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ANALOGUES OF THE WIENER TAUBERIAN AND SCHWARTZ THEOREMS FOR RADIAL FUNCTIONS ON SYMMETRIC SPACES

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We prove a Wiener Tauberian theorem for the L^1 spherical functions on a semisimple Lie group of arbitrary real rank. We also establish a Schwartz-type theorem for complex groups. As a corollary we obtain a Wiener Tauberian type result for compactly supported distributions.

Introduction

Two celebrated theorems from classical analysis dealing with translation invariant subspaces are the Wiener Tauberian theorem and the Schwartz theorem. Let $f \in L^1(\mathbb{R})$ and $\tilde{f}$ be its Fourier transform. Then the Wiener Tauberian theorem says that the ideal generated by f is dense in $L^1(\mathbb{R})$ if and only if $\tilde{f}$ is a nowhere vanishing function on the real line.

The result due to L. Schwartz says that every closed translation invariant subspace V of $C^\infty(\mathbb{R})$ is generated by the exponential polynomials in V. In particular, such a V contains the function $x \to e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. Interestingly, this result fails for $\mathbb{R}^n$ if $n \geq 2$. Even though the exact analogue of the Schwartz theorem fails in this case, it follows from the well-known theorem of Brown, Schreiber and Taylor [Brown et al. 1973] that if $V \subset C^\infty(\mathbb{R}^n)$ is a closed subspace that is translation and rotation invariant, then V contains ψ_s for some $s \in \mathbb{C}$, where

$$\psi_s(x) = C J_{n/2-1}(s|x|)/(s|x|)^{n/2-1} = \int_{S^{n-1}} e^{isx.w} d\sigma(w).$$

Here $J_{n/2-1}$ is the Bessel function of the first kind and of order $n/2-1$ and σ is the unique, normalized rotation invariant measure on the sphere S^{n-1}. The constant C is such that $\psi_s(0) = 1$. It also follows from the work in [Brown et al. 1973] that

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V contains all the exponentials $e^{z.x}$ if $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ satisfies $z_1^2 + z_2^2 + \cdots + z_n^2 = s^2$ for nonzero s. For s vanishing, ψ_s is just the constant function one.

Our aim in this paper is to prove analogues of these results in the context of noncompact semisimple Lie groups.

1. Notation and preliminaries

For any unexplained terminology we refer to [Helgason 1994]. Let G be a connected noncompact semisimple Lie group with finite center and K a fixed maximal compact subgroup of G. Fix an Iwasawa decomposition $G = KAN$ and let $\mathfrak{a}$ be the Lie algebra of A. Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbb{C}}$ its complexification. Let ρ be the half sum of positive roots for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$, the Lie algebra of G. The Killing form induces a positive definite form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{a}^* \times \mathfrak{a}^*$. Extend this form to a bilinear form on $\mathfrak{a}^*_{\mathbb{C}}$. We will use the same notation for the extension as well. Let W be the Weyl group of the symmetric space G/K. Then there is a natural action of W on $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$, and $\langle \cdot\,, \cdot \rangle$ is invariant under this action.

For each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let φ_λ be the elementary spherical function associated with λ. Recall that φ_λ is given by the formula

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))} dk \quad \text{for } x \in G.$$

See [Helgason 1994] for more details. It is known that $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' = \tau\lambda$ for some $\tau \in W$. Let ℓ be the dimension of $\mathfrak{a}$ and F denote the set (in $\mathbb{C}^\ell$)

$$F = \mathfrak{a}^* + iC_\rho \quad \text{where } C_\rho = \text{convex hull of } \{s\rho : s \in W\}.$$

Then it is a well-known theorem of Helgason and Johnson that φ_λ is bounded if and only if $\lambda \in F$.

Let $I(G)$ be the set of all complex valued spherical functions on G, that is,

$$I(G) = \{f : f(k_1 x k_2) = f(x) \ \text{ for } k_1, k_2 \in K,\ x \in G\}.$$

Fix a Haar measure dx on G, and let $I_1(G) = I(G) \cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution and that the maximal ideal space of $I_1(G)$ can be identified with F/W.

