = r \in A \otimes A$ and $r$ is a solution of the PA equations. Then $T_r$ is a homomorphism of prealternative bialgebras from the prealternative bialgebra given in Example 6.6 to $(A, \prec, \succ, \alpha, \beta)$.

**Proof.** Note that $(1 \otimes \alpha)r = r_{12} \prec r_{13}$ and $(\alpha \otimes 1)r = r_{13} \prec r_{23}$. Denote by $\prec_*$ and $\succ_*$ the prealternative algebras structure on $A^*$ induced by $\alpha^*$ and $\beta^*$, respectively, and define the prealternative algebra structure $\prec, \succ$ on $A$ by the linear maps $\gamma^*, \delta^*: A \otimes A \to A$, respectively. Then

$$T_r(a^* \prec_* b^*) = (1 \otimes (a^* \prec_* b^*), r) = (1 \otimes a^* \otimes b^*, (1 \otimes \alpha)r)$$

$$= (1 \otimes a^* \otimes b^*, r_{12} \prec r_{13}) = T_r(a^*) \prec T_r(b^*).$$

$$(T_r \otimes T_r)\gamma(a^*) = (1 \otimes 1 \otimes a^*, r_{13} \prec r_{23}) = (1 \otimes 1 \otimes a^*)(\alpha \otimes 1)r = \alpha(T_r(a^*)),$$

where $a^*, b^* \in A^*$. Similarly we have

$$T_r(a^* \succ_* b^*) = T_r(a^*) \succ T_r(b^*) \quad \text{and} \quad (T_r \otimes T_r)\delta(a^*) = \beta(T_r(a^*)) .$$

**8. PA equations and their properties**

The simplest way to satisfy the conditions of Theorem 7.7 is given as follows.

**Proposition 8.1.** Let $(A, \prec, \succ)$ be a prealternative algebra and $r \in A \otimes A$ be symmetric. Let $\alpha, \beta : A \to A \otimes A$ be two linear maps defined by (7-1). Then $(A, \prec, \succ, \alpha, \beta)$ is a prealternative bialgebra if $r$ satisfies PA-equations.

**Proposition 8.2.** Let $(A, \prec, \succ)$ be a prealternative algebra and $(\text{Alt}(A), \circ)$ be the associated alternative algebra. Let $r \in A \otimes A$ be a symmetric solution of the PA equations in $A$. Then the prealternative algebra structure $\prec_*, \succ_*$ on the Drinfeld symplectic double $\mathfrak{P}\mathfrak{A}\mathfrak{D}(A)$ is given as

$$(8-1) \quad a^* \prec \bullet b^* = a^* \prec_* b^* = \ell_\prec(T_r(b^*))a^* - \ell_\prec(T_r(a^*))b^*,$$

$$(8-2) \quad a^* \succ \bullet b^* = a^* \succ_* b^* = \ell_\succ(T_r(a^*))b^* - \ell_\succ(T_r(b^*))a^*,$$

$$(8-3) \quad x \prec \bullet a^* = x \prec T_r(a^*) + T_r(\ell_\prec(x))a^* - \ell_\prec(x)a^*,$$

$$(8-4) \quad x \succ \bullet a^* = \ell_\succ(x)a^* - T_r(\ell_\succ(x)a^*) + x \succ T_r(a^*),$$

$$(8-5) \quad a^* \prec \bullet x = -T_r(\ell_\succ(x)a^*) + T_r(a^*) \prec x + \ell_\succ(x)a^*,$$

$$= (1 \otimes a^* \otimes b^*, r_{12} \prec r_{13}) = T_r(a^*) \prec T_r(b^*),$$

$$(T_r \otimes T_r)\gamma(a^*) = (1 \otimes 1 \otimes a^*, r_{13} \prec r_{23}) = (1 \otimes 1 \otimes a^*)(\alpha \otimes 1)r = \alpha(T_r(a^*)),$$

where $a^*, b^* \in A^*$. Similarly we have

$$T_r(a^* \succ_* b^*) = T_r(a^*) \succ T_r(b^*) \quad \text{and} \quad (T_r \otimes T_r)\delta(a^*) = \beta(T_r(a^*)) .$$

