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We introduce a function $w_k(\theta)$ for closed convex plane curves, and then prove a geometric inequality involving $w_k(\theta)$ and the area A enclosed by the curve. As a by-product, we give a new proof of the classical isoperimetric inequality. Finally, we give some properties of convex curves with $w_k(\theta)$ being constant and pose an open problem motivated by the elegant Blaschke–Lebesgue theorem.

1. Introduction

Geometric inequalities involving convex sets have received much attention during the last centuries; see for example [Bonnesen and Fenchel 1934; Burago and Zalgaller 1988; Schneider 1993]. Among them the isoperimetric inequalities are of special interest; see [Ball 1991; Blaschke 1956; Bonnesen 1929; Osserman 1978; 1979; Pan and Zhang 2007; Schneider 1993] and references therein. For convex curves in the Euclidean plane $\mathbb{R}^2$, there are also many interesting inequalities involving their geometric quantities such as inradius, outradius, width, area, length and curvature or radius of curvature; see for example [Chernoff 1969; Gage 1983; Green and Osher 1999; Hernández Cifre 2000; Ma and Cheng 2009; Ma and Zhu 2008; Pan and Yang 2008; Sholander 1952].

Chernoff [1969] got an area-width inequality for convex plane curves. Let α be a closed convex curve in the Euclidean plane $\mathbb{R}^2$ with area A and width function $w(\theta)$. Then the geometric inequality

$$A \leq \frac{1}{2}\int_0^{\pi/2} w(\theta)w(\theta+\tfrac{1}{2}\pi)\,d\theta$$

holds, with equality if and only if α is a circle. To our knowledge, this beautiful inequality has not been generalized yet. One purpose of this note is to make some generalization of the Chernoff inequality. To this end, we introduce for convex

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plane curves a function $w_k(\theta)$ for integer $k \geq 2$. This function is defined in (3-1) below and is a generalization of the usual width of a convex curve. Then Chernoff's area-width inequality generalizes as

$$A \leq \frac{1}{k}\int_0^{\pi/k} w_k(\theta)w_k(\theta+\tfrac{1}{k}\pi)d\theta, \tag{1-1}$$

with equality if and only if α is a circle. Moreover, we can calculate

$$\lim_{k\to\infty}\frac{1}{k}\int_0^{\pi/k} w_k(\theta)w_k(\theta+\tfrac{1}{k}\pi)d\theta = \frac{L^2}{4\pi}. \tag{1-2}$$

Thus (1-1) and (1-2) give a new proof of the classical isoperimetric inequality $L^2 \geq 4\pi A$ with equality if and only if the curve is a circle.

Another purpose here is to give some properties of closed convex curves with $w_k(\theta)$ being constant. We will get in Theorem 3.2 an analogue of Barbier's theorem and in Theorem 3.3 we will characterize the support function of such curves. In particular, we pose an open problem that was motivated by the elegant Blaschke–Lebesgue theorem.

2. Preliminaries

Henceforth suppose without loss of generality that α is a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$. Take a point O inside α as the origin of our frame. Let p be the oriented perpendicular distance from O to the tangent at a point on α, and θ the oriented angle from the positive x_1-axis to this perpendicular ray. Clearly, p, as a function of θ, is single-valued and 2π-periodic. We usually call $p(\theta)$ *Minkowski's support function* of α.

One can check that α can be parametrized in terms of θ and $p(\theta)$ as

$$\alpha(\theta) = (\alpha_1(\theta), \alpha_2(\theta)) = (p(\theta)\cos\theta - p'(\theta)\sin\theta,\ p(\theta)\sin\theta + p'(\theta)\cos\theta);$$

see for instance [Hsiung 1981]. The curvature κ of α can be calculated according to $\kappa(\theta) = d\theta/ds = 1/(p(\theta)+p''(\theta)) > 0$. Denote by L and A the length of α and the area it bounds. Then one can get

$$L = \int_\alpha ds = \int_0^{2\pi} \rho(\theta)d\theta = \int_0^{2\pi} p(\theta)d\theta, \tag{2-1}$$

$$\begin{aligned} A = \frac{1}{2}\int_\alpha p(\theta)ds &= \frac{1}{2}\int_0^{2\pi} p(\theta)(p(\theta)+p''(\theta))d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (p^2(\theta) - p'^2(\theta))d\theta. \end{aligned} \tag{2-2}$$

These are known as *Cauchy's formula* and *Blaschke's formula*, respectively.