For $f \in I_1(G)$, define its spherical Fourier transform $\hat{f}$ on F by

$$\hat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx.$$

Then $\hat{f}$ is a W-invariant bounded function on F that is holomorphic in the interior F^0 of F and is continuous on F. Also $\widehat{f * g} = \hat{f}\hat{g}$, where the convolution of f

and g is defined by

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy.$$

Next, we define the L^1-Schwartz space of K-biinvariant functions on G, which will be denoted by $S(G)$. Let $x \in G$. Then $x = k \exp X$ for $k \in K$ and $X \in \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Put $\sigma(x) = \|X\|$, where $\|\cdot\|$ is the norm on $\mathfrak{p}$ induced by the Killing form. For any left-invariant differential operator D on G and any integer $r \geq 0$, we define for a smooth K-biinvariant function f

$$p_{D,r}(f) = \sup_{x \in G} (1 + \sigma(x))^r |\varphi_0(x)|^{-2} |Df(x)|,$$

where φ_0 is the elementary spherical function corresponding to $\lambda = 0$. Define

$$S(G) = \{f : p_{D,r}(f) < \infty \text{ for all } D, r\}.$$

Then $S(G)$ becomes a Fréchet space when equipped with the topology induced by the family of seminorms $p_{D,r}$.

Let $\mathcal{P} = \mathcal{P}(\mathfrak{a}_{\mathbb{C}}^*)$ be the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}^*$. Then each $u \in \mathcal{P}$ gives rise to a differential operator $\partial(u)$ on $\mathfrak{a}_{\mathbb{C}}^*$. Let $Z(F)$ be the space of functions f on F such that

(i) f is holomorphic in F^0 (the interior of F) and continuous on F;

(ii) if $u \in P$ and $m \geq 0$ is any integer, then

$$q_{u,m}(f) = \sup_{\lambda \in F^0} (1 + \|\lambda\|^2)^m |\partial(u) f(\lambda)| < \infty;$$

(iii) f is W-invariant.

Then $Z(F)$ is an algebra under pointwise multiplication and a Fréchet space when equipped with the topology induced by the seminorms $q_{u,m}$.

If $a \in Z(F)$, we define the "wave packet" ψ_a on G by

$$\psi_a(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*} a(\lambda) \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda,$$

where $c(\lambda)$ is the well-known Harish-Chandra c-function. By the Plancherel theorem of Harish-Chandra, we also know that the map $f \to \hat{f}$ extends to a unitary map from $L^2(K \backslash G / K)$ onto $L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} d\lambda)$. We can now state a result of Trombi and Varadarajan [1971].

Theorem 1.1. (i) *If $f \in S(G)$, then $\hat{f} \in Z(F)$.*

(ii) *If $a \in Z(F)$, then the integral defining the "wave packet" ψ_a converges absolutely, and $\psi_a \in S(G)$. Moreover, $\hat{\psi}_a = a$.*

(iii) *The map* $f \to \hat{f}$ *is a topological linear isomorphism of* $S(g)$ *onto* $Z(F)$.

The plan of this paper is as follows. In Section 2, we prove a Wiener Tauberian theorem for $L^1(K\backslash G/K)$ assuming more symmetry on the generating family of functions. In Section 3, we establish a Schwartz-type theorem for complex semisimple Lie groups. As a corollary we also obtain a Wiener Tauberian-type theorem for compactly supported distributions on G/K.

2. A Wiener Tauberian theorem for $L^1(K\backslash G/K)$

Ehrenpreis and Mautner [1955] observed that an exact analogue of the Wiener Tauberian theorem is not true for the commutative algebra of K-biinvariant functions on the semisimple Lie group $\mathrm{SL}(2, \mathbb{R})$. Here K is the maximal compact subgroup $\mathrm{SO}(2)$. However, they did prove an analogue of the Wiener Tauberian theorem under an additional “not too rapidly decreasing condition” on the spherical Fourier transform: If f is a K-biinvariant integrable function on $G = \mathrm{SL}(2, \mathbb{R})$ whose spherical Fourier transform $\hat{f}$ does not vanish anywhere on the maximal ideal space (which can be identified with a certain strip on the complex plane), then f generates a dense subalgebra of $L^1(K\backslash G/K)$ provided $\hat{f}$ does not vanish too fast at ∞.

There have been a number of attempts to generalize these results to $L^1(K\backslash G/K)$ or $L^1(G/K)$, where G is a noncompact connected semisimple Lie group with finite center. Almost complete results have been obtained when G is a real rank one group. See [Benyamini and Weit 1992; Ben Natan et al. 1996; Sarkar 1998; Sitaram 1988] for results on rank one case. See also [Sarkar 1997] for a result on the whole group $\mathrm{SL}(2, \mathbb{R})$.