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**Proposition 8.2.** Let $(A, \prec, \succ)$ be a prealternative algebra and $(\text{Alt}(A), \circ)$ be the associated alternative algebra. Let $r \in A \otimes A$ be a symmetric solution of the PA equations in $A$. Then the prealternative algebra structure $\prec_*, \succ_*$ on the Drinfeld symplectic double $\mathfrak{P}\mathfrak{A}\mathfrak{D}(A)$ is given as

$$(8-1) \quad a^* \prec \bullet b^* = a^* \prec_* b^* = \ell_\prec(T_r(b^*))a^* - \ell_\prec(T_r(a^*))b^*,$$

$$(8-2) \quad a^* \succ \bullet b^* = a^* \succ_* b^* = \ell_\succ(T_r(a^*))b^* - \ell_\succ(T_r(b^*))a^*,$$

$$(8-3) \quad x \prec \bullet a^* = x \prec T_r(a^*) + T_r(\ell_\prec(x))a^* - \ell_\prec(x)a^*,$$

$$(8-4) \quad x \succ \bullet a^* = \ell_\succ(x)a^* - T_r(\ell_\succ(x)a^*) + x \succ T_r(a^*),$$

$$(8-5) \quad a^* \prec \bullet x = -T_r(\ell_\succ(x)a^*) + T_r(a^*) \prec x + \ell_\succ(x)a^*,$$
(8-6) \[ a^* >_\bullet x = T_r(a^*) > x + T_r(l^*_x(x)a^*) - l^*_x(x)a^*, \]

where \( x \in A \) and \( a^*, b^* \in A^* \), the prealternative algebra structure on \( A^* \) is denoted by \( <_*, >_* \), and the associated alternative algebra structure on \( \text{Alt}(A^*) \) is denoted by \( * \).

**Proof.** Let \( \{e_1, \ldots, e_n\} \) be a basis of \( A \) and \( \{e_1^*, \ldots, e_n^*\} \) be its dual. Suppose that

\[ e_i < e_j = \sum_{i,j} c^{k}_{ij} e_k, \quad e_i > e_j = \sum_{i,j} d^{k}_{ij} e_k, \quad r = \sum_{i,j} a_{ij} e_i \otimes e_j, \quad a_{ij} = a_{ji}. \]

Then \( T_r(e_i^*) = \sum_k a_{ki} e_k \). Thus for any \( k, l \)

\[ e_k^* < e_l^* = \sum_s (e_k^* \otimes e_l^*, \alpha(e_s)) e_s^* = \sum_{i,s} (a_{il}(c^k_{is} + d^k_{is}) - a_{ki}d^l_{is}) e_s^* \]

\[ = \sum_{i,s} (a_{il}(e_i \circ e_s, e_k^* - e_{kl}(e_s > e_i, e_l^*))) e_s^* = l^*_o(T_r(e_i^*)) e_k^* - v^*_o(T_r(e_i^*)) e_l^*. \]

So (8-1) holds. Similarly (8-2) holds. Therefore

\[ l^*_< (e_k^*) e_m = \sum_s (e_m, e_k^* < e_s^*) e_s = \sum_s (e_m, l^*_o(T_r(e_s^*)) e_k^* - v^*_o(T_r(e_k^*)) e_s^*) e_s \]

\[ = \sum_s (T_r(e_k^*) \circ e_m, e_k^*) e_s - (e_m > T_r(e_k^*), e_s^*) e_s \]

\[ = T_r(v^*_o(e_m) e_k^*) - e_m > T_r(e_k^*). \]

Hence (8-4) follows from the fact that \( e_m >_\bullet e_k^* = v^*_o(e_m) e_k^* - l^*_< (e_k^*) e_m \). We can get the other equations similarly. \( \square \)

