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Let p be a real number greater than one and let G be a connected graph of bounded degree. We introduce the p-harmonic boundary of G and use it to characterize the graphs G for which the constant functions are the only p-harmonic functions on G. We show that any continuous function on the p-harmonic boundary of G can be extended to a function that is p-harmonic on G. We also give some properties of this boundary that are preserved under rough-isometries. Now let Γ be a finitely generated group. As an application of our results, we characterize the vanishing of the first reduced ℓ^p-cohomology of Γ in terms of the cardinality of its p-harmonic boundary. We also study the relationship between translation invariant linear functionals on a certain difference space of functions on Γ, the p-harmonic boundary of Γ, and the first reduced ℓ^p-cohomology of Γ.

1. Introduction

Let p be a real number greater than one and let Γ be a finitely generated infinite group. There has been some work done relating various boundaries of Γ and the nonvanishing of the first reduced ℓ^p-cohomology space $\overline{H}^1_{(p)}(\Gamma)$ of Γ (to be defined in Section 7). Gromov [1993, Chapter 8, Section C2] — see also [Elek 1997] — showed that if the ℓ^p-corona of Γ contains more than one element, then $\overline{H}^1_{(p)}(\Gamma) \neq 0$. Puls [2007] showed that if there is a Floyd boundary of Γ containing more than two elements, and if the Floyd admissible function satisfies a certain decay condition, then $\overline{H}^1_{(p)}(\Gamma) \neq 0$. However, it is unknown if the converse of either of these two results is true. The motivation for this paper is to find a boundary for Γ whose cardinality characterizes the vanishing of $\overline{H}^1_{(p)}(\Gamma)$. We will show that the p-harmonic boundary, defined in Section 2.1, does the trick. This boundary gives the desired result because $\overline{H}^1_{(p)}(\Gamma) = 0$ if and only if the only p-harmonic functions on Γ are the constants, [Puls 2006, Theorem 3.5]. We will show in Section 7 that the cardinality of the p-harmonic boundary is 0 or 1 if and

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only if the only p-harmonic functions on Γ are the constants. Hence, $\overline{H}^1_{(p)}(\Gamma) = 0$ if and only if the cardinality of the p-harmonic boundary is 0 or 1.

L_p-cohomology was investigated first in [Gol'dshteĭn et al. 1987] for the case of Riemannian manifolds. Gromov [1993, Chapter 8] has studied ℓ^p-cohomology for finitely generated groups, and in the more general setting of graphs with bounded degree. In particular, Cheeger and Gromov [1986] showed that the first reduced ℓ^2-cohomology of a finitely generated amenable group is zero. Gromov [1993, page 150] conjectured that this result is also true for all real numbers $p > 1$. This is our main justification for choosing to study the p-harmonic boundary in the discrete setting. If enough insight can be gained into this boundary, then we may be able to develop the tools needed to compute the p-harmonic boundary of a finitely generated amenable group. This of course would resolve Gromov's conjecture.

More information about the first reduced L_p-cohomology (and the special case of L_2-cohomology) can be found in [Pansu 1989; 2007; 2008; Tessera 2009] for various manifolds, and in [Bekka and Valette 1997; Bourdon 2004; Bourdon et al. 2005; Elek 1998; Martin and Valette 2007; Puls 2003; 2006; 2007] for finitely generated groups. As implied earlier, there is a strong connection between the vanishing of the first reduced L_p-cohomology and the nonexistence nonconstant p-harmonic functions; for a proof in the case of homogeneous Riemannian manifolds, see [Tessera 2009, Proposition 4.11]. Thus results on p-harmonic functions are useful in trying to determine if the first reduced L_p-cohomology vanishes. The papers [Coulhon et al. 2001; Grigoryan 1987] study p-harmonic functions on manifolds, while [Holopainen and Soardi 1997a; Kim and Lee 2005; 2007; Soardi 1993; Yamasaki 1977] examine p-harmonic functions on graphs.

2. Definitions and statement of main results

Let p be a real number greater than one, and let Γ be a finitely generated infinite group. The definition of the p-harmonic boundary for Γ does not depend on the group law of Γ, so we can define this boundary in the more general setting of a graph. The reason is that we can associate a graph, called the Cayley graph of Γ, with Γ. The vertex set for this graph consists of the elements of Γ, and $x_1, x_2 \in \Gamma$ are joined by an edge if and only if $x_1 = x_2 s^{\pm 1}$ for a generator s of Γ.

2.1. ***The p-harmonic boundary.*** Let G be a graph with vertex set V_G and edge set E_G. We will write V for V_G and E for E_G if it is clear what the graph G is. For $x \in V$, we denote by $\deg(x)$ the number of neighbors of x and by N_x the set of neighbors of x. We say a graph G is of *bounded degree* if there exists a positive integer k such that $\deg(x) \leq k$ for every $x \in V$. A path in G is a sequence of vertices $x_1, x_2, \ldots, x_n$ for which $x_{i+1} \in N_{x_i}$ for $1 \leq i \leq n-1$. A graph G is connected if any two given vertices of G are joined by a path. All graphs considered in this paper will be countably infinite, connected, of bounded degree with no self-loops. Assign length one to each edge in E_G; then the graph G is a metric space with respect to the shortest path metric. Let $d_G(\cdot\,,\cdot)$ denote this metric. So if $x, y \in V$, then $d_G(x, y)$ is the length of the shortest path joining x and y. We will drop the subscript G from $d_G(\cdot\,,\cdot)$ when it is clear what graph G we are working with. Finally, if $x \in V$, then $B_n(x)$ will denote the metric ball that contains all elements of V that have distance less than n from x.

Let G be a graph with vertex set V, and let p be a real number greater than one. To construct the p-harmonic boundary of G, we need to first define the space of bounded p-Dirichlet finite functions on G. For any $S \subset V$, the outer boundary ∂S of S is the set of vertices in $V \setminus S$ with at least one neighbor in S. For a real-valued function f on $S \cup \partial S$, we define the p-th power of the *gradient*, the *p-Dirichlet sum*, and the *p-Laplacian* of $x \in S$ by

$$\begin{aligned} |Df(x)|^p &= \sum_{y \in N_x} |f(y) - f(x)|^p, \\ I_p(f, S) &= \sum_{x \in S} |Df(x)|^p, \\ \Delta_p f(x) &= \sum_{y \in N_x} |f(y) - f(x)|^{p-2}(f(y) - f(x)). \end{aligned}$$

In the case $1 < p < 2$, we make the convention that

$$|f(y) - f(x)|^{p-2}(f(y) - f(x)) = 0 \quad \text{if } f(y) = f(x).$$

Let $S \subseteq V$. We say a function f is p-harmonic on S if $\Delta_p f(x) = 0$ for all $x \in S$, and *p-Dirichlet finite* if $I_p(f, V) < \infty$. We denote the set of all p-Dirichlet finite functions on G by $D_p(G)$. Under the norm

$$\|f\|_{D_p} = (I_p(f, V) + |f(o)|^p)^{1/p},$$

$D_p(G)$ is a reflexive Banach space, where o is a fixed vertex of G and $f \in D_p(G)$. Denote by $\mathrm{HD}_p(G)$ the set of p-harmonic functions on V contained in $D_p(G)$. Let $\ell^\infty(G)$ denote the set of bounded functions on V, and let $\|f\|_\infty = \sup_V |f|$ for $f \in \ell^\infty(G)$. Set $\mathrm{BD}_p(G) = D_p(G) \cap \ell^\infty(G)$. The set $\mathrm{BD}_p(G)$ is a Banach space under the norm

$$\|f\|_{\mathrm{BD}_p} = (I_p(f, V))^{1/p} + \|f\|_\infty,$$

where $f \in \mathrm{BD}_p(G)$. Set $\mathrm{BHD}_p(G) = \mathrm{HD}_p(G) \cap \mathrm{BD}_p(G)$. It turns out that $\mathrm{BD}_p(G)$ is closed under pointwise multiplication. To see this, let $f, h \in \mathrm{BD}_p(G)$ and set $a = \sup_V |f|$ and $b = \sup_V |h|$. It follows from Minkowski's inequality that

$$(I_p(fh, V))^{1/p} \le b(I_p(f, V))^{1/p} + a(I_p(h, V))^{1/p}. \tag{2-1}$$

Thus $fh \in \mathrm{BD}_p(G)$. Using the inequality above, we obtain

$$\|fh\|_{\mathrm{BD}_p} \leq ((I_p(f, V))^{1/p} + a)((I_p(h, V))^{1/p} + b) = \|f\|_{\mathrm{BD}_p} \|h\|_{\mathrm{BD}_p}.$$

Hence $\mathrm{BD}_p(G)$ is an abelian Banach algebra. A character on $\mathrm{BD}_p(G)$ is a nonzero homomorphism from $\mathrm{BD}_p(G)$ into the complex numbers. Let $\mathrm{Sp}(\mathrm{BD}_p(G))$ be the set of characters on $\mathrm{BD}_p(G)$; it is known as the spectrum of $\mathrm{BD}_p(G)$. With respect to the weak $*$-topology, $\mathrm{Sp}(\mathrm{BD}_p(G))$ is a compact Hausdorff space. Let $C(\mathrm{Sp}(\mathrm{BD}_p(G)))$ denote the set of continuous functions on $\mathrm{Sp}(\mathrm{BD}_p(G))$. For each $f \in \mathrm{BD}_p(G)$, we define a continuous function $\hat{f}$ on $\mathrm{Sp}(\mathrm{BD}_p(G))$ by $\hat{f}(\tau) = \tau(f)$. The map $f \to \hat{f}$ is known as the Gelfand transform.

Define a map $i : V \to \mathrm{Sp}(\mathrm{BD}_p(G))$ by $(i(x))(f) = f(x)$. For $x \in V$, define δ_x by $\delta_x(v) = 0$ if $v \neq x$ and $\delta_x(x) = 1$. Let $x, y \in V$ and suppose $i(x) = i(y)$; then $(i(x))(\delta_x) = (i(y))(\delta_x)$, which implies $\delta_x(x) = \delta_x(y)$. Thus i is an injection. If f is a nonzero function in $\mathrm{BD}_p(G)$, then there exists an $x \in V$ such that $\hat{f}(i(x)) \neq 0$ since $\hat{f}(i(x)) = f(x)$. Hence $\mathrm{BD}_p(G)$ is semisimple. Then [Taylor and Lay 1986, Theorem 4.6 on page 408] tells us that $\mathrm{BD}_p(G)$ is isomorphic to a subalgebra of $C(\mathrm{Sp}(\mathrm{BD}_p(G)))$ via the Gelfand transform. Since the Gelfand transform separates points of $\mathrm{Sp}(\mathrm{BD}_p(G))$ and the constant functions are contained in $\mathrm{BD}_p(G)$, the Stone–Weierstrass theorem yields that $\mathrm{BD}_p(G)$ is dense in $C(\mathrm{Sp}(\mathrm{BD}_p(G)))$ with respect to the supremum norm. The following proposition shows that $i(V)$ is dense in $\mathrm{Sp}(\mathrm{BD}_p(G))$; see [Elek 1997, Proposition 1.1(ii)] for the proof.

