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MÖBIUS ISOPARAMETRIC HYPERSURFACES WITH THREE DISTINCT PRINCIPAL CURVATURES, II

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Using the method of moving frames and the algebraic techniques of T. E. Cecil and G. R. Jensen that were developed while they classified the Dupin hypersurfaces with three principal curvatures, we extend Hu and Li's main theorem in *Pacific J. Math.* 232:2 (2007), 289–311 by giving a complete classification for all Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with three distinct principal curvatures.

1. Introduction

Let $x : M^n \to \mathbb{S}^{n+1}$ be a connected smooth hypersurface in the $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$ without umbilic point. We choose a local orthonormal basis $\{e_1, \dots, e_n\}$ with respect to the induced metric $I = dx \cdot dx$, and let $\{\theta_1, \dots, \theta_n\}$ be the dual basis. Let $h = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form of x, with squared length $\|h\|^2 = \sum_{i,j}(h_{ij})^2$ and mean curvature $H = (1/n)\sum_i h_{ii}$. Define $\rho^2 = n/(n-1)\cdot(\|h\|^2 - nH^2)$. Then the positive definite form $g = \rho^2 dx \cdot dx$ is Möbius invariant and is called the Möbius metric of $x : M^n \to \mathbb{S}^{n+1}$. The Möbius second fundamental form $\mathbf{B}$, another basic Möbius invariant of x, together with g determine completely a hypersurface of $\mathbb{S}^{n+1}$ up to Möbius equivalence; see Theorem 2.2 below.

An important class of hypersurfaces for Möbius differential geometry is the so-called Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$. According to [Li et al. 2002], a Möbius isoparametric hypersurface of $\mathbb{S}^{n+1}$ is an umbilic-free hypersurface of $\mathbb{S}^{n+1}$ such whose Möbius-invariant 1-form

$$\Phi = -\rho^{-1}\sum_i \Big(e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)\Big)\theta_i$$

vanishes and whose Möbius principal curvatures are all constant. These curvatures are the eigenvalues of the Möbius shape operator $\Psi := \rho^{-1}(\boldsymbol{S} - H\,\mathrm{id})$ with respect to g, where $\boldsymbol{S}$ denotes the shape operator of $x : M^n \to \mathbb{S}^{n+1}$. This definition

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of Möbius isoparametric hypersurfaces is meaningful. Indeed, comparing it with that of (Euclidean) isoparametric hypersurfaces in $\mathbb{S}^{n+1}$, we see that the images of all hypersurfaces of the sphere with constant mean curvature and constant scalar curvature under the Möbius transformation satisfy $\Phi \equiv 0$, and the Möbius-invariant operator Ψ plays the role in Möbius geometry that $\boldsymbol{S}$ does in Euclidean geometry; see Theorem 2.2 below. The two conditions of a Möbius isoparametric hypersurface, namely, that it has vanishing Möbius form and has constant Möbius principal curvatures, are independent and also closely related; for detailed discussion, see [Hu and Tian 2009]. Standard examples of Möbius isoparametric hypersurfaces are the images of (Euclidean) isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ under Möbius transformations. But there are other examples which cannot be obtained by this way; for example, one occurs in our classification for hypersurfaces of $\mathbb{S}^{n+1}$ with *parallel* Möbius second fundamental form, that is, those whose Möbius second fundamental form is parallel with respect to the Levi-Civita connection of the Möbius metric g; see [Hu and Li 2004; Li et al. 2002] for details. On the other hand, it was proved in [Li et al. 2002] that any Möbius isoparametric hypersurface is in particular a Dupin hypersurface, which implies from [Thorbergsson 1983] that for a compact Möbius isoparametric hypersurface embedded in $\mathbb{S}^{n+1}$, the number γ of distinct principal curvatures can only take the values $\gamma = 2, 3, 4, 6$. A characterization of Möbius isoparametric hypersurfaces in terms of Dupin hypersurfaces was given in [Li et al. 2002] and was obtained very recently also by L. A. Rodriques and K. Tenenblat [2009]; this characterization states that a Möbius isoparametric hypersurface is either a cyclide of Dupin or a Dupin hypersurface whose Möbius curvatures are constant. Hence the problem of investigating Möbius isoparametric hypersurfaces reduces to that of investigating Dupin hypersurfaces with constant Möbius curvatures.