**Proposition 8.3.** Let \( (A, <, >) \) be a prealternative algebra and \( (\text{Alt}(A), \circ) \) be the associated alternative algebra. Let \( r \in A \otimes A \) be symmetric. Then \( r \) is a solution of one of the \( \text{PA}^j \)-equations for \( i = 1, 2 \) and \( j = 1, 2, 3 \) if and only if \( T_r \) satisfies

(8-7) \[ T_r(a^*) \circ T_r(b^*) = T_r(v^*_< (T_r(a^*)) b^* + l^*_< (T_r(b^*)) a^*) \] for all \( a^*, b^* \in A^* \),

that is, \( T_r \) is an \( \mathcal{O} \)-operator of \( \text{Alt}(A) \) associated to the bimodule \( (A^*, v^*_<, l^*_<) \). So in this case the \( \text{PA}^j \) equations for \( i = 1, 2 \) and \( j = 1, 2, 3 \) are all equivalent. Moreover, if \( r \) is a solution of one of the \( \text{PA}^j \) equations for \( i = 1, 2 \) and \( j = 1, 2, 3 \), then there is a prealternative algebra structure on \( A^* \) given by

(8-8) \[ a^* b^* = l^*_o(T_r(b^*)) a^* \quad \text{and} \quad a^* b^* = v^*_o(T_r(a^*)) b^* \] for all \( a^*, b^* \in A^* \).

The associated alternative algebra structure \( \text{Alt}(A^*) \) is the same as the one given by (8-1) and (8-2) that is induced by \( r \) in the sense of coboundary prealternative bialgebras.

**Proof.** It is similar to the proof of Proposition 3.6. \( \square \)
Definition 8.4. Let \((A, <, >)\) be a prealternative algebra. We call a bilinear form \(\mathcal{B} : A \otimes A \rightarrow \mathbb{k}\) a 2-cocycle of \(A\) if
\[
\mathcal{B}(x \circ y, z) = \mathcal{B}(x, y > z) + \mathcal{B}(y, z < x) \quad \text{for all } x, y, z \in A.
\]

Proposition 8.5. Let \((A, <, >)\) be a prealternative algebra and \((\text{Alt}(A), \circ)\) be the associated alternative algebra. Let \(\mathcal{B}\) be a 2-cocycle of \((A, <, >)\). Then the bilinear form \(\omega\) defined by
\[
(8-9) \quad \omega(x, y) = \mathcal{B}(x, y) - \mathcal{B}(y, x) \quad \text{for all } x, y \in A
\]
is a closed form on \(\text{Alt}(A)\).

Proof. Straightforward. \(\square\)

Proposition 8.6. Let \((A, <, >)\) be a prealternative algebra and let \(r \in A \otimes A\). Suppose \(r\) is symmetric and nondegenerate. Then \(r\) is a solution of one of the \(PA_i^j\) equations for \(i = 1, 2\) and \(j = 1, 2, 3\) in \((A, <, >)\) if and only if the (nondegenerate) bilinear form \(\mathcal{B}\) induced by \(r\) through (1-8) is a 2-cocycle of \((A, <, >)\).

