METABELIAN SL\(^{(n, \mathbb{C})}\) REPRESENTATIONS OF KNOT GROUPS, II: FIXED POINTS

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Given a knot \(K\) in an integral homology sphere \(\Sigma\) with exterior \(N_K\), there is a natural action of the cyclic group \(\mathbb{Z}/n\) on the space of SL\((n, \mathbb{C})\) representations of the knot group \(\pi_1(N_K)\), which induces an action on the SL\((n, \mathbb{C})\) character variety. We identify the fixed points of this action in terms of characters of metabelian representations, and we apply this in order to show that the twisted Alexander polynomial \(\Delta^\alpha_{K,1}(t)\) associated to an irreducible metabelian SL\((n, \mathbb{C})\) representation \(\alpha\) is actually a polynomial in \(t^n\).

1. Introduction

The study of metabelian representations and metabelian quotients of knot groups goes back to the pioneering work of Neuwirth [1965], de Rham [1967], Burde [1967], and Fox [1970]; see also [Burde and Zieschang 2003, Section 14]. The theory was further developed by many authors, including Hartley [1979; 1983], Livingston [1995], Letsche [2000], Lin [2001], Nagasato [2007], and Jebali [2008]. In [Boden and Friedl 2008], we proved a classification theorem for irreducible metabelian representations and in this paper we continue our study of metabelian representations of knot groups.

Throughout this paper, when we say that \(K\) is a knot, we will always understand that \(K\) is an oriented, simple closed curve in an integral homology 3-sphere \(\Sigma\). We write \(N_K = \Sigma^3 \setminus \tau(K)\), where \(\tau(K)\) denotes an open tubular neighborhood of \(K\).

Given a topological space \(M\), let \(R_n(M)\) be the space of SL\((n, \mathbb{C})\) representations of \(\pi_1(M)\), and let \(X_n(M)\) be the associated character variety. We use \(\xi_\alpha\) to denote the character of the representation \(\alpha : \pi_1(M) \to \text{SL}(n, \mathbb{C})\). We will often make use of the important fact that two irreducible representations determine the same character if and only if they are conjugate; see [Lubotzky and Magid 1985, Corollary 1.33].

Now suppose \(K\) is a knot. The group \(\mathbb{Z}/n\) has an action on the representation variety \(R_n(N_K)\), given by twisting by the \(n\)-th roots of unity \(\omega^k = e^{2\pi ik/n} \in \mathbb{U}(1)\).
(This is a special case of the more general twisting operation described in [Lubotzky and Magid 1985, Chapter 5].) More precisely, we write \( \mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle \) and set \((\sigma \cdot \alpha)(g) = \omega^\varepsilon(g) \alpha(g)\) for each \( g \in \pi_1(N_K) \), where \( \varepsilon : \pi_1(N_K) \to H_1(N_K) = \mathbb{Z} \) is determined by the given orientation of the knot.

This constructs an action of \( \mathbb{Z}/n \) on \( R_n(N_K) \) which, in turn, descends to an action on the character variety \( X_n(N_K) \). Our main result identifies the fixed points of \( \mathbb{Z}/n \) in \( X_n^*(N_K) \) — the irreducible characters — as those associated to metabelian representations.

**Theorem 1.** The character \( \xi_\alpha \) of an irreducible representation \( \alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) is fixed under the \( \mathbb{Z}/n \) action if and only if \( \alpha \) is metabelian.

In proving this result, we will actually characterize the entire fixed point set \( X_n(N_K)^{\mathbb{Z}/n} \) in terms of characters \( \xi_\alpha \) of the metabelian representations \( \alpha = \alpha(n, \chi) \) described in Section 2.3 (see Theorem 4). When \( n = 2 \), it turns out that every metabelian \( \text{SL}(2, \mathbb{C}) \) representation is dihedral. For this case, Theorem 1 was first proved by F. Nagasato and Y. Yamaguchi [2008, Proposition 4.8].

As an application of Theorem 1, we prove a result about the twisted Alexander polynomials associated to metabelian representations. This result was first shown by C. Herald, P. Kirk and C. Livingston [2010] using completely different methods. Our approach is elementary and natural, and is explained in Section 3.2, where we apply it to give an answer to a question raised by Hirasawa and Murasugi [2009].

### 2. The classification of metabelian representations of knot groups

We recall some results from [Boden and Friedl 2008] regarding the classification of metabelian representations of knot groups.

