GRAPHS OF BOUNDED DEGREE
AND THE $p$-HARMONIC BOUNDARY

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Let $p$ be a real number greater than one and let $G$ be a connected graph of bounded degree. We introduce the $p$-harmonic boundary of $G$ and use it to characterize the graphs $G$ for which the constant functions are the only $p$-harmonic functions on $G$. We show that any continuous function on the $p$-harmonic boundary of $G$ can be extended to a function that is $p$-harmonic on $G$. We also give some properties of this boundary that are preserved under rough-isometries. Now let $\Gamma$ be a finitely generated group. As an application of our results, we characterize the vanishing of the first reduced $\ell^p$-cohomology of $\Gamma$ in terms of the cardinality of its $p$-harmonic boundary. We also study the relationship between translation invariant linear functionals on a certain difference space of functions on $\Gamma$, the $p$-harmonic boundary of $\Gamma$, and the first reduced $\ell^p$-cohomology of $\Gamma$.

1. Introduction

Let $p$ be a real number greater than one and let $\Gamma$ be a finitely generated infinite group. There has been some work done relating various boundaries of $\Gamma$ and the nonvanishing of the first reduced $\ell^p$-cohomology space $\overline{H}_{(p)}^1(\Gamma)$ of $\Gamma$ (to be defined in Section 7). Gromov [1993, Chapter 8, Section C2]—see also [Elek 1997]—showed that if the $\ell^p$-corona of $\Gamma$ contains more than one element, then $\overline{H}_{(p)}^1(\Gamma) \neq 0$. Puls [2007] showed that if there is a Floyd boundary of $\Gamma$ containing more than two elements, and if the Floyd admissible function satisfies a certain decay condition, then $\overline{H}_{(p)}^1(\Gamma) \neq 0$. However, it is unknown if the converse of either of these two results is true. The motivation for this paper is to find a boundary for $\Gamma$ whose cardinality characterizes the vanishing of $\overline{H}_{(p)}^1(\Gamma)$. We will show that the $p$-harmonic boundary, defined in Section 2.1, does the trick. This boundary gives the desired result because $\overline{H}_{(p)}^1(\Gamma) = 0$ if and only if the only $p$-harmonic functions on $\Gamma$ are the constants, [Puls 2006, Theorem 3.5]. We will show in Section 7 that the cardinality of the $p$-harmonic boundary is 0 or 1 if and
only if the only $p$-harmonic functions on $\Gamma$ are the constants. Hence, $H_{(p)}^1(\Gamma) = 0$ if and only if the cardinality of the $p$-harmonic boundary is 0 or 1.

$L_p$-cohomology was investigated first in [Gol’dshte˘ın et al. 1987] for the case of Riemannian manifolds. Gromov [1993, Chapter 8] has studied $\ell^p$-cohomology for finitely generated groups, and in the more general setting of graphs with bounded degree. In particular, Cheeger and Gromov [1986] showed that the first reduced $\ell^2$-cohomology of a finitely generated amenable group is zero. Gromov [1993, page 150] conjectured that this result is also true for all real numbers $p > 1$. This is our main justification for choosing to study the $p$-harmonic boundary in the discrete setting. If enough insight can be gained into this boundary, then we may be able to develop the tools needed to compute the $p$-harmonic boundary of a finitely generated amenable group. This of course would resolve Gromov’s conjecture.

More information about the first reduced $L_p$-cohomology (and the special case of $L_2$-cohomology) can be found in [Pansu 1989; 2007; 2008; Tessera 2009] for various manifolds, and in [Bekka and Valette 1997; Bourdon 2004; Bourdon et al. 2005; Elek 1998; Martin and Valette 2007; Puls 2003; 2006; 2007] for finitely generated groups. As implied earlier, there is a strong connection between the vanishing of the first reduced $L_p$-cohomology and the nonexistence nonconstant $p$-harmonic functions; for a proof in the case of homogeneous Riemannian manifolds, see [Tessera 2009, Proposition 4.11]. Thus results on $p$-harmonic functions are useful in trying to determine if the first reduced $L_p$-cohomology vanishes. The papers [Coulhon et al. 2001; Grigoryan 1987] study $p$-harmonic functions on manifolds, while [Holopainen and Soardi 1997a; Kim and Lee 2005; 2007; Soardi 1993; Yamasaki 1977] examine $p$-harmonic functions on graphs.

2. Definitions and statement of main results

Let $p$ be a real number greater than one, and let $\Gamma$ be a finitely generated infinite group. The definition of the $p$-harmonic boundary for $\Gamma$ does not depend on the group law of $\Gamma$, so we can define this boundary in the more general setting of a graph. The reason is that we can associate a graph, called the Cayley graph of $\Gamma$, with $\Gamma$. The vertex set for this graph consists of the elements of $\Gamma$, and $x_1, x_2 \in \Gamma$ are joined by an edge if and only if $x_1 = x_2 s^{\pm 1}$ for a generator $s$ of $\Gamma$.

2.1. The $p$-harmonic boundary. Let $G$ be a graph with vertex set $V_G$ and edge set $E_G$. We will write $V$ for $V_G$ and $E$ for $E_G$ if it is clear what the graph $G$ is. For $x \in V$, we denote by $\deg(x)$ the number of neighbors of $x$ and by $N_x$ the set of neighbors of $x$. We say a graph $G$ is of bounded degree if there exists a positive integer $k$ such that $\deg(x) \leq k$ for every $x \in V$. A path in $G$ is a sequence of vertices $x_1, x_2, \ldots, x_n$ for which $x_{i+1} \in N_{x_i}$ for $1 \leq i \leq n - 1$. A graph $G$ is connected if any two given vertices of $G$ are joined by a path. All graphs considered in this
paper will be countably infinite, connected, of bounded degree with no self-loops. Assign length one to each edge in $E_G$; then the graph $G$ is a metric space with respect to the shortest path metric. Let $d_G(\cdot, \cdot)$ denote this metric. So if $x, y \in V$, then $d_G(x, y)$ is the length of the shortest path joining $x$ and $y$. We will drop the subscript $G$ from $d_G(\cdot, \cdot)$ when it is clear what graph $G$ we are working with. Finally, if $x \in V$, then $B_n(x)$ will denote the metric ball that contains all elements of $V$ that have distance less than $n$ from $x$.

Let $G$ be a graph with vertex set $V$, and let $p$ be a real number greater than one. To construct the $p$-harmonic boundary of $G$, we need to first define the space of bounded $p$-Dirichlet finite functions on $G$. For any $S \subset V$, the outer boundary $\partial S$ of $S$ is the set of vertices in $V \setminus S$ with at least one neighbor in $S$. For a real-valued function $f$ on $S \cup \partial S$, we define the $p$-th power of the gradient, the $p$-Dirichlet sum, and the $p$-Laplacian of $x \in S$ by

$$|Df(x)|^p = \sum_{y \in N_x} |f(y) - f(x)|^p,$$

$$I_p(f, S) = \sum_{x \in S} |Df(x)|^p,$$

$$\Delta_p f(x) = \sum_{y \in N_x} |f(y) - f(x)|^{p-2}(f(y) - f(x)).$$

In the case $1 < p < 2$, we make the convention that

$$|f(y) - f(x)|^{p-2}(f(y) - f(x)) = 0 \quad \text{if} \quad f(y) = f(x).$$

Let $S \subset V$. We say a function $f$ is $p$-harmonic on $S$ if $\Delta_p f(x) = 0$ for all $x \in S$, and $p$-Dirichlet finite if $I_p(f, V) < \infty$. We denote the set of all $p$-Dirichlet finite functions on $G$ by $D_p(G)$. Under the norm

$$\|f\|_{D_p} = \left( I_p(f, V) + \|f(o)\|^p \right)^{1/p},$$

$D_p(G)$ is a reflexive Banach space, where $o$ is a fixed vertex of $G$ and $f \in D_p(G)$. Denote by $\text{HD}_p(G)$ the set of $p$-harmonic functions on $V$ contained in $D_p(G)$. Let $\ell^\infty(G)$ denote the set of bounded functions on $V$, and let $\|f\|_{\infty} = \sup_V |f|$ for $f \in \ell^\infty(G)$. Set $\text{BD}_p(G) = D_p(G) \cap \ell^\infty(G)$. The set $\text{BD}_p(G)$ is a Banach space under the norm

$$\|f\|_{\text{BD}_p} = \left( I_p(f, V) \right)^{1/p} + \|f\|_{\infty},$$

where $f \in \text{BD}_p(G)$. Set $\text{BHD}_p(G) = \text{HD}_p(G) \cap \text{BD}_p(G)$. It turns out that $\text{BD}_p(G)$ is closed under pointwise multiplication. To see this, let $f, h \in \text{BD}_p(G)$ and set $a = \sup_V |f|$ and $b = \sup_V |h|$. It follows from Minkowski’s inequality that

$$\begin{equation}
(I_p(fh, V))^{1/p} \leq b(I_p(f, V))^{1/p} + a(I_p(h, V))^{1/p}.
\end{equation}$$
Thus \( fh \in BD_p(G) \). Using the inequality above, we obtain
\[
\|fh\|_{BD_p} \leq ((I_p(f, V))^{1/p} + a)((I_p(h, V))^{1/p} + b) = \|f\|_{BD_p}\|h\|_{BD_p},
\]
Hence \( BD_p(G) \) is an abelian Banach algebra. A character on \( BD_p(G) \) is a nonzero homomorphism from \( BD_p(G) \) into the complex numbers. Let \( \text{Sp}(BD_p(G)) \) be the set of characters on \( BD_p(G) \); it is known as the spectrum of \( BD_p(G) \). With respect to the weak \( * \)-topology, \( \text{Sp}(BD_p(G)) \) is a compact Hausdorff space. Let \( C(\text{Sp}(BD_p(G))) \) denote the set of continuous functions on \( \text{Sp}(BD_p(G)) \). For each \( f \in BD_p(G) \), we define a continuous function \( \hat{f} \) on \( \text{Sp}(BD_p(G)) \) by \( \hat{f}(\tau) = \tau(f) \). The map \( f \to \hat{f} \) is known as the Gelfand transform.

Define a map \( i : V \to \text{Sp}(BD_p(G)) \) by \( (i(x))(f) = f(x) \). For \( x \in V \), define \( \delta_x \) by \( \delta_x(v) = 0 \) if \( v \neq x \) and \( \delta_x(x) = 1 \). Let \( x, y \in V \) and suppose \( i(x) = i(y) \); then \( (i(x))(\delta_x) = (i(y))(\delta_x) \), which implies \( \delta_x(x) = \delta_x(y) \). Thus \( i \) is an injection. If \( f \) is a nonzero function in \( BD_p(G) \), then there exists an \( x \in V \) such that \( \hat{f}(i(x)) \neq 0 \) since \( \hat{f}(i(x)) = f(x) \). Hence \( BD_p(G) \) is semisimple. Then [Taylor and Lay 1986, Theorem 4.6 on page 408] tells us that \( BD_p(G) \) is isomorphic to a subalgebra of \( C(\text{Sp}(BD_p(G))) \) via the Gelfand transform. Since the Gelfand transform separates points of \( \text{Sp}(BD_p(G)) \) and the constant functions are contained in \( BD_p(G) \), the Stone–Weierstrass theorem yields that \( BD_p(G) \) is dense in \( C(\text{Sp}(BD_p(G))) \) with respect to the supremum norm. The following proposition shows that \( i(V) \) is dense in \( \text{Sp}(BD_p(G)) \); see [Elek 1997, Proposition 1.1(ii)] for the proof.