Sitaram [1980] proved that under suitable conditions on the spherical Fourier transform of a single function f, an analogue of the Wiener Tauberian theorem holds for $L^1(K\backslash G/K)$ with no assumptions on the rank of G. Recently, Narayanan [2009] improved this result to include the case of a family of functions rather than just a single function. One difference between rank one results and those of higher rank has been the precise form of the “not too rapid decay condition”. In [Sitaram 1980; Narayanan 2009], this condition on the spherical Fourier transform of a function is assumed to be true on the whole maximal domain, while for rank one groups it suffices impose this condition on $\mathfrak{a}^*$; see [Benyamini and Weit 1992; Sarkar 1998]. (An important corollary of this is that in the rank one case one can get a Wiener Tauberian-type theorem for a wide class of functions purely in terms of the nonvanishing of the spherical Fourier transform in a certain domain, without having to check any decay conditions; see [Mohanty et al. 2004, Theorem 5.5].) In the first part of this paper we show that such a stronger result is true for the higher rank case as well, provided we assume more symmetry on the generating family of functions, and again as a corollary we get a result of the type alluded to in the parenthesis above.

If $\dim \mathfrak{a}^* = \ell$, then $\mathfrak{a}^*_{\mathbb{C}}$ may be identified with $\mathbb{C}^\ell$ and a point $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ will be denoted $\lambda = (\lambda_1, \lambda_2, \dots \lambda_\ell)$. Denote by $r(\lambda)$ its radius $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_\ell^2)^{1/2}$. Let B_R denote the ball of radius R centered at the origin in $\mathfrak{a}^*$, and let F_R denote the domain in $\mathfrak{a}^*_{\mathbb{C}}$ defined by

$$F_R = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \|\mathrm{Im}(\lambda)\| < R\}.$$

For $a > 0$, let I_a denote the strip in the complex plane defined by

$$I_a = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < a\}.$$

Now, suppose that f is a holomorphic function on F_R and that f depends only on $r(\lambda)$. Then it is easy to see that $g(s) = f(\lambda_1, \lambda_2, \dots \lambda_\ell)$, where $s^2 = r(\lambda)^2$ defines an even holomorphic function on I_R and vice versa.

We will need some lemmas. Let $A(I_a)$ denote the collection of functions g such that

(i) g is even, bounded and holomorphic on I_a,

(ii) g is continuous on $\bar{I}_a$, and

(iii) $\lim_{|s|\to\infty} g(s) = 0$.

Then $A(I_a)$ with the supremum norm is a Banach algebra under pointwise multiplication.

Lemma 2.1. *Let $\{g_\alpha : \alpha \in \Lambda\}$ be a collection of functions in $A(I_a)$. Assume that there is no $s \in \bar{I}_a$ such that $g_\alpha(s) = 0$ for all $\alpha \in I$. Further assume that there exists $\alpha_0 \in I$ such that g_{α_0} does not decay very rapidly on $\mathbb{R}$, that is,*

$$\limsup_{|s|\to\infty} |g_{\alpha_0}(s)| e^{ke^{|s|}} > 0 \quad \textit{for all } k > 0.$$

Then the closed ideal generated by $\{g_\alpha : \alpha \in I\}$ is the whole of $A(I_a)$.

Proof. Let ψ be a suitable biholomorphic map that maps the strip I_a onto the unit disc; see [Benyamini and Weit 1992]. Let $h_\alpha(z) = g_\alpha(\psi(z))$. Then $h_\alpha \in A_0(D)$, where $A_0(D)$ is the collection of even holomorphic functions h on the unit disc that are continuous up to the boundary and satisfy $h(i) = h(-i) = 0$. The not too rapid decay condition on $\mathbb{R}$ is precisely what is needed to apply the Beurling–Rudin theorem to complete the proof. See the proofs of [Benyamini and Weit 1992, Theorem 1.1 and Lemma 1.2] for the details. □

Let p_t denote the K-biinvariant function defined by $\hat{p}_t(\lambda) = e^{-t\langle\lambda,\lambda\rangle}$. It is easy to see that $p_t \in S(G)$.