In [Li et al. 2002], the authors classified locally all Möbius isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ with $\gamma = 2$. By relaxing the restriction that $\gamma = 2$, local classifications for all Möbius isoparametric hypersurfaces in $\mathbb{S}^4$, $\mathbb{S}^5$ and $\mathbb{S}^6$ were established in [Hu and Li 2005], [Hu et al. 2007] and [Hu and Zhai 2008], respectively. It was shown that a Möbius isoparametric hypersurface in $\mathbb{S}^4$ is either of parallel Möbius second fundamental form or Möbius equivalent to the Euclidean isoparametric hypersurface in $\mathbb{S}^4$ with three distinct principal curvatures, that is, a tube of constant radius over a standard Veronese embedding of $\mathbb{R}P^2$ into $\mathbb{S}^4$. Similarly, a hypersurface in $\mathbb{S}^5$ is Möbius isoparametric if and only if either it has parallel Möbius second fundamental form; or it is Möbius equivalent to the preimage of the stereographic projection of the cone $\tilde{x} : N^3 \times \mathbb{R}^+ \to \mathbb{R}^5$ defined by $\tilde{x}(x, t) = tx$, where $t \in \mathbb{R}^+$ and $x : N^3 \to \mathbb{S}^4 \hookrightarrow \mathbb{R}^5$ is the Cartan isoparametric immersion in $\mathbb{S}^4$ with three principal curvatures; or it is Möbius equivalent to the Euclidean isoparametric hypersurfaces in $\mathbb{S}^5$ with four distinct principal curvatures.

All these results remind us of their counterparts in Dupin hypersurfaces; see [Cecil and Jensen 1998; 2000; Cecil et al. 2007; Miyaoka and Ozawa 1989; Niebergall 1991; Pinkall 1985; Pinkall and Thorbergsson 1989].

Hence, the classification of Möbius isoparametric hypersurfaces by Möbius transformation group equivalence can be compared with that of the Dupin hypersurfaces by Lie sphere transformation group equivalence. Note that the Lie sphere transformation group contains the Möbius transformation group in $\mathbb{S}^{n+1}$ as a subgroup and the dimension difference is $n+3$. Thus, Möbius differential geometry for hypersurfaces in sphere should, in some sense, be very different from Lie sphere geometry in many respects, and therefore is worthwhile to pay more attention.

Inspired by the close similarity between Dupin hypersurfaces under the Lie sphere transformation group and Möbius isoparametric hypersurfaces under the Möbius transformation group, and by T. E. Cecil and G. R. Jensen's result [1998] that any locally irreducible Dupin hypersurface in $\mathbb{S}^n$ with three distinct principal curvatures is equivalent by Lie sphere transformation to an isoparametric hypersurface in $\mathbb{S}^n$, we started in [Hu and Li 2007] a program of classifying all Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with three distinct Möbius principal curvatures. There, we were able to obtain the classification under the additional condition that one of the Möbius principal curvatures is of multiplicity one. The purpose of this paper is to extend that result to the general case:

Classification theorem. *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures. Then x is Möbius equivalent to an open part of one of the following hypersurfaces in $\mathbb{S}^{n+1}$:*

(i) *The preimage of the stereographic projection of the warped product embedding*
$$\tilde{x} : \mathbb{S}^p(a) \times \mathbb{S}^q(\sqrt{1-a^2}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1}$$
with $p \geq 1,\ q \geq 1,\ p+q \leq n-1$ and $0 < a < 1$, defined by
$$\tilde{x}(u', u'', t, u''') = (tu', tu'', u'''),$$
where $u' \in \mathbb{S}^p(a),\ u'' \in \mathbb{S}^q(\sqrt{1-a^2}),\ t \in \mathbb{R}^+$ and $u''' \in \mathbb{R}^{n-p-q-1}$.