**Proposition 2.1.** The image of \( V \) under \( i \) is dense in \( \text{Sp}(BD_p(G)) \).

When the context is clear we will abuse notation and write \( V \) for \( i(V) \) and \( x \) for \( i(x) \), where \( x \in V \). The compact Hausdorff space \( \text{Sp}(BD_p(G)) \setminus V \) is known as the \( p \)-Royden boundary of \( G \), which we will denote by \( R_p(G) \). When \( p = 2 \), this is simply known as the Royden boundary of \( G \). Let \( \mathbb{R}G \) be the set of real-valued functions on \( V \) with finite support, and let \( B(\mathbb{R}G)_{D_p} = (\mathbb{R}G)_{D_p} \cap \ell^\infty(G) \). Suppose \( (f_n) \) is a sequence in \( B(\mathbb{R}G)_{D_p} \) that converges to a bounded function \( f \) in the \( BD_p(G) \) norm. It follows from \( \|f - f_n\|_{D_p} \leq \|f - f_n\|_{BD_p} \) that \( f \in (\mathbb{R}G)_{D_p} \). Thus \( B(\mathbb{R}G)_{D_p} \) is closed in \( BD_p(G) \) with respect to the \( BD_p(G) \) norm. We are now ready to define the main object of study for this paper.

The \( p \)-harmonic boundary of \( G \) is the subset
\[
\partial_p(G) := \{ x \in R_p(G) \mid \hat{f}(x) = 0 \text{ for all } f \in B(\mathbb{R}G)_{D_p} \}
\]
of the \( p \)-Royden boundary. When \( p = 2 \), the \( p \)-harmonic boundary is known as the harmonic boundary. Our definition of \( p \)-harmonic boundary directly generalizes that of harmonic boundary. A good reference for the Royden and harmonic boundaries of graphs is [Soardi 1994, Chapter VI].
An important fact about $B(\overline{R}G)_{D_p}$ is that it is an ideal in $BD_p(G)$. To see this, let $f \in B(\overline{R}G)_{D_p}$ and $h \in BD_p(G)$. We need to show that $fh \in B(\overline{R}G)_{D_p}$. We claim that there exists a sequence $(f_n)$ in $\overline{R}G$ converging pointwise to $f$, for which there exists a constant $M$ with $|f_n(x)| \leq M$ for all $n$ and for all $x \in V$, and for which $I_p(f_n, V)$ is bounded. Let $(u_n)$ be a sequence in $\overline{R}G$ that converges to $f$ in $D_p(G)$ and let $M = \sup_{x \in V} |f(x)|$. Set $f_n = \max(\min(u_n, M), -M)$. The sequence $(f_n)$ satisfies the claim above since $I_p(u_n, V)$ is bounded and $I_p(f_n, V) \leq I_p(u_n, V)$. Also $(f_nh)$ is a sequence in $\overline{R}G$ that converges pointwise to $fh$. By (2-1), we see that

$$I_p(f_nh, V) \leq (b(I_p(f_n, V))^{1/p} + M(I_p(h, V))^{1/p})^p,$$

where $b = \sup_{x \in V} |h(x)|$. Since $I_p(f_nh, V)$ is bounded, [Taylor and Lay 1986, Theorem 10.6, page 177] says, by passing to a subsequence if necessary, that $(f_nh)$ converges weakly to a function $\overline{fh}$. Since $B(\overline{R}G)_{D_p}$ is closed, it follows that $\overline{fh} \in B(\overline{R}G)_{D_p}$. Because point evaluations by elements of $V$ are continuous linear functionals on $BD_p(G)$, $(f_nh)$ also converges pointwise to $\overline{fh}$. Hence, $\overline{fh} = fh$ and $fh \in B(\overline{R}G)_{D_p}$.

2.2. Statement of main results. Recall that $p$ is a real number greater than one and that $o$ is a fixed vertex of $V$. By $\#(A)$, we mean the cardinality of a set $A$, and $1_V$ will denote the function on $V$ that always takes the value one. Furthermore, $\ell_p(G)$ will be the set that consists of the functions on $V$ for which $\sum_{x \in V} |f(x)|^p < \infty$. The $\ell_p$-norm for $f \in \ell_p(G)$ is given by $\|f\|_p = (\sum_{x \in V} |f(x)|^p)^{1/p}$. In Section 4, we give a quick review of some results about $p$-harmonic functions on graphs. In Section 4 we prove several results concerning $BD_p(G)$ and $\partial_p(G)$; when $BHD_p(G)$ consists precisely of the constant functions and a neighborhood base is given for the topology on $\partial_p(G)$, we characterize when $\partial_p(G) = \emptyset$.

Before we stating some of our main results, we need a theorem that will allow us to classify graphs in a nice way. We start by giving the following definition. The $p$-capacity of a finite subset $A$ of $V$ is defined by

$$\operatorname{Cap}_p(A, \infty, V) = \inf_u I_p(u, V),$$

where the infimum is taken over all finitely supported functions $u$ on $V$ such that $u = 1$ on $A$. The following theorem will allow us to classify a graph $G$ in terms of the $p$-capacity of a finite set.

**Theorem 2.2** [Yamasaki 1977, Theorem 3.1]. Let $A$ be a finite, nonempty subset of $V$. Then

$$\operatorname{Cap}_p(A, \infty, V) = 0 \iff 1_V \in B(\overline{R}G)_{D_p}.$$

**Corollary 2.3.** Let $A$ and $B$ be nonempty finite subsets of $V$. Then

$$\operatorname{Cap}_p(A, \infty, V) = 0 \iff \operatorname{Cap}_p(B, \infty, V) = 0.$$
We say that a graph $G$ is $p$-parabolic if there exists a finite subset $A$ of $V$ such that $\text{Cap}_p(A, \infty, V) = 0$. If $G$ is not $p$-parabolic, we shall say that $G$ is $p$-hyperbolic. If $G$ is $p$-hyperbolic, then $\text{Cap}_p(A, \infty, V) > 0$ for all finite subsets $A$ of $V$.

In Section 5 we will prove the following results. The first reduces to [Soardi 1994, Theorem 4.6] in the case $p = 2$ and also generalizes [Kim and Lee 2005, Theorem 4.2].

**Theorem 2.4.** Let $p$ be a real number greater than one, and let $G$ be a graph. If $G$ is $p$-parabolic, then all $p$-harmonic functions on $G$ are constant functions.

Identify the constant functions on $V$ with $\mathbb{R}$. By combining this theorem with [Holopainen and Soardi 1997a, Lemma 4.4] and Theorem 4.10 we get a Liouville-type theorem for $p$-harmonic functions:

**Theorem 2.5.** Let $p$ be a real number greater than one. Then $\text{HD}_p(G) = \mathbb{R}$ if and only if the cardinality of $p(G)$ is either zero or one.

**Theorem 2.6.** Let $p$ be a real number greater than one and let $G$ be a graph. If $f$ is a continuous function on $p(G)$, then there exists a $p$-harmonic function $h$ on $V$ such that $\lim_{n \to \infty} h(x_n) = f(x)$, where $x \in p(G)$ and $(x_n)$ is any sequence in $V$ that converges to $x$.

By combining this theorem with the maximum principle and Corollary 4.9 we obtain the following corollary, which generalizes both [Kim and Lee 2005, Theorem 4.3] and [Kim and Lee 2007, Theorem 1.1].

**Corollary 2.7.** Let $p$ be a real number greater than one and let $G$ be a graph. Assume that the $p$-harmonic boundary of $G$ is a finite set $\{x_1, x_2, \ldots, x_n\}$ of points. Then given real numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$, there exists a bounded $p$-harmonic function $h$ that satisfies

$$h(x_i) = a_i \quad \text{for } i = 1, 2, \ldots, n.$$  

Conversely, each bounded $p$-harmonic function is uniquely determined by its values in (2-2).

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A map $\phi : X \to Y$ is said to be a rough isometry if it satisfies the following two conditions:

1. There exist constants $a \geq 1$ and $b \geq 0$ such that for $x_1, x_2 \in X$

   $$(1/a)d_X(x_1, x_2) - b \leq d_Y(\phi(x_1), \phi(x_2)) \leq ad_X(x_1, x_2) + b.$$

2. There exists a positive constant $c$ such that for each $y \in Y$, there exists an $x \in X$ that satisfies $d_Y(\phi(x), y) < c$. 


For a rough isometry \( \phi \), there exists a rough isometry \( \psi : Y \to X \) such that if \( x \in X \) and \( y \in Y \), then \( d_X((\psi \circ \phi)(x), x) \leq a(c + b) \) and \( d_Y((\phi \circ \psi)(y), y) \leq c \). The map \( \psi \), which is not unique, is said to be a rough inverse for \( \phi \). Whenever we refer to a rough inverse to a rough isometry, it will always satisfy the conditions above. In Section 6, we prove the following two results:

**Theorem 2.8.** Let \( p \) be a real number greater than one and let \( G \) and \( H \) be graphs. If there is a rough isometry from \( G \) to \( H \), then \( \partial_p(G) \) is homeomorphic to \( \partial_p(H) \).

**Theorem 2.9.** Let \( p \) be a real number greater than one and let \( G \) and \( H \) be graphs. If there is a rough isometry from \( G \) to \( H \), then there is a bijection from \( \text{BHD}_p(G) \) to \( \text{BHD}_p(H) \).

The main result of [Soardi 1993] is that if \( G \) and \( H \) are roughly isometric graphs, then \( \text{HD}_p(G) = \mathbb{R} \) if and only if \( \text{HD}_p(H) = \mathbb{R} \). By [Holopainen and Soardi 1997a, Lemma 4.4], this is equivalent to \( \text{BHD}_p(G) = \mathbb{R} \) if and only if \( \text{BHD}_p(H) = \mathbb{R} \). Both Theorem 2.8 and Theorem 2.9 are generalizations of this result.

We now return to the case of a finitely generated group \( \Gamma \). In Section 7, we define the first reduced \( \ell^p \)-cohomology space \( \overline{H}^1_{(p)}(\Gamma) \) of \( \Gamma \). Then we will use our results on \( p \)-harmonic boundaries to prove this:

**Theorem 2.10.** Let \( 1 < p \in \mathbb{R} \). Then \( \overline{H}^1_{(p)}(\Gamma) \neq 0 \) if and only if \( #(\partial_p(\Gamma)) > 1 \).

It appears there are not many explicit examples of the \( p \)-Royden boundary \( R_p(G) \) for a given graph \( G \). Wysoczanski [1996] provided the only example we know of by giving an explicit description of \( R_2(\mathbb{Z}) \). We will conclude Section 7 by using Theorem 2.10 to compute the \( p \)-harmonic boundary for the case \( \Gamma = \mathbb{Z}^n \). We will also compute the \( p \)-Royden boundary of nonamenable groups with infinite center, and of the groups \( \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n \) for \( n \geq 2 \), where each \( F_i \) is finitely generated and at least one of the \( \Gamma_i \) is nonamenable.

