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# LEWIS–ZAGIER CORRESPONDENCE FOR HIGHER-ORDER FORMS

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**The Lewis–Zagier correspondence attaches period functions to Maaß wave forms. We extend this correspondence to wave forms of higher order, which are higher-order invariants of the Fuchsian group in question. The key ingredient, an identification of higher-order invariants with ordinary invariants of unipotent twists, makes it possible to apply standard methods of automorphic forms to higher-order forms.**

## Introduction

The Lewis–Zagier correspondence — [Lewis 1997; Lewis and Zagier 1997; 2001]; see also [Bruggeman 1997] — is a bijection between the space of Maaß wave forms of a fixed Laplace-eigenvalue $\lambda$ and the space of real-analytic functions on the line, satisfying a functional equation that involves the eigenvalue. The latter functions are called period functions. In [Deitmar and Hilgert 2007], this correspondence was extended to subgroups $\Gamma$ of finite index in the full modular group $\Gamma(1)$. One can assume $\Gamma$ to be normal in $\Gamma(1)$. The central idea of the latter paper is to consider the action of the finite group $\Gamma(1)/\Gamma$ and, in this way, to consider Maaß forms for $\Gamma$ as vector-valued Maaß forms for $\Gamma(1)$. This technique can be applied to higher-order forms as well [Chinta et al. 2002; Deitmar 2009; Deitmar and Diamantis 2009; Diamantis and Sreekantan 2006; Diamantis et al. 2006; Diamantis and O'Sullivan 2008; Diamantis and Sim 2008], turning the somewhat unfamiliar notion of a higher-order invariant into the notion of a classical invariant of a twist by unipotent representation. In the case of Eisenstein series, this viewpoint has already been used in [Jorgenson and O'Sullivan 2008]. The general framework of higher-order invariants and unipotent twists is described in Section 1. This way of viewing higher-order forms has the advantage that it allows techniques of classical automorphic forms to be applied in the context of higher-order forms. The example of the trace formula will be treated in forthcoming work. This paper only applies this technique to extend the Lewis–Zagier correspondence to higher-order forms.

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We define the corresponding spaces of automorphic forms of higher order in Section 2. Various authors have defined holomorphic forms of higher order. Maaß forms are more subtle, as it is not immediately clear how to establish the $L^2$ structure on higher-order invariants. Deitmar and Diamantis [2009] resorted to the obvious $L^2$ structure for the quotient spaces of consecutive higher-order forms; however, this is unsatisfactory because one wishes to view $L^2$ higher-order forms as higher-order invariants themselves. In this paper this flaw is remedied: we give a space of locally square-integrable functions on the universal cover of the Borel–Serre compactification, whose higher-order invariants give the sought-for $L^2$ invariants.

We use a common definition of higher-order forms that insists on full invariance under parabolic elements. In the language of Section 1, that means that we take the subgroup $P$ to be the subgroup generated by all parabolic elements. It is an open question whether the results here can be extended to the case $P = \{1\}$, that is, to full higher-order invariants. In this case, Fourier expansions have to be replaced with Fourier–Taylor expansions, and thus it is unclear how the correspondence should be defined.

## 1. Higher-order invariants and unipotent representations

We describe higher-order forms by using invariants in unipotent representations.

Let $\Gamma$ be a group and let $W$ be a $\mathbb{C}[\Gamma]$-module. We take the field $\mathbb{C}$ of complex numbers as the base ring. Most of the general theory works over any ring, but our applications are over $\mathbb{C}$. Let $I_\Gamma$ be the augmentation ideal in $\mathbb{C}[\Gamma]$; that is, the kernel of the *augmentation homomorphism*

$$A : \mathbb{C}[\Gamma] \to \mathbb{C}, \quad \sum_\gamma c_\gamma \gamma \mapsto \sum_\gamma c_\gamma.$$

The ideal $I_\Gamma$ is a vector space with basis $\{\gamma - 1\}_{\gamma \in \Gamma \setminus \{1\}}$.