(ii) *The Euclidean isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with three distinct principal curvatures. Thus all the principal curvatures must have the same multiplicity $m \in \{1, 2, 4, 8\}$, and the isoparametric hypersurface must be a tube of constant radius over a standard Veronese embedding of a projective plane $\mathbb{F}P^2$ into $\mathbb{S}^{3m+1}$, where $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), $\mathbb{O}$ (the Cayley numbers) for $m = 1, 2, 4, 8$, respectively.*

(iii) *The minimal hypersurfaces defined by*
$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^{3m} \times \mathbb{H}^{n-3m}\Big(-\frac{n-1}{6mn}\Big) \to \mathbb{S}^{n+1},$$

with

$$\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0, \quad y_0 \in \mathbb{R}^+, \quad y_1 \in \mathbb{R}^{3m+2}, \quad y_2 \in \mathbb{R}^{n-3m},$$

where $y_1 : N^{3m} \to \mathbb{S}^{3m+1}(\sqrt{6mn/(n-1)}) \hookrightarrow \mathbb{R}^{3m+2}$ *is Cartan's minimal isoparametric hypersurface with scalar curvature* $\tilde{R}_1 = 3(m-1)(n-1)/2n$ *and principal curvatures*

$$\sqrt{\frac{n-1}{2mn}}, \qquad 0, \qquad -\sqrt{\frac{n-1}{2mn}} \tag{1-1}$$

which have the same multiplicity m, *where* $m = 1, 2, 4$ *or* 8, *and*

$$(y_0, y_2) : \mathbb{H}^{n-3m}\left(-\frac{n-1}{6mn}\right) \hookrightarrow \mathbb{L}^{n-3m+1}$$

is the standard embedding of the hyperbolic space of sectional curvature $-(n-1)/(6mn)$ *into the* $(n-3m+1)$*-dimensional Lorentz space with*

$$-y_0^2 + y_2^2 = -\frac{6mn}{n-1}.$$

Remark 1.1. All hypersurfaces in (i) are of parallel Möbius second fundamental form and have three distinct Möbius principal curvatures with arbitrary multiplicities p, q and $n-p-q$, respectively. The hypersurfaces in (ii) and (iii) are of nonparallel Möbius second fundamental form. For hypersurfaces in (iii), the multiplicities of the three Möbius principal curvatures are m, m and $n-2m > m$.

Remark 1.2. In the cases that $n = 3$, 4 and 5, the classification theorem was proved in [Hu and Li 2005; Hu et al. 2007; Hu and Li 2007], respectively. The theorem extends the main theorem of [Hu and Li 2007], where it was assumed that the Möbius isoparametric hypersurface M^n for $n \geq 5$ has three distinct Möbius principal curvatures and one of which is simple. The extension is successfully achieved by using the wonderful techniques developed by T. E. Cecil and G. R. Jensen [1998] in their classification of Dupin hypersurfaces with three principal curvatures.

Remark 1.3. As a counterpart to the Cecil–Ryan conjecture for Dupin hypersurfaces, which states that a compact embedded Dupin hypersurface in a space form is Lie equivalent to an Euclidean isoparametric hypersurface, C. P. Wang conjectured that *any compact embedded Möbius isoparametric hypersurface in* $\mathbb{S}^{n+1}$ *is Möbius equivalent to an Euclidean isoparametric hypersurface*. Pinkall and Thorbergsson [1989] and Miyaoka and Ozawa [1989], have constructed counterexamples to the Cecil–Ryan conjecture, but we point out that the classifications of Möbius isoparametric hypersurfaces in [Hu and Li 2007; 2005; Hu et al. 2007; Hu and Zhai 2008; Li et al. 2002] and this paper strengthen Wang's conjecture.

This paper consists of six sections. In Section 2, we first review the elementary facts of Möbius geometry for hypersurfaces in $\mathbb{S}^{n+1}$, and then we recall the classification for hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form [Hu and Li 2004] and the classification for hypersurfaces of $\mathbb{S}^{n+1}$ with two distinct constant Blaschke eigenvalues [Li and Zhang 2007]. In Section 3, we treat the Möbius isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ with nonparallel Möbius second fundamental form and three distinct Möbius principal curvatures. We first present several important properties of the Möbius second fundamental form, and then we divide the discussion into two cases and state the main results, Theorem 3.1 and Theorem 3.2. We prove Theorem 3.1 in Section 4. In Section 5, we prove Theorem 5.1, which gives a preliminary classification for Möbius isoparametric hypersurfaces with three distinct Möbius principal curvatures whose multiplicities are not equal. By the analysis of the Möbius invariants of the hypersurfaces that appear in Theorem 5.1 we obtain Propositions 5.3 — 5.5, from which Theorem 3.2 follows. In Section 6, we complete the proof of the classification theorem.