Proof. Let \(r = \sum_i a_i \otimes b_i\). Since \(r\) is symmetric, we have \(\sum_i a_i \otimes b_i = \sum_i b_i \otimes a_i\). Therefore \(T_r(v^*) = \sum_j \langle v^*, a_i \rangle b_i = \sum_i \langle v^*, b_i \rangle a_i\) for any \(v^* \in A^*\). Since \(r\) is nondegenerate, for any \(x, y, z \in A\) there exist \(u^*, v^*, w^* \in A^*\) such that \(x = T_r(u^*), y = T_r(v^*)\), and \(z = T_r(w^*)\). Therefore
\[
\mathcal{B}(x, z \circ y) = \langle u^*, T_r(w^*) \circ T_r(v^*) \rangle = \sum_{i,j} \langle u^*, b_i \rangle \langle v^*, b_j \rangle \langle u^*, a_i \circ a_j \rangle = \langle u^* \otimes v^* \otimes w^*, r_{13} \circ r_{12} \rangle.
\]
\[
\mathcal{B}(y, x < z) = \langle v^*, T_r(u^*) \prec T_r(w^*) \rangle = \sum_{i,j} \langle v^*, b_i \rangle \langle w^*, b_j \rangle \langle v^*, a_i \prec a_j \rangle = \langle u^* \otimes v^* \otimes w^*, r_{12} \prec r_{23} \rangle.
\]
\[
\mathcal{B}(y > x, z) = \langle T_r(v^*) \succ T_r(u^*), w^* \rangle = \sum_{i,j} \langle v^*, b_i \rangle \langle u^*, b_j \rangle \langle w^*, a_i \succ a_j \rangle = \langle u^* \otimes v^* \otimes w^*, r_{23} \succ r_{13} \rangle.
\]
Hence \(\mathcal{B}\) is a 2-cocycle of \((A, <, >)\) if and only if the second of (7-3) holds. By Proposition 8.3, the conclusion follows. \(\square\)

Corollary 8.7. Let \((A, <, >)\) be a prealternative algebra and \(r \in A \otimes A\). Assume \(r\) is symmetric and there exists a nondegenerate symmetric bilinear form \(h(x, y)\) on \(A\) that is associative in that
\[
(8-10) \quad h(x < y, z) = h(x, y > z) \quad \text{for all } x, y, z \in A.
\]
Define a linear map $\varphi : A \to A^*$ by $\langle \varphi(x), y \rangle = h(x, y)$. Then $\tilde{T}_r = T_r \varphi : A \to A$ is an $\mathbb{C}$-operator associated to the bimodule $(A, l_-, r_-)$ if and only if $r$ is a symmetric solution of the PA equations. In this case, $\tilde{T}_r$ satisfies the equation

$$\tilde{T}_r(x) \circ \tilde{T}_r(y) = \tilde{T}_r(\tilde{T}_r(x) > y + x < \tilde{T}_r(y)).$$

So we can define a prealternative algebra structure on $A$ by

$$x < y = x < \tilde{T}_r(y),$$

$$x > y = \tilde{T}_r(x) > y.$$

Proof. It follows from a proof similar to that of Corollary 3.7. □

Remark 8.8. A symmetric bilinear form on a prealternative algebra $(A, <, >)$ satisfying (8-10) is a 2-cocycle of $(A, <, >)$.

By a proof similar to that of Proposition 3.9, we have this:

Proposition 8.9. Let $(A, \circ)$ be an alternative algebra. Let $(V, L, R)$ be a bimodule of $A$ and $(V^*, R^*, L^*)$ be the dual bimodule. Suppose that $T : V \to A$ is an $\mathbb{C}$-operator associated to $(V, L, R)$. Then $r = T + \sigma(T)$ is a symmetric solution of the PA equations in $T(V) \ltimes_{0, L^*, R^*, 0} V^*$, where $T(V) \subset A$ is a prealternative algebra given by (2-9) and $(V^*, 0, L^*, R^*, 0)$ is a bimodule of $T(V)$.

Corollary 8.10. Let $(A, <, >)$ be a prealternative algebra. Then

$$r = \sum_i (e_i \otimes e_i^* + e_i^* \otimes e_i)$$

is a symmetric solution of the PA equations in $A \ltimes_{0, L^*, R^*, 0} A^*$, where $\{e_1, \ldots, e_n\}$ is a basis of $A$ and $\{e_1^*, \ldots, e_n^*\}$ is its dual. Moreover, $r$ is nondegenerate and the induced 2-cocycle $\mathcal{B}$ of $A \ltimes_{0, L^*, R^*, 0} A^*$ is given by (3-2).