We will need two simple properties of the augmentation ideal that, for the reader's convenience, we will prove in the next lemma. A set $S$ of generators of the group $\Gamma$ is called *symmetric* when $s \in S$ if and only if $s^{-1} \in S$. It is easy to see that $\mathbb{C}[\Gamma] = \mathbb{C} \oplus I_\Gamma$ and that $I_\Gamma = \sum_{s \in S} \mathbb{C}[\Gamma](s - 1)$ for any set of generators $S$ of $\Gamma$. We also fix a normal subgroup $P$ of $\Gamma$. We let $I_P$ denote the augmentation ideal of $P$, and $\tilde{I}_P = \mathbb{C}[\Gamma]\, I_P$. As $P$ is normal, $\tilde{I}_P$ is a two-sided ideal of $\mathbb{C}[\Gamma]$. For any integer $q \geq 0$, we set $J_q = I_\Gamma^q + \tilde{I}_P$. The set of $\Gamma$-invariants $W^\Gamma = H^0(\Gamma, W)$ in $W$ can be described as the set of all $w \in W$ with $I_\Gamma w = 0$. For $q = 1, 2, \ldots$, we define the set of invariants of type $P$ and order $q$ to be

$$H^0_{q,P}(\Gamma, W) = \{w \in W : J_q w = 0\}.$$

Then $H^0_{q,P} = H^0_{q,P}(\Gamma, W)$ is a submodule of $W$, and we have a natural filtration

$$0 \subset H^0_{1,P} \subset H^0_{2,P} \subset \cdots \subset H^0_{q,P} \subset \cdots.$$

Since $I_\Gamma H^0_{q,P} \subset H^0_{q-1,P}$, the group $\Gamma$ acts trivially on $H^0_{q,P}/H^0_{q-1,P}$.

A representation $(\eta, V_\eta)$ of $\Gamma$ on a complex vector space $V_\eta$ is called a *unipotent length-q representation* if $V_\eta$ has a $\Gamma$-stable filtration $0 \subset V_{\eta,1} \subset \cdots \subset V_{\eta,q} = V_\eta$ such that $\Gamma$ acts trivially on each quotient $V_{\eta,k}/V_{\eta,k-1}$, where $k = 1, \dots, q$ and $V_{\eta,0} = 0$.

Let $(\eta, V_\eta)$ be a unipotent length-$q$ representation. Assume that it is $P$-trivial, that is, its restriction to the subgroup $P$ is the trivial representation. There is a natural map $\Phi_\eta : \mathrm{Hom}_\Gamma(V_\eta, W) \otimes V_\eta \to W$ given by $\alpha \otimes v \mapsto \alpha(v)$.

**Lemma 1.1.** *Let $W$ be a $\mathbb{C}[\Gamma]$-module. The submodule $H^0_{q,P}(\Gamma, W)$ constitutes a $P$-trivial, unipotent length-$q$ representation of $\Gamma$. If the group $\Gamma$ is finitely generated, then the space $H^0_{q,P}(\Gamma, W)$ is the sum of all images $\Phi_\eta$, when $\eta$ runs over the set of all $P$-trivial, unipotent length-$q$ representations that are finite-dimensional over $\mathbb{C}$.*

*Proof.* The first assertion is clear. Assume now that $\Gamma$ is finitely generated. The space $H^0_{q,P}(\Gamma, W)$ needn't be finite-dimensional. We use induction on $q$ to show that for each $w \in H^0_{q,\mathrm{par}}(\Gamma, W)$ the complex vector space $\mathbb{C}[\Gamma]v$ is finite-dimensional. For $q = 1$, we have $\mathbb{C}[\Gamma]w = \mathbb{C}w$, and the claim follows. Next, let $w \in H^0_{q+1,P}(\Gamma, W)$ and let $S$ be a finite set of generators of $\Gamma$. Then

$$\mathbb{C}[\Gamma]w = \mathbb{C}w + I_\Gamma w = \mathbb{C}w + \sum_{s \in S} \mathbb{C}[\Gamma](s-1)w.$$

Since $(s-1)w \in H^0_{q,P}(\Gamma, W)$, the claim follows from the induction hypothesis. □

From now on, assume that $\Gamma$ is finitely generated. The philosophy pursued in the rest of the paper is this:

> *Once you know* $\mathrm{Hom}_\Gamma(V_\eta, W)$ *for every $P$-trivial finite-dimensional unipotent length-$q$ representation, you know the space $H^0_{q,P}(\Gamma, W)$.*

So, instead of investigating $H^0_{q,P}(\Gamma, W)$, one should rather look at

$$\mathrm{Hom}_\Gamma(V_\eta, W) \cong (V_\eta^* \otimes W)^\Gamma,$$

which is often easier to handle. In fact, it is enough to restrict to a generic set of $\eta$.