2. Möbius invariants for hypersurfaces in $\mathbb{S}^{n+1}$

In this section we define the Möbius invariants and recall the structure equations for hypersurfaces in the unit sphere $\mathbb{S}^{n+1}$. We refer to [Wang 1998] for more details. Let $\mathbb{L}^{n+3}$ be the Lorentz space, namely $\mathbb{R}^{n+3}$ with inner product $\langle\cdot,\cdot\rangle_1$ defined by

$$\langle x, w\rangle_1 = -x_0w_0 + x_1w_1 + \cdots + x_{n+2}w_{n+2}$$

for $x = (x_0, x_1, \ldots, x_{n+2})$, $w = (w_0, w_1, \ldots, w_{n+2}) \in \mathbb{R}^{n+3}$.

Let $x : M^n \to \mathbb{S}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ be an immersed hypersurface of $\mathbb{S}^{n+1}$ without umbilics. We define the Möbius position vector $Y : M^n \to \mathbb{L}^{n+3}$ of x by

$$Y = \rho(1, x) \quad \text{and} \quad \rho^2 = \frac{n}{n-1}(\|h\|^2 - nH^2) > 0. \tag{2-1}$$

Theorem 2.1 [Wang 1998]. *Two hypersurfaces $x, \tilde{x} : M^n \to \mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+2, 1)$ such that $Y = \tilde{Y}T$ on M^n.*

It follows immediately that $g = \langle dY, dY\rangle_1 = \rho^2 dx \cdot dx$ is a Möbius invariant, which is defined as the *Möbius metric* of $x : M^n \to \mathbb{S}^{n+1}$. Let Δ be the Beltrami–Laplace operator of g. Define $N = -\Delta Y/n - \langle \Delta Y, \Delta Y\rangle_1 Y/(2n^2)$. Then one can show that

$$\langle \Delta Y, Y\rangle_1 = -n, \qquad \langle \Delta Y, dY\rangle_1 = 0, \qquad \langle \Delta Y, \Delta Y\rangle_1 = 1 + n^2R, \tag{2-2}$$

$$\langle Y, Y\rangle_1 = 0, \qquad \langle N, Y\rangle_1 = 1, \qquad \langle N, N\rangle_1 = 0, \tag{2-3}$$

where R is the normalized scalar curvature of g and is called the normalized Möbius scalar curvature of $x : M^n \to \mathbb{S}^{n+1}$.

Let $\{E_1, \dots, E_n\}$ be a local orthonormal basis for (M^n, g), and let $\{\omega_1, \dots, \omega_n\}$ be the dual basis. Write $Y_i = E_i(Y)$, then it follows from (2-1), (2-2) and (2-3) that

$$\langle Y_i, Y\rangle_1 = \langle Y_i, N\rangle_1 = 0, \quad \langle Y_i, Y_j\rangle_1 = \delta_{ij} \quad \text{for } 1 \le i, j \le n.$$

Let V be the orthogonal complement to the subspace $\mathrm{Span}\{Y, N, Y_1, \dots, Y_n\}$ in $\mathbb{L}^{n+3}$. Then along M we have the orthogonal decomposition

$$\mathbb{L}^{n+3} = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \dots, Y_n\} \oplus V.$$

V is called the Möbius normal bundle of $x : M^n \to \mathbb{S}^{n+1}$. A local unit vector basis $E = E_{n+1}$ for V can be written as

$$E = E_{n+1} := (H, Hx + e_{n+1}).$$

Then, $\{Y, N, Y_1, \dots, Y_n, E\}$ forms a moving frame along M^n in $\mathbb{L}^{n+3}$.

In the rest of this paper, we will use the range $1 \le i, j, k, l, t \le n$ of indices.