Proposition 2.1. *The image of V under i is dense in* $\mathrm{Sp}(\mathrm{BD}_p(G))$.

When the context is clear we will abuse notation and write V for $i(V)$ and x for $i(x)$, where $x \in V$. The compact Hausdorff space $\mathrm{Sp}(\mathrm{BD}_p(G)) \setminus V$ is known as the p-Royden boundary of G, which we will denote by $R_p(G)$. When $p = 2$, this is simply known as the Royden boundary of G. Let $\mathbb{R}G$ be the set of real-valued functions on V with finite support, and let $B(\overline{\mathbb{R}G})_{D_p} = (\overline{\mathbb{R}G})_{D_p} \cap \ell^\infty(G)$. Suppose (f_n) is a sequence in $B(\overline{\mathbb{R}G})_{D_p}$ that converges to a bounded function f in the $\mathrm{BD}_p(G)$ norm. It follows from $\|f - f_n\|_{D_p} \leq \|f - f_n\|_{\mathrm{BD}_p}$ that $f \in (\overline{\mathbb{R}G})_{D_p}$. Thus $B(\overline{\mathbb{R}G})_{D_p}$ is closed in $\mathrm{BD}_p(G)$ with respect to the $\mathrm{BD}_p(G)$ norm. We are now ready to define the main object of study for this paper.

The *p-harmonic boundary* of G is the subset

$$\partial_p(G) := \{x \in R_p(G) \mid \hat{f}(x) = 0 \text{ for all } f \in B(\overline{\mathbb{R}G})_{D_p}\}$$

of the p-Royden boundary. When $p = 2$, the p-harmonic boundary is known as the harmonic boundary. Our definition of p-harmonic boundary directly generalizes that of harmonic boundary. A good reference for the Royden and harmonic boundaries of graphs is [Soardi 1994, Chapter VI].

An important fact about $B(\overline{\mathbb{R}G})_{D_p}$ is that it is an ideal in $\mathrm{BD}_p(G)$. To see this, let $f \in B(\overline{\mathbb{R}G})_{D_p}$ and $h \in \mathrm{BD}_p(G)$. We need to show that $fh \in B(\overline{\mathbb{R}G})_{D_p}$. We claim that there exists a sequence (f_n) in $\mathbb{R}G$ converging pointwise to f, for which there exists a constant M with $|f_n(x)| \le M$ for all n and for all $x \in V$, and for which $I_p(f_n, V)$ is bounded. Let (u_n) be a sequence in $\mathbb{R}G$ that converges to f in $D_p(G)$ and let $M = \sup_{x \in V} |f(x)|$. Set $f_n = \max(\min(u_n, M), -M)$. The sequence (f_n) satisfies the claim above since $I_p(u_n, V)$ is bounded and $I_p(f_n, V) \le I_p(u_n, V)$. Also $(f_n h)$ is a sequence in $\mathbb{R}G$ that converges pointwise to fh. By (2-1), we see that

$$I_p(f_n h, V) \le (b(I_p(f_n, V))^{1/p} + M(I_p(h, V))^{1/p})^p,$$

where $b = \sup_{x \in V} |h(x)|$. Since $I_p(f_n h, V)$ is bounded, [Taylor and Lay 1986, Theorem 10.6, page 177] says, by passing to a subsequence if necessary, that $(f_n h)$ converges weakly to a function $\overline{fh}$. Since $B(\overline{\mathbb{R}G})_{D_p}$ is closed, it follows that $\overline{fh} \in B(\overline{\mathbb{R}G})_{D_p}$. Because point evaluations by elements of V are continuous linear functionals on $\mathrm{BD}_p(G)$, $(f_n h)$ also converges pointwise to $\overline{fh}$. Hence, $\overline{fh} = fh$ and $fh \in B(\overline{\mathbb{R}G})_{D_p}$.

2.2. ***Statement of main results.*** Recall that p is a real number greater than one and that o is a fixed vertex of V. By $\#(A)$, we mean the cardinality of a set A, and 1_V will denote the function on V that always takes the value one. Furthermore, $\ell^p(G)$ will be the set that consists of the functions on V for which $\sum_{x \in V} |f(x)|^p < \infty$. The ℓ^p-norm for $f \in \ell^p(G)$ is given by $\|f\|_p = (\sum_{x \in V} |f(x)|^p)^{1/p}$. In Section 3, we give a quick review of some results about p-harmonic functions on graphs. In Section 4 we prove several results concerning $\mathrm{BD}_p(G)$ and $\partial_p(G)$; when $\mathrm{BHD}_p(G)$ consists precisely of the constant functions and a neighborhood base is given for the topology on $\partial_p(G)$, we characterize when $\partial_p(G) = \varnothing$,

Before we stating some of our main results, we need a theorem that will allow us to classify graphs in a nice way. We start by giving the following definition. The *p-capacity* of a finite subset A of V is defined by

$$\mathrm{Cap}_p(A, \infty, V) = \inf_u I_p(u, V),$$

where the infimum is taken over all finitely supported functions u on V such that $u = 1$ on A. The following theorem will allow us to classify a graph G in terms of the p-capacity of a finite set.

Theorem 2.2 [Yamasaki 1977, Theorem 3.1]. *Let A be a finite, nonempty subset of V. Then*

$$\mathrm{Cap}_p(A, \infty, V) = 0 \quad \textit{if and only if} \quad 1_V \in B(\overline{\mathbb{R}G})_{D_p}.$$

Corollary 2.3. *Let A and B be nonempty finite subsets of V. Then*

$$\mathrm{Cap}_p(A, \infty, V) = 0 \quad \textit{if and only if} \quad \mathrm{Cap}_p(B, \infty, V) = 0.$$

We say that a graph G is *p-parabolic* if there exists a finite subset A of V such that $\mathrm{Cap}_p(A, \infty, V) = 0$. If G is not p-parabolic, we shall say that G is *p-hyperbolic*. If G is p-hyperbolic, then $\mathrm{Cap}_p(A, \infty, V) > 0$ for all finite subsets A of V.

In Section 5 we will prove the following results. The first reduces to [Soardi 1994, Theorem 4.6] in the case $p = 2$ and also generalizes [Kim and Lee 2005, Theorem 4.2].

Theorem 2.4. *Let p be a real number greater than one, and let G be a graph. If G is p-parabolic, then all p-harmonic functions on G are constant functions.*

Identify the constant functions on V with $\mathbb{R}$. By combining this theorem with [Holopainen and Soardi 1997a, Lemma 4.4] and Theorem 4.10 we get a Liouville-type theorem for p-harmonic functions:

Theorem 2.5. *Let p be a real number greater than one. Then* $\mathrm{HD}_p(G) = \mathbb{R}$ *if and only if the cardinality of $\partial_p(G)$ is either zero or one.*

Theorem 2.6. *Let p be a real number greater than one and let G be a graph. If f is a continuous function on $\partial_p(G)$, then there exists a p-harmonic function h on V such that* $\lim_{n\to\infty} h(x_n) = f(x)$, *where $x \in \partial_p(G)$ and (x_n) is any sequence in V that converges to x.*

By combining this theorem with the maximum principle and Corollary 4.9 we obtain the following corollary, which generalizes both [Kim and Lee 2005, Theorem 4.3] and [Kim and Lee 2007, Theorem 1.1].

Corollary 2.7. *Let p be a real number greater than one and let G be a graph. Assume that the p-harmonic boundary of G is a finite set $\{x_1, x_2, \dots, x_n\}$ of points. Then given real numbers $a_1, a_2, \dots, a_n \in \mathbb{R}$, there exists a bounded p-harmonic function h that satisfies*

$$h(x_i) = a_i \quad \text{for } i = 1, 2, \dots, n. \tag{2-2}$$

Conversely, each bounded p-harmonic function is uniquely determined by its values in (2-2).

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\phi : X \to Y$ is said to be a *rough isometry* if it satisfies the following two conditions:

(1) There exist constants $a \geq 1$ and $b \geq 0$ such that for $x_1, x_2 \in X$

$$(1/a)d_X(x_1, x_2) - b \leq d_Y(\phi(x_1), \phi(x_2)) \leq a d_X(x_1, x_2) + b.$$

(2) There exists a positive constant c such that for each $y \in Y$, there exists an $x \in X$ that satisfies $d_Y(\phi(x), y) < c$.

For a rough isometry ϕ, there exists a rough isometry $\psi : Y \to X$ such that if $x \in X$ and $y \in Y$, then $d_X((\psi \circ \phi)(x), x) \leq a(c+b)$ and $d_Y((\phi \circ \psi)(y), y) \leq c$. The map ψ, which is not unique, is said to be a rough inverse for ϕ. Whenever we refer to a rough inverse to a rough isometry, it will always satisfy the conditions above. In Section 6, we prove the following two results:

Theorem 2.8. *Let p be a real number greater than one and let G and H be graphs. If there is a rough isometry from G to H, then $\partial_p(G)$ is homeomorphic to $\partial_p(H)$.*

Theorem 2.9. *Let p be a real number greater than one and let G and H be graphs. If there is a rough isometry from G to H, then there is a bijection from* $\mathrm{BHD}_p(G)$ *to* $\mathrm{BHD}_p(H)$.

The main result of [Soardi 1993] is that if G and H are roughly isometric graphs, then $\mathrm{HD}_p(G) = \mathbb{R}$ if and only if $\mathrm{HD}_p(H) = \mathbb{R}$. By [Holopainen and Soardi 1997a, Lemma 4.4], this is equivalent to $\mathrm{BHD}_p(G) = \mathbb{R}$ if and only if $\mathrm{BHD}_p(H) = \mathbb{R}$. Both Theorem 2.8 and Theorem 2.9 are generalizations of this result.

We now return to the case of a finitely generated group Γ. In Section 7, we define the first reduced ℓ^p-cohomology space $\overline{H}^1_{(p)}(\Gamma)$ of Γ. Then we will use our results on p-harmonic boundaries to prove this:

Theorem 2.10. *Let $1 < p \in \mathbb{R}$. Then $\overline{H}^1_{(p)}(\Gamma) \neq 0$ if and only if $\#(\partial_p(\Gamma)) > 1$.*

It appears there are not many explicit examples of the p-Royden boundary $R_p(G)$ for a given graph G. Wysoczański [1996] provided the only example we know of by giving an explicit description of $R_2(\mathbb{Z})$. We will conclude Section 7 by using Theorem 2.10 to compute the p-harmonic boundary for the case $\Gamma = \mathbb{Z}^n$. We will also compute the p-Royden boundary of nonamenable groups with infinite center, and of the groups $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ for $n \geq 2$, where each F_i is finitely generated and at least one of the Γ_i is nonamenable.