Let \( E \) be a normed space of functions on a finitely generated group \( \Gamma \). Let \( f \in E \) and let \( x \in \Gamma \). The right translation of \( f \) by \( x \), denoted by \( f_x \), is the function \( f_x(g) = f(gx^{-1}) \), where \( g \in \Gamma \). Assume that if \( f \in E \), then \( f_x \in E \) for all \( x \in \Gamma \), that is, that \( E \) is right translation invariant. For the rest of this paper translation invariant will mean right translation invariant. We shall say that \( T \) is a translation invariant linear functional (TILF) on \( E \) if \( T(f_x) = T(f) \) for \( f \in E \) and \( x \in \Gamma \). We will use TILFs to denote translation invariant linear functionals. A common question to ask is, If \( T \) is a TILF on \( E \), then is \( T \) continuous? For background about the problem of automatic continuity, see [Meisters 1983; Saeki 1984; Willis 1988; Woodward 1974]. Define

\[
\text{Diff}(E) := \text{linear span}\{f_x - f \mid f \in E, x \in \Gamma\}.
\]

It is clear that \( \text{Diff}(E) \) is contained in the kernel of any TILF on \( E \). In Section 8 we study TILFs on \( D_p(\Gamma)/\mathbb{R} \), and prove the following:
Theorem 2.11. Let $\Gamma$ be a finitely generated infinite group and let $1 < p \in \mathbb{R}$. Then $\#(\partial_p(\Gamma)) > 1$ if and only if there exists a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$.

Willis [1986] showed that if $\Gamma$ is nonamenable, then the only TILF on $\ell^p(\Gamma)$ is the zero functional. (Consequently every TILF is automatically continuous!) We will conclude Section 8 by showing that this result is not true for $D_p(\Gamma)/\mathbb{R}$.

3. Review of $p$-harmonic functions on graphs

The four results below are from [Holopainen and Soardi 1997a, Section 3], where a more comprehensive treatment, including proofs, is given.

- **Existence.** Let $S$ be a finite subset of $V$. For any function $f$ on $\partial S$, there exists an unique function $h$ on $S \cup \partial S$ that is $p$-harmonic on $S$ and equals $f$ on $\partial S$. In the proof of existence, it was shown that the $p$-harmonic function $h$ satisfies $\min_{y \in \partial S} f(y) \leq h(x) \leq \max_{y \in \partial S} f(y)$ for all $x \in S$.

- **Minimizer property.** Let $h$ be a $p$-harmonic function on a finite subset $S$ of $V$. Then $I_p(h, S) \leq I_p(f, S)$ for all functions $f$ on $S \cup \partial S$ satisfying $f = h$ on $\partial S$.

- **Convergence.** Let $(S_n)$ be an increasing sequence of finite connected subsets of $V$ and let $U = \bigcup_i S_i$. Let $(h_i)$ be a sequence of functions on $U \cup \partial U$ such that $h_i(x) \to h(x) < \infty$ for every $x \in U \cup \partial U$. If $h_i$ is $p$-harmonic on $S_i$ for all $i$, then $h$ is $p$-harmonic on $U$.

- **Comparison principle.** Let $h$ and $u$ be $p$-harmonic functions on a finite subset $S$ of $V$. If $h \geq u$ on $\partial S$, then $h \geq u$ on $S$.

We also prove the maximum principle for bounded $p$-harmonic functions on $V$:

**Lemma 3.1.** Let $h$ be a $p$-harmonic function on $V$. If there exists an $x \in V$ such that $h(x) \geq h(y)$ for all $y \in V$, then $h$ is constant on $V$.

**Proof.** Let $x \in V$ such that $h(x) \geq h(x')$ for all $x' \in V$. Because

$$\sum_{y \in N_x} |h(y) - h(x)|^{p-2}h(y) = \sum_{y \in N_x} |h(y) - h(x)|^{p-2}h(x),$$

we see that $h(x) = h(y)$ for all $y \in N_x$. Thus $h(x) = h(z)$ for all $z \in V$ since $G$ is connected.

4. Preliminary results

In this section we will give some results about $\partial_p(G)$ and $BD_p(G)$. Most of the results given in Propositions 4.2 through 4.8 are given in the first two sections of [Soardi 1994, Chapter VI] for the case of $p = 2$. However, our presentation and some of our proofs are different. Recall that $o$ is a fixed vertex of the graph $G$. 

Lemma 4.1. If \( x \in \partial_p(G) \) and \( (x_n) \) is a sequence in \( V \) that converges to \( x \), then \( d(o, x_n) \to \infty \) as \( n \to \infty \).

Proof. Let \( x \in \partial_p(G) \) and suppose \( (x_n) \to x \), where \( (x_n) \) is a sequence in \( V \). Let \( B \) be a positive real number. Define a function \( \chi_B \) on \( V \) by \( \chi_B(y) = 1 \) if \( d(o, y) \leq B \) and \( \chi_B(y) = 0 \) if \( d(o, y) > B \). Since \( \chi_B \) has finite support it is an element of \( \mathbb{R}G \). Suppose there exists a real number \( M \) such that \( d(o, x_n) \leq M \) for all \( n \). Then \( \chi_M(x) = \lim_{n \to \infty} \chi_M(x_n) = 1 \), a contradiction. Thus \( d(o, x_n) \to \infty \) as \( n \to \infty \). \( \square \)

We now characterize \( p \)-parabolic graphs in terms of \( \partial_p(G) \).

Proposition 4.2. Let \( G \) be a graph and let \( 1 < p \in \mathbb{R} \). Then \( \partial_p(G) = \emptyset \) if and only if \( G \) is \( p \)-parabolic.

Proof. Assume \( G \) is \( p \)-parabolic and suppose \( \partial_p(G) \neq \emptyset \). Let \( x \in \partial_p(G) \) and let \( (x_n) \) be a sequence in \( V \) that converges to \( x \). Then \( \chi(x) = \lim_{n \to \infty} \chi(x_n) = 1 \). By Theorem 2.2, we have \( 1_V \in B(\mathbb{R}G)_D \), which says that \( 1_V(x) = 0 \), a contradiction. Hence if \( G \) is \( p \)-parabolic, then \( \partial_p(G) = \emptyset \).

Now suppose that \( G \) is \( p \)-hyperbolic. Then \( \chi(x) \not\in B(\mathbb{R}G)_D \). Since \( B(\mathbb{R}G)_D \) is an ideal in the commutative ring \( \mathbb{R}G \), there exists a maximal ideal \( M \) in \( \mathbb{R}G \) containing \( B(\mathbb{R}G)_D \). Using the correspondence between maximal ideals in \( \mathbb{R}G \) and \( Sp(\mathbb{R}G)_D \), there is an \( x \in Sp(\mathbb{R}G)_D \) that satisfies \( ker(x) = M \). So \( \hat{f}(x) = f(x) = 0 \) for all \( f \in B(\mathbb{R}G)_D \). For each \( y \in V \), there exists an \( f \in \mathbb{R}G \) (in particular \( \delta_x \)) such that \( y(f) = f(y) \neq 0 \), which means that \( x \) cannot be in \( V \). Also, if \( x \in R_p(G) \setminus \partial_p(G) \), then there exists an \( f \in B(\mathbb{R}G)_D \) for which \( \hat{f}(x) \neq 0 \). This implies that \( B(\mathbb{R}G)_D \) is not contained in \( M \). Therefore \( x \in \partial_p(G) \). \( \square \)

For the rest of this paper, we will assume that \( 1_V \not\in B(\mathbb{R}G)_D \) unless otherwise stated, that is, we assume \( G \) is \( p \)-hyperbolic.

Let \( f \) and \( h \) be elements in \( BD_p(G) \) and let \( 1 < p \in \mathbb{R} \). Define

\[
\langle \Delta_p h, f \rangle := \sum_{x \in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-2}(h(y) - h(x))(f(y) - f(x)).
\]

This sum exists since \( \sum_{x \in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-2}(h(y) - h(x))^{q} = I_p(h, V) \) is finite, where \( 1/p + 1/q = 1 \). The next few lemmas will help show the uniqueness of the decomposition of \( BD_p(G) \) that will be given in Theorem 4.6.

Lemma 4.3. Let \( f_1 \) and \( f_2 \) be functions in \( D_p(G) \). Then \( \langle \Delta_p f_1 - \Delta_p f_2, f_1 - f_2 \rangle \) is zero if and only if \( f_1 - f_2 \) is constant on \( V \).

Proof. Let \( f_1, f_2 \in D_p(G) \) and assume there exists an \( x \in V \) with a \( y \in N_x \) such that \( f_1(x) - f_1(y) \neq f_2(x) - f_2(y) \). Define a function \( f : [0, 1] \to \mathbb{R} \) by

\[
f(t) = \sum_{x \in V} \sum_{y \in N_x} |f_1(y) - f_1(x) + t((f_2(y) - f_2(x)) - (f_1(y) - f_1(x)))|^p.
\]
Observe that \( f(0) = I(f_1, V) \) and \( f(1) = I(f_2, V) \). A derivative calculation gives
\[
f'(0) = p\langle \Delta_p f_1, f_2 - f_1 \rangle = -p\langle \Delta_p f_1, f_1 - f_2 \rangle.
\]
It follows from [Ekeland and Témam 1999, Proposition 5.4] that \( I_p(f_1, V) > I_p(f_1, V) - p\langle \Delta_p f_1, f_1 - f_2 \rangle \). Similarly, \( I_p(f_1, V) > I_p(f_2, V) - p\langle \Delta_p f_2, f_2 - f_1 \rangle \). Hence, \( p\langle \Delta_p f_1 - \Delta_p f_2, f_1 - f_2 \rangle > 0 \) if there exists an \( x \in V \) with \( y \in N_x \) that satisfies \( f_1(x) - f_1(y) \neq f_2(x) - f_2(y) \). Conversely, suppose \( f_1 - f_2 \) is constant on \( V \). We immediately see that \( \langle \Delta_p f_1 - \Delta_p f_2, f_1 - f_2 \rangle = 0 \).

**Lemma 4.4.** Let \( h \in BD_p(G) \). Then \( h \in BHD_p(G) \) if and only if \( \langle \Delta_p h, \delta_x \rangle = 0 \) for all \( x \in V \).

**Proof.** Let \( x \in V \) and let \( h \in BD_p(G) \). The lemma follows from
\[
\langle \Delta_p h, \delta_x \rangle = -2(\deg(x)) \sum_{y \in N_x} |h(x) - h(y)|^{p-2}(h(y) - h(x)).
\]
The lemma implies that if \( h \in BHD_p(G) \), then \( \langle \Delta_p h, f \rangle = 0 \) for all \( f \in \mathbb{R}G \).

**Lemma 4.5.** If \( h \in BHD_p(G) \) and \( f \in B(\overline{\ell^p(G)})D_p \), then \( \langle \Delta_p h, f \rangle = 0 \).

