HOMOLOGY SEQUENCE AND EXCISION THEOREM FOR EULER CLASS GROUP

YONG YANG

Let $R$ be a Noetherian commutative ring with $\dim R = d$ and let $I$ be an ideal of $R$. For an integer $n$ such that $2n \geq d + 3$, we define a relative Euler class group $E^n(R, I; R)$. Using this group, in analogy to homology sequence of the $K_0$-group, we construct an exact sequence

$$E^n(R, I; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(p)} E^n(R/I; R/I),$$

called the homology sequence of the Euler class group. The excision theorem in $K$-theory has a corresponding theorem for the Euler class group. An application is that for polynomial and Laurent polynomial rings, we get short split exact sequences

$$0 \to E^n(R[t], (t); R[t]) \xrightarrow{E(p_2)} E^n(R[t]; R[t]) \xrightarrow{E(p)} E^n(R; R) \to 0$$

and

$$0 \to E^n(R[t, t^{-1}], (t - 1); R[t, t^{-1}]) \xrightarrow{E(p_2)} E^n(R[t, t^{-1}]; R[t, t^{-1}]) \xrightarrow{E(p)} E^n(R; R) \to 0.$$  

1. Introduction

Let $R$ be a Noetherian commutative ring of dimension $d$. The notion of the Euler class group of $R$ was introduced by Nori around 1990, with the aim of developing an obstruction theory for algebraic vector bundles over smooth affine varieties [Mandal 1992]. Later, the Nori’s definition was extended by S. M. Bhatwadekar and Raja Sridharan [2000]. Given a Noetherian commutative ring $R$ with dimension $d \geq 2$, they defined an obstruction group $E^d(R; R)$ also called the Euler class group (ECG). For $\mathbb{Q} \subseteq R$ and any projective $R$-module $P$ of rank $d$ with orientation $\chi : R \cong \wedge^d P$, they defined an obstruction class $e(P; \chi) \in E^d(R; R)$ and proved that $P \cong Q \oplus R$ if and only if $e(P; \chi) = 0$. After that, much work
on ECGs and weak ECGs was done. K. D. Das [2003; 2006] defined the ECG $E^d(R[t]; R[t])$ for a Noetherian commutative ring $R$ with $\dim R = d$, and proved that for a general such ring, the ECG $E^d(R; R)$ of $R$ is a direct summand of the ECG $E^d(R[t]; R[t])$, whereas if $R$ is a smooth affine domain over some perfect field $k$, then $E^d(R[t]; R[t]) \cong E^d(R; R)$. The ECG $E^d(R[t, t^{-1}]; R[t, t^{-1}])$ of a Laurent polynomial ring $R[t, t^{-1}]$ was defined in [Keshari 2007].

On the other hand, in K-theory we have a homology theory for the category of projective $R$-modules. The $K_0$-group $K_0(R)$ is closely related to the ECG of $R$. For example, Murthy’s Chern class of the projective $R$-module $P$, which is one of the sources of Euler class theory, is defined by an element in the $K_0$-group. For any Noetherian commutative ring $R$ of dimension $d$, there is a subgroup $F^dK_0(R)$ of $K_0(R)$ [Mandal 1998]. If $R$ is regular and contains the field of rational numbers $\mathbb{Q}$, we have a Riemann–Roch theorem saying that $E^d_0(R) \otimes \mathbb{Q} \cong F^dK_0(R) \otimes \mathbb{Q} \cong CH^d(R) \otimes \mathbb{Q}$, in which $E^d_0(R)$ is the weak ECG of $R$ and $CH^d(R)$ is the Chow group of codimension $d$ of $\text{Spec}(R)$ [Das and Mandal 2006].

Now let $l \subseteq R$ be an ideal of $R$. K-theory gives for $K_0$-groups the homology sequence $K_0(R, l) \rightarrow K_0(R) \rightarrow K_0(R/l)$. If the ring homomorphism $\rho : R \rightarrow R/l$ is split, then this homology sequence reduces to the short split exact sequence

$$0 \rightarrow K_0(R, l) \rightarrow K_0(R) \rightarrow K_0(R/l) \rightarrow 0,$$

which is said to be the excision sequence for $K_0$-groups.

Inspired by the correspondence between K-theory and ECGs, in this paper we establish ECG counterparts to the homology sequence and excision theorem of $K_0$-groups. These counterparts are Theorem 4.2 and Theorem 4.3, respectively.

Let $R$ be a Noetherian commutative ring with dimension $d$, and let $l$ be an ideal of $R$ with $\dim R/l = d - m$. For any integer $n$ such that $2n \geq d + 3$, we define in Section 3 a group homomorphism $E(\rho) : E^n(R; R) \rightarrow E^n(R/l; R/l)$, called the restriction map of the ECG. In analogy to the relative $K_0$-group $K_0(R, l)$ (denoted by $K_0(l)$ in [Rosenberg 1994]), we define in Section 4 the relative ECG $E^n(R, l; R)$ and the relative weak ECG $E^n_0(R, l)$. In particular, when $l = R$ the relative ECG $E^n(R, R; R)$ and the relative weak ECG $E^n_0(R, l)$ are the same as the generalized ECG $E^n(R; R)$ and the weak ECG $E^n_0(R)$, respectively. Using these groups, we construct an exact sequence

$$E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l),$$

the homology sequence of the ECG. If the ring homomorphism $\rho : R \rightarrow R/l$ has a splitting $\beta$ satisfying a dimensional condition (see Theorem 4.3 and Remark 4.4), then the homology sequence above reduces to the short split exact sequence

$$0 \rightarrow E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l) \rightarrow 0,$$
called the excision sequence of the ECG. Under these conditions, we have an isomorphism $E^n(R; R) \cong E^n(R, l; R) \oplus E^n(R/l; R/l)$. In Section 5, we use the results in Section 4 to get the excision sequences of the ECG for the polynomial extension and the Laurent polynomial extension:

$$0 \to E^n(R[t], (t); R[t]) \xrightarrow{E(p_2)} E^n(R[t]; R[t]) \xrightarrow{E(\rho)} E^n(R; R) \to 0$$

and

$$0 \to E^n(R[t, t^{-1}], (t - 1); R[t, t^{-1}]) \xrightarrow{E(p_2)} E^n(R[t, t^{-1}]; R[t, t^{-1}]) \xrightarrow{E(\rho)} E^n(R; R) \to 0.$$

Both of these are split exact; see Corollaries 5.1 and 5.3.

### 2. Some preliminary results

Here, recall the definition of the generalized ECG and collect some related results.

**Definition 2.1.** Let $R$ be a Noetherian commutative ring, $n$ be an integer such that $2n \geq d + 3$. In [Bhatwadekar and Sridharan 2002], the **generalized Euler class group** $E^n(R; R)$ is defined as follows:

Let $J \subseteq R$ be an ideal of height $n$, such that $J/J^2$ is generated by $n$ elements. Two surjections $\alpha$ and $\beta$ from $(R/J)^n$ to $J/J^2$ are said to be related if and only if there exists an elementary matrix $\delta \in \mathcal{E}_J(R/J)$ such that $\alpha\delta = \beta$. This defines an equivalence relation on the set of surjections from $(R/J)^n$ to $J/J^2$.

- Let $G^n$ be the free Abelian group on the set of pairs $(J; \omega_J)$, where $J \subseteq R$ is an ideal of height $n$, having the property that $\text{Spec}(R/J)$ is connected and $J/J^2$ is generated by $n$ elements, and $\omega_J : (R/J)^n \to J/J^2$ is an equivalence class of surjections.