We can write the structure equations as

$$dY = \sum_i Y_i\omega_i, \quad dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_i B_{ij}\omega_j E, \tag{2-4}$$

$$dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_i C_i\omega_i E, \quad dE = -\sum_i C_i\omega_i Y - \sum_{i,j} B_{ij}\omega_j Y_i, \tag{2-5}$$

where ω_{ij} is the connection form of the Möbius metric g and is defined by the structure equations $d\omega_i = \sum_j \omega_{ij}\wedge\omega_j$ and $\omega_{ij}+\omega_{ji}=0$. The tensors $\boldsymbol{A}=\sum_{i,j} A_{ij}\omega_i\otimes\omega_j$, $\Phi = \sum_i C_i\omega_i$ and $\boldsymbol{B} = \sum_{i,j} B_{ij}\,\omega_i\otimes\omega_j$ are called the Blaschke tensor, the Möbius form and the Möbius second fundamental form of $x : M^n \to \mathbb{S}^{n+1}$, respectively. The relations between Φ, $\boldsymbol{B}$, $\boldsymbol{A}$ and the Euclidean invariants of x are given by [Wang 1998]

$$C_i = -\rho^{-2}\big(e_i(H) + \textstyle\sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)\big),$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}), \tag{2-6}$$

$$A_{ij} = -\rho^{-2}\big(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}\big) - \tfrac{1}{2}\rho^{-2}(|\nabla\log\rho|^2 - 1 + H^2)\delta_{ij}, \tag{2-7}$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the orthonormal basis $\{e_i\}$ of $dx\cdot dx$.

The covariant derivatives of C_i, A_{ij}, B_{ij} are defined by

$$\textstyle\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2-8}$$

$$\textstyle\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2-9}$$

$$\textstyle\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2-10}$$

The integrability conditions for the structure equations (2-4) and (2-5) are

$$A_{ij,k}-A_{ik,j}=B_{ik}C_j-B_{ij}C_k, \tag{2-11}$$

$$C_{i,j}-C_{j,i}=\textstyle\sum_k(B_{ik}A_{kj}-A_{ik}B_{kj}), \tag{2-12}$$

$$B_{ij,k}-B_{ik,j}=\delta_{ij}C_k-\delta_{ik}C_j, \tag{2-13}$$

and

$$R_{ijkl}=B_{ik}B_{jl}-B_{il}B_{jk}+\delta_{ik}A_{jl}+\delta_{jl}A_{ik}-\delta_{il}A_{jk}-\delta_{jk}A_{il}, \tag{2-14}$$

$$\textstyle\sum_i B_{ii}=0, \quad \sum_{i,j}(B_{ij})^2=\dfrac{n-1}{n}, \quad \operatorname{tr}\boldsymbol{A}=\sum_i A_{ii}=\dfrac{1}{2n}(1+n^2R). \tag{2-15}$$

Here R_{ijkl} denote the components of the curvature tensor of g, which are defined by the structure equations

$$d\omega_{ij}-\textstyle\sum_k\omega_{ik}\wedge\omega_{kj}=-\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l, \quad R_{ijkl}=-R_{ijlk}. \tag{2-16}$$

The normalized Möbius scalar curvature of $x:M^n\to\mathbb{S}^{n+1}$ is

$$R=\frac{1}{n(n-1)}\textstyle\sum_{i,j}R_{ijij}.$$

The second covariant derivative of B_{ij} is defined by

$$\textstyle\sum_l B_{ij,kl}\,\omega_l=dB_{ij,k}+\sum_l B_{lj,k}\,\omega_{li}+\sum_l B_{il,k}\,\omega_{lj}+\sum_l B_{ij,l}\,\omega_{lk}. \tag{2-17}$$

From exterior differentiation of (2-10), we have the Ricci identity

$$B_{ij,kl}-B_{ij,lk}=\textstyle\sum_t B_{tj}R_{tikl}+\sum_t B_{it}R_{tjkl}. \tag{2-18}$$

From (2-6), we see that the Möbius shape operator of $x:M^n\to\mathbb{S}^{n+1}$ takes the form $\Psi=\rho^{-1}(\boldsymbol{S}-H\,\mathrm{id})=\sum_{i,j}B_{ij}\omega_iE_j$, which implies that for an umbilic-free hypersurface in $\mathbb{S}^{n+1}$, the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures.