**Proof.** Let \( h \) and \( f \) be as stated. Then there exists a sequence \( (f_n) \) in \( \mathbb{R}G \) such that \( \|f - f_n\|_{D_p} \to 0 \) as \( n \to \infty \) since \( \overline{(\mathbb{R}G)}_{D_p} = (\overline{\ell^p(G)})_{D_p} \). Now
\[
0 \leq \|\langle \Delta_p h, f \rangle\| = \|\langle \Delta_p h, f - f_n \rangle\| = \left| \sum_{x \in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-2}(h(y) - h(x))(f - f_n)(x) - (f - f_n)(y) \right|
\leq \sum_{x \in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-1}|(f - f_n)(x) - (f - f_n)(y)|
\leq \left( \sum_{x \in V} \sum_{y \in N_x} (|h(y) - h(x)|^{p-1})^q \right)^{1/q} (I_p(f - f_n, V))^{1/p} \to 0
\]
as \( n \to \infty \). The last inequality follows from Hölder’s inequality. □

Clarkson’s inequality will be needed in the next proof. Let \( f_1 \) and \( f_2 \) be elements of \( D_p(G) \). If \( 2 \leq p \in \mathbb{R} \), then
\[
I_p(f_1 + f_2) + I_p(f_1 - f_2) \leq 2^{p-1}(I_p(f_1) + I_p(f_2))
\]
and if \( 1 < p \leq 2 \), then
\[
(I_p(f_1 + f_2))^{1/(p-1)} + (I_p(f_1 - f_2))^{1/(p-1)} \leq 2(I_p(f_1) + I_p(f_2))^{1/(p-1)}.
\]

The following decomposition of \( BD_p(G) \) will be crucial:

**Theorem 4.6.** Let \( 1 < p \in \mathbb{R} \) and suppose \( f \in BD_p(G) \). Then there exists a unique \( u \in B(\overline{\ell^p(G)})_{D_p} \) and a unique \( h \in BHD_p(G) \) such that \( f = u + h \).
Proof. Our assumption remains that \(1_V \notin B(\ell^p(G))_{D_P}\). Let \(f \in BD_P(G)\). Since \(f\) is bounded there exists real numbers \(a\) and \(b\) for which \(a \leq f(x) \leq b\) is satisfied by all \(x \in V\). Denote by \(h_n\) the function that is \(p\)-harmonic on \(B_n(o)\) and equal to \(f\) on \(V \setminus B_n(o)\). Because \(\min_{y \in \partial B_n(o)} f(y) \leq h_n(x) \leq \max_{y \in \partial B_n(o)} f(y)\) for all \(x \in B_n(o)\), we have \(a \leq h_n \leq b\) for each \(n \in \mathbb{N}\). Furthermore, if \(m > n\), then \(I_p(h_m) \leq I_p(h_n)\). Set \(r_n = I_p(h_n)\) and denote the limit of the bounded decreasing sequence \((r_n)\) by \(r\). We are still assuming that \(m > n\). By the minimizing property of \(p\)-harmonic functions, \(I_p(h_m, V) \leq I_p((h_n + h_m)/2, V)\) since \((h_n + h_m)/2 = h_m\) on \(V \setminus B_m(o)\). Using Clarkson’s inequality we obtain for \(2 \leq p \in \mathbb{R}\),

\[
r_m \leq I_p\left(\frac{1}{2}(h_n + h_m), V\right)
\leq I_p\left(\frac{1}{2}(h_n + h_m), V\right) + I_p\left(\frac{1}{2}(h_n - h_m), V\right)
\leq 2^{p-1}(I_p\left(\frac{1}{2}h_n, V\right) + I_p\left(\frac{1}{2}h_m, V\right))
= \frac{1}{2}(I_p(h_n, V) + I_p(h_m, V))
\]

and for \(1 < p \leq 2\),

\[
r_m^{1/(p-1)} \leq (I_p\left(\frac{1}{2}(h_n + h_m), V\right))^{1/(p-1)}
\leq (I_p\left(\frac{1}{2}(h_n + h_m), V\right))^{1/(p-1)} + (I_p\left(\frac{1}{2}(h_n - h_m), V\right))^{1/(p-1)}
\leq 2(I_p\left(\frac{1}{2}h_n, V\right) + I_p\left(\frac{1}{2}h_m, V\right))^{1/(p-1)}.
\]

Letting \(m, n \to \infty\), we have \(I_p\left(\frac{1}{2}(h_n + h_m), V\right) \to r\) and \(I_p\left(\frac{1}{2}(h_n - h_m), V\right) \to 0\). Also, \((|h_n(o)|)\) is a bounded sequence; thus \((h_n)\) is a Cauchy sequence in \(D_P(G)\). Set \(h\) equal to the limit function of the sequence \((h_n)\) in \(D_P(G)\). Because \((h_n)\) also converges pointwise to \(h\), the convergence property says that \(h\) is \(p\)-harmonic. Clearly, \(a \leq h \leq b\) on \(V\), so \(h \in BHD_p(G)\). Let \(u\) be the limit function in \(D_P(G)\) of the Cauchy sequence \((f - h_n)\). Since \(f - h_n \in \mathbb{R}G\) for each \(n\), we see that \(u \in B(\mathbb{R}G)\). Thus \(f = u + h\).

To show that this decomposition is unique, suppose \(f = u_1 + h_1 = u_2 + h_2\), where \(u_1, u_2 \in B(\ell^p(G))_{D_P}\) and \(h_1, h_2 \in BHD_p(G)\). Lemma 4.5 says that

\[
\langle \Delta_P h_1 - \Delta_P h_2, h_1 - h_2 \rangle = \langle \Delta_P h_1 - \Delta_P h_2, u_2 - u_1 \rangle = 0
\]

since \(u_1 - u_2 \in B(\ell^p(G))_{D_P}\). However, \(u_1 - u_2 = 0\) since \(1_V \notin B(\ell^p(G))_{D_P}\). \(\square\)

**Theorem 4.7** (maximum principle). *Let \(h\) be a nonconstant function in \(BHD_p(G)\) and suppose \(a\) and \(b\) are real numbers for which \(a \leq h \leq b\) on \(\partial_P(G)\). Then \(a < h < b\) on \(V\).*

**Proof.** Since \(\hat{h}\) is continuous on the compact space \(Sp(BD_p(G))\), there is a number \(c > 0\) such that \(b - \hat{h} \geq -c\) on \(Sp(BD_p(G))\). Let \(\epsilon > 0\) and set \(F_\epsilon\) to be the set of \(x \in Sp(BD_p(G))\) such that \(b - h + \epsilon \leq 0\). To prove the theorem, we will first show
that there exists an \( f \in B(\overline{RG})_{D_p} \) with \( \hat{f} = 1 \) on \( F_\epsilon \) and \( 0 \leq \hat{f} \leq 1 \) on \( \text{Sp}(BD_p(G)) \). This \( f \) will yield the inequality

\[
(4-1) \quad cf + b - h + \epsilon \geq 0 \quad \text{on} \quad \text{Sp}(BD_p(G)).
\]

We will then show that \( b - h + \epsilon \geq 0 \) on \( V \). Combining this with Lemma 3.1 and the assumption that \( h \) is nonconstant will give \( h < b \) on \( V \).

Observe that \( F_\epsilon \cap \partial_p(G) = \emptyset \) and \( F_\epsilon \) is a closed subset of \( \text{Sp}(BD_p(G)) \). For each \( x \in F_\epsilon \) there exists an \( f_x \in B(\overline{RG})_{D_p} \) for which \( \hat{f}_x(x) \neq 0 \). Since \( B(\overline{RG})_{D_p} \) is an ideal, we may assume that \( f_x \geq 0 \) on \( V \) and \( \hat{f}_x(x) > 0 \). Let \( U_x \) be a neighborhood of \( x \) in \( \text{Sp}(BD_p(G)) \) that satisfies \( f_x(y) > 0 \) for all \( y \in U_x \). By compactness there exists \( x_1, \ldots, x_n \) for which \( F_\epsilon \subseteq \bigcup_{j=1}^n U_{x_j} \). Set

\[
g = \sum_{j=1}^n f_{x_j} \quad \text{and} \quad \alpha = \inf \{ g(x) \mid x \in F_\epsilon \}.
\]

Clearly \( \alpha > 0 \) and \( g \in B(\overline{RG})_{D_p} \). Now define a function \( f \) on \( \text{Sp}(BD_p(G)) \) by \( f = \min(1, \alpha^{-1}g) \). Note that \( 0 \leq \hat{f} \leq 1 \) on \( \text{Sp}(BD_p(G)) \) and \( \hat{f} = 1 \) and \( F_\epsilon \). We still need to show that \( f \in B(\overline{RG})_{D_p} \). Let \( (g_n) \) be a sequence in \( RG \) that converges to \( g \) in \( D_p(G) \), so \( I_p((g - g_n), V) \to 0 \) as \( n \to \infty \). Set \( f_n = \min(1, \alpha^{-1}g_n) \). The sequence \( (f_n) \) converges pointwise to \( f \). Furthermore, by passing to a subsequence if necessary, \( (f_n) \) converges weakly to a function \( \tilde{f} \) in \( D_p(G) \) since \( I_p(f_n, V) \) is bounded. Clearly \( \tilde{f} \) is bounded, so \( \tilde{f} \in B(\overline{RG})_{D_p} \). It is also true \( (f_n) \) converges pointwise to \( \tilde{f} \) because point evaluations by elements of \( V \) are continuous linear functionals on \( BD_p(G) \). Hence, \( \tilde{f} = f \) and \( f \in BD_p(G) \). Inequality (4-1) is now established.

Next we will show that \( b - h + \epsilon \geq 0 \) on \( V \). Put \( v_\epsilon = cf + b - h + \epsilon \) and denote by \( h_n \) the unique function that is \( p \)-harmonic on \( B_n(o) \) and agrees with \( v_\epsilon \) on \( V \setminus B_n(o) \). We claim that \( h_n \geq 0 \) on \( B_n(o) \). Supposing otherwise, there exists an \( x \in B_n(o) \) for which \( h_n(x) < 0 \). Define a function \( h_n^{*} \) by

\[
h_n^{*} = \begin{cases} v_\epsilon & \text{if } x \in V \setminus B_n(o), \\ \max(h_n, 0) & \text{if } x \in B_n(o). \end{cases}
\]

Now \( I_p(h_n^{*}, B_n(o)) < I_p(h_n, B_n(o)) \), but this contradicts the minimizer property of \( p \)-harmonic functions. This proves the claim. By using the argument used in the proof of Theorem 4.6, we see that \( (h_n) \) converges to a bounded \( p \)-harmonic function \( \tilde{h} \) and that there exists a \( v \in B(\overline{RG})_{D_p} \) such that \( v_\epsilon = v + \tilde{h} \). Furthermore \( \tilde{h} \geq 0 \) on \( V \) because \( h_n \geq 0 \) for each \( n \). The uniqueness part of Theorem 4.6 says that \( v = cf \) and \( \tilde{h} = b - h + \epsilon \). Hence \( b \geq h - \epsilon \) on \( V \). Thus \( h < b \) on \( V \).

A similar argument shows that \( a < h \) on \( V \). Therefore, \( a < h < b \) on \( V \).

We now characterize the functions in \( BD_p(G) \) that vanish on \( \partial_p(G) \).
Theorem 4.8. Let \( f \in BD_p(G) \). Then \( f \in B(\overline{\ell^p(G)})_{D_p} \) if and only if \( \hat{f}(x) = 0 \) for all \( x \in \partial_p(G) \).

Proof. Since \( B(\overline{\ell^p(G)})_{D_p} = B(\overline{\mathbb{R}G})_{D_p} \) it follows immediately that \( \hat{f}(x) = 0 \) for all \( f \in B(\overline{\ell^p(G)})_{D_p} \) and all \( x \in \partial_p(G) \).