Let E be a normed space of functions on a finitely generated group Γ. Let $f \in E$ and let $x \in \Gamma$. The right translation of f by x, denoted by f_x, is the function $f_x(g) = f(gx^{-1})$, where $g \in \Gamma$. Assume that if $f \in E$, then $f_x \in E$ for all $x \in \Gamma$, that is, that E is right translation invariant. For the rest of this paper translation invariant will mean right translation invariant. We shall say that T is a translation invariant linear functional (TILF) on E if $T(f_x) = T(f)$ for $f \in E$ and $x \in \Gamma$. We will use TILFs to denote translation invariant linear functionals. A common question to ask is, If T is a TILF on E, then is T continuous? For background about the problem of automatic continuity, see [Meisters 1983; Saeki 1984; Willis 1988; Woodward 1974]. Define

$$\mathrm{Diff}(E) := \text{linear span}\{f_x - f \mid f \in E, x \in \Gamma\}.$$

It is clear that $\mathrm{Diff}(E)$ is contained in the kernel of any TILF on E. In Section 8 we study TILFs on $D_p(\Gamma)/\mathbb{R}$, and prove the following:

Theorem 2.11. *Let Γ be a finitely generated infinite group and let $1 < p \in \mathbb{R}$. Then $\#(\partial_p(\Gamma)) > 1$ if and only if there exists a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$.*

Willis [1986] showed that if Γ is nonamenable, then the only TILF on $\ell^p(\Gamma)$ is the zero functional. (Consequently every TILF is automatically continuous!) We will conclude Section 8 by showing that this result is not true for $D_p(\Gamma)/\mathbb{R}$.

3. Review of p-harmonic functions on graphs

The four results below are from [Holopainen and Soardi 1997a, Section 3], where a more comprehensive treatment, including proofs, is given.

- *Existence.* Let S be a finite subset of V. For any function f on ∂S, there exists an unique function h on $S \cup \partial S$ that is p-harmonic on S and equals f on ∂S. In the proof of existence, it was shown that the p-harmonic function h satisfies $\min_{y \in \partial S} f(y) \leq h(x) \leq \max_{y \in \partial S} f(y)$ for all $x \in S$.
- *Minimizer property.* Let h be a p-harmonic function on a finite subset S of V. Then $I_p(h, S) \leq I_p(f, S)$ for all functions f on $S \cup \partial S$ satisfying $f = h$ on ∂S.
- *Convergence.* Let (S_n) be an increasing sequence of finite connected subsets of V and let $U = \bigcup_i S_i$. Let (h_i) be a sequence of functions on $U \cup \partial U$ such that $h_i(x) \to h(x) < \infty$ for every $x \in U \cup \partial U$. If h_i is p-harmonic on S_i for all i, then h is p-harmonic on U.
- *Comparison principle.* Let h and u be p-harmonic functions on a finite subset S of V. If $h \geq u$ on ∂S, then $h \geq u$ on S.

We also prove the maximum principle for bounded p-harmonic functions on V:

Lemma 3.1. *Let h be a p-harmonic function on V. If there exists an $x \in V$ such that $h(x) \geq h(y)$ for all $y \in V$, then h is constant on V.*

Proof. Let $x \in V$ such that $h(x) \geq h(x')$ for all $x' \in V$. Because

$$\sum_{y \in N_x} |h(y) - h(x)|^{p-2} h(y) = \sum_{y \in N_x} |h(y) - h(x)|^{p-2} h(x),$$

we see that $h(x) = h(y)$ for all $y \in N_x$. Thus $h(x) = h(z)$ for all $z \in V$ since G is connected. □

4. Preliminary results

In this section we will give some results about $\partial_p(G)$ and $\mathrm{BD}_p(G)$. Most of the results given in Propositions 4.2 through 4.8 are given in the first two sections of [Soardi 1994, Chapter VI] for the case of $p = 2$. However, our presentation and some of our proofs are different. Recall that o is a fixed vertex of the graph G.

Lemma 4.1. *If $x \in \partial_p(G)$ and (x_n) is a sequence in V that converges to x, then $d(o, x_n) \to \infty$ as $n \to \infty$.*

Proof. Let $x \in \partial_p(G)$ and suppose $(x_n) \to x$, where (x_n) is a sequence in V. Let B be a positive real number. Define a function χ_B on V by $\chi_B(y) = 1$ if $d(o, y) \le B$ and $\chi_B(y) = 0$ if $d(o, y) > B$. Since χ_B has finite support it is an element of $\mathbb{R}G$. Suppose there exists a real number M such that $d(o, x_n) \le M$ for all n. Then $\widehat{\chi_M}(x) = \lim_{n\to\infty} \chi_M(x_n) = 1$, a contradiction. Thus $d(o, x_n) \to \infty$ as $n \to \infty$. □

We now characterize p-parabolic graphs in terms of $\partial_p(G)$.

Proposition 4.2. *Let G be a graph and let $1 < p \in \mathbb{R}$. Then $\partial_p(G) = \varnothing$ if and only if G is p-parabolic.*

Proof. Assume G is p-parabolic and suppose $\partial_p(G) \neq \varnothing$. Let $x \in \partial_p(G)$ and let (x_n) be a sequence in V that converges to x. Then $\widehat{1_V}(x) = \lim_{n\to\infty} \widehat{1_V}(x_n) = 1$. By Theorem 2.2, we have $1_V \in B(\overline{\mathbb{R}G})_{D_p}$, which says that $\widehat{1_V}(x) = 0$, a contradiction. Hence if G is p-parabolic, then $\partial_p(G) = \varnothing$.

Now suppose that G is p-hyperbolic. Then $1_V \notin B(\overline{\mathbb{R}G})_{D_p}$. Since $B(\overline{\mathbb{R}G})_{D_p}$ is an ideal in the commutative ring $\mathrm{BD}_p(G)$, there exists a maximal ideal M in $\mathrm{BD}_p(G)$ containing $B(\overline{\mathbb{R}G})_{D_p}$. Using the correspondence between maximal ideals in $\mathrm{BD}_p(G)$ and $\mathrm{Sp}(\mathrm{BD}_p(G))$, there is an $x \in \mathrm{Sp}(\mathrm{BD}_p(G))$ that satisfies $\ker(x) = M$. So $\hat{f}(x) = x(f) = 0$ for all $f \in B(\overline{\mathbb{R}G})_{D_p}$. For each $y \in V$, there exists an $f \in \mathbb{R}G$ (in particular δ_y) such that $y(f) = f(y) \neq 0$, which means that x cannot be in V. Also, if $x \in R_p(G) \setminus \partial_p(G)$, then there exists an $f \in B(\overline{\mathbb{R}G})_{D_p}$ for which $\hat{f}(x) \neq 0$. This implies that $B(\overline{\mathbb{R}G})_{D_p}$ is not contained in M. Therefore $x \in \partial_p(G)$. □

For the rest of this paper, we will assume that $1_V \notin B(\overline{\mathbb{R}G})_{D_p}$ unless otherwise stated, that is, we assume G is p-hyperbolic.

Let f and h be elements in $\mathrm{BD}_p(G)$ and let $1 < p \in \mathbb{R}$. Define

$$\langle \triangle_p h, f\rangle := \sum_{x\in V} \sum_{y\in N_x} |h(y) - h(x)|^{p-2}(h(y) - h(x))(f(y) - f(x)).$$

This sum exists since $\sum_{x\in V} \sum_{y\in N_x} \big||h(y)-h(x)|^{p-2}(h(y)-h(x))\big|^q = I_p(h, V)$ is finite, where $1/p + 1/q = 1$. The next few lemmas will help show the uniqueness of the decomposition of $\mathrm{BD}_p(G)$ that will be given in Theorem 4.6.

Lemma 4.3. *Let f_1 and f_2 be functions in $D_p(G)$. Then $\langle \triangle_p f_1 - \triangle_p f_2, f_1 - f_2\rangle$ is zero if and only if $f_1 - f_2$ is constant on V.*

Proof. Let $f_1, f_2 \in D_p(G)$ and assume there exists an $x \in V$ with a $y \in N_x$ such that $f_1(x) - f_1(y) \neq f_2(x) - f_2(y)$. Define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(t) = \sum_{x\in V} \sum_{y\in N_x} |f_1(y) - f_1(x) + t((f_2(y) - f_2(x)) - (f_1(y) - f_1(x)))|^p.$$

Observe that $f(0) = I(f_1, V)$ and $f(1) = I(f_2, V)$. A derivative calculation gives

$$f'(0) = p\langle \triangle_p f_1, f_2 - f_1\rangle = -p\langle \triangle_p f_1, f_1 - f_2\rangle.$$

It follows from [Ekeland and Témam 1999, Proposition 5.4] that $I_p(f_2, V) > I_p(f_1, V) - p\langle \triangle_p f_1, f_1 - f_2\rangle$. Similarly, $I_p(f_1, V) > I_p(f_2, V) - p\langle \triangle_p f_2, f_2 - f_1\rangle$. Hence, $p\langle \triangle_p f_1 - \triangle_p f_2, f_1 - f_2\rangle > 0$ if there exists an $x \in V$ with $y \in N_x$ that satisfies $f_1(x) - f_1(y) \neq f_2(x) - f_2(y)$. Conversely, suppose $f_1 - f_2$ is constant on V. We immediately see that $\langle \triangle_p f_1 - \triangle_p f_2, f_1 - f_2\rangle = 0$. □

Lemma 4.4. *Let $h \in \mathrm{BD}_p(G)$. Then $h \in \mathrm{BHD}_p(G)$ if and only if $\langle \triangle_p h, \delta_x\rangle = 0$ for all $x \in V$.*

Proof. Let $x \in V$ and let $h \in \mathrm{BD}_p(G)$. The lemma follows from

$$\langle \triangle_p h, \delta_x\rangle = -2(\deg(x)) \sum_{y \in N_x} |h(x) - h(y)|^{p-2}(h(y) - h(x)). \qquad \square$$

The lemma implies that if $h \in \mathrm{BHD}_p(G)$, then $\langle \triangle_p h, f\rangle = 0$ for all $f \in \mathbb{R}G$.

