Pacific Journal of Mathematics

METABELIAN SL(n, $\mathbb{C}$) REPRESENTATIONS OF KNOT GROUPS, II: FIXED POINTS

HANS U. BODEN AND STEFAN FRIEDL

Volume 249 No. 1 January 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 249, No. 1, 2011

METABELIAN SL(n, $\mathbb{C}$) REPRESENTATIONS OF KNOT GROUPS, II: FIXED POINTS

HANS U. BODEN AND STEFAN FRIEDL

Given a knot K in an integral homology sphere Σ with exterior N_K, there is a natural action of the cyclic group $\mathbb{Z}/n$ on the space of SL(n, $\mathbb{C}$) representations of the knot group $\pi_1(N_K)$, which induces an action on the SL(n, $\mathbb{C}$) character variety. We identify the fixed points of this action in terms of characters of metabelian representations, and we apply this in order to show that the twisted Alexander polynomial $\Delta^{\alpha}_{K,1}(t)$ associated to an irreducible metabelian SL(n, $\mathbb{C}$) representation α is actually a polynomial in t^n.

1. Introduction

The study of metabelian representations and metabelian quotients of knot groups goes back to the pioneering work of Neuwirth [1965], de Rham [1967], Burde [1967], and Fox [1970]; see also [Burde and Zieschang 2003, Section 14]. The theory was further developed by many authors, including Hartley [1979; 1983], Livingston [1995], Letsche [2000], Lin [2001], Nagasato [2007], and Jebali [2008]. In [Boden and Friedl 2008], we proved a classification theorem for irreducible metabelian representations and in this paper we continue our study of metabelian representations of knot groups.

Throughout this paper, when we say that K is a knot, we will always understand that K is an oriented, simple closed curve in an integral homology 3-sphere Σ. We write $N_K = \Sigma^3 \setminus \tau(K)$, where $\tau(K)$ denotes an open tubular neighborhood of K.

Given a topological space M, let $R_n(M)$ be the space of SL(n, $\mathbb{C}$) representations of $\pi_1(M)$, and let $X_n(M)$ be the associated character variety. We use ξ_α to denote the character of the representation $\alpha : \pi_1(M) \to \mathrm{SL}(n, \mathbb{C})$. We will often make use of the important fact that two irreducible representations determine the same character if and only if they are conjugate; see [Lubotzky and Magid 1985, Corollary 1.33].

Now suppose K is a knot. The group $\mathbb{Z}/n$ has an action on the representation variety $R_n(N_K)$, given by twisting by the n-th roots of unity $\omega^k = e^{2\pi i k/n} \in \mathrm{U}(1)$.

MSC2000: primary 57M25; secondary 20C15.
Keywords: metabelian representation, knot group, character variety, group action, fixed point.

(This is a special case of the more general twisting operation described in [Lubotzky and Magid 1985, Chapter 5].) More precisely, we write $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle$ and set $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)} \alpha(g)$ for each $g \in \pi_1(N_K)$, where $\varepsilon : \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$ is determined by the given orientation of the knot.

This constructs an action of $\mathbb{Z}/n$ on $R_n(N_K)$ which, in turn, descends to an action on the character variety $X_n(N_K)$. Our main result identifies the fixed points of $\mathbb{Z}/n$ in $X_n^*(N_K)$ — the irreducible characters — as those associated to metabelian representations.

Theorem 1. *The character ξ_α of an irreducible representation $\alpha : \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is fixed under the $\mathbb{Z}/n$ action if and only if α is metabelian.*

In proving this result, we will actually characterize the entire fixed point set $X_n(N_K)^{\mathbb{Z}/n}$ in terms of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Section 2.3 (see Theorem 4). When $n = 2$, it turns out that every metabelian SL(2, $\mathbb{C}$) representation is dihedral. For this case, Theorem 1 was first proved by F. Nagasato and Y. Yamaguchi [2008, Proposition 4.8].

As an application of Theorem 1, we prove a result about the twisted Alexander polynomials associated to metabelian representations. This result was first shown by C. Herald, P. Kirk and C. Livingston [2010] using completely different methods. Our approach is elementary and natural, and is explained in Section 3.2, where we apply it to give an answer to a question raised by Hirasawa and Murasugi [2009].