Conversely, suppose \( f \in BD_p(G) \) and \( \hat{f}(x) = 0 \) for all \( x \in \partial_p(G) \). Theorem 4.6 allows us to write \( f = u + h \), where \( u \in B(\overline{\ell^p(G)})_{D_p} \) and \( h \in BHD_p(G) \). Now \( \hat{h}(x) = 0 \) for all \( x \in \partial_p(G) \) since \( \hat{u}(x) = 0 \). Therefore, \( h = 0 \) by the maximum principle.

\[ \square \]

Corollary 4.9. Every function in \( BHD_p(G) \) is uniquely determined by its values on \( \partial_p(G) \).

Proof. Let \( h_1 \) and \( h_2 \) be elements of \( BHD_p(G) \) with \( \hat{h}_1(x) = \hat{h}_2(x) \) for all \( x \in \partial_p(G) \). Then \( h_1 - h_2 \in B(\overline{\ell^p(G)})_{D_p} \). Let \( (f_n) \) be a sequence in \( \ell^p(G) \) that converges to \( h_1 - h_2 \). Using Lemma 4.5, we obtain

\[ \langle \Delta_p h_1 - \Delta_p h_2, h_1 - h_2 \rangle = \lim_{n \to \infty} \langle \Delta_p h_1 - \Delta_p h_2, f_n \rangle = 0. \]

It now follows from Lemma 4.3 that \( h_1 - h_2 = 0 \).

We can now characterize when \( BHD_p(G) \) is precisely the constant functions.

Theorem 4.10. Let \( 1 < p \in \mathbb{R} \). Then \( BHD_p(G) \neq \mathbb{R} \) if and only if \( \#(\partial_p(G)) > 1 \).

Proof. Suppose that \( \#(\partial_p(G)) = 1 \) and that \( x \in \partial_p(G) \). Let \( h \in BHD_p(G) \). Then \( \hat{h}(x) = c \) for some constant \( c \). It follows from Corollary 4.9 that the function \( h(x) = c \) for all \( x \in V \) is the only function in \( BHD_p(G) \) with \( \hat{h}(x) = c \). Hence \( BHD_p(G) = \mathbb{R} \).

Conversely, suppose \( \#(\partial_p(G)) > 1 \). Let \( x, y \in \partial_p(G) \) such that \( x \neq y \) and pick an \( f \in BD_p(G) \) that satisfies \( x(f) \neq y(f) \). By Theorem 4.8, \( f \notin B(\overline{\ell^p(G)})_{D_p} \). It now follows from Theorem 4.6 and Theorem 4.8 that there exists an \( h \in BHD_p(G) \) with \( \hat{h}(z) = \hat{f}(z) \) for all \( z \in \partial_p(G) \). Since \( V \) is dense in \( \text{Sp}(BD_p(G)) \), there exist sequences \( (x_n) \) and \( (y_n) \) in \( V \) such that \( (x_n)(h) \to x(h) \) and \( (y_n)(h) \to y(h) \). Hence \( \lim_{n \to \infty} h(x_n) = x(h) \neq y(h) = \lim_{n \to \infty} h(y_n) \). Hence \( h \) is not constant on \( V \).

We now define the important concept of a \( D_p \)-massive subset of a graph. An infinite connected subset \( U \) of \( V \) with \( \partial U \neq \emptyset \) is called a \( D_p \)-massive subset if there exists a nonnegative function \( u \in BD_p(G) \) such that

(a) \( \Delta_p u(x) = 0 \) for all \( x \in U \),

(b) \( u(x) = 0 \) for \( x \in \partial U \), and

(c) \( \sup_{x \in U} u(x) = 1 \).

We call any \( u \) that satisfies these conditions an inner potential of the \( D_p \)-massive subset \( U \). The following will be needed in the proof of Lemma 5.1.
Hence one point of $\partial_p(G)$.

Proof. We will write $\overline{U}$ for $i(U)$, where the closure is taken in $\text{Sp}(\text{BD}_p(G))$. Assume $\overline{U} \cap \partial_p(G) = \emptyset$ and let $u$ be an inner potential for $U$. We may and do assume that $u = 0$ on $V \setminus U$. By the existence property for $p$-harmonic functions, there exists a $p$-harmonic function $h_n$ on $B_n(o)$ such that $h_n = u$ on $\partial B_n(o)$ for each natural number $n$. Also $0 \leq \min_{y \in \partial B_n(o)} u(y) \leq h_n \leq \max_{y \in \partial B_n(o)} u(y) \leq 1$ on $B_n(o)$. Extend $h_n$ to all of $V$ by setting $h_n = u$ on $V \setminus B_n(o)$. By the minimizing property of $p$-harmonic functions, $I_p(h_n, B_n(o)) \leq I_p(u, B_n(o))$, and so $I_p(h_n, V) \leq I_p(u, V)$. Both $h_n$ and $u$ are $p$-harmonic on $U \cap B_n(o)$, and we have $u(x) \leq h_n(x)$ for all $x \in \partial(U \cap B_n(o))$. The comparison principle says that $u \leq h_n$ on $U \cap B_n(o)$. On $B_n(o) \setminus U$ we have $u = 0$, so $u \leq h_n \leq 1$ for each $n$. By taking a subsequence if needed, we assume that $(h_n)$ converges pointwise to a function $h$. Now $u \leq h \leq 1$ on $V$, so $\sup_{x \in U} h(x) = 1$. By the convergence property for $p$-harmonic functions, $h$ is $p$-harmonic and $h \in \text{BHD}_p(G)$ since $I_p(h_n, V) \leq I_p(u, V) < \infty$ for all $n$.

Let $x \in \partial_p(G)$. Since $u - h_n = 0$ on $V \setminus B_n(o)$, we see that $\hat{u}(x) - \hat{h}_n(x) = 0$ for all $n$; thus $\hat{u} - \hat{h} = 0$ on $\partial_p(G)$. According to Theorem 4.8, $u - h \in B(\overline{\ell^p(G)})_{\text{BD}_p}$. Hence $u = f + h$, where $f \in B(\overline{\ell^p(G)})_{\text{BD}_p}$. Another appeal to Theorem 4.8 shows that $\hat{u} = \hat{h}$ on $\partial_p(G)$. If $x \in \partial_p(G)$, then $\hat{u}(x) = 0$ because if $(x_n)$ is a sequence in $V$ converging to $x$, then $u(x_n) = 0$ for all but a finite number of $n$ since we are assuming $\overline{U} \cap \partial_p(G) = \emptyset$. So $\hat{h}(x) = 0$ for all $x \in \partial_p(G)$. Hence $\hat{h} = 0$ on $V$ by the maximum principle, which contradicts $\sup_U h = 1$. Therefore, if $U$ is a $D_p$-massive subset of $V$, then $\overline{U}$ contains at least one point of $\partial_p(G)$. 

It would be nice to know if the converse of Proposition 4.11 is true. That is, if $x \in \partial_p(G)$, does there exist a $D_p$-massive subset $U$ of $V$ such that $x \in \overline{U}$? The next result leads to a partial converse and also describes a base of neighborhoods for open sets in $\partial_p(G)$.

Proposition 4.12. Let $x \in \partial_p(G)$ and let $O$ be an open set in $\partial_p(G)$ containing $x$. Then there exists a subset $U$ of $V$ such that

(a) $U = \bigcup_{\alpha \in I} A_{\alpha}$, where each $A_{\alpha}$ is a $D_p$-massive subset of $V$ and $I$ is an index set, and $A_{\alpha} \cap A_{\beta} = \emptyset$ if $\alpha \neq \beta$, and

(b) $x \in \overline{U} \cap \partial_p(G) \subseteq O$.

Proof. Let $x \in \partial_p(G)$, and let $O$ be an open set of $\partial_p(G)$ containing $x$. By Urysohn’s lemma there exists an $f \in C(\text{Sp}(\text{BD}_p(G)))$ with $0 \leq f \leq 1$, $f(x) = 1$ and $f = 0$ on $\partial_p(G) \setminus O$. Since the Gelfand transform of $\text{BD}_p(G)$ is dense in $C(\text{Sp}(\text{BD}_p(G)))$ with respect to the supremum norm, we will assume $f \in \text{BD}_p(G)$. By Theorem 4.6 we have the decomposition $f = w + h$, where $w \in B(\overline{\ell^p(G)})_{\text{BD}_p}$ and $h \in \text{BHD}_p(G)$. Since $\hat{w} = 0$ on $\partial_p(G)$, it follows that $\hat{h}(x) = 1$ and $\hat{h} = 0$.
on \( \partial_p(G) \setminus O \). Also, \( 0 \leq \hat{h} \leq 1 \) on \( \partial_p(G) \), so \( 0 < h < 1 \) on \( V \) by the maximum principle and \( 0 \leq \hat{h} \leq 1 \) on \( \text{Sp}(BD_p(G)) \) due to the density of \( V \). Fix \( \epsilon \) with \( 0 < \epsilon < 1 \) and set \( U = \{ x \in V \mid h(x) > \epsilon \} \). Let \( A \) be a component of \( U \). It now follows from the comparison principle that \( A \) is infinite. Define a function \( v \) on \( V \) by \( v = (h - \epsilon)/(1 - \epsilon) \). There exists a \( p \)-harmonic function \( u_n \) on \( B_n(o) \cap A \) taking the values \( \max\{0, v\} \) on \( V \setminus (B_n(o) \cap A) \) and such that \( 0 \leq u_n \leq 1 \) on \( B_n(o) \cap A \). By passing to a subsequence if necessary, we may assume that \( (u_n) \) converges pointwise to a function \( u \). By the convergence property, \( u \) is \( p \)-harmonic on \( A \). Also \( v \leq u_n \leq 1 \) on \( B_n(o) \), so by replacing \( u \) by a suitable scalar multiple if necessary, we have \( \sup_{a \in A} u(a) = 1 \). Also, \( u = 0 \) on \( \partial A \) because \( h \leq \epsilon \) on \( \partial A \). Since \( h \in BD_p(G) \), it follows that \( u \in BD_p(G) \). Thus \( A \) is a \( D_p \)-massive subset with inner potential \( u \). Hence, each component of \( U \) is a \( D_p \)-massive subset in \( V \). So \( U = \bigcup_{a \in A} A_a \), where each \( A_a \) is \( D_p \)-massive. The proof of part (a) is complete.

Clearly \( x \in \overline{U} \). We will show that \( \overline{U} \cap \partial_p(G) \subseteq O \). Let \( y \in \overline{U} \cap \partial_p(G) \) and let \( (y_k) \) be a sequence in \( U \) that converges to \( y \). Then \( f(y) = \hat{h}(y) = \lim_{k \to \infty} h(y_k) \geq \epsilon \). Hence \( y \in O \) since \( f = 0 \) on \( \partial_p(G) \setminus O \).

The following partial converse to Proposition 4.11 is a direct consequence of Proposition 4.12.

**Corollary 4.13.** If \( \#(\partial_p(G)) \) is finite, then for each \( x \in \partial_p(G) \) there exists a \( D_p \)-massive subset \( U \) of \( V \) such that \( x \in \overline{U} \).

## 5. Proofs of Theorem 2.4 and Theorem 2.6

The key ingredient in the proof of Theorem 2.4 is the following.

**Lemma 5.1.** Let \( 1 < p \in \mathbb{R} \) and suppose that \( G \) is a \( p \)-parabolic graph. If \( f \) is a nonconstant function in \( BHD_p(G) \), then \( \sup_V f > \lim \sup_{d(o,x) \to \infty} f \).