As an example of this approach, consider the case $q = 2$. For each group homomorphism $\chi : \Gamma/P \to (\mathbb{C}, +)$, one gets a $P$-trivial unipotent length-$q$ representation $\eta_\chi$ on $\mathbb{C}^2$ given by

$$\eta_\chi(\gamma) = \begin{pmatrix} 1 & \chi(\gamma) \\ 0 & 1 \end{pmatrix}.$$

We introduce the notation

$$\overline{H}^0_{q,P} = \overline{H}^0_{q,P}(\Gamma, W) = H^0_{q,P}(\Gamma, W)/H^0_{q-1,P}(\Gamma, W) = H^0_{q,P}/H^0_{q-1,P}.$$

**Proposition 1.2.** *The space* $H^0_{2,P}(\Gamma, W)$ *is the sum of all images* $\Phi_{\eta_\chi}$ *when* $\chi$ *ranges over* $\mathrm{Hom}(\Gamma/P, \mathbb{C}) \setminus \{0\}$. *For any two* $\chi \neq \chi'$, *one has*

$$\mathrm{Im}(\Phi_{\eta_\chi}) \cap \mathrm{Im}(\Phi_{\eta_{\chi'}}) = H^0(\Gamma, W).$$

*In other words,*

$$\overline{H}^0_1 = \bigoplus_\chi \mathrm{Im}(\Phi_{\eta_\chi})/H^0.$$

*Proof.* We the *order-lowering operator*

$$\Lambda : \overline{H}^0_{q,P} \to \mathrm{Hom}(\Gamma/P, \overline{H}^0_{q-1,P}) \cong \mathrm{Hom}(\Gamma/P, \mathbb{C}) \otimes \overline{H}^0_{q-1,P},$$

where the last isomorphism exists because $\Gamma$ is finitely generated. This operator is defined by $\Lambda(w)(\gamma) = (\gamma - 1)w$. One sees that it is indeed a homomorphism in $\gamma$ by using the fact that $(\gamma\tau - 1) \equiv (\gamma - 1) + (\tau - 1) \mod I^2$ for any two $\gamma, \tau \in \Gamma$. The map $\Lambda$ is clearly injective.

Pick now $w \in \mathrm{Im}(\Phi_{\eta_\chi}) \cap \mathrm{Im}(\Phi_{\eta_{\chi'}})$ for $\chi \neq \chi'$. Then $\Lambda(w) \in \chi \otimes H^0 \cap \chi' \otimes H^0$, and the latter space is zero since $\chi \neq \chi'$. For surjectivity, pick $w \in H^0_1$. Then $\Lambda(w) = \sum_{i=1}^n \chi_i \otimes w_i$ with $w_i \in H^0$, and so $w \in \sum_{i=1}^n \mathrm{Im}(\varphi_{\eta_{\chi_i}})$. □

## 2. Higher-order forms

We now remedy the aforementioned shortcoming of [Deitmar and Diamantis 2009]. We also give a guide on how to set up higher-order $L^2$ invariants in more general cases (like general lattices in locally compact groups) when there is no gadget such as the Borel–Serre compactification around. In that case, Lemma 2.1 tells you how to define the $L^2$ structure once you have chosen a fundamental domain for the group action.

Let $G$ denote the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ and $K$ its maximal compact subgroup $\mathrm{PSO}(2) = \mathrm{SO}(2)/\{\pm 1\}$. Let $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$ be the full modular group and $\Gamma \subset \Gamma(1)$ be a normal subgroup of finite index that is torsion-free. For every cusp $c$ of $\Gamma$, fix $\sigma_c \in \Gamma(1)$ such that $\sigma_c \infty = c$ and

$$\sigma_c^{-1} \Gamma_c \sigma_c = \pm \begin{pmatrix} 1 & N_c\mathbb{Z} \\ 0 & 1 \end{pmatrix}.$$

The number $N_c \in \mathbb{N}$ is uniquely determined and is called the *width* of the cusp $c$. Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane, and let $\mathbb{O}(\mathbb{H})$ be the set of holomorphic functions on $\mathbb{H}$. We fix a weight $k \in 2\mathbb{Z}$ and define a (right) action of $G$ on functions $f$ on $\mathbb{H}$ by

$$f|_k\,\gamma(z) = (cz+d)^{-k} f\Big(\frac{az+b}{cz+d}\Big) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We define the space $\mathbb{O}^M_{\Gamma,k}(\mathbb{H})$ to be the set of all $f \in \mathbb{O}(\mathbb{H})$ such that for every cusp $c$ of $\Gamma$, the function $f|_k\sigma_c$ is bounded on the domain $\{\mathrm{Im}(z) > 1\}$ by a constant times $\mathrm{Im}(z)^A$ for some $A > 0$.