One can easily show that all coefficients in (2-4) and (2-5) are determined by $\{g,\Psi\}$. Thus:

Theorem 2.2 [Wang 1998; Akivis and Goldberg 1997]. *For $n\geq 3$, two hypersurfaces $x:M^n\to\mathbb{S}^{n+1}$ and $\tilde{x}:\tilde{M}^n\to\mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists a diffeomorphism $F:M^n\to\tilde{M}^n$ that preserves the Möbius metric and the Möbius shape operator.*

An umbilic-free hypersurface $x:M^n\to\mathbb{S}^{n+1}$ is said to have parallel Möbius second fundamental form if $B_{ij,k}=0$ for all i,j,k. Hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form have now been completely classified. A special case of the classification can be stated as follows.

Theorem 2.3 [Hu and Li 2004]. *For $n \geq 2$, let $x : M^n \to \mathbb{S}^{n+1}$ be an immersed umbilic-free hypersurface with parallel Möbius second fundamental form and with three distinct Möbius principal curvatures. Then x is Möbius equivalent to an open part of the image of σ of the warped product embedding*

$$\tilde{x} : \mathbb{S}^p(a) \times \mathbb{S}^q(\sqrt{1-a^2}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1}$$

with $p \geq 1,\ q \geq 1,\ p+q \leq n-1$ and $0 < a < 1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', u'''),$$

for

$$u' \in \mathbb{S}^p(a), \quad u'' \in \mathbb{S}^q(\sqrt{1-a^2}), \quad t \in \mathbb{R}^+, \quad u''' \in \mathbb{R}^{n-p-q-1},$$

where the conformal diffeomorphism $\sigma : \mathbb{R}^{n+1} \to \mathbb{S}^{n+1} \setminus \{(-1, 0, \dots, 0)\}$ is the inverse of the stereographic projection and is defined by

$$\sigma(u) = \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right) \quad \textit{for } u \in \mathbb{R}^{n+1}.$$

To prove our main theorem, we also need the following partial classification for umbilic-free hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct Blaschke eigenvalues, due to Li and Zhang [2007]; see also [Hu and Li 2007]

Theorem 2.4. *For $n \geq 3$, let $x : M^n \to \mathbb{S}^{n+1}$ be an immersed umbilic-free hypersurface with two distinct constant Blaschke eigenvalues and vanishing Möbius form. If x has three distinct Möbius principal curvatures, then it is locally Möbius equivalent to either of the following two families of hypersurfaces in $\mathbb{S}^{n+1}$:*

(1) *Minimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^p \times \mathbb{H}^{n-p}(-r^{-2}) \to \mathbb{S}^{n+1}$$

with $r > 0$ and

$$\begin{aligned} &\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0, \\ &y_0 \in \mathbb{R}^+, \qquad y_1 \in \mathbb{R}^{p+2}, \quad y_2 \in \mathbb{R}^{n-p} \quad \textit{for } 2 \leq p \leq n-1, \end{aligned}$$

where $y_1 : N^p \to \mathbb{S}^{p+1}(r) \hookrightarrow \mathbb{R}^{p+2}$ is an umbilic-free minimal hypersurface immersed into the $(p+1)$-dimensional sphere of radius r and constant scalar curvature

$$\tilde{R}_1 = \frac{np(p-1)-(n-1)r^2}{nr^2},$$

and $(y_0, y_2) : \mathbb{H}^{n-p}(-r^{-2}) \to \mathbb{L}^{n-p+1}$ is the standard embedding of hyperbolic space of sectional curvature $-r^{-2}$ into the $(n-p+1)$-dimensional Lorentz space with $-y_0^2 + y_2^2 = -r^2$.

(2) *Nonminimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^p \times \mathbb{S}^{n-p}(r) \to \mathbb{S}^{n+1}$$

with $r > 0$ *and*

$$\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0,$$
$$y_0 \in \mathbb{R}^+, \qquad y_1 \in \mathbb{R}^{p+1}, \quad y_2 \in \mathbb{R}^{n-p+1} \quad \textit{for } 2 \le p \le n-1,$$

where $(y_0, y_1) : N^p \to \mathbb{H}^{p+1}(-r^{-2}) \hookrightarrow \mathbb{L}^{p+2}$, *with* $-y_0^2 + y_1^2 = -r^2$, *is an umbilic free minimal hypersurface immersed into* $(p+1)$*-dimensional hyperbolic space of sectional curvature* $-r^{-2}$ *and constant scalar curvature*

$$\tilde{R}_1 = -\frac{np(p-1) + (n-1)r^2}{nr^2},$$

and $y_2 : \mathbb{S}^{n-p}(r) \to \mathbb{R}^{n-p+1}$ *is the standard embedding of the* $(n-p)$*-sphere of radius* r.