Lemma 4.5. *If $h \in \mathrm{BHD}_p(G)$ and $f \in B(\overline{\ell^p(G)})_{D_p}$, then $\langle \triangle_p h, f\rangle = 0$.*

Proof. Let h and f be as stated. Then there exists a sequence (f_n) in $\mathbb{R}G$ such that $\|f - f_n\|_{D_p} \to 0$ as $n \to \infty$ since $(\overline{\mathbb{R}G})_{D_p} = (\overline{\ell^p(G)})_{D_p}$. Now

$$\begin{aligned}
0 &\leq |\langle \triangle_p h, f\rangle| = |\langle \triangle_p h, f - f_n\rangle| \\
&= \Big|\sum_{x \in V}\sum_{y \in N_x} |h(y) - h(x)|^{p-2}(h(y) - h(x))((f - f_n)(x) - (f - f_n)(y))\Big| \\
&\leq \sum_{x \in V}\sum_{y \in N_x} |h(y) - h(x)|^{p-1}|(f - f_n)(x) - (f - f_n)(y)| \\
&\leq \Big(\sum_{x \in V}\sum_{y \in N_x} (|h(y) - h(x)|^{p-1})^q\Big)^{1/q}(I_p(f - f_n, V))^{1/p} \to 0
\end{aligned}$$

as $n \to \infty$. The last inequality follows from Hölder's inequality. □

Clarkson's inequality will be needed in the next proof. Let f_1 and f_2 be elements of $D_p(G)$. If $2 \leq p \in \mathbb{R}$, then

$$I_p(f_1 + f_2) + I_p(f_1 - f_2) \leq 2^{p-1}(I_p(f_1) + I_p(f_2))$$

and if $1 < p \leq 2$, then

$$(I_p(f_1 + f_2))^{1/(p-1)} + (I_p(f_1 - f_2))^{1/(p-1)} \leq 2(I_p(f_1) + I_p(f_2))^{1/(p-1)}.$$

The following decomposition of $\mathrm{BD}_p(G)$ will be crucial:

Theorem 4.6. *Let $1 < p \in \mathbb{R}$ and suppose $f \in \mathrm{BD}_p(G)$. Then there exists a unique $u \in B(\overline{\ell^p(G)})_{D_p}$ and a unique $h \in \mathrm{BHD}_p(G)$ such that $f = u + h$.*

Proof. Our assumption remains that $1_V \notin B(\overline{\ell^p(G)})_{D_p}$. Let $f \in \mathrm{BD}_p(G)$. Since f is bounded there exists real numbers a and b for which $a \le f(x) \le b$ is satisfied by all $x \in V$. Denote by h_n the function that is p-harmonic on $B_n(o)$ and equal to f on $V \setminus B_n(o)$. Because $\min_{y\in\partial B_n(o)} f(y) \le h_n(x) \le \max_{y\in\partial B_n(o)} f(y)$ for all $x \in B_n(o)$, we have $a \le h_n \le b$ for each $n \in \mathbb{N}$. Furthermore, if $m > n$, then $I_p(h_m) \le I_p(h_n)$. Set $r_n = I_p(h_n)$ and denote the limit of the bounded decreasing sequence (r_n) by r. We are still assuming that $m > n$. By the minimizing property of p-harmonic functions, $I_p(h_m, V) \le I_p((h_n+h_m)/2, V)$ since $(h_n+h_m)/2 = h_m$ on $V \setminus B_m(o)$. Using Clarkson's inequality we obtain for $2 \le p \in \mathbb{R}$,

$$\begin{aligned} r_m &\le I_p(\tfrac{1}{2}(h_n+h_m), V) \\ &\le I_p(\tfrac{1}{2}(h_n+h_m), V) + I_p(\tfrac{1}{2}(h_n-h_m), V) \\ &\le 2^{p-1}(I_p(\tfrac{1}{2}h_n, V) + I_p(\tfrac{1}{2}h_m, V)) \\ &= \tfrac{1}{2}(I_p(h_n, V) + I_p(h_m, V)) \end{aligned}$$

and for $1 < p \le 2$,

$$\begin{aligned} r_m^{1/(p-1)} &\le (I_p(\tfrac{1}{2}(h_n+h_m), V))^{1/(p-1)} \\ &\le (I_p(\tfrac{1}{2}(h_n+h_m), V))^{1/(p-1)} + (I_p(\tfrac{1}{2}(h_n-h_m), V))^{1/(p-1)} \\ &\le 2(I_p(\tfrac{1}{2}h_n, V) + I_p(\tfrac{1}{2}h_m, V))^{1/(p-1)}. \end{aligned}$$

Letting $m, n \to \infty$, we have $I_p(\frac{1}{2}(h_n+h_m), V) \to r$ and $I_p(\frac{1}{2}(h_n-h_m), V) \to 0$. Also, $(|h_n(o)|)$ is a bounded sequence; thus (h_n) is a Cauchy sequence in $D_p(G)$. Set h equal to the limit function of the sequence (h_n) in $D_p(G)$. Because (h_n) also converges pointwise to h, the convergence property says that h is p-harmonic. Clearly, $a \le h \le b$ on V, so $h \in \mathrm{BHD}_p(G)$. Let u be the limit function in $D_p(G)$ of the Cauchy sequence $(f - h_n)$. Since $f - h_n \in \mathbb{R}G$ for each n, we see that $u \in B(\overline{\mathbb{R}G})_{D_p}$. Thus $f = u + h$.

To show that this decomposition is unique, suppose $f = u_1 + h_1 = u_2 + h_2$, where $u_1, u_2 \in B(\overline{\ell^p(G)})_{D_p}$ and $h_1, h_2 \in \mathrm{BHD}_p(G)$. Lemma 4.5 says that

$$\langle \triangle_p h_1 - \triangle_p h_2, h_1 - h_2\rangle = \langle \triangle_p h_1 - \triangle_p h_2, u_2 - u_1\rangle = 0$$

since $u_1 - u_2 \in B(\overline{\ell^p(G)})_{D_p}$. However, $u_1 - u_2 = 0$ since $1_V \notin B(\overline{\ell^p(G)})_{D_p}$. □

Theorem 4.7 (maximum principle). *Let h be a nonconstant function in* $\mathrm{BHD}_p(G)$ *and suppose a and b are real numbers for which $a \le \hat{h} \le b$ on $\partial_p(G)$. Then $a < h < b$ on V.*

Proof. Since $\hat{h}$ is continuous on the compact space $\mathrm{Sp}(\mathrm{BD}_p(G))$, there is a number $c > 0$ such that $b - \hat{h} \ge -c$ on $\mathrm{Sp}(\mathrm{BD}_p(G))$. Let $\epsilon > 0$ and set F_ϵ to be the set of $x \in \mathrm{Sp}(\mathrm{BD}_p(G))$ such that $b - h + \epsilon \le 0$. To prove the theorem, we will first show that there exists an $f \in B(\overline{\mathbb{R}G})_{D_p}$ with $\hat{f} = 1$ on F_ϵ and $0 \le \hat{f} \le 1$ on $\mathrm{Sp}(\mathrm{BD}_p(G))$. This f will yield the inequality

$$cf + b - h + \epsilon \ge 0 \quad \text{on } \mathrm{Sp}(\mathrm{BD}_p(G)). \tag{4-1}$$

We will then show that $b - h + \epsilon \ge 0$ on V. Combining this with Lemma 3.1 and the assumption that h is nonconstant will give $h < b$ on V.

Observe that $F_\epsilon \cap \partial_p(G) = \varnothing$ and F_ϵ is a closed subset of $\mathrm{Sp}(\mathrm{BD}_p(G))$. For each $x \in F_\epsilon$ there exists an $f_x \in B(\overline{\mathbb{R}G})_{D_p}$ for which $\hat{f}_x(x) \neq 0$. Since $B(\overline{\mathbb{R}G})_{D_p}$ is an ideal, we may assume that $f_x \ge 0$ on V and $\hat{f}_x(x) > 0$. Let U_x be a neighborhood of x in $\mathrm{Sp}(\mathrm{BD}_p(G))$ that satisfies $f_x(y) > 0$ for all $y \in U_x$. By compactness there exists $x_1, \dots, x_n$ for which $F_\epsilon \subseteq \bigcup_{j=1}^n U_{x_j}$. Set

$$g = \sum_{j=1}^n f_{x_j} \quad \text{and} \quad \alpha = \inf\{g(x) \mid x \in F_\epsilon\}.$$

Clearly $\alpha > 0$ and $g \in B(\overline{\mathbb{R}G})_{D_p}$. Now define a function f on $\mathrm{Sp}(\mathrm{BD}_p(G))$ by $f = \min(1, \alpha^{-1}g)$. Note that $0 \le \hat{f} \le 1$ on $\mathrm{Sp}(\mathrm{BD}_p(G))$ and $\hat{f} = 1$ and F_ϵ. We still need to show that $f \in B(\overline{\mathbb{R}G})_{D_p}$. Let (g_n) be a sequence in $\mathbb{R}G$ that converges to g in $D_p(G)$, so $I_p((g - g_n), V) \to 0$ as $n \to \infty$. Set $f_n = \min(1, \alpha^{-1}g_n)$. The sequence (f_n) converges pointwise to f. Furthermore, by passing to a subsequence if necessary, (f_n) converges weakly to a function $\bar{f}$ in $D_p(G)$ since $I_p(f_n, V)$ is bounded. Clearly $\bar{f}$ is bounded, so $\bar{f} \in B(\overline{\mathbb{R}G})_{D_p}$. It is also true (f_n) converges pointwise to $\bar{f}$ because point evaluations by elements of V are continuous linear functionals on $\mathrm{BD}_p(G)$. Hence, $\bar{f} = f$ and $f \in \mathrm{BD}_p(G)$. Inequality (4-1) is now established.

Next we will show that $b - h + \epsilon \ge 0$ on V. Put $v_\epsilon = cf + b - h + \epsilon$ and denote by h_n the unique function that is p-harmonic on $B_n(o)$ and agrees with v_ϵ on $V \setminus B_n(o)$. We claim that $h_n \ge 0$ on $B_n(o)$. Supposing otherwise, there exists an $x \in B_n(o)$ for which $h_n(x) < 0$. Define a function h_n^* by

$$h_n^* = \begin{cases} v_\epsilon & \text{if } x \in V \setminus B_n(o), \\ \max(h_n, 0) & \text{if } x \in B_n(o). \end{cases}$$

Now $I_p(h_n^*, B_n(o)) < I_p(h_n, B_n(o))$, but this contradicts the minimizer property of p-harmonic functions. This proves the claim. By using the argument used in the proof of Theorem 4.6, we see that (h_n) converges to a bounded p-harmonic function $\bar{h}$ and that there exists a $v \in B(\overline{\mathbb{R}G})_{D_p}$ such that $v_\epsilon = v + \bar{h}$. Furthermore $\bar{h} \ge 0$ on V because $h_n \ge 0$ for each n. The uniqueness part of Theorem 4.6 says that $v = cf$ and $\bar{h} = b - h + \epsilon$. Hence $b \ge h - \epsilon$ on V. Thus $h < b$ on V.

A similar argument shows that $a < h$ on V. Therefore, $a < h < b$ on V. □

We now characterize the functions in $\mathrm{BD}_p(G)$ that vanish on $\partial_p(G)$.

Theorem 4.8. *Let* $f \in \mathrm{BD}_p(G)$. *Then* $f \in B(\overline{\ell^p(G)})_{D_p}$ *if and only if* $\hat{f}(x) = 0$ *for all* $x \in \partial_p(G)$.