Further, we consider the space $\mathbb{O}^S_{\Gamma,k}(\mathbb{H})$ of all $f \in \mathbb{O}(\mathbb{H})$ such that for every cusp $c$ of $\Gamma$, the function $f|_k\,\sigma_c$ is bounded on the domain $\{\mathrm{Im}(z) > 1\}$ by a constant times $e^{-A\,\mathrm{Im}(z)}$ for some $A > 0$.

These two spaces are preserved not only by $\Gamma$, but also by the action of the full modular group $\Gamma(1)$.

The normal subgroup $P$ of $\Gamma$ will be the subgroup $\Gamma_{\mathrm{par}}$ generated by all parabolic elements. We write $H^0_{q,\mathrm{par}}$ for $H^0_{q,P}$, and we consider the space

$$M_{k,q}(\Gamma) = H^0_{q,\mathrm{par}}(\Gamma, \mathbb{O}^M_{\Gamma,k}(\mathbb{H}))$$

of modular functions of weight $k$ and order $q$, as well as the corresponding space

$$S_{k,q}(\Gamma) = H^0_{q,\mathrm{par}}(\Gamma, \mathbb{O}^S_{\Gamma,k}(\mathbb{H}))$$

of cusp forms. Then, every $f \in M_{k,q}(\Gamma)$ possesses a *Fourier expansion* at each cusp $c$ of the form

$$f|_k\,\sigma_c(z) = \sum_{n=0}^{\infty} a_{c,n}\, e^{2\pi i(n/N_c)z}.$$

A function $f \in M_{k,q}(\Gamma)$ belongs to the subset $S_{k,q}(\Gamma)$ if and only if $a_{c,0} = 0$ for every cusp $c$ of $\Gamma$.

Since the group $\Gamma$ is normal in $\Gamma(1)$, the latter acts on the finite-dimensional spaces $M_{k,q}(\Gamma)$ and $S_{k,q}(\Gamma)$. These therefore give examples of finite-dimensional representations of $\Gamma(1)$ that become unipotent length-$q$ when restricted to $\Gamma$.

By a *Maaß wave form* for the group $\Gamma$ and parameter $\nu \in \mathbb{C}$, we mean a function $u \in L^2(\Gamma\backslash\mathbb{H})$ that is twice continuously differentiable and satisfies

$$\Delta u = (\tfrac{1}{4} - \nu^2)u.$$

By the regularity of solutions of elliptic differential equations, this condition implies that $u$ is real analytic. Let $\mathcal{M}_\nu = \mathcal{M}_\nu(\Gamma)$ be the space of all Maaß wave forms for $\Gamma$. Sometimes in the definition of Maaß forms, instead of the $L^2$ condition, a weaker condition on the growth at the cusps is imposed.

Next, we define Maaß wave forms of higher order. First, we need the higher-order version of the Hilbert space $L^2(\Gamma\backslash\mathbb{H})$. For this, recall the construction of the Borel–Serre [1973] compactification $\overline{\Gamma\backslash\mathbb{H}}$ of $\Gamma\backslash\mathbb{H}$. One constructs a space

$\mathbb{H}_\Gamma \supset \mathbb{H}$ by attaching to $\mathbb{H}$ a real line at each cusp $c$ of $\Gamma$, and then one equips this set with a suitable topology such that $\Gamma$ acts properly discontinuously. The quotient $\Gamma\backslash\mathbb{H}_\Gamma$ is the Borel–Serre compactification.

The space $\mathbb{H}_\Gamma$ is constructed so that, for a given (closed) fundamental domain $D \subset \mathbb{H}$ of $\Gamma\backslash\mathbb{H}$ with finitely many geodesic sides, the closure $\overline{D}$ in $\mathbb{H}_\Gamma$ is a fundamental domain for $\Gamma\backslash\mathbb{H}_\Gamma$. By the discontinuity of the group action, this has the consequence that for every compact set $K \subset \mathbb{H}_\Gamma$, there exists a finite set $F \subset \Gamma$ such that $K \subset F\overline{D} = \bigcup_{\gamma\in F} \gamma\overline{D}$.