- Now assume that $J \subseteq R$ be an ideal of height $n$ and $J/J^2$ is generated by $n$ elements. By [Bhatwadekar and Sridharan 2002, Lemma 4.1], $J$ has a unique decomposition $J = \bigcap_{i=1}^{r} J_i$ where ideals $J_i$ are pairwise comaximal and $\text{Spec}(R/J_i)$ is connected. Let $\omega_J : (R/J)^n \to J/J^2$ be a surjection. Then $\omega_J$ gives rise in a natural way to surjections $\omega_{J_i} : (R/J_i)^n \to J_i/J_i^2$. By $(J; \omega_J)$ we mean the element $\sum_{i=1}^{r} (J_i; \omega_{J_i})$ in $G^n$, and $(J; \omega_J)$ is called a local orientation.

- Let $H^n$ be the subgroup of $G^n$ generated by set of pairs $(J; \omega_J)$, where $J$ is an ideal of height $n$ generated by $n$ elements and $\omega_J : (R/J)^n \to J/J^2$ has the property that $\omega_J$ can be lifted to a surjection $\Theta : R^n \to J$. The **generalized Euler class group** $E^n(R; R)$ is defined by $E^n(R; R) = G^n/H^n$. 
Let \( \{e_i\} \) be the standard basis of \( \mathbb{R}^n \) and \( \alpha : \mathbb{R}^n \rightarrow J/J^2 \) be a surjection from \( \mathbb{R}^n \) to \( J/J^2 \) that sends \( \tilde{e}_i \) to \( \tilde{a}_i \) for \( 1 \leq i \leq n \), where \( a_i \in J \) and \( \{\tilde{a}_i\} \) generate \( J/J^2 \). In rest of this paper, we always use \((a_1, \ldots, a_n)\) to denote \( \alpha \).

The generalized weak ECG was defined in [Mandal and Yang 2010]:

- Let \( L_0^n \) denote the set of all ideals \( J \) of height \( n \) such that \( \text{Spec}(J/J^2) \) is connected and there is a surjection \( \alpha : (R/J)^n \rightarrow J/J^2 \). Let \( G_0^n \) be the free group generated on the set \( L_0^n \).
- For any ideal \( J \subseteq R \) of height \( n \) such that \( J/J^2 \) is generated by \( n \) elements, there is a unique decomposition \( J = \bigcap_{i=1}^r J_i \), where the ideals \( J_i \) are pairwise comaximal and \( \text{Spec}(R/J_i) \) is connected. By \( (J) \) we mean the element \( \sum_{i=1}^r (J_i) \) in \( G_0^n \).
- Let \( H_0^n \) be the subgroup of \( G_0^n \) generated by \( (J) \), where \( J \) could be generated by \( n \) elements. Then the generalized weak Euler class group is defined by \( E_0^n = G_0^n/H_0^n \).

**Theorem 2.2** [Bhatwadekar and Sridharan 2002, Theorem 4.2]. Suppose \( R \) is a \( d \)-dimensional Noetherian commutative ring, and let \( n \) be an integer with \( 2n \geq d + 3 \). Let \( J \subseteq R \) be an ideal of height \( n \) such that \( J/J^2 \) is generated by \( n \) elements, and let \( \omega_J : (R/J)^n \rightarrow J/J^2 \) be an equivalence class of surjections. Suppose that \( (J; \omega_J) \) is zero in the ECG \( E^n(R; R) \). Then, \( J \) is generated by \( n \) elements and \( \omega_J \) can be lifted to a surjection \( \Theta : \mathbb{R}^n \rightarrow J \).

The following lemma is easy to prove, so we omit the proof.

**Lemma 2.3.** Let \((I; \omega_I)\) and \((J; \omega_J)\) be two elements of \( E^n(R; R) \). Let surjections

\[
\omega_I : \mathbb{R}^n \xrightarrow{(a_1, \ldots, a_n)} I/I^2 \quad \text{and} \quad \omega_J : \mathbb{R}^n \xrightarrow{(b_1, \ldots, b_n)} J/J^2
\]

be representatives of the equivalence classes of \( \omega_I \) and \( \omega_J \), respectively. Suppose \( I \) and \( J \) are comaximal ideals of \( R \). Then by the Chinese remainder theorem, we can find a unique surjection

\[
\omega_{I \cap J} : \mathbb{R}^n \xrightarrow{(c_1, \ldots, c_n)} I \cap J/(I \cap J)^2,
\]

where the \( c_i \) are elements of \( I \cap J \) such that \( c_i = a_i \pmod{I^2} \) and \( c_i = b_i \pmod{J^2} \). Then \((I \cap J; \omega_{I \cap J}) \in E^n(R; R) \) is independent of choice representative in the equivalence classes \( \omega_I \) and \( \omega_J \), and \((I; \omega_I) + (J; \omega_J) = (I \cap J; \omega_{I \cap J}) \in E^n(R; R) \).

The next lemma is an adapted version of [Mandal and Yang 2010, Lemma 4.3]. We give a proof for this new form.

**Lemma 2.4** (transversal lemma). Suppose \( R \) is a Noetherian commutative ring with \( \dim R = d \). Assume \( I \) is an ideal of \( R \) with height \( I = n \) and \( \omega : \mathbb{R}^n \rightarrow I/J \) is
a surjection in which \( J \) is an ideal of \( R \) contained in \( I^2 \). Let \( l_1, \ldots, l_r \) be finitely many ideals of \( R \). Then we can find a surjective lift \( v : R^n \to I \cap K \) such that

\[
K + J = R, \quad \text{height } K \geq n, \quad \text{height}((K + l_i)/l_i) \geq n \quad \text{for any } l_i \text{ with } 1 \leq i \leq r.
\]

**Proof.** We use standard generalized dimension theory. First, there is a lift \( v_0 : R^n \to I \) of \( \omega \). Then \( I = (v_0(R^n), a) \) for some \( a \in J \).

Let \( \mathcal{P}_{n-1} \subseteq \text{Spec}(R) \) be the set of all prime ideals \( p \) with height \( p \leq n - 1 \) and \( a \notin p \). For any \( l_i \) such that \( 1 \leq i \leq r \), let \( \mathcal{P}_{i,n-1} \subseteq \text{Spec}(R) \) be the set of all prime ideals \( p \) such that \( p \supseteq l_i \) and \( a \notin p \), and \( \text{height}(p/l_i) \leq n - 1 \). Write \( \mathcal{P} = \bigcup_{i=1}^r \mathcal{P}_{i,n-1} \cup \mathcal{P}_{n-1} \).

Let \( d_0 : \mathcal{P}_{n-1} \to \mathbb{N} \) be the restriction of the usual dimension function and let \( d_i : \mathcal{P}_{i,n-1} \to \mathbb{N} \) be the dimension function induced by that on \( \text{Spec}(R_{a/l_{i,a}}) \) for \( 1 \leq i \leq r \). Then \( d_0 \) and \( d_i \) for \( 1 \leq i \leq r \) induce a generalized dimension function \( d : \mathcal{P} \to \mathbb{N} \); see [Mandal 1997] or [Plumstead 1983].

Now \( (v_0, a) \in R^n \otimes R \) is a basic element on \( \mathcal{P} \). Since \( \text{rank}(R^n) = n > d(p) \) for all \( p \in \mathcal{P} \), there is a \( \phi \in R^{n*} \) such that \( v = v_0 + a\phi \) is basic on \( \mathcal{P} \). Clearly, \( v \) is a lift of \( \omega \) and \( I = (v(R^n), a) \).

Since \( v \) is a lift of \( \omega \), we can write \( v(R^n) = I \cap K \), such that \( K + J = R \). It is routine to check that \( \text{height}(K) \geq n \) and \( \text{height}((K + l_i)/l_i) \geq n \) for \( 1 \leq i \leq r \).