Lemma 2.2. *Let $J \subset L^1(K\backslash G/K)$ be a closed ideal. If $p_t \in J$ for some $t > 0$, then $J = L^1(K\backslash G/K)$.*

Proof. Since $\hat{p}_t$ has no zeros and does not decay too rapidly, this immediately follows from the main result in [Narayanan 2009] or [Sitaram 1980]. ☐

We say a function $f \in L^1(K\backslash G/K)$ is *radial* if the spherical Fourier transform $\hat{f}(\lambda)$ is a function of $r(\lambda)$. Notice that, if the group G is of real rank one, then the class of radial functions is precisely the class of K-biinvariant functions in $L^1(G)$. When the group G is complex, it is possible to describe the class of radial functions (see the next section). The following is our main theorem in this section:

Theorem 2.3. *Let $\{f_\alpha : \alpha \in I\}$ be a collection of radial functions in $L^1(K\backslash G/K)$. Assume that the spherical transform $\hat{f}_\alpha$ extends as a bounded holomorphic function to the bigger domain F_R, where $R > \|\rho\|$ with $\lim_{|\lambda|\to\infty} \hat{f}_\alpha(\lambda) = 0$ for all α and that there exists no $\lambda \in F_R$ such that $\hat{f}_\alpha(\lambda) = 0$ for all α. Further assume that there exists an α_0 such that $\hat{f}_{\alpha_0}$ does not decay too rapidly on $\mathfrak{a}^*$, that is,*

$$\limsup_{|\lambda|\to\infty} |\hat{f}_{\alpha_0}(\lambda)| \exp(ke^{|\lambda|}) > 0 \quad \textit{for all } k > 0.$$

Then the closed ideal generated by $\{f_\alpha : \alpha \in I\}$ is all of $L^1(K\backslash G/K)$.

Proof. Since f_α is radial, each $\hat{f}_\alpha$ gives rise to an even bounded holomorphic function $g_\alpha(s)$ on the strip I_R. If $|\rho| < a < R$, then the collection $\{g_\alpha(s) : \alpha \in I\}$ satisfies the hypotheses in Lemma 2.1 on the domain I_a. It follows that the family $\{g_\alpha\}$ generates $A(I_a)$. In particular, we have a sequence

$$h_1^n(s) g_{\alpha_1(n)}(s) + h_2^n(s) g_{\alpha_2(n)}(s) + \cdots + h_k^n(s) g_{\alpha_k(n)}(s) \to e^{-s^2/2}$$

uniformly on $\bar{I}_a$, where $g_{\alpha_j(n)}$ are in the given family and $h_j^n(s) \in A(I_a)$.

Each h_j^n can be viewed as a holomorphic function on the domain F_a contained in $\mathfrak{a}_{\mathbb{C}}^*$ that depends only on $r(\lambda)$. Since the h_j^n are bounded and $|\rho| < a$ it can be easily checked that $e^{-\langle\lambda,\lambda\rangle/2} h_j^n(\lambda) \in Z(F)$. Again, an application of the Cauchy integral formula says that

$$e^{-\langle\lambda,\lambda\rangle/2} h_1^n(\lambda) \hat{f}_{\alpha_1(n)}(\lambda) + e^{-\langle\lambda,\lambda\rangle/2} h_2^n(\lambda) \hat{f}_{\alpha_2(n)}(\lambda) + \cdots e^{-\langle\lambda,\lambda\rangle/2} h_k^n(\lambda) \hat{f}_{\alpha_k(n)}(\lambda)$$

converges to $e^{-\langle\lambda,\lambda\rangle}$ in the topology of $Z(F)$; see the proof of [Benyamini and Weit 1992, Theorem 1.1]. By Theorem 1.1, this simply means that the ideal generated by $\{f_\alpha : \alpha \in I\}$ in $L^1(K\backslash G/K)$ contains the function p , where $\hat{p}(\lambda) = e^{-\langle\lambda,\lambda\rangle}$. We finish the proof by appealing to Lemma 2.2. ☐

Corollary 2.4. *Let $\{f_\alpha : \alpha \in I\}$ be a family of radial functions satisfying the hypotheses of Theorem 2.3. Then the closed subspace spanned by the left G-translates of the this family is all of $L^1(G/K)$.*

Proof. Let J be the closed subspace generated by the left translates of the given family. By Theorem 2.3, $L^1(K\backslash G/K) \subset J$. Now, it is easy to see that J has to be equal to $L^1(G/K)$. □

Corollary 2.5. *Let* $\{f_\alpha : \alpha \in I\}$ *be a family of* L^1*-radial functions. Assume that each* $\hat{f}_\alpha$ *extends to a bounded holomorphic function on the bigger domain* F_R *for some* $R > \|\rho\|$. *Assume further that* $\lim_{\|\lambda\|\to\infty} \hat{f}_\alpha(\lambda) \to 0$. *If there exists an* α_0 *such that* f_{α_0} *is* not *equal to a real analytic function almost everywhere*, *then the left* G*-translates of the family above span a dense subset of* $L^1(G/K)$.