Since the support function of a given convex curve α is always continuous, bounded and 2π-periodic, it has a Fourier series of the form

$$p(\theta)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta), \tag{2-3}$$

where

$$\begin{aligned} a_0=\frac{1}{\pi}\int_0^{2\pi}p(\theta)d\theta,\quad a_n&=\frac{1}{\pi}\int_0^{2\pi}p(\theta)\cos n\theta d\theta,\\ b_n&=\frac{1}{\pi}\int_0^{2\pi}p(\theta)\sin n\theta d\theta\quad\text{for } n\geq 1. \end{aligned} \tag{2-4}$$

We wish to express L and A in terms of the Fourier coefficients of $p(\theta)$. From (2-1) and the first equation of (2-4) one easily sees that $L=\pi a_0$. Then differentiating (2-3) with respect to θ gives us $p'(\theta)=\sum_{n=1}^{\infty}n(-a_n\sin n\theta+b_n\cos n\theta)$. By the Parseval equality and (2-2), we get

$$A=\tfrac{1}{4}\pi a_0^2+\tfrac{1}{2}\pi\sum_{n=2}^{\infty}(1-n^2)(a_n^2+b_n^2), \tag{2-5}$$

The width $w(\theta)$ of α in direction $\boldsymbol{u}(\theta)=(\cos\theta,\sin\theta)$ is defined to be the distance between two tangents to α perpendicular to $\boldsymbol{u}(\theta)$. It is clear that

$$w(\theta)=p(\theta)+p(\theta+\pi). \tag{2-6}$$

The closed convex curve α is said to be of constant width if its width in any direction is a positive constant w_0, and in this case, the constant w_0 is called the width of α. If α is a constant width curve with width w_0, then $p(\theta)+p(\theta+\pi)=w_0$ for any $\theta\in[0,2\pi]$.

It is obvious that a circle is a constant width curve. There are, however,- many other noncircular curves of constant width; see for example [Burke 1966; Hsiung 1981; Rabinowitz 1997]. Among them, the most famous example is the Reuleaux triangle, which has been used in the design of piston for the Wankel engine. A famous result about constant width curves due to Barbier [1860] states that all closed convex curves of constant width w_0 have the same perimeter πw_0. Another elegant result is the Blaschke–Lebesgue theorem, which says that among all closed convex curves with constant width w_0, the Reuleaux triangles of the same constant width have the smallest area.

Our Theorem 3.2 bears analogy to Barbier's theorem. The open problem posed in the next section is motivated by the Blaschke–Lebesgue theorem first proved by Blaschke [1915] and Lebesgue [1914; 1921]. Harrell [2002] gives a new proof of this theorem and some historic remarks on it. The higher dimensional Blaschke–Lebesgue problem appears to be very difficult to solve and remains open. For partial results, see [Anciaux and Georgiou 2009; Anciaux and Guilfoyle 2010] and the literature therein.

3. Main results

For an integer $k \geq 2$, we introduce for a convex curve α a function $w_k(\theta)$ by

$$w_k(\theta) = p(\theta) + p(\theta + 2\pi/k) + \cdots + p(\theta + (2(k-1)\pi)/k). \tag{3-1}$$

Since

$$1 + \cos(2\pi/k) + \cdots + \cos(2(k-1)\pi/k) = 0,$$
$$\sin(2\pi/k) + \cdots + \sin(2(k-1)\pi/k) = 0,$$

the function $w_k(\theta)$ is independent of the choice of the origin O (inside α) and thus is well-defined. It is clear that $w_k(\theta)$ is a periodic function with period $2\pi/k$.

If $k = 2$, $w_2(\theta)$ is the usual width (see (2-6)) of a curve, that is, our $w_k(\theta)$ is a generalization of the usual width function $w(\theta)$. In this case, Chernoff [1969] got a nice area-width inequality. For general k, we can generalize:

Theorem 3.1. *Let α be a closed convex plane curve, bounding a region of area A. Then*

$$A \leq \frac{1}{k} \int_0^{\pi/k} w_k(\theta) w_k(\theta + \pi/k)\, d\theta,$$

where the equality holds if and only if α is a circle.