**Lemma 2.5** (avoid lemma). Let \( R \) be a Noetherian commutative ring such that \( \text{dim } R = d \), and let \( l \subseteq R \) be an ideal of \( R \). Assume that \( I \) is an ideal of \( R \) and \( \phi : R^n \to I/I^2 \) is a surjective map. If there is a surjective map \( \psi : R^n \to (I + l)/l \) such that \( \tilde{\phi} = \psi \otimes (R/(I + l)) = \psi \otimes (\overline{R}/\overline{I}) \), in which the bar denotes the reduction modulo \( l \), and \( \tilde{\phi} \) is the surjective map \( \tilde{\phi} : R^n \to (I + l)/(I^2 + l) \cong \overline{I}/\overline{I^2} \) induced by \( \phi \), then we can find a surjective lift \( \tilde{\phi} : R^n \to I/(I^2 l) \) of \( \phi \).

**Proof.** Let \( \phi_1 : R^n \to I \) and \( \psi_1 : R^n \to I \) be lifts of \( \phi \) and \( \psi \), respectively. Since \( \tilde{\phi} = \psi \otimes R/(I + l) \), we have \( \phi_1 - \psi_1 \in \text{Hom}(R^n, I^2 + l) \). Then we can find \( \alpha \in \text{Hom}(R^n, I^2) \) and \( \beta \in \text{Hom}(R^n, l) \) such that \( \phi_1 - \psi_1 = \alpha + \beta \). This can be seen from the commutative diagram

\[
\begin{array}{ccc}
R^n & \xrightarrow{(\alpha, \beta)} & R^n \\
\downarrow{\phi_1 - \psi_1} & & \\
I^2 \oplus l & \longrightarrow & I^2 + l,
\end{array}
\]

in which \( (\alpha, \beta) \in \text{Hom}(R^n, I^2 \oplus l) \) is a lift of \( \phi_1 - \psi_1 \).

Now we construct a map \( \phi_2 = \phi_1 - \alpha \in \text{Hom}(R^n, I) \). Of course, \( \phi_2 \) is still a lift of \( \phi \). Let the bar denote the reduction modulo \( l \). Then since \( \phi_2 = \phi_1 - \alpha = \psi_1 + \beta \), and \( \beta \in \text{Hom}(R^n, l) \), we have \( \overline{\phi_2} = \overline{\psi_1} = \psi \). Recall that \( \psi \) is surjective, so it is clear that \( \phi_2(R^n) + I \cap l = I \). Now consider the ideal \( \phi_2(R^n) + I^2 l \). Since \( \phi_2(R^n) + I \cap l = \phi_2(R^n) + I^2 = I \), it follows that any prime ideal \( p \) of \( R \) contains
Moreover by Lemma 2.4, we can find \( \varphi \) is a surjective lift of \( \omega \). So we get \( \varphi_2(R^n) + I^2l = I \).

Now let \( \hat{\varphi} : R^n \twoheadrightarrow I/(I^2l) \) be the map induced by \( \varphi_2 \). It’s obvious that \( \hat{\varphi} \) is a surjective lift of \( \varphi \).

\[ \square \]

### 3. Restriction and extension map

In this section, we construct two group homomorphisms, the restriction map and extension map for the ECG.

Let \( \varphi : R \to A \) be a ring homomorphism and \( I \subseteq R \) be an ideal of \( R \). In the rest of the paper, \( \varphi(I) \) without special decorations will always denote the ideal \( \varphi(I)A \), which is the ideal of \( A \) generated by \( \varphi(I) \).

**Definition 3.1** (restriction map). Let \( R \) be a Noetherian commutative ring with \( \dim R = d \), and \( l \subseteq R \) be an ideal of \( R \) with \( \dim R/l = d - m \). Let the bar denote the reduction modulo \( l \), and let \( \rho : R \to R/l \) denote the natural ring homomorphism. For an integer \( n \) such that \( 2n \geq d + 3 \), let \( E^n(R; R) \) and \( E^n(R/l; R/l) \) denote the generalized ECG of \( R \) and \( R/l \), respectively, as defined in [Bhatwadekar and Sridharan 2002]. Then we can define a group homomorphism \( E(\rho) : E^n(R; R) \to E^n(R/\bar{I}; \bar{R}) \), called the restriction map of ECG, as follows:

For any element \( x \in E^n(R; R) \), from the properties of the group \( E^n(R; R) \), we know that \( x \) can be written as a pair of \( (I; \omega_I) \in E^n(R; R) \), where \( I \) is an ideal of \( R \) with height \( I \geq n \), and \( \omega_I \) is an equivalence class of surjections \( \omega_I : R^n \twoheadrightarrow I/I^2 \). Moreover by Lemma 2.4, we can find \( (I; \omega_I) = (I'; \omega_{I'}) \in E^n(R; R) \) such that height \( I' + l \geq n \) in \( \bar{R} \). Then we define \( E(\rho)(I; \omega_I) = (I' + l; \omega_{\bar{I}' + l}) \in E^n(\bar{R}; \bar{R}) \), in which \( \omega_{\bar{I}' + l} \) is the equivalence class of induced surjection defined as:

\[
\omega_{\bar{I}' + l} : R^n \xrightarrow{\omega_{I'}} \frac{I'}{I'^2} \xrightarrow{\tilde{\gamma}} \frac{I' + l}{I'^2 + l} \cong \frac{(I' + l)}{(I' + l)^2},
\]

where \( \tilde{\gamma} \) is the natural map from \( I'/I'^2 \) to \( (I' + l)/(I'^2 + l) \), and \( \omega_{I'} \) is any representative of the equivalence class \( \omega_{I'} \).

(1) Since the map \( \varphi \) is surjective, we know that the element \( E(\rho)(I; \omega_I) \) is independent of choice of the representative of \( \omega_{I'} \).

(2) If \( (I; \omega_I) = 0 \in E^n(R; R) \), then \( E(\rho)(I; \omega_I) = 0 \in E^n(\bar{R}; \bar{R}) \).

**Proof of (2).** Since \( (I; \omega_I) = 0 \in E^n(R; R) \), there exists by Theorem 2.2 a surjective lift of \( \omega_{I'} \), denoted by \( v_{I'} \). Then it is easy to check that \( v_{I'} : R^n \twoheadrightarrow I' \twoheadrightarrow (I' + l)/l \) is a surjective lift of \( \omega_{\bar{I}' + l} \). So we have \( E(\rho)(I; \omega_I) = 0 \in E^n(\bar{R}; \bar{R}) \).

(3) \( E(\rho)(I; \omega_I) \) is independent of choice of the element \( (I'; \omega_{I'}) \).
Proof of (3). If there is another element \((I''; \omega_{I''}) \in E^n(R; R)\) such that \((I''; \omega_{I''}) = (I; \omega_I)\), and height \(I'' + l \geq n\) in \(R\), then by Lemma 2.4 we can find \((K; \omega_K)\) in \(E^n(R; R)\) such that

\[
K + I = K + I' = K + I'' = R,
\]

height \(K + l \geq n\),

\((K; \omega_K) + (I; \omega_I) = 0 \in E^n(R; R).
\]

By Lemma 2.3, \((K; \omega_K) + (I'; \omega_{I'}) = (K \cap I'; \omega_{K \cap I'}) = (K; \omega_K) + (I''; \omega_{I''}) = (K \cap I''; \omega_{K \cap I''}) = 0 \in E^n(R; R)\). Then, from the properties of the ECG and the result (2) above, it can be easily checked that

\[
E(\rho)(K \cap I'; \omega_{K \cap I'}) = (K \cap I' + l; \omega_{K \cap I' + l}) = (K + l; \omega_{K + l}) + (I' + l; \omega_{I' + l}) = E(\rho)(K; \omega_K) + E(\rho)(I'; \omega_{I'}) = E(\rho)(K \cap I''; \omega_{K \cap I''}) = (K \cap I'' + l; \omega_{K \cap I'' + l}) = E(\rho)(K; \omega_K) + E(\rho)(I''; \omega_{I''}) = (K + l; \omega_{K + l}) + (I'' + l; \omega_{I'' + l}),
\]

which is equal to zero. Therefore, \(E(\rho)(I'; \omega_{I'}) = E(\rho)(I''; \omega_{I''})\). This shows that \(E(\rho)(I, \omega_I)\) is independent of the choice of the element \((I'; \omega_{I'})\).