Proof. This follows exactly as in [Mohanty et al. 2004, Theorem 5.5]. □

3. Schwartz theorem for complex groups

When G is a connected noncompact semisimple Lie group of real rank one with finite center, a Schwartz-type theorem was proved by Bagchi and Sitaram [1979]. Let K be a maximal compact subgroup of G. Then their result states the following: Let V be a closed subspace of $C^\infty(K\backslash G/K)$ with the property that $f \in V$ implies $w * f \in V$ for every compactly supported K-biinvariant distribution w on G/K. Then V contains an elementary spherical function φ_λ for some $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. This was proved by establishing a one-one correspondence between ideals in $C^\infty(K\backslash G/K)$ and those of $C^\infty(\mathbb{R})_{\text{even}}$. This also proves that a similar result cannot hold for higher rank groups.

Going back to $\mathbb{R}^n$, we notice that if $f \in C^\infty(\mathbb{R}^n)$ is radial, then the translation invariant subspace V_f generated by f is also rotation invariant. It follows from [Brown et al. 1973] that V_f contains ψ_s for some $s \in \mathbb{C}$, where ψ_s is the Bessel function defined in the introduction. Our aim in this section is to prove a similar result for the complex semisimple Lie groups. Our definition of radiality, taken from [Volchkov and Volchkov 2008], coincides with the definition in the previous section when the function is in $L^1(K\backslash G/K)$.

Throughout this section we assume that G is a complex semisimple Lie group. Let $\mathrm{Exp} : \mathfrak{p} \to G/K$ denote the map $P \to (\exp P)K$. Then Exp is a diffeomorphism. If dx denotes the G-invariant measure on G/K, then

$$\int_{G/K} f(x)dx = \int_{\mathfrak{p}} f(\mathrm{Exp}\, P)J(P)dP, \tag{1}$$

where

$$J(P) = \det\Big(\frac{\sinh adP}{adP}\Big).$$

Since G is a complex group, the elementary spherical functions are given by a simple formula:

$$\varphi_\lambda(\mathrm{Exp}\, P) = J(P)^{-1/2} \int_K e^{i\langle A_\lambda, Ad(k)P\rangle} dk \quad \text{for } P \in \mathfrak{p}. \tag{2}$$

Here A_λ is the unique element in $\mathfrak{a}_\mathbb{C}$ such that $\lambda(H) = \langle A, A_\lambda\rangle$ for all $H \in \mathfrak{a}_\mathbb{C}$.

Let $E(K\backslash G/K)$ be the strong dual of $C^\infty(K\backslash G/K)$. Then $E(K\backslash G/K)$ can be identified with the space of compactly supported K-biinvariant distributions on G/K. If w is such a distribution, then $\hat{w}(\lambda) = w(\varphi_\lambda)$ is well defined and is called the spherical Fourier transform of w. By the Paley–Wiener theorem, we know that $\lambda \to \hat{w}(\lambda)$ is an entire function of exponential type. Similarly, $E(\mathbb{R}^\ell)$ will denote the space of compactly supported distribution on $\mathbb{R}^\ell$ and $E^W(\mathbb{R}^\ell)$ consists of the Weyl group invariant ones. From the work of Gangolli and others, as noted in [Bagchi and Sitaram 1979], we know that the Abel transform

$$S : E(K\backslash G/K) \to E^W(\mathbb{R}^\ell)$$

is an isomorphism and $\widetilde{S(w)}(\lambda) = \hat{w}(\lambda)$ for $w \in E(K\backslash G/K)$, where $\widetilde{S(w)}(\lambda)$ is the Euclidean Fourier transform of the distribution $S(w)$.