Now we extend the hyperbolic measure to $\mathbb{H}_\Gamma$ so that the boundary $\partial\mathbb{H}_\Gamma = \mathbb{H}_\Gamma\backslash\mathbb{H}$ is a nullset. Let $L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)$ be the space of local $L^2$ functions on $\mathbb{H}_\Gamma$. Then, $\Gamma$ acts on $L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)$. Since $\Gamma$ acts discontinuously with compact quotient on $\mathbb{H}_\Gamma$, one has

$$L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)^\Gamma = L^2(\Gamma\backslash\mathbb{H}).$$

We define $L^2_q(\Gamma\backslash\mathbb{H})$ as the space of all $f \in L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)$ such that $J_q f = 0$; in other words,

$$L^2_q(\Gamma\backslash\mathbb{H}) = H^0_{q,\mathrm{par}}(\Gamma, L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)).$$

Then, $L^2_1(\Gamma\backslash\mathbb{H}) = L^2(\Gamma\backslash\mathbb{H})$ is a Hilbert space in a natural way.

We want to endow the spaces $L^2_q(\Gamma\backslash\mathbb{H})$ with Hilbert space structures when $q \geq 2$ as well. For this, we introduce the space $F_q$ of all measurable functions $f : \mathbb{H} \to \mathbb{C}$ such that $J_q f = 0$ modulo nullfunctions. Then, $L^2_q(\Gamma\backslash\mathbb{H})$ is a subset of $F_q$.

**Lemma 2.1.** *Let $S \subset \Gamma$ be a finite set of generators, assumed to be symmetric and to contain the unit element. Let $D \subset \mathbb{H}$ be a closed fundamental domain of $\Gamma$ with finitely many geodesic sides.*

*Any $f \in F_q$ is uniquely determined by its restriction to*

$$S^{q-1}D = \bigcup_{s_1,\dots,s_{q-1}\in S} s_1 \dots s_{q-1}D.$$

*Further, one has*

$$F_q \cap L^2(S^{q-1}D) = L^2_q(\Gamma\backslash\mathbb{H}),$$

*where on both sides we mean the restriction to $S^qD$, which is unambiguous by the first assertion. In this way the space $L^2_q(\Gamma\backslash\mathbb{H})$ is a closed subspace of the Hilbert space $L^2(S^{q-1}D)$. The induced Hilbert-space topology on $L^2_q(\Gamma\backslash\mathbb{H})$ is independent of the choices of $S$ and $D$, although the inner product is not. The action of the group $\Gamma(1)$ on $L^2_q(\Gamma\backslash\mathbb{H})$ is continuous, but not unitary unless $q = 1$.*

*Proof.* We have to show that any $f \in L_q = L^2_q(\Gamma\backslash\mathbb{H})$ that vanishes on $S^{q-1}D$ is zero. We use induction on $q$. The case $q = 1$ is clear. Take $q \geq 2$ and write $\bar{L}_q = L_q/L_{q-1}$. Consider the *order-lowering operator*

$$\Lambda : L_q \to \mathrm{Hom}(\Gamma, \bar{L}_{q-1}) \cong \mathrm{Hom}(\Gamma, \mathbb{C}) \otimes \bar{L}_{q-1},$$

given by

$$\Lambda(f)(\gamma) = (\gamma - 1)f.$$

The kernel of $\Lambda$ is $L_{q-1}$. Assume $f(S^{q-1}D) = 0$. Then, for every $s \in S$, we have $(s-1)f(S^{q-2}D) = 0$, and hence by the induction hypothesis we conclude that $(s-1)f = 0$. However, since $S$ generates $\Gamma$, this means that $\Lambda(f) = 0$ and so $f \in L_{q-1}$, so again by the induction hypothesis, we get $f = 0$.

We next show that

$$F_q \cap L^2(S^{q-1}D) = F_q \cap L^2(S^{q-1+j}D) \quad \text{for every } j \geq 0.$$

The inclusion "$\supset$" is clear. We show the other inclusion by induction on $q$ and $j$. For $q = 1$ or $j = 0$, there is no problem. So, assume the claim proved for $q$. Let $f \in F_{q+1} \cap L^2(S^{q+j}D)$ and $s \in S$. Then $f(sz) = f(z) + f(sz) - f(z)$, the function $f(z)$ is in $L^2(S^{q+j}D)$, and the function $f(sz) - f(z)$ is in $F_q \cap L^2(S^{q+j-1}D) = L^2(S^{q+j}D)$ by the induction hypothesis. It follows that $f \in L^2(s^{-1}S^{q+j}D)$, and since this holds for every $s$, we get $f \in L^2(S^{q+j+1}D)$ as claimed.