Proof. The proof is divided into two steps.

Step 1. We first show that

$$\int_0^{\pi/k} w_k(\theta) w_k(\theta + \pi/k)\, d\theta = \frac{1}{2} \sum_{m=1}^{k} \int_0^{2\pi} p(\theta) p(\theta + (2m-1)\pi/k)\, d\theta. \tag{3-2}$$

To see this, write

$$a_{ij} = \int_0^{\pi/k} p\Big(\theta + \frac{(2i-1)\pi}{k}\Big) p\Big(\theta + \frac{2(j-1)\pi}{k}\Big) d\theta \quad \text{for } i, j = 1, 2, \ldots, 2k.$$

Then

$$\begin{aligned} &\int_0^{\pi/k} w_k(\theta) w_k\Big(\theta + \frac{\pi}{k}\Big) d\theta \\ &= \int_0^{\pi/k} \Big(p(\theta) + p\Big(\theta + \frac{2\pi}{k}\Big) + \cdots + p\Big(\theta + \frac{2(k-1)\pi}{k}\Big)\Big) \\ &\qquad \cdot \Big(p\Big(\theta + \frac{\pi}{k}\Big) + p\Big(\theta + \frac{3\pi}{k}\Big) + \cdots + p\Big(\theta + \frac{(2k-1)\pi}{k}\Big)\Big) d\theta = \sum_{i,j=1}^{k} a_{ij}. \end{aligned} \tag{3-3}$$

Since p is a 2π-periodic function, we get

$$a_{i+k,j} = a_{ij} = a_{i,j+k} \quad \text{for } i,\, j = 1, 2, \dots, k. \tag{3-4}$$

Now, we claim that

$$\sum_{m=1}^{k}\sum_{l=1}^{k} a_{m+l-1,l} = \sum_{m=1}^{k}\sum_{l=1}^{k} a_{l,m+l} = \sum_{i,j=1}^{k} a_{ij}. \tag{3-5}$$

The sum at left can be treated as

$$\sum_{l=1}^{k}\Bigl(\sum_{m=1}^{k-l+1} a_{m+l-1,l} + \sum_{m=k-l+2}^{k} a_{m+l-k-1,l}\Bigr) = \sum_{l=1}^{k}\Bigl(\sum_{i=l}^{k} a_{il} + \sum_{i=1}^{l-1} a_{il}\Bigr) = \sum_{l=1}^{k}\sum_{i=1}^{k} a_{il},$$

while the middle sum becomes

$$\sum_{l=1}^{k}\Bigl(\sum_{m=1}^{k-l} a_{l,m+l} + \sum_{m=k-l+1}^{k} a_{l,m+l-k}\Bigr) = \sum_{l=1}^{k}\Bigl(\sum_{i=l+1}^{k} a_{li} + \sum_{i=1}^{l} a_{li}\Bigr) = \sum_{l=1}^{k}\sum_{i=1}^{k} a_{li}.$$

Thus, we get

$$\int_0^{\pi/k} w_k(\theta) w_k(\theta + \pi/k)\, d\theta = \sum_{i,j=1}^{k} a_{ij} = \frac{1}{2}\sum_{m=1}^{k}\Bigl(\sum_{l=1}^{k}(a_{m+l-1,l} + a_{l,m+l})\Bigr). \tag{3-6}$$

Next, we shall show that, for $1 \le m \le k$,

$$\sum_{l=1}^{k}(a_{m+l-1,l} + a_{l,m+l}) = \int_0^{2\pi} p(\theta)\, p(\theta + (2m-1)\pi/k)\, d\theta. \tag{3-7}$$