Proof. Since $B(\overline{\ell^p(G)})_{D_p} = B(\overline{\mathbb{R}G})_{D_p}$ it follows immediately that $\hat{f}(x) = 0$ for all $f \in B(\overline{\ell^p(G)})_{D_p}$ and all $x \in \partial_p(G)$.

Conversely, suppose $f \in \mathrm{BD}_p(G)$ and $\hat{f}(x) = 0$ for all $x \in \partial_p(G)$. Theorem 4.6 allows us to write $f = u + h$, where $u \in B(\overline{\ell^p(G)})_{D_p}$ and $h \in \mathrm{BHD}_p(G)$. Now $\hat{h}(x) = 0$ for all $x \in \partial_p(G)$ since $\hat{u}(x) = 0$. Therefore, $h = 0$ by the maximum principle. □

Corollary 4.9. *Every function in* $\mathrm{BHD}_p(G)$ *is uniquely determined by its values on* $\partial_p(G)$.

Proof. Let h_1 and h_2 be elements of $\mathrm{BHD}_p(G)$ with $\widehat{h_1}(x) = \widehat{h_2}(x)$ for all $x \in \partial_p(G)$. Then $h_1 - h_2 \in B(\overline{\ell^p(G)})_{D_p}$. Let (f_n) be a sequence in $\ell^p(G)$ that converges to $h_1 - h_2$. Using Lemma 4.5, we obtain

$$\langle \Delta_p h_1 - \Delta_p h_2, h_1 - h_2 \rangle = \lim_{n \to \infty} \langle \Delta_p h_1 - \Delta_p h_2, f_n \rangle = 0.$$

It now follows from Lemma 4.3 that $h_1 - h_2 = 0$. □

We can now characterize when $\mathrm{BHD}_p(G)$ is precisely the constant functions.

Theorem 4.10. *Let* $1 < p \in \mathbb{R}$. *Then* $\mathrm{BHD}_p(G) \neq \mathbb{R}$ *if and only if* $\#(\partial_p(G)) > 1$.

Proof. Suppose that $\#(\partial_p(G)) = 1$ and that $x \in \partial_p(G)$. Let $h \in \mathrm{BHD}_p(G)$. Then $\hat{h}(x) = c$ for some constant c. It follows from Corollary 4.9 that the function $h(x) = c$ for all $x \in V$ is the only function in $\mathrm{BHD}_p(G)$ with $\hat{h}(x) = c$. Hence $\mathrm{BHD}_p(G) = \mathbb{R}$.

Conversely, suppose $\#(\partial_p(G)) > 1$. Let $x, y \in \partial_p(G)$ such that $x \neq y$ and pick an $f \in \mathrm{BD}_p(G)$ that satisfies $x(f) \neq y(f)$. By Theorem 4.8, $f \notin B(\overline{\ell^p(G)})_{D_p}$. It now follows from Theorem 4.6 and Theorem 4.8 that there exists an $h \in \mathrm{BHD}_p(G)$ with $\hat{h}(z) = \hat{f}(z)$ for all $z \in \partial_p(G)$. Since V is dense in $\mathrm{Sp}(\mathrm{BD}_p(G))$, there exist sequences (x_n) and (y_n) in V such that $(x_n)(h) \to x(h)$ and $(y_n)(h) \to y(h)$. Hence $\lim_{n\to\infty} h(x_n) = x(h) \neq y(h) = \lim_{n\to\infty} h(y_n)$. Hence h is not constant on V. □

We now define the important concept of a D_p-massive subset of a graph. An infinite connected subset U of V with $\partial U \neq \varnothing$ is called a D_p-massive subset if there exists a nonnegative function $u \in \mathrm{BD}_p(G)$ such that

(a) $\Delta_p u(x) = 0$ for all $x \in U$,

(b) $u(x) = 0$ for $x \in \partial U$, and

(c) $\sup_{x \in U} u(x) = 1$.

We call any u that satisfies these conditions an *inner potential* of the D_p-massive subset U. The following will be needed in the proof of Lemma 5.1.

Proposition 4.11. *If U is a D_p-massive subset of V, then $\overline{i(U)}$ contains at least one point of $\partial_p(G)$.*

Proof. We will write $\overline{U}$ for $\overline{i(U)}$, where the closure is taken in $\mathrm{Sp}(\mathrm{BD}_p(G))$. Assume $\overline{U} \cap \partial_p(G) = \varnothing$ and let u be an inner potential for U. We may and do assume that $u = 0$ on $V \setminus U$. By the existence property for p-harmonic functions, there exists a p-harmonic function h_n on $B_n(o)$ such that $h_n = u$ on $\partial B_n(o)$ for each natural number n. Also $0 \leq \min_{y \in \partial B_n(o)} u(y) \leq h_n \leq \max_{y \in \partial B_n(o)} u(y) \leq 1$ on $B_n(o)$. Extend h_n to all of V by setting $h_n = u$ on $V \setminus B_n(o)$. By the minimizing property of p-harmonic functions, $I_p(h_n, B_n(o)) \leq I_p(u, B_n(o))$, and so $I_p(h_n, V) \leq I_p(u, V)$. Both h_n and u are p-harmonic on $U \cap B_n(o)$, and we have $u(x) \leq h_n(x)$ for all $x \in \partial(U \cap B_n(o))$. The comparison principle says that $u \leq h_n$ on $U \cap B_n(o)$. On $B_n(o) \setminus U$ we have $u = 0$, so $u \leq h_n \leq 1$ for each n. By taking a subsequence if needed, we assume that (h_n) converges pointwise to a function h. Now $u \leq h \leq 1$ on V, so $\sup_{x \in U} h(x) = 1$. By the convergence property for p-harmonic functions, h is p-harmonic and $h \in \mathrm{BHD}_p(G)$ since $I_p(h_n, V) \leq I_p(u, V) < \infty$ for all n.

Let $x \in \partial_p(G)$. Since $u - h_n = 0$ on $V \setminus B_n(o)$, we see that $\hat{u}(x) - \widehat{h_n}(x) = 0$ for all n; thus $\widehat{u - h} = 0$ on $\partial_p(G)$. According to Theorem 4.8, $u - h \in B(\overline{\ell^p(G)})_{D_p}$. Hence $u = f + h$, where $f \in B(\overline{\ell^p(G)})_{D_p}$. Another appeal to Theorem 4.8 shows that $\hat{u} = \hat{h}$ on $\partial_p(G)$. If $x \in \partial_p(G)$, then $\hat{u}(x) = 0$ because if (x_n) is a sequence in V converging to x, then $u(x_n) = 0$ for all but a finite number of n since we are assuming $\overline{U} \cap \partial_p(G) = \varnothing$. So $\hat{h}(x) = 0$ for all $x \in \partial_p(G)$. Hence $h = 0$ on V by the maximum principle, which contradicts $\sup_U h = 1$. Therefore, if U is a D_p-massive subset of V, then $\overline{U}$ contains at least one point of $\partial_p(G)$. $\square$

It would be nice to know if the converse of Proposition 4.11 is true. That is, if $x \in \partial_p(G)$, does there exist a D_p-massive subset U of V such that $x \in \overline{U}$? The next result leads to a partial converse and also describes a base of neighborhoods for open sets in $\partial_p(G)$.

Proposition 4.12. *Let $x \in \partial_p(G)$ and let O be an open set in $\partial_p(G)$ containing x. Then there exists a subset U of V such that*

(a) $U = \bigcup_{\alpha \in I} A_\alpha$, *where each A_α is a D_p-massive subset of V and I is an index set, and $A_\alpha \cap A_\beta = \varnothing$ if $\alpha \neq \beta$, and*

(b) $x \in \overline{U} \cap \partial_p(G) \subseteq O$.

Proof. Let $x \in \partial_p(G)$, and let O be an open set of $\partial_p(G)$ containing x. By Urysohn's lemma there exists an $f \in C(\mathrm{Sp}(\mathrm{BD}_p(G)))$ with $0 \leq f \leq 1$, $f(x) = 1$ and $f = 0$ on $\partial_p(G) \setminus O$. Since the Gelfand transform of $\mathrm{BD}_p(G)$ is dense in $C(\mathrm{Sp}(\mathrm{BD}_p(G)))$ with respect to the supremum norm, we will assume $f \in \mathrm{BD}_p(G)$. By Theorem 4.6 we have the decomposition $f = w + h$, where $w \in B(\overline{\ell^p(G)})_{D_p}$ and $h \in \mathrm{BHD}_p(G)$. Since $\hat{w} = 0$ on $\partial_p(G)$, it follows that $\hat{h}(x) = 1$ and $\hat{h} = 0$

on $\partial_p(G) \setminus O$. Also, $0 \le \hat{h} \le 1$ on $\partial_p(G)$, so $0 < h < 1$ on V by the maximum principle and $0 \le \hat{h} \le 1$ on $\mathrm{Sp}(\mathrm{BD}_p(G))$ due to the density of V. Fix ϵ with $0 < \epsilon < 1$ and set $U = \{x \in V \mid h(x) > \epsilon\}$. Let A be a component of U. It now follows from the comparison principle that A is infinite. Define a function v on V by $v = (h - \epsilon)/(1 - \epsilon)$. There exists a p-harmonic function u_n on $B_n(o) \cap A$ taking the values $\max\{0, v\}$ on $V \setminus (B_n(o) \cap A)$ and such that $0 \le u_n \le 1$ on $B_n(o) \cap A$. By passing to a subsequence if necessary, we may assume that (u_n) converges pointwise to a function u. By the convergence property, u is p-harmonic on A. Also $v \le u_n \le 1$ on $B_n(o)$, so by replacing u by a suitable scalar multiple if necessary, we have $\sup_{a \in A} u(a) = 1$. Also, $u = 0$ on ∂A because $h \le \epsilon$ on ∂A. Since $h \in \mathrm{BD}_p(G)$, it follows that $u \in \mathrm{BD}_p(G)$. Thus A is a D_p-massive subset with inner potential u. Hence, each component of U is a D_p-massive subset in V. So $U = \bigcup_{\alpha \in I} A_\alpha$, where each A_α is D_p-massive. The proof of part (a) is complete.

Clearly $x \in \overline{U}$. We will show that $\overline{U} \cap \partial_p(G) \subseteq O$. Let $y \in \overline{U} \cap \partial_p(G)$ and let (y_k) be a sequence in U that converges to y. Then $f(y) = \hat{h}(y) = \lim_{k \to \infty} h(y_k) \ge \epsilon$. Hence $y \in O$ since $f = 0$ on $\partial_p(G) \setminus O$. □

The following partial converse to Proposition 4.11 is a direct consequence of Proposition 4.12.