Proposition 3.1 [Bagchi and Sitaram 1979]. *There exists a linear topological isomorphism T from $C^\infty(K\backslash G/K)$ onto $C^\infty(\mathbb{R}^\ell)^W$ such that*

$$S(w)(T(f)) = w(f)$$

for all $w \in E(K\backslash G/K)$ and $f \in C^\infty(K\backslash G/K)$. We also have

$$S(w') * T(w * f) = T(w' * w * f)$$

for all $w, w' \in E(K\backslash G/K)$ and $f \in C^\infty(K\backslash G/K)$. Moreover,

$$T(\varphi_\lambda)(x) = \frac{1}{|W|}\sum_{\tau\in W} \exp(i\langle \tau.\lambda, x\rangle).$$

A K-biinvariant function f is called *radial* if it is of the form

$$f(x) = J(\operatorname{Exp}^{-1} x)^{-1/2} u(d(0, x)),$$

where d is the Riemannian distance induced by the Killing form on G/K and u is a function on $[0, \infty)$. Then [Volchkov and Volchkov 2008, Theorem 4.6] shows that this definition of radiality coincides with the one in the previous section if the function is integrable. That is, $f \in L^1(K\backslash G/K)$ has the above form if and only if the spherical Fourier transform $\hat{f}(\lambda)$ depends only on $r(\lambda)$. We denote the class of smooth radial functions by $C^\infty(K\backslash G/K)_{\mathrm{rad}}$, and $C_c^\infty(K\backslash G/K)_{\mathrm{rad}}$ will consist of compactly supported functions in $C^\infty(K\backslash G/K)_{\mathrm{rad}}$.

For $f \in C^\infty(K\backslash G/K)$ define

$$f^\#(\operatorname{Exp} P) = J(P)^{-1/2}\int_{\mathrm{SO}(\mathfrak{p})} J(\sigma.P)^{1/2} f(\sigma.P)\, d\sigma,$$

where $\mathrm{SO}(\mathfrak{p})$ is the special orthogonal group on $\mathfrak{p}$ and $d\sigma$ is the Haar measure on $\mathrm{SO}(\mathfrak{p})$. Here, by $f(P)$ we mean $f(\operatorname{Exp} P)$. Clearly, $f \to f^{\#}$ is the projection from $C^\infty(K\backslash G/K)$ onto $C^\infty(K\backslash G/K)_{\mathrm{rad}}$.

Proposition 3.2. (a) *The space* $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ *is reflexive.*

(b) *The strong dual* $E(K\backslash G/K)_{\mathrm{rad}}$ *of* $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ *is given by*

$$\{w \in E(K\backslash G/K) : \hat{w}(\lambda) \text{ is a function of } r(\lambda)\}.$$

(c) $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ *is invariant under convolution by* $w \in E(K\backslash G/K)_{\mathrm{rad}}$.

Proof. (a) The space $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ is a closed subspace of $C^\infty(K\backslash G/K)$, which is a reflexive Fréchet space.

(b) Define $B_\lambda = \varphi_\lambda^{\#}$, the projection of φ_λ into $C^\infty(K\backslash G/K)_{\mathrm{rad}}$. A simple computation shows that

$$B_\lambda(\operatorname{Exp} P) = J(P)^{-1/2} \int_{\mathrm{SO}(\mathfrak{p})} e^{i\langle A_\lambda, \sigma.P\rangle} d\sigma.$$

It is clear that B_λ as a function of λ depends only on $r(\lambda)$. Now, let $w \in E(K\backslash G/K)$. Define a distribution $w^{\#}$ by $w^{\#}(f) = w(f^{\#})$. It is easy to see that $w^{\#}$ is a compactly supported K-biinvariant distribution. Clearly, if $w \in E(K\backslash G/K)_{\mathrm{rad}}$, then $w = w^{\#}$. It follows that $\hat{w}(\lambda) = w(\varphi_\lambda) = w(B_\lambda)$. Consequently, $\hat{w}(\lambda)$ is a function of $r(\lambda)$. It also follows that $E(K\backslash G/K)_{\mathrm{rad}}$ is reflexive.

(c) If $w \in E(K\backslash G/K)_{\mathrm{rad}}$ and $g \in C_c^\infty(K\backslash G/K)_{\mathrm{rad}}$, then $w * g \in C_c^\infty(K\backslash G/K)_{\mathrm{rad}}$. This follows from (b) above and [Volchkov and Volchkov 2008, Theorem 4.6]. Next, if g is arbitrary, we may approximate g with $g_n \in C_c^\infty(K\backslash G/K)_{\mathrm{rad}}$. □

We can now state our main result in this section. Let V be a closed subspace of $C^\infty(K\backslash G/K)_{\mathrm{rad}}$. We say V is an ideal in $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ if $f \in V$ and $w \in E(K\backslash G/K)_{\mathrm{rad}}$ implies that $w * f \in V$.