Proof. Use Proposition 8.9 with $V = A$, $(L, R) = (l_-, r_-)$ and $T = \text{id}$. □

Corollary 8.11. Let $(A, <, >)$ be a prealternative algebra and $(\text{Alt}(A), \circ)$ be the associated alternative algebra. If $r$ is a nondegenerate symmetric solution of the PA equations in $A$, then there is a new compatible prealternative algebra structure on $\text{Alt}(A)$ given by

$$x <' y = T_r(l^*_r(y)T_r^{-1}(x)),$$

$$x >' y = T_r(r^*_r(x)T_r^{-1}(y)) \text{ for all } x, y \in A,$

which is just the prealternative algebra structure given by

$$\mathcal{B}(x <' y, z) = \mathcal{B}(x, y > z),$$

$$\mathcal{B}(x >' y, z) = \mathcal{B}(y, z < x) \text{ for all } x, y, z \in A,$$

where $\mathcal{B}$ is the nondegenerate symmetric 2-cocycle of $A$ induced by $r$ through (1-8).
Proposition 8.12. Let \((A, <, >, \alpha, \beta)\) be a prealternative bialgebra arising from a symmetric solution \(r\) of the PA equations and let the corresponding matched pair of prealternative algebras be \((A^*, -t^*_>, \ell^*_>, r^*_>, -\ell^*_>, -r^*_>, \ell^*_<, r^*_<, -\ell^*_<)\).

(1) As prealternative algebras,
\[A \bowtie -t^*_>, \ell^*_>, r^*_> -\ell^*_<, r^*_< A^*\text{ and } A \ltimes -t^*_>, \ell^*_>, r^*_> -\ell^*_<, r^*_< A^*\]
are isomorphic.

(2) The symmetric solutions of the PA equations are in one-to-one correspondence with linear maps \(T_r : A^* \to A\) whose graphs are Lagrangian prealternative subalgebras (with respect to the bilinear form (3-10) of \(A^* \ltimes -t^*_>, \ell^*_>, r^*_> -\ell^*_< A^*\).

Proof. It is similar to the proof of Proposition 3.13. \qed

9. Comparison between alternative D-bialgebras and prealternative bialgebras

The results in the previous sections allow us to compare alternative D-bialgebras (see the appendix) and prealternative bialgebras in terms of matched pairs of alternative algebras; alternative algebra structures on the direct sum of the alternative algebras in the matched pairs; bilinear forms on the direct sum of the alternative algebras in the matched pairs; double structures on the direct sum of the alternative algebras in the matched pairs; algebraic equations associated to coboundary cases, nondegenerate solutions; \(\mathcal{O}\)-operators of alternative algebras; and constructions from prealternative algebras. See Table 1.

From the table, we observe that there is a clear analogy between alternative D-bialgebras and prealternative bialgebras. Due to the correspondences between certain symmetries and skew-symmetries appearing in the table, we regard it as a kind of duality.

Appendix: Another approach to alternative D-bialgebras

In this section we prove the main results of [Goncharov 2007] by using a slightly different method (in fact, we will prove some results that are a little stronger than those there). There will be some results that were not presented there, such as the Drinfeld double theorem for an alternative D-bialgebra (Theorem A.10) and a homomorphism property of an alternative D-bialgebra (Theorem A.11). We omit the proofs since they are quite similar to the study of prealternative bialgebras.

Theorem A.1. Let \((A, \circ)\) be an alternative algebra and \((A^*, \ast)\) be an alternative algebra induced by a linear map \(\Delta : A \to A \otimes A\). Then \((A, \circ, \Delta)\) is an alternative D-bialgebra if and only if \((A, A^*, \ell^*_>, \ell^*_<, r^*_>, r^*_<)\) is a matched pair of alternative algebras.
Table 1. Comparison between alternative D-bialgebras and prealternative bialgebras