**Proof.** Suppose that \( \lim \sup_{d(o,x) \to \infty} f(x) = \sup_V f = M \). Since \( f \) is nonconstant, there exists an \( \epsilon > 0 \) such that the set \( W = \{ x \in V \mid f(x) > M - \epsilon \} \) is a proper infinite subset of \( V \). Let \( U \) be a component of \( W \). If \( U \) is finite, then we can construct a unique \( p \)-harmonic function \( w \) on \( U \) that agrees with \( f \) on \( \partial U \). Since \( f \) is \( p \)-harmonic, \( f = w \) on \( U \) by uniqueness. But if \( x \in U \), then

\[
 w(x) \leq \max_{y \in \partial U} f(y) \leq M - \epsilon < f(x),
\]

a contradiction. Thus \( U \) is infinite. Now set \( h = (f - M + \epsilon)/\epsilon \). There is an number \( N \in \mathbb{N} \) such that \( B_n(o) \cap U \neq \emptyset \) for \( n > N \). For \( n > N \), let \( u_n \) be a \( p \)-harmonic function on \( B_n(o) \cap U \) that takes the values \( \max\{0, h\} \) on \( V \setminus (B_n(o) \cap U) \). Note that \( u_n \geq 0 \). Since \( h \) is \( p \)-harmonic on \( B_n(o) \cap U \), it follows from the comparison principle that \( h \leq u_n \leq 1 \) on \( B_n(o) \cap U \). By taking a subsequence if necessary, we may assume that the sequence \( (u_n) \) converges pointwise to a function \( u \). By
the convergence property, \( u \) is \( p \)-harmonic on \( U \). If \( x \in \partial U \), then \( f(x) \leq M - \epsilon \). Therefore, \( u_n(x) = 0 \) for all \( n \), which implies \( u(x) = 0 \). Thus \( u = 0 \) on \( \partial U \). Since \( \sup_U h = 1 \), we see that \( \sup_U u = 1 \). We can show using the minimizing property for \( p \)-harmonic functions that \( I_p(u_n, U \cap B_n(o)) \leq I_p(\max\{0, h\}, U \cap B_n(o)) \), and it follows from this inequality that \( I_p(u_n, U) \leq I_p(h, U) \). Hence \( I_p(u, U) < \infty \) because \( I_p(h, V) < \infty \). Thus \( U \) is a \( D_p \)-massive subset of \( V \).

By Proposition 4.11, we have \( \overline{U} \cap \partial_p(G) \neq \emptyset \), which contradicts Proposition 4.2 since we are assuming \( G \) is \( p \)-parabolic. Hence \( \sup_V f > \lim sup_{d(o,x) \to \infty} f \). \( \square \)

**Proof of Theorem 2.4.** Let \( h \in \text{BHD}_p(G) \) and suppose that \( h \) is nonconstant. Since \( h \) is bounded, \( \sup_V h = B < \infty \). Lemma 5.1 says that there exists an \( x \in V \) such that \( h(x) = B \). By the maximum principle, \( h \) is constant on \( V \), a contradiction. Hence \( \text{BHD}_p(G) \) consists of only the constant functions. Therefore, \( \text{HD}_p(G) \) is precisely the constant functions by [Holopainen and Soardi 1997a, Lemma 4.4]. \( \square \)

**Proof of Theorem 2.6.** Let \( f \) be a continuous function on \( \partial_p(G) \). By Tietze’s extension theorem, there exists a continuous extension of \( f \), which we also denote by \( f \), to all of \( \text{Sp}(<BD_p(G))) \). Let \( (f_n) \) be a sequence in \( \text{BD}_p(G) \) converging to \( f \) in the supremum norm. For each \( n \in \mathbb{N} \) and each \( r \in \mathbb{N} \), let \( h_{n,r} \) be a function on \( V \) that is \( p \)-harmonic on \( B_r(o) \) and takes the values \( f_n \) on \( V \setminus B_r(o) \). The function \( h_{n,r} \in \text{BD}_p(G) \) since \( B_r(o) \) is finite, and \( |h_{n,r}| \leq \sup_V |f_n| \) because

\[
\min_{y \in \partial B_r(o)} f_n(y) \leq h_{n,r} \leq \max_{y \in \partial B_r(o)} f_n(y) \quad \text{on} \quad B_r(o).
\]

By the Ascoli–Arzela theorem, there exists a subsequence of \( (h_{n,r}) \), which we also denote by \( (h_{n,r}) \), that converges uniformly on all finite subsets of \( V \) to a function \( h \) as \( r \) goes to infinity. The function \( h_n \) is \( p \)-harmonic on \( V \) by the convergence property. For each \( r \), the minimizing property of \( p \)-harmonic functions gives \( I_p(h_{n,r}, B_r(o)) \leq I_p(f_n, B_r(o)) \), so \( I_p(h_{n,r}, V) \leq I_p(f_n, V) \), which implies \( h_n \in \text{BHD}_p(G) \).

Let \( \epsilon > 0 \). Since \( (f_n) \to f \) in the supremum norm, there exists a number \( N \) such that \( \sup_V |f_n - f_m| < \epsilon \) for \( n, m \geq N \). It follows that \( \sup_{\partial B_r(o)} |h_{n,r} - h_{m,r}| < \epsilon \) for all \( r \in \mathbb{N} \) because \( f_n = h_{n,r} \) on \( V \setminus B_r(o) \). Both \( h_{n,r} \) and \( h_{m,r} + \epsilon \) are \( p \)-harmonic on \( B_r(o) \) and \( h_{m,r} - \epsilon \leq h_{n,r} \leq h_{m,r} + \epsilon \) on \( \partial B_r(o) \), so by applying the comparison principle, we obtain \( \sup_{B_r(o)} |h_{n,r} - h_{m,r}| < \epsilon \) for all \( r \). It now follows that \( \sup_{B_r(o)} |h_n - h_m| < 3\epsilon \) for all \( r \). Thus \( \sup_V |h_n - h_m| \leq 3\epsilon \). Hence, the Cauchy sequence \( (h_n) \) converges uniformly on finite subsets of \( V \) to a function \( h \), which is \( p \)-harmonic by the convergence property.

Let \( \epsilon > 0 \). There exists an \( N \in \mathbb{N} \) such that \( \sup_V |f_n - f| < \epsilon \) and \( \sup_V |h_n - h| < \epsilon \) if \( n \geq N \). Let \( x \in \partial_p(G) \). Since \( f_n(x) = h_n(x) \), there exists a neighborhood \( U \) of \( x \) such that \( |h_n(y) - f_n(x)| < \epsilon \) for all \( y \in U \). Therefore, \( \lim_{k \to \infty} h(x_k) = f(x) \), where \( (x_k) \) is a sequence in \( V \) that converges to \( x \). \( \square \)
6. Proofs of Theorem 2.8 and Theorem 2.9

Let \( G \) and \( H \) be graphs with vertex sets \( V_G \) and \( V_H \), respectively. Fix a vertex \( o_G \) in \( G \) and a vertex \( o_H \) in \( H \). Let \( \phi : G \to H \) be a rough isometry, and let \( \phi^* \) denote the map from \( \ell^\infty(H) \) to \( \ell^\infty(G) \) given by \( \phi^* f(x) = f(\phi(x)) \). We start by defining a map \( \tilde{\phi} : \partial_p(G) \to \partial_p(H) \). Let \( x \in \partial_p(G) \). Then there exists a sequence \( (x_n) \) in \( V_G \) such that \( (x_n) \to x \). Now \( (\phi(x_n)) \) is a sequence in the compact Hausdorff space \( Sp(BD_p(H)) \). By passing to a subsequence, if necessary we may assume that \( (\phi(x_n)) \) converges to a unique limit \( y \in Sp(BD_p(H)) \). Now define \( \tilde{\phi}(x) = y \).

Before we show that \( y \in \partial_p(H) \) and \( \tilde{\phi} \) is well defined, we need a lemma.

**Lemma 6.1.** Let \( G \) and \( H \) be graphs. If \( \phi : G \to H \) is a rough isometry, then

(a) \( \phi^* \) maps \( BD_p(H) \) to \( BD_p(G) \),

(b) \( \phi^* \) maps \( \ell^p(H) \) to \( \ell^p(G) \), and

(c) \( \phi^* \) maps \( B(\ell^p(H))_{D_p} \) to \( B(\ell^p(G))_{D_p} \).

**Proof.** We will only prove part (a) since the proofs of parts (b) and (c) are similar. Let \( f \in BD_p(H) \). We will now show that \( \phi^* f \in BD_p(G) \). Let \( x \in V_G \) and \( w \in N_x \), so \( x \) and \( w \) are neighbors in \( G \) but \( \phi(w) \) and \( \phi(x) \) are not necessarily neighbors in \( H \). However, by the definition of rough isometry there exists constants \( a \geq 1 \) and \( b \geq 0 \) such that \( d_H(\phi(w), \phi(x)) \leq a + b \). Set \( h_1 = \phi(x) \) and \( h_l = \phi(w) \), and let \( h_1, \ldots, h_l \) be a path in \( H \) with length at most \( a + b \). Thus

\[
|\phi^* f(w) - \phi^* f(x)|^p = |f(\phi(w)) - f(\phi(x))|^p \\
\leq |a + b|^{p-1} \sum_{j=1}^{l-1} |f(h_{j+1}) - f(h_j)|^p.
\]

The inequality follows from Jensen’s inequality applied to the function \( x^p \) for \( x > 0 \).

Let \( y \in V_H \) and \( z \in N_y \). We claim that there is at most a finite number of paths in \( H \) of length at most \( a + b \) that contain the edge \( y, z \) and have the endpoints \( \phi(x) \) and \( \phi(w) \). To see this, let \( U \) be the set of all elements in \( V_G \) such that the four distances \( d_H(\phi(x), y) \), \( d_H(\phi(x), z) \), \( d_H(\phi(w), y) \) and \( d_H(\phi(w), z) \) are all at most \( a + b \). Let \( x, x' \in U \). By the triangle inequality, \( d_H(\phi(x'), y) + d_H(\phi(x), y) \). It now follows from the definition of rough isometry that \( d_G(x', x) \leq 2a^2 + 3ab \). Thus the metric ball \( B(x, 2a^2 + 3ab + 1) \) contains \( U \) as a subset. Hence the cardinality of \( U \) is bounded above by some constant \( k \), which is independent of \( y \) and \( z \). Since \( f \in BD_p(H) \) it follows from (6.1) that

\[
\sum_{x \in V_G} \sum_{w \in N_x} |\phi^* f(w) - \phi^* f(x)|^p \leq |a + b|^{p-1} \sum_{y \in V_H} \sum_{z \in N_y} |f(z) - f(y)|^p < \infty. \quad \square
\]

**Proposition 6.2.** The map \( \tilde{\phi} \) is well defined from \( \partial_p(G) \) to \( \partial_p(H) \).
Proof. Let \(x, y\) and \((x_n)\) be as above. We first show that \(y \in \partial_p(H)\). Lemma 4.1 tells us that \(d_G(o_G, x_n) \to \infty\) as \(n \to \infty\). The element \(\phi(o_G)\) is fixed in \(H\), so it follows from the definition of rough isometry that \(d_H(\phi(o_G), \phi(x_n)) \to \infty\) as \(n \to \infty\). Thus \(y \in \text{Sp}(BD_p(H)) \setminus H\) since \(y = \lim_{n \to \infty} \phi(x_n) \notin H\). Let \(f \in B(\ell^p(H))_{D_p}\) and suppose \(f(y) \neq 0\). Then \(0 \neq \lim_{n \to \infty} f(\phi(x_n)) = \phi^* f(x)\). By Lemma 6.1(c), \(\phi^* f \in B(\ell^p(G))_{D_p}\) and Theorem 4.8 says that \(\phi^* f(x) = 0\), a contradiction. Hence \(\tilde{f}(y) = 0\) for all \(f \in B(\ell^p(H))_{D_p}\), so \(y \in \partial_p(H)\).