#### 2.1. Preliminaries.**

Given a group \( \pi \), we write \( \pi^{(n)} \) for the \( n \)-th term of the derived series of \( \pi \). These subgroups are defined inductively by setting \( \pi^{(0)} = \pi \) and \( \pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}] \). The group \( \pi \) is called **metabelian** if \( \pi^{(2)} = \{e\} \).

Suppose \( V \) is a finite-dimensional vector space over \( \mathbb{C} \). A representation \( \varrho : \pi \to \text{Aut}(V) \) is called **metabelian** if \( \varrho \) factors through \( \pi/\pi^{(2)} \). The representation \( \varrho \) is called **reducible** if there exists a proper subspace \( U \subset V \) invariant under \( \varrho(\gamma) \) for all \( \gamma \in \pi \). Otherwise, \( \varrho \) is called **irreducible** or **simple**. If \( \varrho \) is the direct sum of simple representations, then \( \varrho \) is called **semisimple**.

Two representations \( \varrho_1 : \pi \to \text{Aut}(V) \) and \( \varrho_2 : \pi \to \text{Aut}(W) \) are called **isomorphic** if there exists an isomorphism \( \varphi : V \to W \) such that \( \varphi^{-1} \circ \varrho_1(g) \circ \varphi = \varrho_2(g) \) for all \( g \in \pi \).

#### 2.2. Metabelian quotients of knot groups.

Let \( K \subset \Sigma^3 \) be a knot in an integral homology 3-sphere. Denote by \( \tilde{N}_K \) the infinite cyclic cover of \( N_K \) corresponding
to the abelianization $\pi_1(N_K) \to H_1(N_K) \cong \mathbb{Z}$. Thus, $\pi_1(\tilde{N}_K) = \pi_1(N_K)^{(1)}$ and $H_1(N_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\tilde{N}_K) \cong \pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)}$.

The $\mathbb{Z}[t^{\pm 1}]$-module structure is given on the right hand side by $t^n \cdot g := \mu^{-n} g \mu^n$, where $\mu$ is a meridian of $K$.

For a knot $K$, we set $\pi := \pi_1(N_K)$ and consider the short exact sequence $1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1$.

Since $\pi/\pi^{(1)} = H_1(N_K) \cong \mathbb{Z}$, this sequence splits, and we get isomorphisms

$$\pi/\pi^{(2)} \cong \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \cong H_1(N_K; \mathbb{Z}[t^{\pm 1}]),$$

where the semidirect products are taken with respect to the $\mathbb{Z}$ actions defined by letting $n \in \mathbb{Z}$ act on $\pi^{(1)}/\pi^{(2)}$ by conjugation by $\mu^n$, and on $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$ by multiplication by $t^n$.

2.3. Irreducible metabelian $SL(n, \mathbb{C})$ representations of knot groups. Let $K$ be a knot and write $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. The discussion of the previous section shows that irreducible metabelian $SL(n, \mathbb{C})$ representations of $\pi_1(N_K)$ correspond precisely to the irreducible $SL(n, \mathbb{C})$ representations of $\mathbb{Z} \rtimes H$.

Let $\chi : H \to \mathbb{C}^*$ be a character that factors through $H/(t^n-1)$, and take $z \in S^1$ with $z^n = (-1)^{n+1}$. It follows from [Boden and Friedl 2008, Section 3] that, for $(j, h) \in \mathbb{Z} \rtimes H$,

$$\alpha_{(\chi, z)}(j, h) = \begin{pmatrix} 0 & 0 & \cdots & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \cdots & 0 \\ 0 & \chi(th) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi(t^{n-1}h) \end{pmatrix}$$

defines an $SL(n, \mathbb{C})$ representation whose isomorphism type does not depend on the choice of $z$. In our notation we will not normally distinguish between metabelian representations of $\pi_1(N_K)$ and representations of $\mathbb{Z} \rtimes H$.

We say that a character $\chi : H \to \mathbb{C}^*$ has order $n$ if it factors through $H/(t^n-1)$ but not through $H/(t^\ell-1)$ for any $\ell < n$. Given a character $\chi : H \to \mathbb{C}^*$, let $t^i \chi$ be the character defined by $(t^i \chi)(h) = \chi(t^i h)$. Any character $\chi : H \to \mathbb{C}^*$ that factors through $H/(t^n-1)$ must have order $k$ for some divisor $k$ of $n$. The next statement is a combination of [Boden and Friedl 2008, Lemma 2.2 and Theorem 3.3].