2. The classification of metabelian representations of knot groups

We recall some results from [Boden and Friedl 2008] regarding the classification of metabelian representations of knot groups.

2.1. *Preliminaries.* Given a group π, we write $\pi^{(n)}$ for the n-th term of the derived series of π. These subgroups are defined inductively by setting $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}]$. The group π is called *metabelian* if $\pi^{(2)} = \{e\}$.

Suppose V is a finite-dimensional vector space over $\mathbb{C}$. A representation $\varrho : \pi \to \mathrm{Aut}(V)$ is called *metabelian* if ϱ factors through $\pi/\pi^{(2)}$. The representation ϱ is called *reducible* if there exists a proper subspace $U \subset V$ invariant under $\varrho(\gamma)$ for all $\gamma \in \pi$. Otherwise, ϱ is called *irreducible* or *simple*. If ϱ is the direct sum of simple representations, then ϱ is called *semisimple*.

Two representations $\varrho_1 : \pi \to \mathrm{Aut}(V)$ and $\varrho_2 : \pi \to \mathrm{Aut}(W)$ are called *isomorphic* if there exists an isomorphism $\varphi : V \to W$ such that $\varphi^{-1} \circ \varrho_1(g) \circ \varphi = \varrho_2(g)$ for all $g \in \pi$.

2.2. *Metabelian quotients of knot groups.* Let $K \subset \Sigma^3$ be a knot in an integral homology 3-sphere. Denote by $\widetilde{N}_K$ the infinite cyclic cover of N_K corresponding to the abelianization $\pi_1(N_K)\to H_1(N_K)\cong\mathbb{Z}$. Thus, $\pi_1(\widetilde{N}_K)=\pi_1(N_K)^{(1)}$ and

$$H_1\big(N_K;\mathbb{Z}[t^{\pm1}]\big)=H_1(\widetilde{N}_K)\cong\pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)}.$$

The $\mathbb{Z}[t^{\pm1}]$-module structure is given on the right hand side by $t^n\cdot g:=\mu^{-n}g\,\mu^n$, where μ is a meridian of K.

For a knot K, we set $\pi:=\pi_1(N_K)$ and consider the short exact sequence

$$1\to\pi^{(1)}/\pi^{(2)}\to\pi/\pi^{(2)}\to\pi/\pi^{(1)}\to 1.$$

Since $\pi/\pi^{(1)}=H_1(N_K)\cong\mathbb{Z}$, this sequence splits, and we get isomorphisms

$$\begin{array}{ccccccc}\pi/\pi^{(2)} & \cong & \pi/\pi^{(1)}\ltimes\pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z}\ltimes\pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z}\ltimes H_1(N_K;\mathbb{Z}[t^{\pm1}]),\\ g & \mapsto & (\mu^{\varepsilon(g)},\ \mu^{-\varepsilon(g)}g) & \mapsto & (\varepsilon(g),\ \mu^{-\varepsilon(g)}g), & & \end{array}$$

where the semidirect products are taken with respect to the $\mathbb{Z}$ actions defined by letting $n\in\mathbb{Z}$ act on $\pi^{(1)}/\pi^{(2)}$ by conjugation by μ^n, and on $H_1(N_K;\mathbb{Z}[t^{\pm1}])$ by multiplication by t^n.

2.3. ***Irreducible metabelian* SL($n,\mathbb{C}$) *representations of knot groups.*** Let K be a knot and write $H=H_1(N_K;\mathbb{Z}[t^{\pm1}])$. The discussion of the previous section shows that irreducible metabelian SL($n,\mathbb{C}$) representations of $\pi_1(N_K)$ correspond precisely to the irreducible SL($n,\mathbb{C}$) representations of $\mathbb{Z}\ltimes H$.

Let $\chi:H\to\mathbb{C}^*$ be a character that factors through $H/(t^n-1)$, and take $z\in S^1$ with $z^n=(-1)^{n+1}$. It follows from [Boden and Friedl 2008, Section 3] that, for $(j,h)\in\mathbb{Z}\ltimes H$,

$$\alpha_{(\chi,z)}(j,h)=\begin{pmatrix}0&0&\cdots&z\\ z&0&\cdots&0\\ \vdots&\ddots&\ddots&\vdots\\ 0&\cdots&z&0\end{pmatrix}^{j}\begin{pmatrix}\chi(h)&0&\cdots&0\\ 0&\chi(th)&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&\chi(t^{n-1}h)\end{pmatrix}$$

defines an SL($n,\mathbb{C}$) representation whose isomorphism type does not depend on the choice of z. In our notation we will not normally distinguish between metabelian representations of $\pi_1(N_K)$ and representations of $\mathbb{Z}\ltimes H$.