In fact,

$$\begin{aligned}
\text{left side of (3-7)} &= \sum_{l=1}^{k}\biggl(\int_0^{\pi/k} p\Bigl(\theta + \frac{2(m+l-1)-1}{k}\pi\Bigr) p\Bigl(\theta + \frac{2(l-1)}{k}\pi\Bigr) d\theta \\
&\qquad + \int_0^{\pi/k} p\Bigl(\theta + \frac{2l-1}{k}\pi\Bigr) p\Bigl(\theta + \frac{2(m+l-1)}{k}\pi\Bigr) d\theta\biggr) \\
&= \sum_{l=1}^{k}\biggl(\int_{2(l-1)\pi/k}^{(2l-1)\pi/k} p(\theta)\, p\Bigl(\theta + \frac{(2m-1)\pi}{k}\Bigr) d\theta \\
&\qquad + \int_{(2l-1)\pi/k}^{2l\pi/k} p(\theta)\, p\Bigl(\theta + \frac{(2m-1)\pi}{k}\Bigr) d\theta\biggr) \\
&= \int_0^{2\pi} p(\theta)\, p\Bigl(\theta + \frac{(2m-1)\pi}{k}\Bigr) d\theta.
\end{aligned}$$

Now, combining (3-3)–(3-7) yields (3-2).

Step 2. After some calculations, we get, for $1 \le m \le k$,

$$\int_0^{2\pi} p(\theta)p\Big(\theta+\frac{(2m-1)\pi}{k}\Big)d\theta = \tfrac{1}{2}\pi a_0^2+\pi\sum_{n=1}^{\infty}(a_n^2+b_n^2)\cos\frac{n(2m-1)\pi}{k}. \tag{3-8}$$

For any integer n not a multiple of k, we have

$$\begin{aligned}\sum_{m=1}^{k}\cos\frac{n(2m-1)\pi}{k} &= \frac{1}{\sin(n\pi/k)}\sum_{m=1}^{k}\Big(\cos\frac{n(2m-1)\pi}{k}\sin\frac{n\pi}{k}\Big)\\ &= \frac{\sin 2n\pi}{2\sin(n\pi/k)} = 0.\end{aligned} \tag{3-9}$$

It follows from (3-2), (3-8), (3-9) and (2-5) that

$$\begin{aligned}\frac{1}{k}\int_0^{\pi/k} w_k(\theta)w_k\Big(\theta+\frac{\pi}{k}\Big)d\theta &= \frac{1}{2k}\sum_{m=1}^{k}\int_0^{2\pi} p(\theta)p\Big(\theta+\frac{(2m-1)\pi}{k}\Big)d\theta\\ &= \tfrac{1}{4}\pi a_0^2+\frac{\pi}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\frac{1}{k}\sum_{m=1}^{k}\cos\frac{n(2m-1)\pi}{k}\\ &= \tfrac{1}{4}\pi a_0^2+\frac{\pi}{2}\sum_{l=1}^{\infty}(a_{kl}^2+b_{kl}^2)\frac{1}{k}\sum_{m=1}^{k}\cos(l(2m-1)\pi)\\ &= \tfrac{1}{4}\pi a_0^2+\frac{\pi}{2}\sum_{l=1}^{\infty}(-1)^l(a_{kl}^2+b_{kl}^2)\\ &= A+\frac{\pi}{2}\Big(\sum_{n=2}^{\infty}(a_n^2+b_n^2)(n^2-1)+\sum_{l=1}^{\infty}(-1)^l(a_{kl}^2+b_{kl}^2)\Big)\\ &\ge A.\end{aligned} \tag{3-10}$$

The equality holds if and only if $a_n = b_n = 0$ for all $n \ge 2$, that is, when the curve is a circle. □

From the continuity of $p(\theta)$, it is easy to see that, for all $\theta_k \in [0, 2\pi/k]$,

$$\lim_{k\to\infty}\frac{2\pi}{k}w_k(\theta_k) = \lim_{k\to\infty}\frac{2\pi}{k}\sum_{m=1}^{k}p\Big(\theta_k+\frac{2m\pi}{k}\Big) = \int_0^{2\pi}p(\theta)d\theta.$$

Moreover, for any $k \in \mathbb{N}$, there exists a $\xi_k \in [0, \pi/k]$ such that

$$\frac{1}{k}\int_0^{\pi/k} w_k(\theta)w_k(\theta+\pi/k)d\theta = \frac{\pi}{k^2}w_k(\xi_k)w_k(\xi_k+\pi/k).$$

Since $\xi_k \in [0, \pi/k] \subset [0, 2\pi/k]$, we have $\xi_k + \pi/k \in [0, 2\pi/k]$. Thus, we obtain

$$\lim_{k\to\infty} \frac{1}{k}\int_0^{\pi/k} w_k(\theta)w_k(\theta+\pi/k)d\theta = \lim_{k\to\infty}\frac{\pi}{k^2}w_k(\xi_k)w_k(\xi_k+\pi/k) = \frac{1}{4\pi}\Big(\int_0^{2\pi} p(\theta)d\theta\Big)^2 = \frac{L^2}{4\pi}.$$

which with (3-10) gives us a new proof of the classical isoperimetric inequality.