\(\square\)

(4) If \((I; \omega_I) = (J; \omega_J) \in E^n(R; R)\), then

\[
E(\rho)(I; \omega_I) = E(\rho)(J; \omega_J) \in E^n(R/l; R/l).
\]

(5) For any elements \(x, y \in E^n(R; R)\), we have

\[
E(\rho)(x) + E(\rho)(y) = E(\rho)(x + y).
\]

Proof of (5). Let \(x = (I; \omega_I), y = (J; \omega_J)\) be two elements of \(E^n(R; R)\). By the method we used above, we may further assume that \(I + J = R\). Now define maps

\[
\omega_I : R^n \xrightarrow{(i_1, \ldots, i_n)} I/I^2, \quad \text{where } i_r \in I \text{ for } 1 \leq r \leq n,
\]

\[
\omega_J : R^n \xrightarrow{(j_1, \ldots, j_n)} J/J^2, \quad \text{where } j_r \in J \text{ for } 1 \leq r \leq n,
\]

Since \(I\) and \(J\) are comaximal, by Lemma 2.3, we can find a surjection

\[
\omega_{I \cap J} : R^n \xrightarrow{(k_1, \ldots, k_n)} \frac{I \cap J}{(I \cap J)^2}
\]

such that \(k_r \in I \cap J\) and \(k_r = i_r \mod I^2\) and \(k_r = j_r \mod J^2\) for \(1 \leq r \leq n\), that is, \(x + y = (I \cap J; \omega_{I \cap J}) \in E^n(R; R)\). Hence, if the bar denotes reduction
modulo l, we get
\[(E(\rho)(x) + E(\rho)(y)) = (\tilde{I}; (\tilde{i}_1, \ldots, \tilde{i}_n)) + (\tilde{J}; (\tilde{j}_1, \ldots, \tilde{j}_n))
= (\tilde{I}; (\tilde{k}_1, \ldots, \tilde{k}_n)) + (\tilde{J}; (\tilde{k}_1, \ldots, \tilde{k}_n)) = (\tilde{I} \cap \tilde{J}; (\tilde{k}_1, \ldots, \tilde{k}_n))
= E(\rho)(\tilde{I} \cap \tilde{J}; \omega_{\tilde{I} \cap \tilde{J}}) = E(\rho)(x + y). \]

(6) If \( n > d - m \), the map \( E(\rho) \) vanishes.

**Proof of (6).** This comes from the fact that in this case \( E^n(R/l; R/l) = 0 \). \( \square \)

By all of the above, the group homomorphism \( E(\rho) : E^n(R; R) \to E^n(R; R) \) is well-defined.

**Definition 3.2** (extension map). Let \( R \) and \( A \) be Noetherian commutative rings with dimension \( d \) and \( s \), respectively. Let \( n \) be an integer with \( 2n \geq d + 3 \) and \( 2n \geq s + 3 \). If there is a ring homomorphism \( \phi : R \to A \) such that

\[ (\star) \quad \text{height } \phi(I) \geq n \text{ for any local } n\text{-orientation } \omega_I : R^n \to I/I^2. \]

then similarly to the above definition, we can construct a group homomorphism \( E(\phi) : E^n(R; R) \to E^n(A; A) \), called the extension map of the ECG, as follows

Let \( x = (I; \omega_I) \in E^n(R; R) \) be any element, and suppose that \( \omega_I \) is the surjective map

\[ R^n \xrightarrow{(i_1, \ldots, i_n)} I/I^2 \quad \text{in which } i_t \in I \text{ for } 1 \leq t \leq n. \]

Then we define \( E(\phi)(x) \) by \( (\phi(I); \omega_{\phi(I)}) \in E^n(A; A) \), where \( \omega_{\phi(I)} \) is the surjection

\[ A^n \xrightarrow{\phi(i_1), \ldots, \phi(i_n)} \phi(I)/\phi(I)^2. \]

By a method similar to the one used in Definition 3.1, it can be checked that \( E(\phi) \) is indeed a group homomorphism.

By forgetting the orientation in Definitions 3.1 and 3.2, we have the following for the weak ECG.

**Definition 3.3.** Let \( R \) and \( l \) be as in Definition 3.1. Let \( \phi : R \to R/l \) be the natural ring homomorphism. For an integer \( n \) with \( 2n \geq d + 3 \), there is a group homomorphism \( E_0(\phi) : E^n_0(R) \to E^n_0(R/l) \), which is called the restriction map of the weak ECG.

Similarly, let \( R, A \) and \( n \) be as in Definition 3.2, and let \( \phi : R \to A \) be a ring homomorphism satisfying the condition (\( \star \)). Then there is a group homomorphism \( E_0(\phi) : E^n_0(R) \to E^n_0(A) \), which is called the extension map of the weak ECG.
4. The relative ECG and homology sequence

In this section, in analogy to related notions in K-theory, we define the relative and
relative weak ECGs. Using these groups, we construct homology sequences for
the ECG, which are the counterparts of homology sequence for $K_0$-groups. Also
we will give excision theorems for the ECG, which are the counterparts of excision
theorem for $K_0$-groups.

**Definition 4.1.** Let $R$ be a Noetherian commutative ring with $\dim R = d$, and let
$l$ be an ideal of $R$. Then we have the double $D(R, l)$ of $R$ along $l$ as the subring
of the Cartesian product $R \times R$, given by

$$D(R, l) = \{(x, y) \in R \times R : x - y \in l\}.$$  

Note that if $p_1$ denotes the projection onto the first coordinate, then there is a split
exact sequence

$$0 \to l \to D(R, l) \xrightarrow{p_1} R \to 0$$

in the sense that $p_1$ is split surjective (with splitting map given by the diagonal
embedding of $R$ in $D(R, l)$ and with $\ker p_1$ identified with $l$.)

Since $D(R, l)$ is finite over the subring $R$ (given by the diagonal embedding),
we get $\dim D(R, l) = \dim R = d$, and $\height(\ker p_1) = 0$ (with $\ker p_1$ being regarded
as an ideal of $D(R, l)$).

Then for any integer $n$ with $2n \geq d + 3$, the relative ECG of $R$ and $l$ is defined
by

$$E^n(R, l; R) = \ker(E(p_1) : E^n(D(R, l); D(R, l)) \to E^n(R; R)).$$

and the relative weak ECG of $R$ and $l$ is defined by

$$E^n_0(R, l) = \ker(E_0(p_1) : E^n_0(D(R, l)) \to E^n_0(R)).$$

in which $E(p_1)$ and $E_0(p_1)$ are the restriction map of the ECG of Definition 3.1
and the restriction map of the weak ECG of Definition 3.3, respectively.

It can be seen easily that when $l = R$, the relative ECG $E^n(R, R; R)$ and the
relative weak ECG $E^n_0(R, R)$ are the same as the generalized ECG $E^n(R; R)$ and
the generalized weak ECG $E^n_0(R)$, respectively.