**Theorem 2.** Suppose $\chi : H \to \mathbb{C}^*$ is a character that factors through $H/(t^n-1)$.

(i) $\alpha_{(n, \chi)} : \mathbb{Z} \rtimes H \to SL(n, \mathbb{C})$ is irreducible if and only if the character $\chi$ has order $n$. 
(ii) Given two characters \( \chi, \chi' : H \to \mathbb{C}^* \) of order \( n \), the representations \( \alpha_{(n,\chi)} \) and \( \alpha_{(n,\chi')} \) are conjugate if and only if \( \chi = t^k \chi' \) for some \( k \).

(iii) For any irreducible representation \( \alpha : \mathbb{Z} \ltimes H \to \text{SL}(n, \mathbb{C}) \), there is a character \( \chi : H \to \mathbb{C}^* \) of order \( n \) such that \( \alpha \) is conjugate to \( \alpha_{(n,\chi)} \).

### 3. Main results

#### 3.1. Metabelian characters as fixed points.

Set \( \omega = e^{2\pi i/n} \) and recall the action of the cyclic group \( \mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle \) on representations \( \alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) obtained by setting \( (\sigma \cdot \alpha)(g) = \omega^{e(g)}\alpha(g) \) for all \( g \in \pi_1(N_K) \), where \( e : \pi_1(N_K) \to H_1(N_K) = \mathbb{Z} \).

**Lemma 3.** Suppose \( \alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C}) \) is a representation whose associated character \( \xi_\alpha \in X_n(N_K) \) is a fixed point of the \( \mathbb{Z}/n \) action. Up to conjugation,

\[
\alpha(\mu) = \begin{pmatrix}
0 & 0 & \cdots & z \\
z & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

for some (in fact, any) \( z \in U(1) \) such that \( z^n = (-1)^{n+1} \).

**Proof.** Let \( c(t) = \det(\alpha(\mu) - tI) \) denote the characteristic polynomial of \( \alpha(\mu) \), which we can write as

\[
c(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + 1.
\]

Note that \( c(t) \) is determined by the character \( \xi_\alpha \in X_n(N_K) \) and so, assuming \( \xi_\alpha \) is a fixed point of the \( \mathbb{Z}/n \) action, we conclude that \( \alpha(\mu) \) and \( \omega^k \alpha(\mu) \) have the same characteristic polynomials for all \( k \). In particular,

\[
c(t) = \det(\omega^{-k} \alpha(\mu) - tI) = \det(\omega^{-1} \alpha(\mu) - tI) = \det(\omega^{-1} \alpha(\mu) - tI) = \det(\alpha(\mu) - \omega tI) = \det(\alpha(\mu) - t \omega I) = c(\omega t).
\]

However, \( \omega^k \neq 1 \) unless \( n \) divides \( k \), and this implies \( 0 = c_{n-1} = c_{n-2} = \cdots = c_1 \) and \( c(t) = (-1)^n t^n + 1 \). In particular, the matrix \( \alpha(\mu) \) and the matrix appearing in Equation (1) have the same set of \( n \) distinct eigenvalues. This implies that the two matrices are conjugate. \( \square \)

To prove Theorem 1, we establish the following more general result:

**Theorem 4.** The fixed point set of the \( \mathbb{Z}/n \) action on \( X_n(N_K) \) consists of characters \( \xi_\alpha \) of the metabelian representations \( \alpha = \alpha_{(n,\chi)} \) described in Section 2.3. In other words, \( X_n(N_K)^{\mathbb{Z}/n} = \{\xi_\alpha \mid \alpha = \alpha_{(n,\chi)} \text{ for } \chi : H_1(N_K; \mathbb{Z}[t^\pm 1]) \to \mathbb{C}^*\} \).
Theorem 1 can be viewed as the special case of Theorem 4 when $\alpha$ is irreducible. (Recall that irreducible representations are conjugate if and only if they define the same character.) Note that not every reducible metabelian representation is of the form $\alpha(n, \chi)$.

Proof. We first show that if $\alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})$ is given as $\alpha(n, \chi)$, then $\sigma \cdot \alpha$ is conjugate to $\alpha$. This of course implies that $\xi_\alpha = \xi_{\sigma \cdot \alpha}$.