We can also get the following generalization of Barbier's theorem.

Theorem 3.2. *All convex curves for $w_k(\theta)$ is equal to a constant Λ have the same length $L = (2\pi/k)\Lambda$.*

Proof. It is easy to see from (2-1) that $L = \int_0^{2\pi/k} w_k(\theta)d\theta = (2\pi/k)\Lambda$. □

Among all curves with the same length L, circles have the greatest area. For constant width curves, the Blaschke–Lebesgue theorem claims that the Reuleaux triangles have the least area.

Question. Among all closed convex curves with $w_k(\theta)$ equal to a fixed consant Λ, which has the least possible area?

Theorem 3.3. *Suppose α is a closed convex plane curve with $w_k(\theta)$ equal to a constant Λ. Then the Fourier expansion of the support function $p(\theta)$ of α is of the form*

$$p(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1,n\neq mk}^{\infty}(a_n\cos n\theta + b_n \sin n\theta),$$

where $a_0 = (1/\pi)\int_0^{2\pi} p(\theta)d\theta = L/\pi = 2\Lambda/k$ and $m \in \mathbb{N}$.

Proof. In terms of the Fourier expansion of the support function $p(\theta)$ of α,

$$\begin{aligned} w_k(\theta) &= \tfrac{1}{2}ka_0+\sum_{n=1}^{\infty}\Big(a_n\cos n\theta+b_n\sin n\theta+a_n\cos\Big(n\theta+\frac{2n\pi}{k}\Big)+b_n\sin\Big(n\theta+\frac{2n\pi}{k}\Big) \\ &\qquad +\cdots+a_n\cos\Big(n\theta+\frac{2n(k-1)\pi}{k}\Big)+b_n\sin\Big(n\theta+\frac{2n(k-1)\pi}{k}\Big)\Big) \\ &= \tfrac{1}{2}ka_0+\sum_{n=1}^{\infty}\Big((a_n\cos n\theta+b_n\sin n\theta)\Big(1+\cos\frac{2n\pi}{k}+\cdots+\cos\frac{2n(k-1)\pi}{k}\Big) \\ &\qquad +(b_n\cos n\theta-a_n\sin n\theta)\Big(\sin\frac{2n\pi}{k}+\sin\frac{4n\pi}{k}+\cdots+\sin\frac{2n(k-1)\pi}{k}\Big)\Big) \\ &= \tfrac{1}{2}ka_0+k\sum_{n=1}^{\infty}(a_{kn}\cos(kn\theta)+b_{kn}\sin(kn\theta)). \end{aligned}$$

If $w_k(\theta) = \Lambda$, then one gets $a_0 = (2/k)\Lambda$ and $a_{kn} = 0 = b_{kn}$. □

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References

[Anciaux and Georgiou 2009] H. Anciaux and N. Georgiou, "The Blaschke–Lebesgue problem for constant width bodies of revolution", preprint, version 1, 2009. arXiv 0903.4284v1

[Anciaux and Guilfoyle 2010] H. Anciaux and B. Guilfoyle, "On the three-dimensional Blaschke–Lebesgue problem", preprint, version 2, 2010. arXiv 0906.3217v2

[Ball 1991] K. Ball, "Volume ratios and a reverse isoperimetric inequality", *J. London Math. Soc.* (2) **44**:2 (1991), 351–359. MR 92j:52013 Zbl 0694.46010

[Barbier 1860] A. Barbier, "Note sur le problème de l'aiguille le et jeu du joint couvert", *J. Math. Pure Appl* (2) **5** (1860), 273–286.