We will now show that \(\tilde{\phi}\) is well-defined. Let \((x_n)\) and \((x'_n)\) be sequences in \(V_G\) that both converge to \(x \in \partial_p(G)\). Now suppose that \(\phi(x_n)\) converges to \(y_1\) and \(\phi(x'_n)\) converges to \(y_2\) in \(\text{Sp}(BD_p(H))\). Assume that \(y_1 \neq y_2\) and let \(f \in BD_p(H)\) such that \(f(y_1) \neq f(y_2)\). By Lemma 6.1(a), we have \(\phi^* f \in BD_p(G)\). Thus

\[
\lim_{n \to \infty} \phi^* f(x_n) = \phi^* f(x) = \lim_{n \to \infty} \phi^* f(x'_n),
\]

which implies \(f(y_1) = f(y_2)\), a contradiction. Hence \(\tilde{\phi}\) is a well-defined map from \(\partial_p(G)\) to \(\partial_p(H)\).

The next lemma will be used to show that \(\tilde{\phi}\) is one-to-one and onto.

**Lemma 6.3.** Let \(\phi : G \to H\) be a rough isometry and let \(\psi\) be a rough inverse of \(\phi\). If \(f \in D_p(G)\), then \(\lim_{d_G(o_G, x) \to \infty} |f((\psi \circ \phi)(x)) - f(x)| = 0\).

**Proof.** Let \(x \in V_G\). Since \(\psi\) is a rough inverse of \(\phi\), there are nonnegative constants \(a, b, c, d\) with \(a \geq 1\) such that \(d_G((\psi \circ \phi)(x), x) \leq a(c + b)\). Let \(x_1, x_2, \ldots, x_n\) be a path in \(V_G\) of length not more than \(a(c + b)\) with \(x_1 = x\) and \(x_n = (\psi \circ \phi)(x)\). So

\[
|f((\psi \circ \phi)(x)) - f(x)|^p = \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)|^p \leq n^{p-1} \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)|^p.
\]

The last sum approaches zero as \(d_G(o_G, x) \to \infty\) since \(f \in D_p(G)\) and \(n \leq a(c + b)\). Thus \(\lim_{d_G(o_G, x) \to \infty} |f((\psi \circ \phi)(x)) - f(x)| = 0\). \(\square\)

**Proposition 6.4.** The function \(\tilde{\phi}\) is a bijection.

**Proof.** Let \(x_1, x_2 \in \partial_p(G)\) with \(x_1 \neq x_2\), and let \(f \in BD_p(G)\) with \(f(x_1) \neq f(x_2)\). There exists sequences \((x_n)\) and \((x'_n)\) in \(V_G\) such that \((x_n) \to x_1\) and \((x'_n) \to x_2\). Assume that

\[
\tilde{\phi}(x_1) = \lim_{n \to \infty} (\phi(x_n)) = \lim_{n \to \infty} (\phi(x'_n)) = \tilde{\phi}(x_2),
\]

so \(\lim_{n \to \infty} f((\psi \circ \phi)(x_n)) = \lim_{n \to \infty} f((\psi \circ \phi)(x'_n))\). It follows from Lemma 6.3 that \(\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(x'_n)\); thus \(f(x_1) = f(x_2)\), a contradiction. Hence \(\tilde{\phi}\) is one-to-one.

We now show that \(\tilde{\phi}\) is onto. Let \(y \in \partial_p(H)\) and let \((y_n)\) be a sequence in \(V_H\) that converges to \(y\). By passing to a subsequence if necessary, we can assume that there is a unique \(x\) in the compact Hausdorff space \(\text{Sp}(BD_p(G))\) such that \((\psi(y_n)) \to x\).
Since \( \lim_{n \to \infty} d_H(o_H, y_n) \to \infty \), we have \( \lim_{n \to \infty} d_G(o_G, \psi(y_n)) \to \infty \), so \( x \not\in G \). Using an argument similar to the first paragraph in the proof of Proposition 6.2, we obtain \( x \in \partial_p(G) \). The proof will be complete once we show that \( \tilde{\phi}(x) = y \). Let \( f \in \text{BD}_p(H) \). By Lemma 6.3, we see that \( \lim_{n \to \infty} | f((\phi \circ \psi)(y_n)) - f(y_n) | = 0 \). Thus \( f(\tilde{\phi}(x)) = f(y) \) for all \( f \in \text{BD}_p(H) \). Hence \( \tilde{\phi}(x) = y \). \( \square \)

We finally show that the bijection \( \tilde{\phi} \) is also a homeomorphism. We only need to show that \( \tilde{\phi} \) is continuous, since both \( \text{Sp}(\text{BD}_p(G)) \) and \( \text{Sp}(\text{BD}_p(H)) \) are compact Hausdorff spaces. Let \( W \) be an open set in \( \partial_p(H) \) and let \( x \in \tilde{\phi}^{-1}(W) \). Choose \( y \in W \) so that \( x = \tilde{\phi}^{-1}(y) \). By Proposition 4.12, there exists a subset \( U \) of \( V_H \) such that \( y \in \overline{U} \) and \( \overline{U} \cap \partial_p(H) \subseteq W \). We saw in the proof of Proposition 4.12 that there is an \( h \in \text{BHD}_p(H) \) for which \( \hat{h}(y) = 1 \) and \( \hat{h} = 0 \) on \( \partial_p(H) \setminus W \) and \( \hat{h} \geq \epsilon \) on \( \overline{U} \), where \( 0 < \epsilon < 1 \). By Lemma 6.1(a), we have \( \phi^*h = h \circ \phi \in \text{BD}_p(G) \). Combining Theorems 4.6 and 4.8, we have an \( \tilde{h} \in \text{BHD}_p(G) \) that satisfies \( \hat{h} = \tilde{h} \circ \tilde{\phi} \) on \( \partial_p(G) \). Let \( O = \{ x' \in \partial_p(G) \mid \hat{h}(x') \geq \epsilon \} \). Now \( O \) is an open set containing \( x \) since \( \tilde{h} \) is continuous on \( \partial_p(G) \) and \( \tilde{h}(x) = 1 \). For \( z \in O \), we see that \( \hat{h}(\tilde{\phi}(z)) = \tilde{h}(z) \geq \epsilon \), thus \( \tilde{\phi}(z) \in W \) for all \( z \in O \). Thus \( O \subseteq \tilde{\phi}^{-1}(W) \). Since our choice of \( x \) was arbitrary, \( \tilde{\phi}^{-1}(W) \) is open and consequently \( \tilde{\phi} \) is continuous. The proof that \( \tilde{\phi} \) is a homeomorphism is complete.

We now prove Theorem 2.9. Let \( \phi \) be a rough isometry from \( G \) to \( H \), and let \( \psi \) be a rough inverse of \( \phi \). Let \( h \in \text{BHD}_p(G) \). By Lemma 6.1(a), \( h \circ \psi \in \text{BD}_p(H) \). Let \( \pi(h \circ \psi) \) be the unique element in \( \text{BHD}_p(H) \) given by Theorem 4.6. We now define a map \( \Phi : \text{BHD}_p(G) \mapsto \text{BHD}_p(H) \) by \( \Phi(h) = \pi(h \circ \psi) \). Theorem 4.8 implies that \( \pi(h \circ \psi)(\tilde{\phi}(x)) = (h \circ \psi)(\tilde{\phi}(x)) \) for all \( x \in \partial_p(G) \), where \( \tilde{\phi} \) is the homeomorphism from \( \partial_p(G) \) to \( \partial_p(H) \) defined earlier in this section. Thus \( \Phi(h)(\tilde{\phi}(x)) = (h \circ \psi)(\tilde{\phi}(x)) = h(x) \) for all \( x \in \partial_p(G) \). We can now show that \( \Phi \) is one-to-one. Let \( h_1, h_2 \in \text{BHD}_p(G) \) and suppose that \( \Phi(h_1) = \Phi(h_2) \). So \( \Phi(h_1)(\tilde{\phi}(x)) = \Phi(h_2)(\tilde{\phi}(x)) \) for all \( x \in \partial_p(G) \), which implies \( h_1(x) = h_2(x) \) for all \( x \in \partial_p(G) \). Hence, \( h_1 = h_2 \) by Corollary 4.9. Thus \( \Phi \) is one-to-one.

We will now show that \( \Phi \) is onto. Let \( f \in \text{BD}_p(H) \). Then \( f \circ \phi \in \text{BD}_p(G) \). Let \( h = \pi(f \circ \phi) \), where \( \pi(f \circ \phi) \) is the unique element in \( \text{BHD}_p(G) \) given by Theorem 4.6. Let \( y \in \partial_p(H) \). Since \( h(x) = (f \circ \phi)(x) \) for all \( x \in \partial_p(G) \) and \( \overline{\psi \circ \phi} \) equals the identity on \( \partial_p(G) \), we see that \( (\Phi(h))(y) = \pi(h \circ \psi)(y) = (h \circ \psi)(y) = f((\phi \circ \psi)(y)) = f(y) \). Thus \( \Phi \) is onto and the proof of Theorem 2.9 is complete.

The map \( \Phi \) is an isomorphism in the case \( p = 2 \) since \( \text{BHD}_2(G) \) and \( \text{BHD}_2(H) \) are linear spaces. However, in general these spaces are not linear if \( p \neq 2 \).

### 7. The first reduced \( \ell^p \)-cohomology of \( \Gamma \)

In the final two sections, \( \Gamma \) will denote a finitely generated group with generating set \( S \). So for a real-valued function \( f \) on \( \Gamma \) the \( p \)-th power of the gradient and the
\( p \)-Laplacian of \( x \in \Gamma \) are

\[
|Df(x)|^p = \sum_{s \in S} |f(xs^{-1}) - f(x)|^p,
\]

\[
\Delta_p f(x) = \sum_{s \in S} |f(xs^{-1}) - f(x)|^{p-2}(f(xs^{-1}) - f(x)).
\]

If \( f \in D_p(\Gamma) \), then \( (\|f\|_{D_p} = I_p(f, \Gamma) + |f(e)|^p)^{1/p} \), where \( e \) is the identity element of \( \Gamma \). Also \( \ell^p(\Gamma) \) is the set that consists of real-valued functions on \( \Gamma \) for which \( \sum_{x \in \Gamma} |f(x)|^p \) is finite. The first reduced \( \ell^p \)-cohomology space of \( \Gamma \) is defined by

\[
H^1_{(p)}(\Gamma) = D_p(\Gamma)/(\ell^p(\Gamma) \oplus \mathbb{R})_{D_p}.
\]

We now prove Theorem 2.10. Suppose \( \partial_p(\Gamma) = \emptyset \). By Proposition 4.2, there exists a sequence \((f_n) \in \mathbb{R}\Gamma \) that satisfies \( \|f_n - 1_\Gamma\|_{D_p} \to 0 \). It follows that \( I_p(f_n, \Gamma) \to 0 \) and \( (f_n(e)) \to 0 \). Thus \( H^1_{(p)}(\Gamma) = 0 \) by [Puls 2003, Theorem 3.2]. We now assume \( \partial_p(G) \neq \emptyset \). It was shown in [Puls 2006, Theorem 3.5] that \( H^1_{(p)}(\Gamma) \neq 0 \) if and only if \( \text{HD}_{p}(\Gamma) \neq \mathbb{R} \). Since \#(S) < \infty, [Holopainen and Soardi 1997a, Lemma 4.4] says that \( \text{BHD}_{p}(\Gamma) = \mathbb{R} \) if and only if \( \text{HD}_{p}(\Gamma) = \mathbb{R} \). Theorem 2.10 now follows from Theorem 4.10.