<table>
<thead>
<tr>
<th>Algebras</th>
<th>Alternative D-bialgebras</th>
<th>Prealternative bialgebras</th>
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<tr>
<td>Matched pairs of alternative algebras</td>
<td>(A, A*, t_o^<em>, l_o, r_o^</em>, l_o)</td>
<td>(Alt(A), Alt(A*), r_o^<em>, l_o, r_o^</em>, l_o)</td>
</tr>
<tr>
<td>Alternative algebra structures on the direct sum of the alternative algebras in the matched pairs</td>
<td>alternative analogues of Manin triples</td>
<td>phase spaces</td>
</tr>
<tr>
<td>Bilinear forms on the direct sum of the alternative algebras in the matched pairs</td>
<td>symmetric</td>
<td>skew-symmetric</td>
</tr>
<tr>
<td>⟨x + a*, y + b*⟩ = ⟨x, b*⟩ + ⟨a*, y⟩</td>
<td>⟨x + a*, y + b*⟩</td>
<td>= −⟨x, b*⟩ + ⟨a*, y⟩</td>
</tr>
<tr>
<td>Double structures on the direct sum of the alternative algebras in the matched pairs</td>
<td>Drinfeld’s doubles</td>
<td>Drinfeld’s symplectic doubles</td>
</tr>
<tr>
<td>Algebraic equations associated to coboundary cases</td>
<td>skew-symmetric solutions</td>
<td>symmetric solutions</td>
</tr>
<tr>
<td>alternative YBEs in alternative algebras</td>
<td>PA-equations in prealternative algebras</td>
<td></td>
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<td>Nondegenerate solutions</td>
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<td>C-operators</td>
<td>associated to (t_o, l_o)</td>
<td>associated to (t_o, l_o)</td>
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<td>Constructions from prealternative algebras</td>
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<td>r = \sum_{i=1}^{n}(e_i \otimes e_i^* - e_i^* \otimes e_i)</td>
<td>\sum_{i=1}^{n}(e_i \otimes e_i^* + e_i^* \otimes e_i)</td>
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<td>induced bilinear forms</td>
<td>induced bilinear forms</td>
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<tr>
<td>⟨x + a*, y + b*⟩</td>
<td>⟨x + a*, y + b*⟩</td>
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<td>= −⟨x, b*⟩ + ⟨a*, y⟩</td>
<td>= ⟨x, b*⟩ + ⟨a*, y⟩</td>
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</table>

**Theorem A.2.** Let \((A, o, \Delta)\) be an alternative algebra equipped with a linear map \(\Delta : A \to A \otimes A\) such that \(\Delta^* : A^* \otimes A^* \to A^*\) induces an alternative algebra structure on \(A^*\). Then \((A, A^*, t_o^*, l_o, r_o^*, l_o)\) is a matched pair of alternative algebras if and
only if the following equations hold:

(A-1) \[ \Delta(x \circ y) = (-l_0(x) \otimes 1 + 1 \otimes \text{Ass}_o(x))\Delta(y) + (\sigma_0(y) \otimes 1)\Delta(x) + (r_0(y) \otimes 1 - 1 \otimes l_0(y))\sigma \Delta(x), \]

(A-2) \[ \Delta(x \circ y) = (\text{Ass}_o(y) \otimes 1 - 1 \otimes r_0(y))\Delta(x) + (1 \otimes l_0(x))\Delta(y) \]

(A-3) \[ \Delta(x \circ y + y \circ x) = (r_0(y) \otimes 1 + 1 \otimes l_0(y))\Delta(x) \]

(A-4) \[ (\Delta + \sigma \Delta)(x \circ y) = (r_0(y) \otimes 1)\Delta(x) + (1 \otimes r_0(y))\sigma \Delta(x) \]

\[ + (l_0(x) \otimes 1)\sigma \Delta(y) + (1 \otimes l_0(x))\Delta(y), \]

where \( x, y \in A \), the multiplication \(*\) is induced by \( \Delta \), and \( \text{Ass}_o = l_0 + r_0 \).

Remark A.3. Equations (A-1) and (A-4) have already appeared as [Goncharov 2007, Lemma 2].

Definition A.4. An alternative D-bialgebra \((A, \circ, \Delta)\) is called coboundary if there exists an \( r \in A \otimes A \) such that \( \Delta \) is given by

(A-5) \[ \Delta(x) = (r_0(x) \otimes 1 - 1 \otimes l_0(x))r \quad \text{for all } x \in A. \]

This definition is the same as Definition 3.3.