We now come to

$$F_q \cap L^2(S^{q-1}D) = L^2_q(\Gamma \backslash \mathbb{H}).$$

Let $f \in F_q \cap L^2(S^{q-1}D)$. For every compact subset $K$ of $\mathbb{H}_\Gamma$, there exists $j \geq 0$ such that $K \subset S^{q-1+j}D$. Therefore $f$ is in $L^2(K)$ for every compact subset $K$ of $H_\Gamma$. Since the latter space is locally compact, $f$ is in $L^2_{\text{loc}}(\mathbb{H}_\Gamma)$. Since $I^q_\Gamma f = 0$, we get $f \in L^2_q(\Gamma \backslash \mathbb{H})$. For the other inclusion, let $f \in L^2_q(\Gamma \backslash \mathbb{H})$. Since $S^{q-1}D$ is relatively compact in $\mathbb{H}_\Gamma$, it follows that $f \in L^2(S^{q-1}D)$ as claimed.

We next prove the independence of the topology of $S$. Let $S'$ be another set of generators. There exists $l \in \mathbb{N}$ such that $S' \subset S^l$. Hence, it suffices to show that for every $j \geq 0$, the topology from the inclusion $L^2_q(\Gamma \backslash \mathbb{H}) \subset L^2(S^{q-1}D)$ coincides with the topology from the inclusion $L^2_q(\Gamma \backslash \mathbb{H}) \subset L^2(S^{q+j}D)$. If a sequence tends to zero in the latter topology, it clearly tends to zero in the first as well. The other way around is proved by an induction on $j$ similar to the one above. In particular, the continuity of the $\Gamma(1)$-action follows.

Finally, we prove the independence of $D$. Let $D'$ be another closed fundamental domain with finitely many geodesic sides. Then there exists $l \in \mathbb{N}$ such that $D' \subset S^l D$, and the claim follows along the same lines as before. $\square$

We define the space $\mathcal{M}_{\nu,q} = \mathcal{M}_{\nu,q}(\Gamma)$ of *Maaß wave forms of order $q$* to be the space of all $u \in L^2_q(\Gamma \backslash \mathbb{H})$ that are twice continuously differentiable and satisfy

$$\Delta u = (\tfrac{1}{4} - \nu^2)u.$$

Fix a finite-dimensional representation $(\eta, V_\eta)$ of $\Gamma(1)$ that is $\Gamma_{\text{par}}$-trivial and that becomes a unipotent length-$q$ representation on restriction to $\Gamma$.

We set $\mathcal{M}_{\nu,q,\eta}$ equal to $(V_\eta \otimes \mathcal{M}_{\nu,q})^{\Gamma(1)}$. Likewise, we define $\tilde{\mathcal{M}}_{\nu,q}(\Gamma) = \tilde{\mathcal{M}}_{\nu,q}$ as the space of all $u \in F_q(\Gamma)$ that are twice continuously differentiable and satisfy $\Delta u = (\frac{1}{4} - \nu^2)u$, and we set $\tilde{\mathcal{M}}_{\nu,q,\eta} = (V_\eta \otimes \tilde{\mathcal{M}}_{\nu,q})^{\Gamma(1)}$.

**Lemma 2.2.** *If $\mathcal{D}'_\nu$ is the space of all distributions $u$ on $\mathbb{H}$ with $\Delta u = (\frac{1}{4} - \nu^2)u$, then*

$$\tilde{\mathcal{M}}_{\nu,q,\eta} = (V_\eta \otimes \tilde{\mathcal{M}}_{\nu,q})^{\Gamma(1)} = (V_\eta \otimes \mathcal{D}'_\nu)^{\Gamma(1)} \quad \textit{and}$$
$$\mathcal{M}_{\nu,q,\eta} = (V_\eta \otimes \mathcal{M}_{\nu,q})^{\Gamma(1)} = (V_\eta \otimes (\mathcal{D}'_\nu \cap L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)))^{\Gamma(1)}.$$

*Proof.* The inclusion "$\subset$" is obvious in both cases. We show "$\supset$". In the first case, the space on the left can be described as the space of all smooth functions $u : \mathbb{H} \to V_\eta$ satisfying $\Delta u = (\frac{1}{4} - \nu^2)u$, as well as $J_{q+1}u = 0$ and $u(\gamma z) = \eta(\gamma)u(z)$ for every $\gamma \in \Gamma(1)$. Let $u \in (V_\eta \otimes \mathcal{D}'_\nu)^{\Gamma(1)}$. As $u$ satisfies an elliptic differential equation with smooth coefficients, $u$ is a smooth function with $\Delta u = (\frac{1}{4} - \nu^2)u$. The condition $u(\gamma z) = \eta(\gamma)u(z)$ is obvious. Finally, the condition $J_{q+1}u = 0$ follows from the fact that $\eta|_\Gamma$ satisfies $\eta(J_{q+1}) = 0$ since it is $\Gamma_{\mathrm{par}}$-trivial and unipotent of length $q$. Hence, the first claim is proven. The second is similar. □