We now use Theorem 2.10 to compute \( \partial_p(\Gamma) \) and \( R_p(\Gamma) \) for some special cases of \( \Gamma \). By [Holopainen and Soardi 1997b, Corollary 1.10], \( \text{BHD}_{p}(\Gamma) = \mathbb{R} \) when \( \Gamma \) has polynomial growth and \( 1 < p \in \mathbb{R} \). Thus, if \( \Gamma \) has polynomial growth, then \( H^1_{(p)}(\Gamma) = 0 \) and \( \partial_p(\Gamma) \) is either the empty set or contains exactly one element. It would be nice to know when a group with polynomial growth is \( p \)-parabolic or \( p \)-hyperbolic. This has been worked out for the case \( \Gamma = \mathbb{Z}^n \), where \( n \) is a positive integer. Yamasaki [1977, Example 4.1] showed that \( \mathbb{Z} \) is \( p \)-parabolic for \( p > 1 \), and thus \( \partial_p(\mathbb{Z}) = \emptyset \) for \( p > 1 \). The main result of [Maeda 1977] says that \( \mathbb{Z}^n \) with \( n \geq 2 \) is \( p \)-parabolic if and only if \( p \leq n \). Hence, \( \partial_p(\mathbb{Z}^n) = \emptyset \) if \( p \geq n \) and \( \partial_p(\mathbb{Z}^n) \) consists of exactly one point if \( 1 < p < n \).

There is a one-to-one correspondence between the maximal ideals of \( \text{BD}_{p}(\Gamma) \) and the points of \( \text{Sp}(\text{BD}_{p}(\Gamma)) \). If \( \tau \in R_p(\Gamma) \), then \( \ker(\tau) \) is the maximal ideal of \( \text{BD}_{p}(\Gamma) \) corresponding to \( \tau \). For each \( x \in \Gamma \), we have \( \delta_x \in \ker(\tau) \). By the continuity of \( \tau \), we see that \( \ell^p(\Gamma) \subseteq \ker(\tau) \). Assume that \( \Gamma \) is nonamenable. Then \( \ell^p(\Gamma) \) is closed in \( D_p(\Gamma) \) by [Guichardet 1977, Corollary 1]. Hence \( (\ell^p(\Gamma))_{D_p} = \ell^p(\Gamma) \). Also, \( (\ell^p(\Gamma))_{\text{BD}_{p}} = \ell^p(\Gamma) \) because \( \ell^p(\Gamma) \subseteq \text{BD}_{p} \). Thus \( \check{f}(\tau) = 0 \) for every \( f \in (\ell^p(\Gamma))_{D_p} \). Therefore, \( R_p(\Gamma) = \partial_p(\Gamma) \) when \( \Gamma \) is nonamenable. Consequently, \( R_p(\Gamma) \) contains exactly one point when \( \Gamma \) is nonamenable and \( \overline{H^1_{(p)}}(\Gamma) = 0 \). Some groups that satisfy this last condition for \( 1 < p \in \mathbb{R} \) are nonamenable groups with infinite center [Martin and Valette 2007, Theorem 4.2], and \( \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n \) for \( n \geq 2 \), each \( \Gamma_i \) is finitely generated, and at least one of the \( \Gamma_i \) is nonamenable [Martin and Valette 2007, Theorem 4.7].
8. Translation invariant linear functionals

Recall that \( \Gamma \) denotes a finitely generated group with generating set \( S \). In this section we will study TILFs on \( D_p(\Gamma)/\mathbb{R} \). By definition we have the inclusions

\[
\text{Diff}(\ell^p(\Gamma)) \subseteq \text{Diff}(D_p(\Gamma)/\mathbb{R}) \subseteq \ell^p(\Gamma) \subseteq D_p(\Gamma)/\mathbb{R}.
\]

The set \( D_p(\Gamma)/\mathbb{R} \) is a Banach space under the norm induced from \( I_p(\cdot, \Gamma) \). Thus if \( [f] \) if a class from \( D_p(G)/\mathbb{R} \), then its norm is given by

\[
\| [f] \|_{D_p} = \left( \sum_{x \in \Gamma} \sum_{s \in S} |f(xs^{-1}) - f(x)|^p \right)^{1/p}.
\]

We will write \( \| f \|_{D_p} \) for \( \|[f]\|_{D_p} \). Now \( (\ell^p(\Gamma))_{D_p} = D_p(\Gamma)/\mathbb{R} \) if and only if \( (\ell^p(\Gamma) \oplus \mathbb{R})_{D_p} = D_p(\Gamma) \). So \( \overline{H}_{(p)}(\Gamma) = 0 \) if and only if \( (\ell^p(\Gamma))_{D_p} = D_p(\Gamma)/\mathbb{R} \).

**Lemma 8.1.** \( (\text{Diff}(D_p(\Gamma)/\mathbb{R}))_{D_p} = (\ell^p(\Gamma))_{D_p} \).

**Proof.** Let \( f \in \ell^p(\Gamma) \). By [Woodward 1974, Lemma 1], there is a sequence \( (f_n) \) in \( \text{Diff}(\ell^p(\Gamma)) \) that converges to \( f \) in the \( \ell^p \)-norm. It follows from Minkowski’s inequality that for \( s \in S \),

\[
\| (f - f_n)_s - (f - f_n) \|_p = \sum_{x \in \Gamma} |f(xs^{-1}) - f_n(xs^{-1}) - (f(x) - f_n(x))|^p \to 0
\]

as \( n \to \infty \). Hence \( f \in (\text{Diff}(\ell^p(\Gamma)))_{D_p} \), implying \( \ell^p(\Gamma) \subseteq (\text{Diff}(\ell^p(\Gamma)))_{D_p} \). The result now follows. \( \square \)

**Theorem 8.2.** Let \( 1 < p \in \mathbb{R} \). Then \( \overline{H}^1_{(p)}(\Gamma) \neq 0 \) if and only if there exists a nonzero continuous TILF on \( D_p(\Gamma)/\mathbb{R} \).

**Proof.** If \( \overline{H}^1_{(p)}(\Gamma) \neq 0 \), then \( (\ell^p(\Gamma))_{D_p} \neq D_p(\Gamma)/\mathbb{R} \). It now follows from the Hahn–Banach theorem that there exists a nonzero continuous linear functional \( T \) on \( D_p(\Gamma)/\mathbb{R} \) such that \( (\ell^p(\Gamma))_{D_p} \) is contained in the kernel of \( T \). Thus \( T \) is translation invariant by Lemma 8.1.

Conversely, if \( T \) is a continuous TILF on \( D_p(\Gamma)/\mathbb{R} \), then \( T(f) = 0 \) for all \( f \in (\ell^p(\Gamma))_{D_p} \). So if there exists a nonzero continuous TILF on \( D_p(\Gamma)/\mathbb{R} \), then \( (\ell^p(\Gamma))_{D_p} \neq D_p(\Gamma)/\mathbb{R} \). \( \square \)

Theorem 2.11 now follows by combining Theorems 8.2 and 2.10.

If \( h \in D_p(\Gamma)/\mathbb{R} \), then \( \langle \Delta_p h, \cdot \rangle \) is a well-defined continuous linear functional on \( D_p(\Gamma)/\mathbb{R} \) since equivalent functions in \( D_p(\Gamma)/\mathbb{R} \) differ by a constant. It was shown in [Puls 2006, Proposition 3.4] that if \( h \in \text{HD}_p(\Gamma)/\mathbb{R} \) and \( f \in (\ell^p(\Gamma))_{D_p} \), then \( \langle \Delta_p h, f \rangle = 0 \). Consequently, if \( h \in \text{HD}_p(\Gamma)/\mathbb{R} \), then \( \langle \Delta_p h, \cdot \rangle \) defines a continuous TILF on \( D_p(\Gamma)/\mathbb{R} \). Thus there are no nonzero continuous TILFs on \( D_p(\Gamma)/\mathbb{R} \) when \( \text{HD}_p(\Gamma) \) only contains the constant functions.
If $\tilde{H}_1^{(p)}(\Gamma) = 0$, then $(\ell^p(\Gamma))_{D_p} = D_p(\Gamma)/\mathbb{R}$. It is known that $\ell^p(\Gamma)$ is closed in $D_p(\Gamma)/\mathbb{R}$ if and only if $\Gamma$ is nonamenable, [Guichardet 1977, Corollary 1]. As was mentioned in Section 2, if $\Gamma$ is nonamenable, then zero is the only TILF on $\ell^p(\Gamma)$. Consequently zero is the only TILF on $D_p(\Gamma)/\mathbb{R}$ when $\Gamma$ is nonamenable and $\tilde{H}_1^{(p)}(\Gamma) = 0$. Summing up:

**Theorem 8.3.** Let $\Gamma$ be an infinite, finitely generated group and let $1 < p \in \mathbb{R}$. The following are equivalent:

1. $\tilde{H}_1^{(p)}(\Gamma) = 0$.
2. Either $\partial_p(\Gamma) = \emptyset$ or $\#(\partial_p(\Gamma)) = 1$.
3. $HD_p(\Gamma) = \mathbb{R}$.
4. $BHD_p(\Gamma) = \mathbb{R}$.
5. The only continuous TILF on $D_p(\Gamma)/\mathbb{R}$ is zero. If $\Gamma$ is also nonamenable, then this is still equivalent to (6):
6. Zero is the only TILF on $D_p(\Gamma)/\mathbb{R}$.

Some examples show zero is not the only TILF on $D_p(\Gamma)/\mathbb{R}$ when $\Gamma$ is nonamenable; this differs from the $\ell^p(\Gamma)$ case. Puls [2006, Corollary 4.3] showed $\tilde{H}_1^{(p)}(\Gamma) \neq 0$ for groups with infinitely many ends and $1 < p \in \mathbb{R}$. Thus by Theorem 8.2 there exists a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$.

If there is a nonzero continuous TILF on $D_r(\Gamma)/\mathbb{R}$ for some nonamenable group $\Gamma$ and some real number $r$, then is it true that there is a nonzero continuous TILF on $D_p(\Gamma)/\mathbb{R}$ for all real numbers $p > 1$? The answer to this question is no. To see this, let $\mathcal{H}^n$ denote hyperbolic $n$-space, and suppose $\Gamma$ is a group that acts properly discontinuously on $\mathcal{H}^n$ by isometries and that the action is cocompact and free. By combining [Bourdon et al. 2005, Theorem 2] and [Puls 2007, Theorem 1.1], we obtain $\tilde{H}_1^{(p)}(\Gamma) \neq 0$ if and only if $p > n - 1$.

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