**Theorem 4.2** (homology sequence). Let $R$ be a Noetherian commutative ring with
$\dim R = d$, and let $l \subseteq R$ be an ideal of $R$ with $\dim R/l = d - m$. Let $p_2$ denote
the projection from $D(R, l)$ to the second coordinate. Then, for any integer $n$ such
that $2n \geq d + 3$, we have the exact sequence

$$E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l),$$

called the homology sequence of the ECG.
Proof. Step I: First, we check that \(E(\rho) \circ E(p_2) = 0\). Let \(\ker p_1\) and \(\ker p_2\) denote the kernels of projections \(p_1\) and \(p_2\). Then \(\height(\ker p_1) = \height(\ker p_2) = 0\). On the other hand, we have ring homomorphisms \(\rho \circ p_1, \rho \circ p_2 : D(R, l) \to R/l\). By the definition of \(D(R, l)\), it can be seen easily that \(\rho \circ p_1 = \rho \circ p_2\). Hence for any element \(x \in E^n(R, l; R)\), by the method of used construction of the restriction map and by Lemma 2.4, we can assume that \(x = (Z; \omega_Z)\), in which \(Z\) is an ideal of \(D(R, l)\) with properties

\[
\height p_1(Z) \geq n, \quad \height p_2(Z) \geq n \quad \text{in } R,
\]
\[
\height \rho \circ p_1(Z) = \height \rho \circ p_2(Z) \geq n \quad \text{in } R/l.
\]

Write \(\omega_Z\) as \((z_1, \ldots, z_n)\), where \(z_i = (x_i, y_i) \in D(R, l)\) for \(1 \leq i \leq n\), and let the bar denote the reduction modulo \(l\). Then it can be seen that

\[
E(\rho) \circ E(p_1)(Z; \omega_Z) = (\bar{p_1(Z)}; (\bar{x}_1, \ldots, \bar{x}_n)),
\]
\[
E(\rho) \circ E(p_2)(Z; \omega_Z) = (\bar{p_2(Z)}; (\bar{y}_1, \ldots, \bar{y}_n)).
\]

Since \(Z\) is an ideal of \(D(R, l)\) and \((x_i, y_i) \in D(R, l)\) for \(1 \leq i \leq n\), we have \(\bar{p_1(Z)} = \bar{p_2(Z)}\) and \(\bar{x}_i = \bar{y}_i\) for \(1 \leq i \leq n\). On the other hand, from the definition of \(E^n(R, l; R)\), we know that

\[
E(\rho) \circ E(p_1)(Z; \omega_Z) = E(\rho)(p_1(Z); (x_1, \ldots, x_n)) = (\bar{p_1(Z)}; (\bar{x}_1, \ldots, \bar{x}_n)) = 0.
\]

Thus we get

\[
E(\rho) \circ E(p_2)(Z; \omega_Z) = (\bar{p_2(Z)}; (\bar{y}_1, \ldots, \bar{y}_n)) = (\bar{p_1(Z)}; (\bar{x}_1, \ldots, \bar{x}_n)) = 0.
\]

This establishes that \(E(\rho) \circ E(p_2) = 0\), that is, \(\ker E(\rho) \supseteq \image E(p_2)\).

Step II: Next we check that the kernel of \(E(\rho)\) is contained in the image of \(E(p_2)\), that is, \(\ker E(\rho) \subseteq \image E(p_2)\).

Let \(x \in E^n(R; R)\) such that \(E(\rho)(x) = 0 \in E^n(R/l; R/l)\). By the method we used in the construction of restriction map, we can assume that \(x = (I; \omega_I)\), in which \(I\) is an ideal of \(R\) such that properties \(\height I \geq n\), and \(\height(I + l)/l \geq n\) in \(\bar{R}\).

By the assumption that \(E(\rho)(x) = 0 \in E^n(R/l; R/l)\), we have \((\bar{I}; \omega_{\bar{I}}) = 0 \in E^n(R/l; R/l)\), in which \(\omega_{\bar{I}} : R^n \to \bar{I}/\bar{I}^2\) is the map induced by \(\omega_I\). By [Bhatwadekar and Sridharan 2000, Theorem 4.2], there exists a surjective map \(v_{\bar{I}} : R^n \to \bar{I}\) such that \(\bar{v}_{\bar{I}} \otimes \bar{R}/\bar{I} = \omega_{\bar{I}}\). So by Lemma 2.4, \(\omega_{\bar{I}} : R^n \to I/I^2\) can be lifted to a surjective map \(\hat{\omega}_{\bar{I}} : R^n \to I/(I^2)\). Then by Lemma 2.3, we can find a surjective lifting \(v : R^n \to I\cap K\) of \(\hat{\omega}_{\bar{I}}\) such that \(K + I^2l = R\) and \(\height K \geq n\). Since \(K + I^2l = R\), \(v\) induces a surjective map \(\omega_K : R^n \to K/K^2\), which defines an element \((K; \omega_K) \in E^n(R; R)\). It can be seen easily that \((K; \omega_K) + (I; \omega_I) = 0 \in E^n(R; R)\).
Now write $\omega_K$ as $(x_1, \ldots, x_n)$, where $x_i \in K$ for $1 \leq i \leq n$. Then we can define an element $(Z; \omega_Z) \in E^n(R, l; R)$ as follows:

- Define $Z \subseteq D(R, l)$ to be the ideal of $D(R, l)$ that is generated by pairs $(r_1, r_2) \in R \times R$ such that $r_2 \in K$ and $r_1 - r_2 \in l$.
- Define the map $\omega_Z : D(R, l)^n \xrightarrow{(z_1, \ldots, z_n)} Z / Z^2$, in which $z_i = (x_i, x_i) \in Z$ for $1 \leq i \leq n$.

By the facts that $\ker p_1 \cap \ker p_2 = (0, l) \cap (l, 0) = 0 \subseteq D(R, l)$ and $p_1(Z) = R$ and height $p_2(Z) = \text{height } K \geq n$, we have height $l \geq n$.

We should check that $\omega_Z$ is surjective. Let $z = (k + l_1, k) \in Z$, where $k \in K$ and $l_1 \in l$. Since $\omega_K$ is surjective, there exist $r_i \in R$ for $1 \leq i \leq n$ and $k_1 \in K^2$, such that $k = \sum_{i=1}^n r_i x_i + k_1$. Since $K^2 + l^2 = R$ contains $l$, there exist $k_2 \in K^2$ and $l_2 \in l^2$ such that $k_2 + l_2 = l_1$. By the fact $k_2 = l_1 - l_2 \in K^2 \cap l = K^2 l$, we get $k_3 \in K^2$, and $l_3 \in l$ for $1 \leq t \leq m$, such that $k_2 = \sum_{t=1}^m l_3 t k_3$. Finally,

$$z = (k + l_1, k) = \sum_{i=1}^n (r_i, r_i)(x_i, x_i) + (k_1, k_1) + (l_1, 0)$$

$$= \sum_{i=1}^n (r_i, r_i)(x_i, x_i) + (k_1, k_1) + (l_2, 0) + \sum_{t=1}^m (l_3 t, 0)(k_3 t, k_3 t).$$

This shows that $\omega_Z$ is surjective.

Since $K + l = R$, we have $p_1(Z) = R$ by the construction of $Z$. This implies that $E(p_1)(Z; \omega_Z) = 0 \in E^n(R; R)$. Putting all of these together, we see $(Z; \omega_Z)$ is indeed an element of $E^n(R, l; R)$.

It can be seen easily that $E(p_2)(Z; \omega_Z) = (K; \omega_K) \in E^n(R; R)$. Now let $y \in E^n(R, l; R)$ be such that $y + (Z; \omega_Z) = 0$. Since

$$E(p_2)(y) + E(p_2)(Z; \omega_Z) = (I; \omega_I) + (K; \omega_K) = 0,$$

we see that $E(p_2)(y) = (I; \omega_I)$. This shows that $\ker E(\rho) \subseteq \text{Im } E(p_2)$.

By steps I and II, the sequence is indeed an exact sequence. \hfill \Box

**Theorem 4.3** (excision theorem). Let $R$ be a Noetherian commutative ring with dim $R = d$, and let $l \subseteq R$ be an ideal of $R$ with dim $R / l = d - m$. Let $p_2$ denote the projection from $D(R, l)$ to the second coordinate. If there exists a splitting $\beta$ of the ring homomorphism $\rho : R \rightarrow R / l$ such that $\beta$ satisfies condition $(\ast)$, then for any integer $n$ with $2n \geq d + 3$, we have the split exact sequence

$$0 \rightarrow E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R / l; R / l) \rightarrow 0,$$

called the excision sequence of the ECG. In particular, we have an isomorphism $E^n(R; R) \cong E^n(R, l; R) \oplus E^n(R / l; R / l)$. 