[Blaschke 1915] W. Blaschke, "Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts", *Math. Ann.* **76**:4 (1915), 504–513. MR 1511839 JFM 45.0731.04

[Blaschke 1956] W. Blaschke, *Kreis und Kugel*, 2nd ed., de Gruyter, Berlin, 1956. MR 17,1123d Zbl 0070.17501

[Bonnesen 1929] T. Bonnesen, *Les problèmes des isopérimètres et des isépiphanes*, Gauthier-Villars, Paris, 1929. JFM 55.0431.08

[Bonnesen and Fenchel 1934] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **1**, Springer, 1934. MR 51 #8954 Zbl 0008.07708 JFM 60.0673.01

[Burago and Zalgaller 1988] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften **285**, Springer, Berlin, 1988. MR 89b:52020 Zbl 0633.53002

[Burke 1966] J. F. Burke, "A curve of constant diameter", *Math. Mag.* **39** (1966), 84–85. MR 33 #1794 Zbl 0136.18903

[Chernoff 1969] P. R. Chernoff, "An area-width inequality for convex curves", *Amer. Math. Monthly* **76** (1969), 34–35. MR 38 #3769 Zbl 0175.19501

[Gage 1983] M. E. Gage, "An isoperimetric inequality with applications to curve shortening", *Duke Math. J.* **50**:4 (1983), 1225–1229. MR 85d:52007 Zbl 0534.52008

[Green and Osher 1999] M. Green and S. Osher, "Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves", *Asian J. Math.* **3**:3 (1999), 659–676. MR 2001j:53089 Zbl 0969.53040

[Harrell 2002] E. M. Harrell, II, "A direct proof of a theorem of Blaschke and Lebesgue", *J. Geom. Anal.* **12**:1 (2002), 81–88. MR 2002k:52009 Zbl 1044.52001

[Hernández Cifre 2000] M. A. Hernández Cifre, "Is there a planar convex set with given width, diameter, and inradius?", *Amer. Math. Monthly* **107**:10 (2000), 893–900. MR 2001j:52002 Zbl 0983.52002

[Hsiung 1981] C. C. Hsiung, *A first course in differential geometry*, Wiley, New York, 1981. MR 83c:53001 Zbl 0458.53001

[Lebesgue 1914] H. Lebesgue, "Sur le problème des isopérimètres et sur les domaines de largeur constante", *Bull. Soc. Math. France C. R.* **7** (1914), 72–76.

[Lebesgue 1921] H. Lebesgue, "Sur quelques questions des minimums, relatives aux courbes orbiformes, et sur leurs rapports avec le calcul de variations", *J. Math. Pure Appl.* **8** (1921), 67–96. JFM 48.0838.01

[Ma and Cheng 2009] L. Ma and L. Cheng, "A non-local area preserving curve flow", preprint, 2009. arXiv 0907.1430

[Ma and Zhu 2008] L. Ma and A. Zhu, "On a length preserving curve flow", preprint, 2008. arXiv 0811.2083

[Osserman 1978] R. Osserman, "The isoperimetric inequality", *Bull. Amer. Math. Soc.* **84**:6 (1978), 1182–1238. MR 58 #18161 Zbl 0411.52006

[Osserman 1979] R. Osserman, "Bonnesen-style isoperimetric inequalities", *Amer. Math. Monthly* **86**:1 (1979), 1–29. MR 80h:52013 Zbl 0404.52012

[Pan and Yang 2008] S. Pan and J. Yang, "On a non-local perimeter-preserving curve evolution problem for convex plane curves", *Manuscripta Math.* **127**:4 (2008), 469–484. MR 2010h:53099 Zbl 1169.35033

[Pan and Zhang 2007] S. Pan and H. Zhang, "A reverse isoperimetric inequality for convex plane curves", *Beiträge Algebra Geom.* **48**:1 (2007), 303–308. MR 2008c:52011 Zbl 1121.52023

[Rabinowitz 1997] S. Rabinowitz, "A polynomial curve of constant width", *Missouri J. Math. Sci.* **9**:1 (1997), 23–27. MR 98d:52002 Zbl 1097.52501

[Schneider 1993] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, 1993. MR 94d:52007 Zbl 0798.52001

[Sholander 1952] M. Sholander, "On certain minimum problems in the theory of convex curves", *Trans. Amer. Math. Soc.* **73** (1952), 139–173. MR 14,787d Zbl 0047.15901

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