Assume then that $\alpha = \alpha(n, \chi)$. We have

$$\alpha(\mu) = \begin{pmatrix} 0 & 0 & \cdots & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z \end{pmatrix},$$

where $z$ satisfies $z^n = (-1)^{n+1}$. Also $\alpha(\mu)$ is diagonal for all $\mu \in [\pi_1(N_K), \pi_1(N_K)]$.

From the definition of $\sigma \cdot \alpha$, we see that

$$(\sigma \cdot \alpha)(\mu) = \omega \alpha(\mu) = \begin{pmatrix} 0 & 0 & \cdots & \omega z \\ \omega z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \omega z & 0 \end{pmatrix}$$

and that $(\sigma \cdot \alpha)(g) = \alpha(g)$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. It follows easily from Theorem 2(ii) that $\sigma \cdot \alpha$ and $\alpha(n, \chi)$ are conjugate; however, it is easy to see this directly too. Simply take

$$P = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{pmatrix},$$

and compute that $\sigma \cdot \alpha = P \alpha P^{-1}$ as claimed.

We now show the other implication, namely, that each point $\xi \in X_n(N_K)^{\mathbb{Z}/n}$ in the fixed point set can be represented as the character $\xi = \xi_\alpha$ of a metabelian representation $\alpha = \alpha(n, \chi)$, where $\chi : H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*$ is a character that factors through $H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^n - 1)$ and hence has order $k$ for some $k$ that divides $n$. (Note that Theorem 2(i) tells us that $\alpha(n, \chi)$ is irreducible if and only if $\chi$ has order $n$.)

From the general results on representation spaces and character varieties (see [Lubotzky and Magid 1985]), it follows that every point in the character variety $X_n(N_K)$ can be represented as $\xi_\alpha$ for some semisimple representation $\alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})$. Further, two semisimple representations $\alpha_1$ and $\alpha_2$ determine
the same character if and only if $\alpha_1$ is conjugate to $\alpha_2$. (This is evident from the fact that the orbits of the semisimple representations under conjugation are closed.)

Given $\xi \in X_n(N_K)^{\mathbb{Z}/n}$, we can therefore suppose that $\xi = \xi_\alpha$ for some semisimple representation $\alpha$. Clearly $\sigma \cdot \alpha$ is also semisimple, and since $\xi_\alpha = \xi_{\sigma \cdot \alpha}$, we conclude from the previous argument that $\alpha$ and $\sigma \cdot \alpha$ are conjugate representations. This means that there exists a matrix $A \in \text{SL}(n, \mathbb{C})$ such that $A \alpha A^{-1} = \sigma \cdot \alpha$. In other words, for all $g \in \pi_1(N_K)$, we have

$$ (2) \quad A \alpha(g) A^{-1} = \omega^{e(g)} \alpha(g). $$

Lemma 3 implies that $\alpha(\mu)$ is conjugate to the matrix in Equation (1). It is convenient to conjugate $\alpha$ so that $\alpha(\mu)$ is diagonal, meaning that

$$ \alpha(\mu) = \begin{pmatrix} z & 0 \\ \omega z & \cdots \\ 0 & \cdots & \omega^{n-1}z \end{pmatrix}, $$

where $z$ satisfies $z^n = (-1)^{n+1}$.

We now apply (2) to the meridian to conclude that

$$ A \alpha(\mu) = \omega \alpha(\mu) A, $$

which implies that $A = (a_{ij})$ satisfies $a_{ij} = 0$ unless $j = i + 1 \mod (n)$. Thus, we see that

$$ A = \begin{pmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{n-1} \\ \lambda_n & 0 & \cdots & 0 & 0 \end{pmatrix} $$

for some $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_1 \cdots \lambda_n = (-1)^{n+1}$.

It is completely straightforward to see that the characteristic polynomial of $A$ is

$$ \det(A - tI) = (-1)^n(t^n - (-1)^{n+1}). $$

From this, we conclude that $A$ has as its eigenvalues the $n$ distinct $n$-th roots of $(-1)^{n+1}$. In particular, the subset of matrices in $\text{SL}(n, \mathbb{C})$ that commute with $A$ is just a copy of the unique maximal torus $T_A \cong (\mathbb{C}^*)^{n-1}$ containing $A$.