Proof. **Step I:** First, we check that $E(\rho)$ is a split surjection. Since the ring homomorphism $\beta : R/l \to R$ satisfies the condition (*), by Definition 3.2 there is a group homomorphism $E(\beta) : E^n(R/l; R/l) \to E^n(R; R)$. By the fact that $\beta$ is a splitting of $\rho$, it is easy to check that $E(\beta)$ has the property $E(\rho) \circ E(\beta) = Id_{E^n(R/l; R/l)}$. This shows that $E(\rho)$ is split surjective.

**Step II:** We check that $E(p_2)$ is injective. Now we have a surjective ring homomorphism $\rho \circ p_2 : D(R, l) \to R/l$ and an exact sequence

$$0 \to l \times l \to D(R, l) \xrightarrow{\rho \circ p_2} R/l \to 0$$

in which $l \times l \subset D(R, l)$ is the ideal of $D(R, l)$ generated by elements $(l_1, l_2) \in R \times R$, where $l_1, l_2 \in l$. Then we have the restriction map $E(\rho \circ p_2) : E^n(R, l; R) \to E^n(R/l; R/l)$ of the ECG. It can be easily checked that $E(\rho) \circ E(\rho \circ p_2) = E(\rho \circ p_2)$.

Let $x = (Z; \omega Z) \in E^n(R, l; R)$ be such that $E(p_2)(x) = 0 \in E^n(R; R)$. By the method we used in the construction of restriction map, we can assume that height $p_1(Z) \geq n$, height $p_2(Z) \geq n$ and height $\rho \circ p_2(Z) \geq n$. On the other hand, since $E(\rho \circ p_2) = E(\rho) \circ E(p_2) = 0$, by the same method we used in the proof of Theorem 4.2, we can find $(K; \omega_K) \in E^n(R, l; R)$ such that

- $(K; \omega_K) + (Z; \omega_Z) = 0$,
- $K + Z^2(l \times l) = D(R, l)$,
- height $K \geq n$, height $p_1(K) \geq n$ and height $p_2(K) \geq n$.

By the assumption that $E(p_2)(Z; \omega_Z) = 0$, we get $E(p_2)(K; \omega_K) = 0 \in E^n(R; R)$.

Now, write $\omega_K : D(R, l)^n \to K/K^2$ as $(k_1, \ldots, k_n)$ where $k_i = (x_i, y_i) \in K$ for $1 \leq i \leq n$. We have the following.

- $E(p_1)(K; \omega_K) = (p_1(K); \omega_{p_1(K)}) = (I_1; \omega_{I_1})$, where $I_1$ denotes the ideal $p_1(K)$, and $\omega_{I_1}$ denotes the surjection induced by $\omega_K$, that is,

$$\omega_{p_1(K)} : R^n \xrightarrow{(x_1, \ldots, x_n)} I_1/I_1^2.$$

- $E(p_2)(K; \omega_K) = (p_2(K); \omega_{p_2(K)}) = (I_2; \omega_{I_2})$, where $I_2$ denotes the ideal $p_2(K)$, and $\omega_{I_2}$ denotes the surjection induced by $\omega_K$, that is,

$$\omega_{p_2(K)} : R^n \xrightarrow{(y_1, \ldots, y_n)} I_2/I_2^2.$$

Since $K + (l \times l) = D(R, l)$, we see that $I_1 + l = I_2 + l = R$.

By the fact that $E(p_1)(K; \omega_K) = E(p_2)(K; \omega_K) = 0 \in E^n(R; R)$, there exist surjective lifts

$$v_{I_1} : R^n \xrightarrow{(x_1', \ldots, x_n')} I_1$$

and

$$v_{I_2} : R^n \xrightarrow{(y_1', \ldots, y_n')} I_2$$

of $\omega_{I_1}$ and $\omega_{I_2}$, respectively, in which $x_i' \in I_1$, and $y_i' \in I_2$ for $1 \leq i \leq n$. 


Since \( v_{I_1} \) is a lift of \( \omega_{I_1} \), we have \( x'_i - x_i \in I_1^2 \) for \( 1 \leq i \leq n \). Let \( x'_i - x_i = a_i \), with \( a_i \in I_1^2 \) for \( 1 \leq i \leq n \). Since \( I_1 = p_1(K) \), there exist \( b_i \in I_2^2 \) such that \( (a_i, b_i) \in K^2 \) for \( 1 \leq i \leq n \). Now let \( y''_i = y_i + b_i \) for \( 1 \leq i \leq n \). It follows from the facts \( (x_i, y_i) \in K \) and \( (a_i, b_i) \in K^2 \) that \( (x'_i, y''_i) = (x_i + a_i, y_i + b_i) \in K \) and \( (x'_i, y''_i) = (x_i + a_i, y_i + b_i) = (x_i, y_i) \mod K^2 \) for \( 1 \leq i \leq n \). So we obtain a surjection

\[
\omega^1_K : R^n \xrightarrow{(k'_1, \ldots, k'_n)} K/K^2
\]
in which \( k'_i = (x'_i, y''_i) \in K \) for \( 1 \leq i \leq n \). Clearly, \((K; \omega^1_K) = (K; \omega_K)\).

By the same method, we can get a surjection

\[
\omega^2_K : R^n \xrightarrow{(k''_1, \ldots, k''_n)} K/K^2
\]
in which \( k''_i = (x''_i, y''_i) \in K \) for \( 1 \leq i \leq n \), such that \((K; \omega^2_K) = (K; \omega_K)\).

Now we construct two elements \((\mathfrak{g}_1; \omega_{\mathfrak{g}_1})\) and \((\mathfrak{g}_2; \omega_{\mathfrak{g}_2})\) of \( E^n(R, l; R) \). Let the bar denote reduction modulo \( l \).

(I) Define \( \mathfrak{g}_1 \) to be the ideal of \( D(R, l) \) that is generated by pairs \((\beta(\bar{a}), b)\), where \((a, b) \in K\).

(II) Define the map \( \omega_{\mathfrak{g}_1} : D(R, l)^n \xrightarrow{(k''_1, \ldots, k''_n)} \mathfrak{g}_1/\mathfrak{g}_1^2 \), where \( \bar{k}''_i = (\beta(\bar{x}''_i), y''_i) \in \mathfrak{g}_1 \) for \( 1 \leq i \leq n \).

(I*) Define \( \mathfrak{g}_2 \) to be the ideal of \( D(R, l) \) that is generated by pairs \((a, \beta(\bar{b}))\), where \((a, b) \in K\).

(II*) Define the map \( \omega_{\mathfrak{g}_2} : D(R, l)^n \xrightarrow{(k''_1, \ldots, k''_n)} \mathfrak{g}_2/\mathfrak{g}_2^2 \), where \( \bar{k}''_i = (x'_i, \beta(\bar{y}''_i)) \in \mathfrak{g}_2 \) for \( 1 \leq i \leq n \).

We check that height \( \mathfrak{g}_1 \geq n \) and height \( \mathfrak{g}_2 \geq n \). By the fact that \( \ker p_1 \cap \ker p_2 = (0, l) \cap (l, 0) = 0 \subset D(R, l) \) and

\[
p_1(\mathfrak{g}_1) = \beta(\bar{I}_1) = R \quad \text{and} \quad \text{height } p_2(\mathfrak{g}_1) = \text{height } I_2 \geq n,
\]
we get height \( \mathfrak{g}_1 \geq n \). Similarly, we have height \( \mathfrak{g}_2 \geq n \).

We check that \((\mathfrak{g}_1; \omega_{\mathfrak{g}_1})\) and \((\mathfrak{g}_2; \omega_{\mathfrak{g}_2})\) equate to zero in \( E^n(R, l; R) \).