We say that a character $\chi:H\to\mathbb{C}^*$ has *order n* if it factors through $H/(t^n-1)$ but not through $H/(t^\ell-1)$ for any $\ell<n$. Given a character $\chi:H\to\mathbb{C}^*$, let $t^i\chi$ be the character defined by $(t^i\chi)(h)=\chi(t^ih)$. Any character $\chi:H\to\mathbb{C}^*$ that factors through $H/(t^n-1)$ must have order k for some divisor k of n. The next statement is a combination of [Boden and Friedl 2008, Lemma 2.2 and Theorem 3.3].

Theorem 2. *Suppose $\chi:H\to\mathbb{C}^*$ is a character that factors through $H/(t^n-1)$.*

(i) $\alpha_{(n,\chi)}:\mathbb{Z}\ltimes H\to\mathrm{SL}(n,\mathbb{C})$ *is irreducible if and only if the character χ has order n.*

(ii) *Given two characters* $\chi, \chi' : H \to \mathbb{C}^*$ *of order* n, *the representations* $\alpha_{(n,\chi)}$ *and* $\alpha_{(n,\chi')}$ *are conjugate if and only if* $\chi = t^k \chi'$ *for some* k.

(iii) *For any irreducible representation* $\alpha : \mathbb{Z} \ltimes H \to \mathrm{SL}(n, \mathbb{C})$, *there is a character* $\chi : H \to \mathbb{C}^*$ *of order* n *such that* α *is conjugate to* $\alpha_{(n,\chi)}$.

3. Main results

3.1. ***Metabelian characters as fixed points.*** Set $\omega = e^{2\pi i/n}$ and recall the action of the cyclic group $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1\rangle$ on representations $\alpha : \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ obtained by setting $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)}\alpha(g)$ for all $g \in \pi_1(N_K)$, where $\varepsilon : \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$.

Lemma 3. *Suppose* $\alpha : \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ *is a representation whose associated character* $\xi_\alpha \in X_n(N_K)$ *is a fixed point of the* $\mathbb{Z}/n$ *action. Up to conjugation,*

$$\alpha(\mu) = \begin{pmatrix} 0 & 0 & \cdots & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z & 0 \end{pmatrix} \tag{1}$$

for some (*in fact*, *any*) $z \in \mathrm{U}(1)$ *such that* $z^n = (-1)^{n+1}$.

Proof. Let $c(t) = \det(\alpha(\mu) - tI)$ denote the characteristic polynomial of $\alpha(\mu)$, which we can write as

$$c(t) = (-1)^n t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + 1.$$

Note that $c(t)$ is determined by the character $\xi_\alpha \in X_n(N_K)$ and so, assuming ξ_α is a fixed point of the $\mathbb{Z}/n$ action, we conclude that $\alpha(\mu)$ and $\omega^k\alpha(\mu)$ have the same characteristic polynomials for all k. In particular,

$$\begin{aligned} c(t) &= \det(\alpha(\mu) - tI) = \det(\omega^{-1}\alpha(\mu) - tI) \\ &= \det(\omega^{-1}\alpha(\mu) - (\omega^{-1}\omega)tI) = \det(\omega^{-1}I)\det(\alpha(\mu) - \omega tI) \\ &= \det(\alpha(\mu) - t\omega I) = c(\omega t). \end{aligned}$$

However, $\omega^k \neq 1$ unless n divides k, and this implies $0 = c_{n-1} = c_{n-2} = \cdots = c_1$ and $c(t) = (-1)^n t^n + 1$. In particular, the matrix $\alpha(\mu)$ and the matrix appearing in Equation (1) have the same set of n distinct eigenvalues. This implies that the two matrices are conjugate. $\square$

To prove Theorem 1, we establish the following more general result:

Theorem 4. *The fixed point set of the* $\mathbb{Z}/n$ *action on* $X_n(N_K)$ *consists of characters* ξ_α *of the metabelian representations* $\alpha = \alpha_{(n,\chi)}$ *described in Section 2.3. In other words,* $X_n(N_K)^{\mathbb{Z}/n} = \{\xi_\alpha \mid \alpha = \alpha_{(n,\chi)}$ *for* $\chi : H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*\}$.