3. Möbius isoparametric hypersurfaces with $\gamma = 3$

Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct principal curvatures B_1, B_2, B_3 of multiplicities m_1, m_2, m_3, respectively. Without loss of generality, we assume that $m_1 \ge m_2 \ge m_3 \ge 1$.

Since x has constant Möbius principal curvatures, we can choose, around each point of M, a local frame field $\{E_i\}_{1 \le i \le n}$ orthonormal with respect to the Möbius metric g such that the matrix (B_{ij}) is diagonalized. Let us write

$$(B_{ij}) = \mathrm{diag}(b_1, \ldots, b_n), \tag{3-1}$$

where $\{b_i\}$ are all constants. From the assumption, we can assume without loss of generality that

$$b_1 = \cdots = b_{m_1} = B_1, \quad b_{m_1+1} = \cdots = b_{m_1+m_2} = B_2, \quad b_{m_1+m_2+1} = \cdots = b_n = B_3.$$

Here B_1, B_2 and B_3 are distinct and, by (2-15), they satisfy the conditions

$$m_1 B_1 + m_2 B_2 + m_3 B_3 = 0, \quad m_1 B_1^2 + m_2 B_2^2 + m_3 B_3^2 = \frac{n-1}{n}. \tag{3-2}$$

From now on, unless stated otherwise we impose the additional index conventions

$$\begin{aligned} 1 \le {} & a, b, c, d \le m_1, \\ m_1 + 1 \le {} & \quad p, q \quad \le m_1 + m_2, \\ m_1 + m_2 + 1 \le {} & \quad \alpha, \beta \quad \le m_1 + m_2 + m_3 = n. \end{aligned} \tag{3-3}$$

With respect to the local frame field $\{E_i\}$, we write the Blaschke tensor as $\boldsymbol{A} = \sum_{i,j} A_{ij}\, \omega_i \otimes \omega_j$. Since the Möbius form Φ vanishes, we see from (2-12) that $\boldsymbol{A}$

and $\boldsymbol{B}$ commute, which implies that $A_{pa} = A_{a\alpha} = A_{p\alpha} = 0$. Moreover, for any fixed point $\xi \in M$, we can choose the local frame field $\{E_i\}$ to guarantee that, in addition to (3-1) around ξ, we have at the pont ξ

$$(A_{ij}) = \operatorname{diag}(A_1, \dots, A_n). \tag{3-4}$$

Here $\{A_i\}_{1 \le i \le n}$ are the eigenvalues of the Blaschke tensor $\boldsymbol{A}$. Obviously, we can further arrange the local frame field $\{E_i\}$ around ξ so that, in addition to (3-1) around ξ, these eigenvalues are ordered at ξ as

$$\begin{gathered} A_1(\xi) \le A_2(\xi) \le \cdots \le A_{m_1}(\xi), \\ A_{m_1+1}(\xi) \le \cdots \le A_{m_1+m_2}(\xi), \\ A_{m_1+m_2+1}(\xi) \le \cdots \le A_n(\xi). \end{gathered} \tag{3-5}$$

In this way, we see that $A_1, \dots, A_n$ are well-defined continuous functions on M. Denote by M^* the set of all such points $\xi \in M$: Around ξ there exists an orthonormal frame field $\{E_i\}$ with respect to which (3-1) and (3-4) hold. Obviously, M^* is an open subset of M. In the computation that follows, we will fix a point $\xi \in M^*$ and then take an open set $U \subset M^*$ containing ξ such that over U there exists an orthonormal frame field $\{E_i\}$ for which (3-1) and (3-4) hold.