For any $g \in [\pi_1(N_K), \pi_1(N_K)]$, we have $\alpha(g) = (\sigma \cdot \alpha)(g)$. Thus it follows that $A \alpha(g) A^{-1} = \alpha(g)$, and this implies that $\alpha(g) \in T_A$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. This shows that the restriction of $\alpha$ to the commutator subgroup $[\pi_1(N_K), \pi_1(N_K)]$ is abelian. We conclude from this that $\alpha$ is indeed metabelian. Notice that this, and an application of Theorem 2(iii), completes the proof in case $\alpha$ is irreducible.
In the general case, it follows from the discussion in Section 2.2 that $\alpha$ factors through $\mathbb{Z} \times H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Let $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Given a character $\chi : H \to \mathbb{C}^*$, we define the associated weight space $V_{\chi}$ by setting

$$V_{\chi} = \{v \in \mathbb{C}^n | \chi(h) \cdot v = \alpha(h) v \text{ for all } h \in H\}.$$ 

Recall that $A \cdot \alpha(h) \cdot A^{-1} = \alpha(h)$ for any $h \in H$. It is straightforward to show that $A$ restricts to an automorphism of $V_{\chi}$. Since $H$ is abelian, there exists at least one character $\chi : H \to \mathbb{C}^*$ such that $V_{\chi}$ is nontrivial. For any $i$, denote by $t^i \chi$ the character given by $(t^i \chi)(h) = \chi(t^i h)$ for $h \in H$.

Note that $A$ has $n$ distinct eigenvalues and therefore is diagonalizable. Since $A$ restricts to an automorphism of $V_{\chi}$, there is an eigenvector $v$ of $A$ that lies in $V_{\chi}$. Let $\lambda$ be the corresponding eigenvalue. By the proof of [Boden and Friedl 2008, Theorem 2.3], the map $\alpha(\mu)$ induces an isomorphism $V_{\chi} \to V_{t^i \chi}$. We now calculate

$$A \cdot \alpha(\mu) v = (A \alpha(\mu) A^{-1}) \cdot A v = \omega \alpha(\mu) \cdot \lambda v = \lambda \omega \alpha(\mu) v;$$

that is, $\alpha(\mu) v \in V_{t^i \chi}$ is an eigenvector of $A$ with eigenvalue $\omega \lambda$.

Iterating this argument, we see that $\alpha(\mu)^i v$ lies in $V_{t^i \chi}$ and is an eigenvector of $A$ with eigenvalue $\omega^i \lambda$. Since $\omega$ is a primitive $n$-th root of unity, the eigenvalues $\lambda, \omega \lambda, \ldots, \omega^{n-1} \lambda$ are all distinct, and this implies that the corresponding eigenvectors $v, \alpha(\mu) v, \ldots, \alpha(\mu)^{n-1} v$ form a basis for $\mathbb{C}^n$.

Let $m$ be the order of $\chi$; that is, $m$ is the minimal number such that $\chi = t^m \chi$. From the previous argument, we see that $\mathbb{C}^n$ is generated by $V_{\chi}, V_{t^\chi}, \ldots, V_{t^{m-1} \chi}$. Since the characters $\chi, t^\chi, \ldots, t^{m-1} \chi$ are pairwise distinct, it follows that $\mathbb{C}^n$ is given as the direct sum $V_{\chi} \oplus V_{t^\chi} \oplus \cdots \oplus V_{t^{m-1} \chi}$.

We write $k = \dim_{\mathbb{C}}(V_{\chi})$ and note that $n = km$. We note further that $\alpha(\mu)^m$ has eigenvalues

$$(3) \quad \{z^m, z^m e^{2\pi i/k}, \ldots, z^m e^{2\pi i(k-1)/k}\},$$

and each eigenvalue has multiplicity $m$. Clearly $\alpha(\mu)^m$ restricts to an automorphism of $V_{t^i \chi}$ for $i = 0, \ldots, m-1$, and equally clearly we see that the restrictions all give conjugate representations. This implies that the restriction of $\alpha(\mu)^m$ to $V_{\chi}$ has eigenvalues in the set (3) above, each occurring with multiplicity 1. In particular, we can find a basis $\{v_1, \ldots, v_k\}$ for $V_{\chi}$ in which the matrix of $\alpha(\mu)^m$ has the form