Theorem 1 can be viewed as the special case of Theorem 4 when α is irreducible. (Recall that irreducible representations are conjugate if and only if they define the same character.) Note that not every reducible metabelian representation is of the form $\alpha_{(n,\chi)}$.

Proof. We first show that if $\alpha : \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is given as $\alpha = \alpha_{(n,\chi)}$, then $\sigma \cdot \alpha$ is conjugate to α. This of course implies that $\xi_\alpha = \xi_{\sigma\cdot\alpha}$.

Assume then that $\alpha = \alpha_{(n,\chi)}$. We have

$$\alpha(\mu) = \begin{pmatrix} 0 & 0 & \cdots & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z & 0 \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$. Also $\alpha(g)$ is diagonal for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. From the definition of $\sigma \cdot \alpha$, we see that

$$(\sigma \cdot \alpha)(\mu) = \omega\,\alpha(\mu) = \begin{pmatrix} 0 & 0 & \cdots & \omega z \\ \omega z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \omega z & 0 \end{pmatrix}$$

and that $(\sigma \cdot \alpha)(g) = \alpha(g)$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. It follows easily from Theorem 2(ii) that $\sigma \cdot \alpha$ and $\alpha_{(n,\chi)}$ are conjugate; however, it is easy to see this directly too. Simply take

$$P = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{pmatrix},$$

and compute that $\sigma \cdot \alpha = P\,\alpha\,P^{-1}$ as claimed.

We now show the other implication, namely, that each point $\xi \in X_n(N_K)^{\mathbb{Z}/n}$ in the fixed point set can be represented as the character $\xi = \xi_\alpha$ of a metabelian representation $\alpha = \alpha_{(n,\chi)}$, where $\chi : H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*$ is a character that factors through $H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^n - 1)$ and hence has order k for some k that divides n. (Note that Theorem 2(i) tells us that $\alpha_{(n,\chi)}$ is irreducible if and only if χ has order n.)

From the general results on representation spaces and character varieties (see [Lubotzky and Magid 1985]), it follows that every point in the character variety $X_n(N_K)$ can be represented as ξ_α for some semisimple representation $\alpha : \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$. Further, two semisimple representations α_1 and α_2 determine the same character if and only if α_1 is conjugate to α_2. (This is evident from the fact that the orbits of the semisimple representations under conjugation are closed.)

Given $\xi \in X_n(N_K)^{\mathbb{Z}/n}$, we can therefore suppose that $\xi = \xi_\alpha$ for some semisimple representation α. Clearly $\sigma \cdot \alpha$ is also semisimple, and since $\xi_\alpha = \xi_{\sigma \cdot \alpha}$, we conclude from the previous argument that α and $\sigma \cdot \alpha$ are conjugate representations. This means that there exists a matrix $A \in \mathrm{SL}(n, \mathbb{C})$ such that $A\alpha A^{-1} = \sigma \cdot \alpha$. In other words, for all $g \in \pi_1(N_K)$, we have

$$A\alpha(g)A^{-1} = \omega^{\varepsilon(g)}\alpha(g). \tag{2}$$

Lemma 3 implies that $\alpha(\mu)$ is conjugate to the matrix in Equation (1). It is convenient to conjugate α so that $\alpha(\mu)$ is diagonal, meaning that

$$\alpha(\mu) = \begin{pmatrix} z & & & 0 \\ & \omega z & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} z \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$.

We now apply (2) to the meridian to conclude that

$$A\alpha(\mu) = \omega\,\alpha(\mu)\,A,$$

which implies that $A = (a_{ij})$ satisfies $a_{ij} = 0$ unless $j = i + 1 \mod (n)$. Thus, we see that

$$A = \begin{pmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{n-1} \\ \lambda_n & 0 & \cdots & 0 & 0 \end{pmatrix}$$

for some $\lambda_1, \dots, \lambda_n$ satisfying $\lambda_1 \cdots \lambda_n = (-1)^{n+1}$.