Theorem 3.3. (a) *If V is a nonzero ideal in* $C^\infty(K\backslash G/K)_{\mathrm{rad}}$, *then there exists a* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *such that* $B_\lambda \in V$.

(b) *If* $f \in C^\infty(K\backslash G/K)_{\mathrm{rad}}$, *then the closed left G-invariant subspace generated by f in* $C^\infty(G/K)$ *contains* φ_λ *for some* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Proof. We closely follow the arguments in [Bagchi and Sitaram 1979].

(a) Notice that the map

$$S : E(K\backslash G/K)_{\mathrm{rad}} \to E(\mathbb{R}^\ell)_{\mathrm{rad}}$$

is a linear topological isomorphism. Using the reflexivity of the spaces involved and arguing as in [Bagchi and Sitaram 1979] we obtain that (as in Proposition 3.1)

$$T : C^\infty(K\backslash G/K)_{\mathrm{rad}} \to C^\infty(\mathbb{R}^\ell)_{\mathrm{rad}}$$

is a linear topological isomorphism, where $C^\infty(\mathbb{R}^\ell)_{\mathrm{rad}}$ stands for the space of C^∞ radial functions on $\mathbb{R}^\ell$ and

$$S(w)(T(f)) = w(f) \quad \text{for all } w \in E(K\backslash G/K)_{\mathrm{rad}},\ f \in C^\infty(K\backslash G/K)_{\mathrm{rad}}.$$

Another application of Proposition 3.1 implies that we have a bijection between the ideals in $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ and $C^\infty(\mathbb{R}^\ell)_{\mathrm{rad}}$. Here, an ideal in $C^\infty(\mathbb{R}^\ell)_{\mathrm{rad}}$ is a closed subspace invariant under convolution by compactly supported radial distributions on $\mathbb{R}^\ell$. From [Bagchi and Sitaram 1990] or [Brown et al. 1973], any ideal in $C^\infty(\mathbb{R}^\ell)_{\mathrm{rad}}$ contains ψ_s (Bessel function) for some $s \in \mathbb{C}$. To complete the proof it suffices to show that under the topological isomorphism T the function B_λ is mapped into ψ_s, where $s^2 = r(\lambda)^2$.

Now, we have $S(w)(T(B_\lambda)) = w(B_\lambda)$. Since $w \in E(K\backslash G/K)_{\mathrm{rad}}$, we know that $w(B_\lambda)$ is nothing but $w(\varphi_\lambda)$, which equals $\widetilde{(Sw)}(\lambda)$. Since S is onto, this implies that $T(B_\lambda) = \psi_s$, where $s^2 = r(\lambda)^2$.

(b) From [Bagchi and Sitaram 1979] we know that $T(\varphi_\lambda) = \Phi_\lambda$ where $\Phi_\lambda(x) = |W|^{-1}\sum_{\tau\in W}\exp(i\tau\lambda.x)$. Let V_f denote the left G-invariant subspace generated by f. Then $T(V_f)$ surely contains the space

$$V_{T(f)} = \{S(w) * T(f) : w \in E(K\backslash G/K)\}.$$

From Proposition 3.2, $T(f)$ is a radial C^∞ function on $\mathbb{R}^\ell$. Hence, from [Brown et al. 1973], the translation invariant subspace $X_{T(f)}$ generated by $T(f)$ in $C^\infty(\mathbb{R}^\ell)$ contains ψ_s for some $s \in \mathbb{C}$. Consequently, if $s \neq 0$, the space $X_{T(f)}$ will contain all the exponentials $e^{iz.x}$, where $z = (z_1, z_2, \ldots, z_\ell)$ satisfies $r(z)^2 = s^2$. If $s = 0$, then $X_{T(f)}$ contains the constant functions. Now, it is easy to see that the map $X_{T(f)} \to V_{T(f)}$, $x \mapsto |W|^{-1}\sum_{\tau\in W} g(\tau.x)$ is surjective. Hence, there exists a $\lambda \in \mathbb{C}^l$ such that $\Phi_\lambda \in V_{T(f)}$. Since $T(\varphi_\lambda) = \Phi_\lambda$, this finishes the proof. □

Our next result is a Wiener Tauberian-type theorem for compactly supported distributions. Let $E(G/K)$ denote the space of compactly supported distributions on G/K. If $g \in G$ and $w \in E(G/K)$, then the left g-translate of w is the compactly supported distribution ${}^g w$ defined by

$${}^g w(f) = w({}^{g^{-1}}f) \quad \text{for } f \in C^\infty(G/K),$$

where ${}^x f(y) = f(x^{-1}y)$.