As in the holomorphic case, every Maaß form $f \in \mathcal{M}_{\nu,q}(\Gamma)$ has a Fourier expansion

$$f(\sigma_c z) = \sum_{n=0}^{\infty} a_{c,n}(y)\, e^{2\pi i (n/N_c)x}$$

at each cusp $c$, with smooth functions $a_{c,n}(y)$.

We define the space $\mathcal{S}_{\nu,q}$ of *Maaß cusp forms* to be the space of all $f \in \mathcal{M}_{\nu,q}$ with $a_{c,0}(y) = 0$ for every cusp $c$. We also set $\mathcal{S}_{\nu,q,\eta} = (V_\eta \otimes \mathcal{S}_{\nu,q})^{\Gamma(1)}$.

Note that since $\eta$ is unipotent of length $q$ on $\Gamma$, we have $\mathcal{S}_{\nu,q,\eta} = (V_\eta \otimes \mathcal{S}_{\nu,q'})^{\Gamma(1)}$ for every $q' \geq q$.

## 3. Setting up the transform

It is the aim of this note to extend the Lewis correspondence [Deitmar and Hilgert 2007; Lewis 1997; Lewis and Zagier 1997; Lewis and Zagier 2001] to the case of higher-order forms. We will first explain our approach in the case of cusp forms.

Throughout, $(\eta, V_\eta)$ will be a finite-dimensional representation of $\Gamma(1)$ that becomes $\Gamma_{\mathrm{par}}$-trivial and unipotent of length $q$ when restricted to $\Gamma$.

For the canonical generators of $\Gamma(1)$, we fix the notation

$$S = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The, $S^2 = \mathbf{1} = (ST)^3$ and $T$ is of infinite order. Note that since $\Gamma(1)/\Gamma$ is a finite group, there exists $N \in \mathbb{N}$ such that $T^N \in \Gamma$; let $N$ be minimal with this property.

Then, $N = N_\infty$ is the width of the cusp $\infty$ of $\Gamma$. We have $\eta(T)^N = \eta(T^N) = 1$ since $\eta$ is trivial on the parabolic elements of $\Gamma$.

Let $\Psi_{\nu,\eta}$ be the space of all holomorphic functions $\psi : \mathbb{C} \setminus (-\infty, 0] \to V_\eta$ satisfying the *Lewis equation*

$$\eta(T)\,\psi(z) = \psi(z+1) + (z+1)^{-2\nu-1}\,\eta(ST^{-1})\,\psi\Big(\frac{z}{z+1}\Big), \tag{1}$$

and the *asymptotic formula*

$$\begin{aligned}0 = e^{+\pi i\nu} \lim_{\mathrm{Im}(z)\to\infty} \Big(\psi(z) + z^{-2\nu-1}\,\eta(S)\psi\Big(\frac{-1}{z}\Big)\Big) \\ + e^{-\pi i\nu} \lim_{\mathrm{Im}(z)\to-\infty} \Big(\psi(z) + z^{-2\nu-1}\,\eta(S)\psi\Big(\frac{-1}{z}\Big)\Big),\end{aligned} \tag{2}$$

where both limits are assumed to exist.

Let $A$ denote the subgroup of $G$ consisting of diagonal matrices, and let $N$ be the subgroup of upper-triangular matrices with $\pm 1$ on the diagonal. As a manifold, the group $G$ is a direct product $G = ANK$. For $\nu \in \mathbb{C}$ and $a = \pm\,\mathrm{diag}(t, t^{-1}) \in A$ with $t > 0$, let $a^\nu = t^{2\nu}$. (We insert the factor 2 for compatibility reasons.)

Let $(\pi_\nu, V_{\pi_\nu})$ denote the principal series representation of $G$ with parameter $\nu$. The representation space $V_{\pi_\nu}$ is the Hilbert space of all functions $\varphi : G \to \mathbb{C}$ with $\varphi(anx) = a^{\nu+1/2}\varphi(x)$ for $a \in A,\ n \in N,\ x \in G$, and $\int_K |\varphi(k)|^2 dk < \infty$ modulo nullfunctions. The representation is $\pi_\nu(x)\varphi(y) = \varphi(yx)$. There is a special vector $\varphi_0$ in $V_{\pi_\nu}$ given by $\varphi_0(ank) = a^{\nu+1/2}$. This vector is called the *basic spherical function* with parameter $\nu$.