In fact we have a map

\[
v_{\mathfrak{g}_1} : D(R, l)^n \xrightarrow{(\bar{k}''_1, \ldots, \bar{k}''_n)} \mathfrak{g}_1,
\]
which is defined by \( v_{\mathfrak{g}_1}(e_i) = \bar{k}''_i \in \mathfrak{g}_1 \), where \( \{e_i\} \) for \( 1 \leq i \leq n \) is the standard basis of \( R^n \). It is obviously a lift of \( \omega_{\mathfrak{g}_1} \). Now let \( (\beta(\bar{a}), b) \in \mathfrak{g}_1 \) for \((a, b) \in K\). Since \( I_2 \) is generated by \( \{y''_i\} \), there exist \( r_i \in R \) for \( 1 \leq i \leq n \) such that \( \sum_{i=1}^n r_i y''_i = b \).
On the other hand, it follows from the facts \((a, b) \in K\) and \((x_i'', y_i') \in K\) that \(\tilde{a} = \tilde{b}\) and \(\tilde{x}_i'' = \tilde{y}_i'\) for \(1 \leq i \leq n\). So we get

\[
\beta(\tilde{a}) = \beta(\tilde{b}) = \sum_{i=1}^{n} \beta(\tilde{r}_i)\beta(\tilde{y}_i') = \sum_{i=1}^{n} \beta(r_i)\beta(x_i'').
\]

Thus \((\beta(\tilde{a}), b) = \sum_{i=1}^{n} (\beta(\tilde{r}_i), r_i)(\beta(\tilde{x}_i''), y_i')\). This shows that \(v_{\beta_1}\) is a surjective lift of \(\omega_{\beta_1}\). So \((\mathcal{J}_1; \omega_{\beta_1}) = 0 \in E^n(R, l; R)\). By the same method, we can prove that \((\mathcal{J}_2; \omega_{\beta_2}) = 0 \in E^n(R, l; R)\).

Next we check that \((\mathcal{J}_1; \omega_{\beta_1}) + (\mathcal{J}_2; \omega_{\beta_2}) = (K; \omega_K)\).

We first check that \(\mathcal{J}_1 + \mathcal{J}_2 = K\). Since \(I_1 + I_2 = I_2 + l = R\), there exists \((i_1, i_2) \in K\) such that \(\beta(\tilde{i}_1) = 1\). So \((1, i_2) \in \mathcal{J}_1\), and \(1 - i_2 \in l\). By the same method, we can find \(i_1' \in I_1\) such that \((i_1', 1) \in \mathcal{J}_2\). So \((0, 1 - i_2)(i_1', 1) = (0, 1 - i_2) \in \mathcal{J}_2\). It follows that \((1, i_2) + (0, 1 - i_2) = (1, 1) \in \mathcal{J}_1 + \mathcal{J}_2\). Hence \(\mathcal{J}_1 + \mathcal{J}_2 = K\).

Second, we check that \(\mathcal{J}_1 \cap \mathcal{J}_2 = K\). Let \((i_1, i_2) \in K\). Then \(\beta(\tilde{i}_1), \beta(\tilde{i}_2) \in \mathcal{J}_1\). Now let \(i_1 - \beta(\tilde{i}_1) = l_1 \in l\). Since \(I_1 + l = R\), there exists \((i_1', i_2') \in K\) such that \(\beta(\tilde{i}_1') = 1\). So we have \((l_1, 0)(\beta(\tilde{i}_1'), i_2') = (l_1, 0) \in \mathcal{J}_1\). Since \((\beta(\tilde{i}_1), i_2) + (l_1, 0) = (i_1, i_2) \in \mathcal{J}_1\), we get \(K \subseteq \mathcal{J}_1\). Similarly, we can prove \(K \subseteq \mathcal{J}_2\). Thus \(K \subseteq \mathcal{J}_1 \cap \mathcal{J}_2\).

Let \((i_1, i_2) \in \mathcal{J}_1\). Then there exist \(r_{1l}, r_{2l} \in R\), with \(r_{1l} - r_{2l} \in l\), and \((x_{1l}, x_{2l}) \in K\) for \(1 \leq t \leq m\), where \(m \in \mathbb{Z}\) is an integer such that

\[
(i_1, i_2) = \sum_{t=1}^{m} (r_{1t}, r_{2t})(\beta(\tilde{x}_{1t}), x_{2t}) = \left(\sum_{t=1}^{m} r_{1t}\beta(\tilde{x}_{1t}), \sum_{t=1}^{m} r_{2t}x_{2t}\right).
\]

So we get \(i_2 \in I_2\). Thus if \((i_1, i_2) \in \mathcal{J}_1 \cap \mathcal{J}_2\), we will also have \(i_1 \in I_1\). Now we have \((\sum_{t=1}^{m} r_{1t}x_{1t}, \sum_{t=1}^{m} r_{2t}x_{2t}) \in K\) and \((i_1, i_2) - (\sum_{t=1}^{m} r_{1t}x_{1t}, \sum_{t=1}^{m} r_{2t}x_{2t}) = (l_1, 0) \in D(R, l)\). It follows that \(i_1 - \sum_{t=1}^{m} r_{1t}x_{1t} = l_1 \in l \cap I_1 = I_1\). Hence there exist \(i_1', i_2' \in I_1, i_2' \in I_2\) and \(l_2 \in l\) such that \(l_2 = l_1\). Thus \((l_1, 0) = \sum_{t=1}^{m} (l_{2t}, 0)(i_{1t}, i_{2t}) \in K\) for \(1 \leq \lambda \leq s\), with \(s \in \mathbb{Z}\). Thus \((l_1, 0) = \sum_{t=1}^{m} (l_{2t}, 0)(i_{1t}, i_{2t}) \in K\) and \((i_1, i_2) = (\sum_{t=1}^{m} r_{1t}x_{1t}, \sum_{t=1}^{m} r_{2t}x_{2t}) + (l_1, 0) \in K\). This shows that \(K \supseteq \mathcal{J}_1 \cap \mathcal{J}_2\). So we get \(\mathcal{J}_1 \cap \mathcal{J}_2 = K\).

Third, we check that

\[
\omega_K \otimes D(R, l)/\mathcal{J}_1 = \omega_{\beta_1} \quad \text{and} \quad \omega_K \otimes D(R, l)/\mathcal{J}_2 = \omega_{\beta_2}.
\]

Since \((K; \omega^K_1) = (K; \omega_K)\), to prove \(\omega_K \otimes D(R, l)/\mathcal{J}_2 = \omega_{\beta_2}\), we only need to show that \((x_i', y_i'')(x_i', \beta(\tilde{y}_i'')) \in \mathcal{J}_2^2\).

In fact let \((x_i', y_i'')(x_i', \beta(\tilde{y}_i'')) = (0, l_1) \in D(R, l)\), where \(l_1 \in l\). Since \(I_2 + l = R\), there exist \(a \in I_2^2\) and \(b \in I_2^2\), such that \((a, b) \in K^2\) and \((a, \beta(\tilde{b})) = (a, 1) \in \mathcal{J}_2^2\). So we have \((0, l_1)(a, 1) = (0, l_1) \in \mathcal{J}_2^2\). This shows that \(\omega_K \otimes D(R, l)/\mathcal{J}_2 = \omega_{\beta_2}\). By the same way, we can prove \(\omega_K \otimes D(R, l)/\mathcal{J}_1 = \omega_{\beta_1}\).
By all of the results above, \((\mathcal{H}_1; \omega_{\mathcal{H}_1}) + (\mathcal{H}_2; \omega_{\mathcal{H}_2}) = (K; \omega_K) = 0\) holds in the group \(E^n(R, l; R)\). This shows that \(E(p_2)\) is injective.

Putting these results together with those from Step I and Step II, we have a split exact sequence

\[
0 \to E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l) \to 0.
\]

In particular, \(E^n(R; R) \cong E^n(R, l; R) \oplus E^n(R/l; R/l)\). The proof is complete. □

**Remark 4.4.** The proof shows that if the ring homomorphism \(\rho : R \to R/l\) has a splitting \(\beta\), which may not satisfy condition (\(*\)), the map \(E(p_2)\) is still injective.