Theorem 3.4. *Suppose $\{w_\alpha : \alpha \in I\}$ is a family of distributions contained in $E(K\backslash G/K)_{\mathrm{rad}}$. Then the left G-translates of this family span a dense subset of $E(G/K)$ if and only if there is no $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $\hat{w}_\alpha(\lambda) = 0$ for all $\alpha \in I$.*

Proof. We start with the "if" part of the theorem. Let J stand for the closed span of the left G-translates of the distributions w_α in $E(G/K)$. It suffices to show that $E(K\backslash G/K) \subset J$. To see this, let $f \in C^\infty(G/K)$ be such that $w(f) = 0$ for all $w \in E(K\backslash G/K)$. Since J is left G-invariant, we also have $w(f_g) = 0$ for all $g \in G$, where f_g is the K-biinvariant function defined by $f_g(x) = \int_K f(gkx)dk$. It follows that $f_g \equiv 0$ for all $g \in G$ and consequently $f \equiv 0$.

Next, we claim that $E(K\backslash G/K) \subset J$ if $E(K\backslash G/K)_{\mathrm{rad}} \subset J$. To prove this it is enough to show that

$$\{g * w : w \in E(K\backslash G/K)_{\mathrm{rad}},\, g \in C_c^\infty(K\backslash G/K)\}$$

is dense in $E(K\backslash G/K)$. By Proposition 3.2, the map S from $E(K\backslash G/K)$ onto $E(\mathbb{R}^\ell)^W$ is a linear topological isomorphism mapping $E(K\backslash G/K)_{\mathrm{rad}}$ onto $E(\mathbb{R}^\ell)_{\mathrm{rad}}$ isomorphically. Hence, it suffices to prove a similar statement for $E(\mathbb{R}^\ell)_{\mathrm{rad}}$ and $E(\mathbb{R}^\ell)^W$ — an easy exercise in distribution theory.

So, to complete the proof of Theorem 3.4 we only need to show that

$$\{g * w_\alpha : \alpha \in I,\, g \in C_c^\infty(K\backslash G/K)_{\mathrm{rad}}\}$$

is dense in $E(K\backslash G/K)_{\mathrm{rad}}$. If not, consider

$$J_{\mathrm{rad}} = \{f \in C^\infty(K\backslash G/K)_{\mathrm{rad}} : (g * w_\alpha)(f) = 0 \text{ for all } g \in C_c^\infty(K\backslash G/K), \alpha \in I\}.$$

This set is clearly a closed subspace of $C^\infty(K\backslash G/K)_{\mathrm{rad}}$ that is invariant under convolution by $C_c^\infty(K\backslash G/K)_{\mathrm{rad}}$. By Theorem 3.3 we have $B_\lambda \in J_{\mathrm{rad}}$ for some $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. It follows that $\hat{w}_\alpha(\lambda) = 0$ for all $\alpha \in I$, which is a contradiction. This finishes the proof.

For the “only if” part, it suffices to observe that if $g \in C_c^\infty(G/K)$ then

$$g * w_\alpha(\varphi_\lambda) = \widehat{g^\#}(\lambda)\hat{w}_\alpha(\lambda), \quad \text{where } g^\#(x) = \int_K g(kx)dk. \qquad \square$$

Remark. A single distribution $w \in E(K\backslash G/K)_{\mathrm{rad}}$ cannot generate the whole of $E(G/K)$ unless w is the measure supported at the identity coset. This is because $\hat{w}$ cannot have zeroes, and so by the Hadamard factorization theorem it has to be an exponential function, which in turn has to be a constant due to the Weyl group invariance.

Remark. A similar theorem for *all* rank one groups (not necessarily complex) may be derived from the results in [Bagchi and Sitaram 1990].

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