Corollary 4.13. *If $\#(\partial_p(G))$ is finite, then for each $x \in \partial_p(G)$ there exists a D_p-massive subset U of V such that $x \in \overline{U}$.*

5. Proofs of Theorem 2.4 and Theorem 2.6

The key ingredient in the proof of Theorem 2.4 is the following.

Lemma 5.1. *Let $1 < p \in \mathbb{R}$ and suppose that G is a p-parabolic graph. If f is a nonconstant function in $\mathrm{BHD}_p(G)$, then $\sup_V f > \limsup_{d(o,x) \to \infty} f$.*

Proof. Suppose that $\limsup_{d(o,x) \to \infty} f(x) = \sup_V f = M$. Since f is nonconstant, there exists an $\epsilon > 0$ such that the set $W = \{x \in V \mid f(x) > M - \epsilon\}$ is a proper infinite subset of V. Let U be a component of W. If U is finite, then we can construct a unique p-harmonic function w on U that agrees with f on ∂U. Since f is p-harmonic, $f = w$ on U by uniqueness. But if $x \in U$, then

$$w(x) \le \max_{y \in \partial U} f(y) \le M - \epsilon < f(x),$$

a contradiction. Thus U is infinite. Now set $h = (f - M + \epsilon)/\epsilon$. There is an number $N \in \mathbb{N}$ such that $B_n(o) \cap U \ne \varnothing$ for $n > N$. For $n > N$, let u_n be a p-harmonic function on $B_n(o) \cap U$ that takes the values $\max\{0, h\}$ on $V \setminus (B_n(o) \cap U)$. Note that $u_n \ge 0$. Since h is p-harmonic on $B_n(o) \cap U$, it follows from the comparison principle that $h \le u_n \le 1$ on $B_n(o) \cap U$. By taking a subsequence if necessary, we may assume that the sequence (u_n) converges pointwise to a function u. By the convergence property, u is p-harmonic on U. If $x \in \partial U$, then $f(x) \leq M - \epsilon$. Therefore, $u_n(x) = 0$ for all n, which implies $u(x) = 0$. Thus $u = 0$ on ∂U. Since $\sup_U h = 1$, we see that $\sup_U u = 1$. We can show using the minimizing property for p-harmonic functions that $I_p(u_n, U \cap B_n(o)) \leq I_p(\max\{0, h\}, U \cap B_n(o))$, and it follows from this inequality that $I_p(u_n, U) \leq I_p(h, U)$. Hence $I_p(u, U) < \infty$ because $I_p(h, V) < \infty$. Thus U is a D_p-massive subset of V.

By Proposition 4.11, we have $\overline{U} \cap \partial_p(G) \neq \varnothing$, which contradicts Proposition 4.2 since we are assuming G is p-parabolic. Hence $\sup_V f > \limsup_{d(o,x)\to\infty} f$. $\square$

Proof of Theorem 2.4. Let $h \in \mathrm{BHD}_p(G)$ and suppose that h is nonconstant. Since h is bounded, $\sup_V h = B < \infty$. Lemma 5.1 says that there exists an $x \in V$ such that $h(x) = B$. By the maximum principle, h is constant on V, a contradiction. Hence $\mathrm{BHD}_p(G)$ consists of only the constant functions. Therefore, $\mathrm{HD}_p(G)$ is precisely the constant functions by [Holopainen and Soardi 1997a, Lemma 4.4]. $\square$

Proof of Theorem 2.6. Let f be a continuous function on $\partial_p(G)$. By Tietze's extension theorem, there exists a continuous extension of f, which we also denote by f, to all of $\mathrm{Sp}(\mathrm{BD}_p(G))$. Let (f_n) be a sequence in $\mathrm{BD}_p(G)$ converging to f in the supremum norm. For each $n \in \mathbb{N}$ and each $r \in \mathbb{N}$, let $h_{n,r}$ be a function on V that is p-harmonic on $B_r(o)$ and takes the values f_n on $V \setminus B_r(o)$. The function $h_{n,r} \in \mathrm{BD}_p(G)$ since $B_r(o)$ is finite, and $|h_{n,r}| \leq \sup_V |f_n|$ because

$$\min_{y \in \partial B_r(o)} f_n(y) \leq h_{n,r} \leq \max_{y \in \partial B_r(o)} f_n(y) \quad \text{on } B_r(o).$$

By the Ascoli–Arzela theorem, there exists a subsequence of $(h_{n,r})$, which we also denote by $(h_{n,r})$, that converges uniformly on all finite subsets of V to a function h_n as r goes to infinity. The function h_n is p-harmonic on V by the convergence property. For each r, the minimizing property of p-harmonic functions gives $I_p(h_{n,r}, B_r(o)) \leq I_p(f_n, B_r(o))$, so $I_p(h_{n,r}, V) \leq I_p(f_n, V)$, which implies $h_n \in \mathrm{BHD}_p(G)$.

Let $\epsilon > 0$. Since $(f_n) \to f$ in the supremum norm, there exists a number N such that $\sup_V |f_n - f_m| < \epsilon$ for $n, m \geq N$. It follows that $\sup_{\partial B_r(o)} |h_{n,r} - h_{m,r}| < \epsilon$ for all $r \in \mathbb{N}$ because $f_n = h_{n,r}$ on $V \setminus B_r(o)$. Both $h_{n,r}$ and $h_{m,r} + \epsilon$ are p-harmonic on $B_r(o)$ and $h_{m,r} - \epsilon \leq h_{n,r} \leq h_{m,r} + \epsilon$ on $\partial B_r(o)$, so by applying the comparison principle, we obtain $\sup_{B_r(o)} |h_{n,r} - h_{m,r}| < \epsilon$ for all r. It now follows that $\sup_{B_r(o)} |h_n - h_m| < 3\epsilon$ for all r. Thus $\sup_V |h_n - h_m| \leq 3\epsilon$. Hence, the Cauchy sequence (h_n) converges uniformly on finite subsets of V to a function h, which is p-harmonic by the convergence property.

Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\sup_V |f_n - f| < \epsilon$ and $\sup_V |h_n - h| < \epsilon$ if $n \geq N$. Let $x \in \partial_p(G)$. Since $f_n(x) = h_n(x)$, there exists a neighborhood U of x such that $|h_n(y) - f_n(x)| < \epsilon$ for all $y \in U$. Therefore, $\lim_{k\to\infty} h(x_k) = f(x)$, where (x_k) is a sequence in V that converges to x. $\square$

6. Proofs of Theorem 2.8 and Theorem 2.9

Let G and H be graphs with vertex sets V_G and V_H, respectively. Fix a vertex o_G in G and a vertex o_H in H. Let $\phi : G \to H$ be a rough isometry, and let ϕ^* denote the map from $\ell^\infty(H)$ to $\ell^\infty(G)$ given by $\phi^* f(x) = f(\phi(x))$. We start by defining a map $\bar{\phi} : \partial_p(G) \to \partial_p(H)$. Let $x \in \partial_p(G)$. Then there exists a sequence (x_n) in V_G such that $(x_n) \to x$. Now $(\phi(x_n))$ is a sequence in the compact Hausdorff space $\mathrm{Sp}(\mathrm{BD}_p(H))$. By passing to a subsequence, if necessary we may assume that $(\phi(x_n))$ converges to a unique limit y in $\mathrm{Sp}(\mathrm{BD}_p(H))$. Now define $\bar{\phi}(x) = y$. Before we show that $y \in \partial_p(H)$ and $\bar{\phi}$ is well defined, we need a lemma.

Lemma 6.1. *Let G and H be graphs. If $\phi : G \to H$ is a rough isometry, then*

(a) *ϕ^* maps $\mathrm{BD}_p(H)$ to $\mathrm{BD}_p(G)$,*

(b) *ϕ^* maps $\ell^p(H)$ to $\ell^p(G)$, and*

(c) *ϕ^* maps $B(\overline{\ell^p(H)})_{D_p}$ to $B(\overline{\ell^p(G)})_{D_p}$.*

Proof. We will only prove part (a) since the proofs of parts (b) and (c) are similar. Let $f \in \mathrm{BD}_p(H)$. We will now show that $\phi^* f \in \mathrm{BD}_p(G)$. Let $x \in V_G$ and $w \in N_x$, so x and w are neighbors in G but $\phi(w)$ and $\phi(x)$ are not necessarily neighbors in H. However, by the definition of rough isometry there exists constants $a \geq 1$ and $b \geq 0$ such that $d_H(\phi(w), \phi(x)) \leq a + b$. Set $h_1 = \phi(x)$ and $h_l = \phi(w)$, and let $h_1, \ldots, h_l$ be a path in H with length at most $a + b$. Thus

$$\begin{aligned} |\phi^* f(w) - \phi^* f(x)|^p &= |f(\phi(w)) - f(\phi(x))|^p \\ &\leq |a+b|^{p-1} \sum_{j=1}^{l-1} |f(h_{j+1}) - f(h_j)|^p. \end{aligned} \tag{6-1}$$

The inequality follows from Jensen's inequality applied to the function x^p for $x > 0$.

Let $y \in V_H$ and $z \in N_y$. We claim that there is at most a finite number of paths in H of length at most $a + b$ that contain the edge y, z and have the endpoints $\phi(x)$ and $\phi(w)$. To see this, let U be the set of all elements in V_G such that the four distances $d_H(\phi(x), y)$, $d_H(\phi(x), z)$, $d_H(\phi(w), y)$ and $d_H(\phi(w), z)$ are all at most $a + b$. Let $x, x' \in U$. By the triangle inequality, $d_H(\phi(x'), \phi(x)) \leq d_H(\phi(x'), y) + d_H(\phi(x), y)$. It now follows from the definition of rough isometry that $d_G(x', x) \leq 2a^2 + 3ab$. Thus the metric ball $B(x, 2a^2 + 3ab + 1)$ contains U as a subset. Hence the cardinality of U is bounded above by some constant k, which is independent of y and z. Since $f \in \mathrm{BD}_p(H)$ it follows from (6-1) that

$$\sum_{x \in V_G} \sum_{w \in N_x} |\phi^* f(w) - \phi^* f(x)|^p \leq |a+b|^{p-1} k \sum_{y \in V_H} \sum_{z \in N_y} |f(z) - f(y)|^p < \infty. \qquad \square$$

Proposition 6.2. *The map $\bar{\phi}$ is well defined from $\partial_p(G)$ to $\partial_p(H)$.*

Proof. Let x, y and (x_n) be as above. We first show that $y \in \partial_p(H)$. Lemma 4.1 tells us that $d_G(o_G, x_n) \to \infty$ as $n \to \infty$. The element $\phi(o_G)$ is fixed in H, so it follows from the definition of rough isometry that $d_H(\phi(o_G), \phi(x_n)) \to \infty$ as $n \to \infty$. Thus $y \in \mathrm{Sp}(\mathrm{BD}_p(H)) \setminus H$ since $y = \lim_{n\to\infty} \phi(x_n) \notin H$. Let $f \in B(\overline{\ell^p(H)})_{D_p}$ and suppose $\hat{f}(y) \neq 0$. Then $0 \neq \lim_{n\to\infty} f(\phi(x_n)) = \phi^* f(x)$. By Lemma 6.1(c), $\phi^* f \in B(\overline{\ell^p(G)})_{D_p}$ and Theorem 4.8 says that $\phi^* f(x) = 0$, a contradiction. Hence $\hat{f}(y) = 0$ for all $f \in B(\overline{\ell^p(H)})_{D_p}$, so $y \in \partial_p(H)$.