It is completely straightforward to see that the characteristic polynomial of A is

$$\det(A - tI) = (-1)^n(t^n - (-1)^{n+1}).$$

From this, we conclude that A has as its eigenvalues the n distinct n-th roots of $(-1)^{n+1}$. In particular, the subset of matrices in $\mathrm{SL}(n, \mathbb{C})$ that commute with A is just a copy of the unique maximal torus $T_A \cong (\mathbb{C}^*)^{n-1}$ containing A.

For any $g \in [\pi_1(N_K), \pi_1(N_K)]$, we have $\alpha(g) = (\sigma \cdot \alpha)(g)$. Thus it follows that $A\alpha(g)A^{-1} = \alpha(g)$, and this implies that $\alpha(g) \in T_A$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. This shows that the restriction of α to the commutator subgroup $[\pi_1(N_K), \pi_1(N_K)]$ is abelian. We conclude from this that α is indeed metabelian. Notice that this, and an application of Theorem 2(iii), completes the proof in case α is irreducible.

In the general case, it follows from the discussion in Section 2.2 that α factors through $\mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Let $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Given a character $\chi : H \to \mathbb{C}^*$, we define the associated weight space V_χ by setting

$$V_\chi = \{v \in \mathbb{C}^n \mid \chi(h) \cdot v = \alpha(h)\, v \text{ for all } h \in H\}.$$

Recall that $A \cdot \alpha(h) \cdot A^{-1} = \alpha(h)$ for any $h \in H$. It is straightforward to show that A restricts to an automorphism of V_χ. Since H is abelian, there exists at least one character $\chi : H \to \mathbb{C}^*$ such that V_χ is nontrivial. For any i, denote by $t^i\chi$ the character given by $(t^i\chi)(h) = \chi(t^i h)$ for $h \in H$.

Note that A has n distinct eigenvalues and therefore is diagonalizable. Since A restricts to an automorphism of V_χ, there is an eigenvector v of A that lies in V_χ. Let λ be the corresponding eigenvalue. By the proof of [Boden and Friedl 2008, Theorem 2.3], the map $\alpha(\mu)$ induces an isomorphism $V_\chi \to V_{t\chi}$. We now calculate

$$A \cdot \alpha(\mu)\, v = (A\,\alpha(\mu) A^{-1}) \cdot Av = \omega\, \alpha(\mu) \cdot \lambda v = \lambda\omega \cdot \alpha(\mu)\, v;$$

that is, $\alpha(\mu)\, v \in V_{t\chi}$ is an eigenvector of A with eigenvalue $\omega\lambda$.

Iterating this argument, we see that $\alpha(\mu)^i v$ lies in $V_{t^i\chi}$ and is an eigenvector of A with eigenvalue $\omega^i\lambda$. Since ω is a primitive n-th root of unity, the eigenvalues $\lambda, \omega\lambda, \ldots, \omega^{n-1}\lambda$ are all distinct, and this implies that the corresponding eigenvectors $v, \alpha(\mu)\, v, \ldots, \alpha(\mu)^{n-1} v$ form a basis for $\mathbb{C}^n$.

Let m be the order of χ; that is, m is the minimal number such that $\chi = t^m\chi$. From the previous argument, we see that $\mathbb{C}^n$ is generated by $V_\chi, V_{t\chi}, \ldots, V_{t^m\chi}$. Since the characters $\chi, t\chi, \ldots, t^m\chi$ are pairwise distinct, it follows that $\mathbb{C}^n$ is given as the direct sum $V_\chi \oplus V_{t\chi} \oplus \cdots \oplus V_{t^{m-1}\chi}$.

We write $k = \dim_{\mathbb{C}}(V_\chi)$ and note that $n = km$. We note further that $\alpha(\mu)^m$ has eigenvalues

$$\{z^m, z^m e^{2\pi i/k}, \ldots, z^m e^{2\pi i(k-1)/k}\}, \tag{3}$$

and each eigenvalue has multiplicity m. Clearly $\alpha(\mu)^m$ restricts to an automorphism of $V_{t^i\chi}$ for $i = 0, \ldots, m-1$, and equally clearly we see that the restrictions all give conjugate representations. This implies that the restriction of $\alpha(\mu)^m$ to V_χ has eigenvalues in the set (3) above, each occurring with multiplicity 1. In particular, we can find a basis $\{v_1, \ldots, v_k\}$ for V_χ in which the matrix of $\alpha(\mu)^m$ has the form

$$\alpha(\mu^m) = \begin{pmatrix} 0 & 0 & \cdots & z^m \\ z^m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z^m & 0 \end{pmatrix}.$$