### 5. Applications for polynomial and Laurent polynomial extensions

In this section, we will use Theorem 4.3 to get excision sequences for Euler class groups of polynomial rings and Laurent polynomial rings.

**Excision sequence of polynomial rings.** Let \(R\) be a Noetherian commutative ring with \(\text{dim } R = d\). Let \(R[t]\) be the polynomial ring over \(R\). Then for any integer \(n\) with \(2n \geq d + 4\), we get the following corollary by setting \(l = (t)\) in Theorem 4.3.

**Corollary 5.1.** Let \(R\) be a Noetherian commutative ring with \(\text{dim } R = d\), and let \(R[t]\) be the polynomial ring over \(R\). Then for any integer \(n\) with \(2n \geq d + 4\), we have the short split exact sequence

\[
0 \to E^n(R[t], (t); R[t]) \xrightarrow{E(p_2)} E^n(R[t]; R[t]) \xrightarrow{E(\rho)} E^n(R; R) \to 0
\]

and an isomorphism

\[
E^n(R[t]; R[t]) \cong E^n(R[t], (t); R[t]) \oplus E^n(R; R).
\]

Das [2006] constructed a map \(\Psi : E^d(R[t]; R[t]) \to E^d(R; R)\) that is split surjective. In fact, this map is the map \(E(\rho)\).

**Proposition 5.2** [Das 2003]. Let \(R\) be a smooth affine domain containing the field of rational numbers, with \(\text{dim } R = d\). Then there is an isomorphism

\[
\Phi : E^d(R[t]; R[t]) \cong E^d(R; R).
\]

Using [Bhatwadekar and Keshari 2003, theorem 4.13], and Corollary 5.1, this result can generalized:

**Corollary 5.3.** Let \(k\) be an infinite perfect field, and let \(R\) be a \(d\)-dimensional regular domain that is essentially of finite type over \(k\). Let \(n\) be an integer such that \(2n \geq d + 4\). Then \(E^n(\rho) : E^n(R[t]; R[t]) \to E^n(R; R)\) is an isomorphism.
Proof. By Corollary 5.1, we only need to prove that $E(p_2)(x) = 0$ for any element $x \in E^n(R[t], (t); R[t])$. Suppose $(Z; \omega_Z) \in E^n(R[t], (t); R[t])$, where $Z$ is an ideal of $D(R[t], (t))$ and $\omega_Z : D(R[t], (t))^n \to Z/Z^2$ is a surjection. Let $p_1, p_2$ denote projections from $D(R[t], (t))$ to respective coordinates. We may assume further that height $p_1(Z) = \text{height } p_2(Z) = n$ in $R[t]$. Let

$$E(p_1)(Z; \omega_Z) = (p_1(Z); \omega_{p_1(Z)}) \quad \text{and} \quad E(p_2)(Z; \omega_Z) = (p_2(Z); \omega_{p_2(Z)}),$$

where $\omega_{p_1(Z)}$ and $\omega_{p_2(Z)}$ are respectively surjections $R[t]^n \to p_1(Z)/p_1(Z)^2$ and $R[t]^n \to p_2(Z)/p_2(Z)^2$ induced by $\omega_Z$. Since $(Z; \omega_Z) \in E^n(R[t], (t); R[t])$, we have $(p_1(Z); \omega_{p_1(Z)}) = 0 \in E^n(R[t]; R[t])$. For any ideal $I \subseteq R[t]$, let $I(0)$ denote the reduction modulo $(t)$, which is the same as setting $t = 0$ in $I$. Now we get $(p_1(Z)(0); \omega_{p_1(Z)(0)}) = 0 \in E^n(R; R)$, where $\omega_{p_1(Z)(0)}$ is the surjection $R^n \to p_1(Z)(0)/(p_1(Z)(0))^2$ that the surjection $\omega_{p_1(Z)}$ induces by setting $t = 0$. By [Bhatwadekar and Sridharan 2002, Theorem 4.2], this means that $\omega_{p_1(Z)(0)}$ can be lifted to a surjection $v_{p_1(Z)(0)} : R^n \to p_1(Z)(0)$. On the other hand, since $(Z; \omega_Z)$ is in $E^n(R[t], (t); R[t])$, we have $p_1(Z)(0) = p_2(Z)(0)$ and $\omega_{p_1(Z)(0)} = \omega_{p_2(Z)(0)}$. So we get

$$\omega_{p_2(Z)} : R[t]^n \to p_2(Z)/p_2(Z)^2 \quad \text{and} \quad v_{p_1(Z)(0)} : R^n \to p_2(Z)(0),$$

such that $\omega_{p_2(Z)} \otimes R[t]/(t) = v_{p_1(Z)(0)} \otimes R/p_2(Z)(0)$. We can find a surjective lift of $\omega_{p_2(Z)}$ by [Bhatwadekar and Keshari 2003, Theorem 4.13]. This shows that $(p_2(Z); \omega_{p_2(Z)}) = E(p_2)(Z; \omega_Z) = 0 \in E^n(R[t]; R[t]).$ 

\[\square\]

**Excision sequence of Laurent polynomial rings.**

**Corollary 5.4.** Let $R$ be a Noetherian commutative ring with dim $R = d$, and let $n$ be an integer such that $2n \geq d + 4$. Let $R[t, t^{-1}]$ be the Laurent polynomial ring over $R$. Then by setting $l = (t - 1)$ in Theorem 4.3, we get the short split exact sequence

$$0 \to E^n(R[t, t^{-1}], (t - 1); R[t, t^{-1}]) \xrightarrow{E(p_2)} E^n(R[t, t^{-1}]; R[t, t^{-1}]) \xrightarrow{E(\rho)} E^n(R; R) \to 0$$

and an isomorphism

$$E^n(R[t, t^{-1}]; R[t, t^{-1}]) \cong E^n(R[t, t^{-1}], (t - 1); R[t, t^{-1}]) \oplus E^n(R; R).$$

**Postscript.** In K-theory, we have the following results: When $R$ is a regular Noetherian commutative ring, the homology sequences for the $K_0$-groups of the polynomial ring $R[t]$ and the Laurent polynomial ring $R[t, t^{-1}]$ reduce to isomorphisms $K_0(R[t]) \cong K_0(R)$ and $K_0(R[t, t^{-1}]) \cong K_0(R)$, respectively [Rosenberg 1994]. In
light of this result and the correspondence between excision sequence for K-theory and the excision sequence for the ECGs, we ask if there exist isomorphisms

\[ E^n(R[t]; R[t]) \cong E^n(R; R) \quad \text{and} \quad E^n(R[t, t^{-1}]; R[t, t^{-1}]) \cong E^n(R; R). \]

For the case of polynomial extension, Corollary 5.3 gives an affirmative answer. For the case of Laurent polynomial extension, we wonder if there is also an isomorphism \( E^n(R[t, t^{-1}]; R[t, t^{-1}]) \cong E^n(R; R) \) if the ring \( R \) satisfies the conditions of Corollary 5.3.

In fact, for the weak ECG, we can prove the following result using Suslin's cancellation theorem [1977] and [Murthy 1994, Theorem 2.2].

Let \( R \) be a smooth affine algebra over some algebraically closed field \( k \), with \( \dim R = d \). Let \( R[t, t^{-1}] \) be the Laurent polynomial ring over \( R \). Then we have \( E^d_0(R[t, t^{-1}]) \otimes \mathbb{Q} \cong E^d_0(R) \otimes \mathbb{Q} \), which corresponds to what the Riemann–Roch theorem tells us in geometry.

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**YONG YANG**

**DEPARTMENT OF MATHEMATICS AND SYSTEMS SCIENCE**

**NATIONAL UNIVERSITY OF DEFENSE TECHNOLOGY**

**CHANGSHA, HUNAN 410073**

**CHINA**

yangmoses@gmail.com
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