Lemma A.5. Let \( A \) be a vector space. Then a linear map \( \Delta : A \to A \otimes A \) induces an alternative algebra structure on \( A^* \) if and only if for any \( x, y \in A \)

\[ (\Delta \otimes 1)\Delta(x) + (\sigma \otimes 1)(\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x) + (\sigma \otimes 1)(1 \otimes \Delta)\Delta(x), \]

\[ (\Delta \otimes 1)\Delta(x) + (1 \otimes \sigma)(\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x) + (1 \otimes \sigma)(1 \otimes \Delta)\Delta(x). \]

Definition A.6. Let \((A, \circ)\) be an alternative algebra. The following equations are called the \( A_i \)-equations in \( A \) for \( i = 1, 2 \):

(A-6) \[ A_1 = r_{23} \circ r_{12} - r_{13} \circ r_{23} - r_{12} \circ r_{13} = 0, \]

\[ A_2 = r_{12} \circ r_{23} - r_{23} \circ r_{13} - r_{13} \circ r_{12} = 0. \]

Note that \( A_1 \) given by (A-6) is exactly \( C_A(r) \) given by (3-5).

Proposition A.7. Let \((A, \circ)\) be an alternative algebra and \( r \in A \otimes A \). If \( r \) is skew-symmetric, then \( A_1 \) and \( A_2 \) are the same.

Proposition A.8. Let \((A, \circ)\) be an alternative algebra. Let \( r \in A \otimes A \) be skew-symmetric. Define a linear map \( \Delta : A \to A \otimes A \) by (A-5). Then \( \Delta \) induces an alternative algebra structure on \( A^* \) if and only if for any \( x \in A \)

\[ -(r_0(x) \otimes 1 \otimes 1)A_1 - (1 \otimes r_0(x) \otimes 1)A_2 + (1 \otimes 1 \otimes l_0(x))(A_1 + A_2) = 0, \]

\[ -(r_0(x) \otimes 1 \otimes 1)(A_1 + A_2) + (1 \otimes l_0(x) \otimes 1)A_2 + (1 \otimes 1 \otimes l_0(x))A_1 = 0. \]
Theorem A.9 [Goncharov 2007, Theorem 2]. Suppose \((A, \circ)\) is an alternative algebra and \(r \in A \otimes A\). Let \(\Delta : A \to A\) be a linear map defined by (A-5). If \(r\) is a skew-symmetric solution of the alternative Yang–Baxter equation in \(A\), then \((A, \circ, \Delta)\) is an alternative \(D\)-bialgebra.

Theorem A.10. Let \((A, \circ, \Delta)\) be an alternative \(D\)-bialgebra. Then there is a canonical alternative bialgebra structure on \(A \oplus A^*\) such that the inclusion \(i_1 : A \to A \oplus A^*\) is a homomorphism of alternative \(D\)-bialgebras, where the alternative bialgebra structure on \(A\) is given by \((A, \circ, -\Delta_A)\) and the inclusion \(i_2\) of \(A^*\) into \(A \oplus A^*\) is a homomorphism of alternative \(D\)-bialgebras, where the alternative bialgebra structure on \(A^*\) is given by \((A^*, \ast, \Delta_B)\), where \(\ast\) is induced by \(\Delta_A\), and where the alternative algebra structure \(\circ\) on \(A\) is induced by \(\Delta_B : A^* \to A^* \otimes A^*\).

Theorem A.11. Let \((A, \circ, \Delta)\) be an alternative \(D\)-bialgebra arising from a solution \(r\) of the alternative Yang–Baxter equation in \(A\). Then \(T_r\) is a homomorphism of alternative \(D\)-bialgebras from \((A^*, \ast, \delta)\) to \((A, \circ, -\Delta)\), where \(\ast\) is induced by \(\Delta\) and the alternative algebra structure \(\circ\) on \(A\) is induced by \(\delta : A^* \to A^* \otimes A^*\).

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References


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