It is now straightforward to verify that, with respect to the ordered basis

$$\left\{\begin{array}{llll} v_1, & z^{-1}\alpha(\mu)\,v_1, & \cdots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_1, \\ v_2, & z^{-1}\alpha(\mu)\,v_2, & \cdots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_2, \\ \vdots & \vdots & & \vdots \\ v_k, & z^{-1}\alpha(\mu)\,v_k, & \cdots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_k \end{array}\right\},$$

α is given by $\alpha(n,\chi)$. □

3.2. ***Application to twisted Alexander polynomials.*** As an application, we prove a the following result regarding twisted Alexander polynomials of knots corresponding to metabelian representations. Denote by $\Delta^{\alpha}_{K,i}(t)$ the i-th twisted Alexander polynomial for a given representation $\alpha:\pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$, as presented in [Friedl and Vidussi 2009].

Proposition 5. *Let α be a metabelian representation of the form $\alpha=\alpha_{(n,\chi)}:\pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$. Then*

$$\Delta^{\alpha}_{K,0}(t)=\begin{cases}1-t^n & \text{if } \chi \text{ is trivial,}\\ 1 & \text{otherwise.}\end{cases}$$

Further, the twisted Alexander polynomial $\Delta^{\alpha}_{K,1}(t)$ is actually a polynomial in t^n.

Remark 6. In their paper, C. Herald, P. Kirk, and C. Livingston prove the same result using an entirely different approach; compare with [Herald et al. 2010, page 10] We also point out that Proposition 5 gives a positive answer to [Hirasawa and Murasugi 2009, Conjecture A].

Proof. The proof of the first statement is not difficult. It is immediate when χ is trivial, and it follows by a direct calculation when χ is nontrivial.

We turn to the proof of the second statement. For $\theta\in U(1)$ and any representation $\beta:\pi_1(N_K)\to \mathrm{GL}(n,\mathbb{C})$, define the θ-twist of β to be the representation that sends $g\in\pi_1(N_K)$ to $\theta^{\varepsilon(g)}\beta(g)$, where $\varepsilon:\pi_1(N_K)\to\mathbb{Z}$ is determined by the orientation of K. We denote the newly obtained representation by $\beta_\theta:\pi_1(N_K)\to \mathrm{GL}(n,\mathbb{C})$. Note that in case $\alpha:\pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ and $\theta=e^{2\pi ik/n}$ is an n-th root of unity, α_θ is again an $\mathrm{SL}(n,\mathbb{C})$ representation. The proof of the proposition relies on the formula

$$\Delta^{\beta_\theta}_{K,1}(t)=\Delta^{\beta}_{K,1}(\theta t). \tag{4}$$

This formula is well known, and follows directly from the definition of the twisted Alexander polynomial. Equation (4) combines with Theorem 1 to complete the proof, as we now explain. Take $\omega=e^{2\pi i/n}$. If $\alpha=\alpha_{(n,\chi)}$ is metabelian, Theorem 1 shows that its conjugacy class is fixed under the $\mathbb{Z}/n$ action. In particular, since α and α_ω are conjugate, Equation (4) shows that

$$\Delta^{\alpha}_{K,1}(t) = \Delta^{\alpha_\omega}_{K,1}(t) = \Delta^{\alpha}_{K,1}(\omega t).$$

Expanding $\Delta^{\alpha}_{K,1}(t) = \sum a_i t^i$ and using the fact that $t^k = (\omega t)^k$ if and only if k is a multiple of n, this shows that $a_k = 0$ unless k is a multiple of n. $\square$

Acknowledgments

We would like to thank Steven Boyer, Christopher Herald, Michael Heusener, Paul Kirk, Charles Livingston, Andrew Nicas, and Adam Sikora for generously sharing their knowledge, wisdom, and insight. We would also like to thank Fumikazu Nagasato and Yoshikazu Yamaguchi for communicating the results of their paper to us.