We will now show that $\bar{\phi}$ is well-defined. Let (x_n) and (x'_n) be sequences in V_G that both converge to $x \in \partial_p(G)$. Now suppose that $(\phi(x_n))$ converges to y_1 and $(\phi(x'_n))$ converges to y_2 in $\mathrm{Sp}(\mathrm{BD}_p(H))$. Assume that $y_1 \neq y_2$ and let $f \in \mathrm{BD}_p(H)$ such that $f(y_1) \neq f(y_2)$. By Lemma 6.1(a), we have $\phi^* f \in \mathrm{BD}_p(G)$. Thus

$$\lim_{n\to\infty} \phi^* f(x_n) = \phi^* f(x) = \lim_{n\to\infty} \phi^* f(x'_n),$$

which implies $f(y_1) = f(y_2)$, a contradiction. Hence $\bar{\phi}$ is a well-defined map from $\partial_p(G)$ to $\partial_p(H)$. □

The next lemma will be used to show that $\bar{\phi}$ is one-to-one and onto.

Lemma 6.3. *Let $\phi : G \to H$ be a rough isometry and let ψ be a rough inverse of ϕ. If $f \in D_p(G)$, then* $\lim_{d_G(o_G,x)\to\infty} |f((\psi \circ \phi)(x)) - f(x)| = 0$.

Proof. Let $x \in V_G$. Since ψ is a rough inverse of ϕ, there are nonnegative constants a, b and c with $a \geq 1$ such that $d_G((\psi \circ \phi)(x), x) \leq a(c + b)$. Let $x_1, x_2, \ldots, x_n$ be a path in V_G of length not more than $a(c+b)$ with $x_1 = x$ and $x_n = (\psi \circ \phi)(x)$. So

$$|f((\psi\circ\phi)(x)) - f(x)|^p = \left|\sum_{k=1}^{n-1} (f(x_{k+1}) - f(x_k))\right|^p \leq n^{p-1} \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)|^p.$$

The last sum approaches zero as $d_G(o_G, x) \to \infty$ since $f \in D_p(G)$ and $n \leq a(c+b)$. Thus $\lim_{d_G(o_G,x)\to\infty} |f((\psi \circ \phi)(x)) - f(x)| = 0$. □

Proposition 6.4. *The function $\bar{\phi}$ is a bijection.*

Proof. Let $x_1, x_2 \in \partial_p(G)$ with $x_1 \neq x_2$, and let $f \in \mathrm{BD}_p(G)$ with $f(x_1) \neq f(x_2)$. There exists sequences (x_n) and (x'_n) in V_G such that $(x_n) \to x_1$ and $(x'_n) \to x_2$. Assume that

$$\bar{\phi}(x_1) = \lim_{n\to\infty} (\phi(x_n)) = \lim_{n\to\infty} (\phi(x'_n)) = \bar{\phi}(x_2),$$

so $\lim_{n\to\infty} f((\psi \circ \phi)(x_n)) = \lim_{n\to\infty} f((\psi \circ \phi)(x'_n))$. It follows from Lemma 6.3 that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x'_n)$; thus $f(x_1) = f(x_2)$, a contradiction. Hence $\bar{\phi}$ is one-to-one.

We now show that $\bar{\phi}$ is onto. Let $y \in \partial_p(H)$ and let (y_n) be a sequence in V_H that converges to y. By passing to a subsequence if necessary, we can assume that there is a unique x in the compact Hausdorff space $\mathrm{Sp}(\mathrm{BD}_p(G))$ such that $(\psi(y_n)) \to x$.

Since $\lim_{n\to\infty} d_H(o_H, y_n) \to \infty$, we have $\lim_{n\to\infty} d_G(o_G, \psi(y_n)) \to \infty$, so $x \notin G$. Using an argument similar to the first paragraph in the proof of Proposition 6.2, we obtain $x \in \partial_p(G)$. The proof will be complete once we show that $\bar{\phi}(x) = y$. Let $f \in \mathrm{BD}_p(H)$. By Lemma 6.3, we see that $\lim_{n\to\infty}|f((\phi \circ \psi)(y_n)) - f(y_n)| = 0$. Thus $f(\bar{\phi}(x)) = f(y)$ for all $f \in \mathrm{BD}_p(H)$. Hence $\bar{\phi}(x) = y$. □

We finally show that the bijection $\bar{\phi}$ is also a homeomorphism. We only need to show that $\bar{\phi}$ is continuous, since both $\mathrm{Sp}(\mathrm{BD}_p(G))$ and $\mathrm{Sp}(\mathrm{BD}_p(H))$ are compact Hausdorff spaces. Let W be an open set in $\partial_p(H)$ and let $x \in \bar{\phi}^{-1}(W)$. Choose $y \in W$ so that $x = \bar{\phi}^{-1}(y)$. By Proposition 4.12, there exists a subset U of V_H such that $y \in \overline{U}$ and $\overline{U} \cap \partial_p(H) \subseteq W$. We saw in the proof of Proposition 4.12 that there is an $h \in \mathrm{BHD}_p(H)$ for which $\hat{h}(y) = 1$ and $\hat{h} = 0$ on $\partial_p(H) \setminus W$ and $\hat{h} \geq \epsilon$ on $\overline{U}$, where $0 < \epsilon < 1$. By Lemma 6.1(a), we have $\phi^* h = h \circ \phi \in \mathrm{BD}_p(G)$. Combining Theorems 4.6 and 4.8, we have an $\bar{h} \in \mathrm{BHD}_p(G)$ that satisfies $\bar{h} = \hat{h} \circ \bar{\phi}$ on $\partial_p(G)$. Let $O = \{x' \in \partial_p(G) \mid \bar{h}(x') > \epsilon\}$. Now O is an open set containing x since $\bar{h}$ is continuous on $\partial_p(G)$ and $\bar{h}(x) = 1$. For $z \in O$, we see that $\hat{h}(\bar{\phi}(z)) = \bar{h}(z) \geq \epsilon$, thus $\bar{\phi}(z) \in W$ for all z in O. Thus $O \subseteq \bar{\phi}^{-1}(W)$. Since our choice of x was arbitrary, $\bar{\phi}^{-1}(W)$ is open and consequently $\bar{\phi}$ is continuous. The proof that $\bar{\phi}$ is a homeomorphism is complete.

We now prove Theorem 2.9. Let ϕ be a rough isometry from G to H, and let ψ be a rough inverse of ϕ. Let $h \in \mathrm{BHD}_p(G)$. By Lemma 6.1(a), $h \circ \psi \in \mathrm{BD}_p(H)$. Let $\pi(h \circ \psi)$ be the unique element in $\mathrm{BHD}_p(H)$ given by Theorem 4.6. We now define a map $\Phi : \mathrm{BHD}_p(G) \mapsto \mathrm{BHD}_p(H)$ by $\Phi(h) = \pi(h \circ \psi)$. Theorem 4.8 implies that $\pi(h \circ \psi)(\bar{\phi}(x)) = (h \circ \psi)(\bar{\phi}(x))$ for all $x \in \partial_p(G)$, where $\bar{\phi}$ is the homeomorphism from $\partial_p(G)$ to $\partial_p(H)$ defined earlier in this section. Thus $\Phi(h)(\bar{\phi}(x)) = (h \circ \psi)(\bar{\phi}(x)) = h(x)$ for all $x \in \partial_p(G)$. We can now show that Φ is one-to-one. Let $h_1, h_2 \in \mathrm{BHD}_p(G)$ and suppose that $\Phi(h_1) = \Phi(h_2)$. So $\Phi(h_1)(\bar{\phi}(x)) = \Phi(h_2)(\bar{\phi}(x))$ for all $x \in \partial_p(G)$, which implies $h_1(x) = h_2(x)$ for all $x \in \partial_p(G)$. Hence, $h_1 = h_2$ by Corollary 4.9. Thus Φ is one-to-one.

We will now show that Φ is onto. Let $f \in \mathrm{BHD}_p(H)$. Then $f \circ \phi \in \mathrm{BD}_p(G)$. Let $h = \pi(f \circ \phi)$, where $\pi(f \circ \phi)$ is the unique element in $\mathrm{BHD}_p(G)$ given by Theorem 4.6. Let $y \in \partial_p(H)$. Since $h(x) = \pi(f \circ \phi)(x)$ for all $x \in \partial_p(G)$ and $\bar{\psi} \circ \bar{\phi}$ equals the identity on $\partial_p(G)$, we see that $(\Phi(h))(y) = \pi(h \circ \psi)(y) = h(\psi(y)) = f((\phi \circ \psi)(y)) = f(y)$. Thus Φ is onto and the proof of Theorem 2.9 is complete.

The map Φ is an isomorphism in the case $p = 2$ since $\mathrm{BHD}_2(G)$ and $\mathrm{BHD}_2(H)$ are linear spaces. However, in general these spaces are not linear if $p \neq 2$.

7. The first reduced ℓ^p-cohomology of Γ

In the final two sections, Γ will denote a finitely generated group with generating set S. So for a real-valued function f on Γ the p-th power of the gradient and the

p-Laplacian of $x \in \Gamma$ are

$$|Df(x)|^p = \sum_{s \in S} |f(xs^{-1}) - f(x)|^p,$$
$$\Delta_p f(x) = \sum_{s \in S} |f(xs^{-1}) - f(x)|^{p-2}(f(xs^{-1}) - f(x)).$$

If $f \in D_p(\Gamma)$, then $(\|f\|_{D_p} = I_p(f, \Gamma) + |f(e)|^p)^{1/p}$, where e is the identity element of Γ. Also $\ell^p(\Gamma)$ is the set that consists of real-valued functions on Γ for which $\sum_{x \in \Gamma} |f(x)|^p$ is finite. The first reduced ℓ^p-cohomology space of Γ is defined by

$$\overline{H}^1_{(p)}(\Gamma) = D_p(\Gamma)/(\overline{\ell^p(\Gamma) \oplus \mathbb{R}})_{D_p}.$$

We now prove Theorem 2.10. Suppose $\partial_p(\Gamma) = \varnothing$. By Proposition 4.2, there exists a sequence (f_n) in $\mathbb{R}\Gamma$ that satisfies $\|f_n - 1_\Gamma\|_{D_p} \to 0$. It follows that $I_p(f_n, \Gamma) \to 0$ and $(f_n(e)) \not\to 0$. Thus $\overline{H}^1_{(p)}(\Gamma) = 0$ by [Puls 2003, Theorem 3.2]. We now assume $\partial_p(G) \neq \varnothing$. It was shown in [Puls 2006, Theorem 3.5] that $\overline{H}^1_{(p)}(\Gamma) \neq 0$ if and only if $\mathrm{HD}_p(\Gamma) \neq \mathbb{R}$. Since $\#(S) < \infty$, [Holopainen and Soardi 1997a, Lemma 4.4] says that $\mathrm{BHD}_p(\Gamma) = \mathbb{R}$ if and only if $\mathrm{HD}_p(\Gamma) = \mathbb{R}$. Theorem 2.10 now follows from Theorem 4.10.