$$\alpha(\mu^m) = \begin{pmatrix} 0 & 0 & \cdots & z^m \\ z^m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z^m \end{pmatrix}.$$
It is now straightforward to verify that, with respect to the ordered basis
\[
\begin{align*}
&v_1, \ z^{-1} \alpha(\mu)v_1, \ \cdots, \ z^{-(m-1)} \alpha(\mu)^{m-1}v_1, \\
v_2, \ z^{-1} \alpha(\mu)v_2, \ \cdots, \ z^{-(m-1)} \alpha(\mu)^{m-1}v_2, \\
&\vdots \hspace{5cm} \vdots \\
v_k, \ z^{-1} \alpha(\mu)v_k, \ \cdots, \ z^{-(m-1)} \alpha(\mu)^{m-1}v_k
\end{align*}
\]
\(\alpha\) is given by \(\alpha(n, \chi)\).

3.2. Application to twisted Alexander polynomials. As an application, we prove the following result regarding twisted Alexander polynomials of knots corresponding to metabelian representations. Denote by \(\Delta^\alpha_{K,i}(t)\) the \(i\)-th twisted Alexander polynomial for a given representation \(\alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})\), as presented in [Friedl and Vidussi 2009].

**Proposition 5.** Let \(\alpha\) be a metabelian representation of the form \(\alpha = \alpha(n, \chi) : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})\). Then
\[
\Delta^\alpha_{K,0}(t) = \begin{cases} 
1 - t^n & \text{if } \chi \text{ is trivial,} \\
1 & \text{otherwise.}
\end{cases}
\]

Further, the twisted Alexander polynomial \(\Delta^\alpha_{K,1}(t)\) is actually a polynomial in \(t^n\).

**Remark 6.** In their paper, C. Herald, P. Kirk, and C. Livingston prove the same result using an entirely different approach; compare with [Herald et al. 2010, page 10]. We also point out that Proposition 5 gives a positive answer to [Hirasawa and Murasugi 2009, Conjecture A].

**Proof.** The proof of the first statement is not difficult. It is immediate when \(\chi\) is trivial, and it follows by a direct calculation when \(\chi\) is nontrivial.

We turn to the proof of the second statement. For \(\theta \in U(1)\) and any representation \(\beta : \pi_1(N_K) \to \text{GL}(n, \mathbb{C})\), define the \(\theta\)-twist of \(\beta\) to be the representation that sends \(g \in \pi_1(N_K)\) to \(\theta^{\epsilon(g)} \beta(g)\), where \(\epsilon : \pi_1(N_K) \to \mathbb{Z}\) is determined by the orientation of \(K\). We denote the newly obtained representation by \(\beta_\theta : \pi_1(N_K) \to \text{GL}(n, \mathbb{C})\). Note that in case \(\alpha : \pi_1(N_K) \to \text{SL}(n, \mathbb{C})\) and \(\theta = e^{2\pi ik/n}\) is an \(n\)-th root of unity, \(\alpha_\theta\) is again an \(\text{SL}(n, \mathbb{C})\) representation. The proof of the proposition relies on the formula
\[
\Delta^\beta_{K,1}(t) = \Delta^\beta_{K,1}(\theta t).
\]
This formula is well known, and follows directly from the definition of the twisted Alexander polynomial. Equation (4) combines with Theorem 1 to complete the proof, as we now explain. Take \(\omega = e^{2\pi i/n}\). If \(\alpha = \alpha(n, \chi)\) is metabelian, Theorem 1
shows that its conjugacy class is fixed under the $\mathbb{Z}/n$ action. In particular, since $\alpha$ and $\alpha_\omega$ are conjugate, Equation (4) shows that
\[ \Delta^{\alpha}_{K,1}(t) = \Delta^{\alpha_\omega}_{K,1}(t) = \Delta^{\alpha}_{K,1}(\omega t). \]
Expanding $\Delta^{\alpha}_{K,1}(t) = \sum a_i t^i$ and using the fact that $t^k = (\omega t)^k$ if and only if $k$ is a multiple of $n$, this shows that $a_k = 0$ unless $k$ is a multiple of $n$. □

Acknowledgments

We would like to thank Steven Boyer, Christopher Herald, Michael Heusener, Paul Kirk, Charles Livingston, Andrew Nicas, and Adam Sikora for generously sharing their knowledge, wisdom, and insight. We would also like to thank Fumikazu Nagasato and Yoshikazu Yamaguchi for communicating the results of their paper to us.

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Received September 20, 2009. Revised February 9, 2010.

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