References

[Boden and Friedl 2008] H. U. Boden and S. Friedl, "Metabelian SL($n,\mathbb{C}$) representations of knot groups", *Pacific J. Math.* **238**:1 (2008), 7–25. MR 2010c:57007 Zbl 1154.57004

[Burde 1967] G. Burde, "Darstellungen von Knotengruppen", *Math. Ann.* **173** (1967), 24–33. MR 35 #3652 Zbl 0146.45602

[Burde and Zieschang 2003] G. Burde and H. Zieschang, *Knots*, 2nd ed., de Gruyter Studies in Mathematics **5**, de Gruyter, Berlin, 2003. MR 2003m:57005 Zbl 1009.57003

[Fox 1970] R. H. Fox, "Metacyclic invariants of knots and links", *Canad. J. Math.* **22** (1970), 193–201. MR 41 #6197 Zbl 0195.54002

[Friedl and Vidussi 2009] S. Friedl and S. Vidussi, "A survey of twisted Alexander polynomials", preprint, 2009. arXiv 0905.0591

[Hartley 1979] R. Hartley, "Metabelian representations of knot groups", *Pacific J. Math.* **82**:1 (1979), 93–104. MR 81a:57007 Zbl 0404.20032

[Hartley 1983] R. Hartley, "Lifting group homomorphisms", *Pacific J. Math.* **105**:2 (1983), 311–320. MR 84i:57004 Zbl 0513.57001

[Herald et al. 2010] C. Herald, P. Kirk, and C. Livingston, "Metabelian representations, twisted Alexander polynomials, knot slicing, and mutation", *Math. Z.* **265**:4 (2010), 925–949. MR 2652542

[Hirasawa and Murasugi 2009] M. Hirasawa and K. Murasugi, "Twisted Alexander polynomials of 2-bridge knots associated to metabelian representations", preprint, 2009. arXiv 0903.1689

[Jebali 2008] H. Jebali, "Module d'Alexander et représentations métabéliennes", *Ann. Fac. Sci. Toulouse Math.* (6) **17**:4 (2008), 751–764. MR 2010i:57018 Zbl 05543508

[Letsche 2000] C. F. Letsche, "An obstruction to slicing knots using the eta invariant", *Math. Proc. Cambridge Philos. Soc.* **128**:2 (2000), 301–319. MR 2001b:57017 Zbl 0957.57007

[Lin 2001] X. S. Lin, "Representations of knot groups and twisted Alexander polynomials", *Acta Math. Sin.* (*Engl. Ser.*) **17**:3 (2001), 361–380. MR 2003f:57018 Zbl 0986.57003

[Livingston 1995] C. Livingston, "Lifting representations of knot groups", *J. Knot Theory Ramifications* **4**:2 (1995), 225–234. MR 96i:57006 Zbl 0907.57008

[Lubotzky and Magid 1985] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **336**, American Mathematical Society, Providence, RI, 1985. MR 87c:20021

[Nagasato 2007] F. Nagasato, "Finiteness of a section of the SL(2, $\mathbb{C}$)-character variety of the knot group", *Kobe J. Math.* **24**:2 (2007), 125–136. MR 2009m:57024 Zbl 1165.57009

[Nagasato and Yamaguchi 2008] F. Nagasato and Y. Yamaguchi, "On the geometry of a certain slice of the character variety of a knot group", preprint, 2008. arXiv 0807.0714

[Neuwirth 1965] L. P. Neuwirth, *Knot groups*, Annals of Mathematics Studies **56**, Princeton University Press, 1965. MR 31 #734 Zbl 0184.48903

[de Rham 1967] G. de Rham, "Introduction aux polynômes d'un nœud", *Enseignement Math.* (2) **13** (1967), 187–194. MR 39 #2149

Received September 20, 2009. Revised February 9, 2010.

HANS U. BODEN
DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
1280 MAIN STREET WEST
HAMILTON, ONTARIO L8S 4K1
CANADA

boden@mcmaster.ca
http://www.math.mcmaster.ca/boden

STEFAN FRIEDL
MATHEMATICS INSTITUTE
ZEEMAN BUILDING
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL
UNITED KINGDOM

friedl@gmail.com
http://www.warwick.ac.uk/~masgaw

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 249 No. 1 January 2011

0030-8730(201101)249:1;1-D