We now use Theorem 2.10 to compute $\partial_p(\Gamma)$ and $R_p(\Gamma)$ for some special cases of Γ. By [Holopainen and Soardi 1997b, Corollary 1.10], $\mathrm{BHD}_p(\Gamma) = \mathbb{R}$ when Γ has polynomial growth and $1 < p \in \mathbb{R}$. Thus, if Γ has polynomial growth, then $\overline{H}^1_{(p)}(\Gamma) = 0$ and $\partial_p(\Gamma)$ is either the empty set or contains exactly one element. It would be nice to know when a group with polynomial growth is p-parabolic or p-hyperbolic. This has been worked out for the case $\Gamma = \mathbb{Z}^n$, where n is a positive integer. Yamasaki [1977, Example 4.1] showed that $\mathbb{Z}$ is p-parabolic for $p > 1$, and thus $\partial_p(\mathbb{Z}) = \varnothing$ for $p > 1$. The main result of [Maeda 1977] says that $\mathbb{Z}^n$ with $n \geq 2$ is p-parabolic if and only if $p \geq n$. Hence, $\partial_p(\mathbb{Z}^n) = \varnothing$ if $p \geq n$ and $\partial_p(\mathbb{Z}^n)$ consists of exactly one point if $1 < p < n$.

There is a one-to-one correspondence between the maximal ideals of $\mathrm{BD}_p(\Gamma)$ and the points of $\mathrm{Sp}(\mathrm{BD}_p(\Gamma))$. If $\tau \in R_p(\Gamma)$, then $\ker(\tau)$ is the maximal ideal of $\mathrm{BD}_p(\Gamma)$ corresponding to τ. For each $x \in \Gamma$, we have $\delta_x \in \ker(\tau)$. By the continuity of τ, we see that $\ell^p(\Gamma) \subseteq \ker(\tau)$. Assume that Γ is nonamenable. Then $\ell^p(\Gamma)$ is closed in $D_p(\Gamma)$ by [Guichardet 1977, Corollary 1]. Hence $(\overline{\mathbb{R}\Gamma})_{D_p} = \ell^p(\Gamma)$. Also, $(\overline{\ell^p(\Gamma)})_{\mathrm{BD}_p} = \ell^p(\Gamma)$ because $(\overline{\ell^p(\Gamma)})_{\mathrm{BD}_p} \subseteq B(\overline{\ell^p(\Gamma)})_{D_p}$. Thus $\hat{f}(\tau) = 0$ for every $f \in (\overline{\mathbb{R}\Gamma})_{D_p}$. Therefore, $R_p(\Gamma) = \partial_p(\Gamma)$ when Γ is nonamenable. Consequently, $R_p(\Gamma)$ contains exactly one point when Γ is nonamenable and $\overline{H}^1_{(p)}(\Gamma) = 0$. Some groups that satisfy this last condition for $1 < p \in \mathbb{R}$ are nonamenable groups with infinite center [Martin and Valette 2007, Theorem 4.2], and $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ for $n \geq 2$, each Γ_i is finitely generated, and at least one of the Γ_i is nonamenable [Martin and Valette 2007, Theorem 4.7].

8. Translation invariant linear functionals

Recall that Γ denotes a finitely generated group with generating set S. In this section we will study TILFs on $D_p(\Gamma)/\mathbb{R}$. By definition we have the inclusions

$$\operatorname{Diff}(\ell^p(\Gamma)) \subseteq \operatorname{Diff}(D_p(\Gamma)/\mathbb{R}) \subseteq \ell^p(\Gamma) \subseteq D_p(\Gamma)/\mathbb{R}.$$

The set $D_p(\Gamma)/\mathbb{R}$ is a Banach space under the norm induced from $I_p(\cdot, \Gamma)$. Thus if $[f]$ if a class from $D_p(G)/\mathbb{R}$, then its norm is given by

$$\|[f]\|_{D(p)} = \Big(\sum_{x\in\Gamma}\sum_{s\in S}|f(xs^{-1}) - f(x)|^p\Big)^{1/p}.$$

We will write $\|f\|_{D(p)}$ for $\|[f]\|_{D(p)}$. Now $(\overline{\ell^p(\Gamma)})_{D(p)} = D_p(\Gamma)/\mathbb{R}$ if and only if $(\overline{\ell^p(\Gamma)\oplus\mathbb{R}})_{D_p} = D_p(\Gamma)$. So $\overline{H}^1_{(p)}(\Gamma) = 0$ if and only if $(\overline{\ell^p(\Gamma)})_{D(p)} = D_p(\Gamma)/\mathbb{R}$.

Lemma 8.1. $(\overline{\operatorname{Diff}(D_p(\Gamma)/\mathbb{R})})_{D(p)} = (\overline{\ell^p(\Gamma)})_{D(p)}$.

Proof. Let $f \in \ell^p(\Gamma)$. By [Woodward 1974, Lemma 1], there is a sequence (f_n) in $\operatorname{Diff}(\ell^p(\Gamma))$ that converges to f in the ℓ^p-norm. It follows from Minkowski's inequality that for $s \in S$,

$$\|(f - f_n)_s - (f - f_n)\|_p^p = \sum_{x\in\Gamma}|f(xs^{-1}) - f_n(xs^{-1}) - (f(x) - f_n(x))|^p \to 0$$

as $n \to \infty$. Hence $f \in (\overline{\operatorname{Diff}(\ell^p(\Gamma))})_{D(p)}$, implying $\ell^p(\Gamma) \subseteq (\overline{\operatorname{Diff}(\ell^p(\Gamma))})_{D(p)}$. The result now follows. □

Theorem 8.2. *Let $1 < p \in \mathbb{R}$. Then $\overline{H}^1_{(p)}(\Gamma) \neq 0$ if and only if there exists a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$.*

Proof. If $\overline{H}^1_{(p)}(\Gamma) \neq 0$, then $(\overline{\ell^p(\Gamma)})_{D(p)} \neq D_p(\Gamma)/\mathbb{R}$. It now follows from the Hahn–Banach theorem that there exists a nonzero continuous linear functional T on $D_p(\Gamma)/\mathbb{R}$ such that $(\overline{\ell^p(\Gamma)})_{D(p)}$ is contained in the kernel of T. Thus T is translation invariant by Lemma 8.1.

Conversely, if T is a continuous TILF on $D_p(\Gamma)/\mathbb{R}$, then $T(f) = 0$ for all $f \in (\overline{\ell^p(\Gamma)})_{D(p)}$. So if there exists a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$, then $(\overline{\ell^p(\Gamma)})_{D(p)} \neq D_p(\Gamma)/\mathbb{R}$. □

Theorem 2.11 now follows by combining Theorems 8.2 and 2.10.

If $h \in D_p(\Gamma)/\mathbb{R}$, then $\langle \triangle_p h, \cdot\rangle$ is a well-defined continuous linear functional on $D_p(\Gamma)/\mathbb{R}$ since equivalent functions in $D_p(\Gamma)/\mathbb{R}$ differ by a constant. It was shown in [Puls 2006, Proposition 3.4] that if $h \in \mathrm{HD}_p(\Gamma)/\mathbb{R}$ and $f \in (\overline{\ell^p(\Gamma)})_{D(p)}$, then $\langle \triangle_p h, f\rangle = 0$. Consequently, if $h \in \mathrm{HD}_p(\Gamma)/\mathbb{R}$, then $\langle \triangle_p h, \cdot\rangle$ defines a continuous TILF on $D_p(\Gamma)/\mathbb{R}$. Thus there are no nonzero continuous TILFs on $D_p(\Gamma)/\mathbb{R}$ when $\mathrm{HD}_p(\Gamma)$ only contains the constant functions.

If $\overline{H}^1_{(p)}(\Gamma)=0$, then $(\overline{\ell^p(\Gamma)})_{D(p)}=D_p(\Gamma)/\mathbb{R}$. It is known that $\ell^p(\Gamma)$ is closed in $D_p(\Gamma)/\mathbb{R}$ if and only if Γ is nonamenable, [Guichardet 1977, Corollary 1]. As was mentioned in Section 2, if Γ is nonamenable, then zero is the only TILF on $\ell^p(\Gamma)$. Consequently zero is the only TILF on $D_p(\Gamma)/\mathbb{R}$ when Γ is nonamenable and $\overline{H}^1_{(p)}(\Gamma)=0$. Summing up:

Theorem 8.3. *Let Γ be an infinite, finitely generated group and let $1<p\in\mathbb{R}$. The following are equivalent*:

(1) $\overline{H}^1_{(p)}(\Gamma)=0$.

(2) *Either $\partial_p(\Gamma)=\varnothing$ or $\#(\partial_p(\Gamma))=1$.*

(3) $\mathrm{HD}_p(\Gamma)=\mathbb{R}$.

(4) $\mathrm{BHD}_p(\Gamma)=\mathbb{R}$.

(5) *The only continuous TILF on $D_p(\Gamma)/\mathbb{R}$ is zero. If Γ is also nonamenable*, *then this is still equivalent to* (6):

(6) *Zero is the only TILF on $D_p(\Gamma)/\mathbb{R}$.*

Some examples show zero is not the only TILF on $D_p(\Gamma)/\mathbb{R}$ when Γ is nonamenable; this differs from the $\ell^p(\Gamma)$ case. Puls [2006, Corollary 4.3] showed $\overline{H}^1_{(p)}(\Gamma)\neq 0$ for groups with infinitely many ends and $1<p\in\mathbb{R}$. Thus by Theorem 8.2 there exists a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$.

If there is a nonzero continuous TILF on $D_r(\Gamma)/\mathbb{R}$ for some nonamenable group Γ and some real number r, then is it true that there is a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$ for all real numbers $p>1$? The answer to this question is no. To see this, let $\mathscr{H}^n$ denote hyperbolic n-space, and suppose Γ is a group that acts properly discontinuously on $\mathscr{H}^n$ by isometries and that the action is cocompact and free. By combining [Bourdon et al. 2005, Theorem 2] and [Puls 2007, Theorem 1.1], we obtain $\overline{H}^1_{(p)}(\Gamma)\neq 0$ if and only if $p>n-1$.

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