For a continuous $G$-representation $(\pi, V_\pi)$ on a topological vector space $V_\pi$, let $\pi^\omega$ denote the subrepresentation on the space of analytic vectors; that is, $V_{\pi^\omega}$ consists of all vectors $v$ in $V_\pi$ such that for every continuous linear map $\alpha : V_\pi \to \mathbb{C}$, the map $g \mapsto \alpha(\pi(g)v)$ is real analytic on $G$. This space comes with a natural topology. Let $\pi^{-\omega}$ be its topological dual. In the case of $\pi = \pi_\nu$, it is known that $\pi_\nu^\omega$ and $\pi_\nu^{-\omega}$ are in perfect duality; that is, they are each other's topological duals. The vectors in $\pi_\nu^{-\omega}$ are called *hyperfunction vectors* of the representation $\pi_\nu$.

As a crucial tool, we will use the space

$$A_{\nu,\eta}^{-\omega} = (\pi_\nu^{-\omega} \otimes \eta)^{\Gamma(1)} = H^0(\Gamma(1), \pi_\nu^{-\omega} \otimes \eta)$$

and call it the space of *$\eta$-automorphic hyperfunctions*.

For an automorphic hyperfunction $\alpha \in A_{\nu,\eta}^{-\omega}$, we consider the function $u : G \to V_\eta$ given by

$$u(g) = \langle \pi_{-\nu}(g)\,\varphi_0, \alpha\rangle\ .$$

Here, $\langle\,\cdot\,,\cdot\,\rangle$ is the canonical pairing $\pi_{-\nu}^\omega \times \pi_{-\nu}^{-\omega} \otimes \eta \to V_\eta$. Then, $u$ is right $K$-invariant, and hence can be viewed as a function on $\mathbb{H}$. As such, it lies in $\tilde{\mathcal{M}}_{\nu,\eta}$ since $\alpha$ is $\Gamma$-equivariant, and the Casimir operator on $G$ (which induces $\Delta$) is scalar on $\pi_\nu$

with eigenvalue $\frac{1}{4}-\nu^2$. The transform $P:\alpha\mapsto u$ is called the *Poisson transform*. It follows from [Schlichtkrull 1984, Theorem 5.4.3] that the Poisson transform

$$P : A^{-\omega}_{\nu,\eta} \ \to \ \tilde{\mathcal{M}}_{\nu,\eta}$$

is an isomorphism for $\nu\notin\frac{1}{2}+\mathbb{Z}$.

For $\alpha\in A^{-\omega}_{\nu,\eta}$, set

$$\psi_\alpha(z)=f_\alpha(z)-z^{-2\nu-1}\,\eta(S)\,f_\alpha\Big(\frac{-1}{z}\Big),$$

with $f_\alpha$ such that the function $z\mapsto(1+z^2)^{\nu+1/2}f_\alpha(z)$ represents the restriction $\alpha|_{\mathbb{R}}$. Then, the *Bruggeman transform* $B:\alpha\mapsto\psi_\alpha$ maps $A^{-\omega}_{\nu,\eta}$ to $\Psi_{\nu,\eta}$. It is a bijection if $\nu\notin\frac{1}{2}+\mathbb{Z}$, as can be seen in a manner similar to [Deitmar and Hilgert 2007, Proposition 2.2].

When $\nu\notin\frac{1}{2}+\mathbb{Z}$, we finally define the *Lewis transform* as the map $L:\mathcal{M}_{\nu,\eta}\to\Psi_{\nu,\eta}$ given by $L=B\circ P^{-1}$.

**Theorem 3.1** (Lewis transform; see [Lewis and Zagier 2001, Theorem 1.1]). *When $\nu\notin\frac{1}{2}+\mathbb{Z}$ and $\Re(\nu)>-\frac{1}{2}$, the Lewis transform is a bijective linear map from the space of Maaß cusp forms $\mathcal{S}_{\nu,q,\eta}$ to the space $\Psi^o_{\nu,\eta}$ of period functions.*

*Proof.* The proof runs, with small obvious changes, along the lines of the corresponding result [Deitmar and Hilgert 2007, Theorem 3.3]. □

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