Applying the condition to (2-11) and (2-13), we see that both $A_{ij,k}$ and $B_{ij,k}$ are totally symmetric tensors. As usual we define

$$\omega_{ij} = \sum_k \Gamma^i_{kj} \omega_k \quad \text{and} \quad \Gamma^i_{kj} = -\Gamma^j_{ki}. \tag{3-6}$$

From (2-10), (3-1) and (3-6) and that $\{b_i\}_{1 \le i \le n}$ consists of constants, we get

$$B_{ij,k} = (b_i - b_j)\Gamma^i_{kj} = (b_j - b_k)\Gamma^j_{ik} = (b_k - b_i)\Gamma^k_{ji} \quad \text{for all } i, j, k. \tag{3-7}$$

Hence we see that

$$B_{ii,j} = B_{ij,i} = B_{ab,j} = B_{pq,j} = B_{\alpha\beta,j} = 0 \quad \text{for all } i, j, a, b, p, q, \alpha, \beta, \tag{3-8}$$

and the only possible nonzero elements in $\{B_{ij,k}\}$ are of the form $B_{pa,\alpha}$.

For the rest of this section, we assume that $B_{ij,k} \not\equiv 0$. We define the nonnegative smooth function f by

$$f = \tfrac{1}{6}|\nabla \boldsymbol{B}|^2 = \frac{1}{6}\sum_{i,j,k} B^2_{ij,k} = \sum_{p,a,\alpha} B^2_{pa,\alpha}.$$

Moreover, we define three arrays of vectors, an $m_2 \times m_3$ array $(\vec{v}_{p\alpha})$ of vectors in $\mathbb{R}^{m_1}$, an $m_1 \times m_3$ array $(\vec{v}_{a\alpha})$ of vectors in $\mathbb{R}^{m_2}$, and an $m_2 \times m_1$ array $(\vec{v}_{pa})$ of vectors in $\mathbb{R}^{m_3}$, by

$$\vec{v}_{p\alpha} = (B_{p\alpha,1}, B_{p\alpha,2}, \dots, B_{p\alpha,m_1}),$$
$$\vec{v}_{a\alpha} = (B_{a\alpha,m_1+1}, B_{a\alpha,m_1+2}, \dots, B_{a\alpha,m_1+m_2}),$$
$$\vec{v}_{pa} = (B_{pa,m_1+m_2+1}, B_{pa,m_1+m_2+2}, \dots, B_{pa,n}).$$

Lemma 3.1. *Let U be an open set of M^* as stated above. Then at each point of U, the arrays $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$ and $(\vec{v}_{pa})$ satisfy*

$$\begin{cases} \vec{v}_{p\alpha}\cdot\vec{v}_{p\beta} = 0 = \vec{v}_{a\alpha}\cdot\vec{v}_{a\beta} & \text{for all } p, a \text{ and any } \alpha\neq\beta, \\ \vec{v}_{p\alpha}\cdot\vec{v}_{q\alpha} = 0 = \vec{v}_{pa}\cdot\vec{v}_{qa} & \text{for all } \alpha, a \text{ and any } p\neq q, \\ \vec{v}_{a\alpha}\cdot\vec{v}_{b\alpha} = 0 = \vec{v}_{pa}\cdot\vec{v}_{pb} & \text{for all } p, \alpha \text{ and } a\neq b; \end{cases} \tag{3-9}$$

$$\begin{cases} \vec{v}_{p\alpha}\cdot\vec{v}_{q\beta} + \vec{v}_{q\alpha}\cdot\vec{v}_{p\beta} = 0 & \text{if } \alpha\neq\beta \text{ and } p\neq q, \\ \vec{v}_{a\alpha}\cdot\vec{v}_{b\beta} + \vec{v}_{b\alpha}\cdot\vec{v}_{a\beta} = 0 & \text{if } \alpha\neq\beta \text{ and } a\neq b, \\ \vec{v}_{pa}\cdot\vec{v}_{qb} + \vec{v}_{qa}\cdot\vec{v}_{pb} = 0 & \text{if } a\neq b \text{ and } p\neq q; \end{cases} \tag{3-10}$$

$$\begin{cases} |\vec{v}_{p\alpha}|^2 + |\vec{v}_{q\beta}|^2 = |\vec{v}_{q\alpha}|^2 + |\vec{v}_{p\beta}|^2 & \text{if } \alpha\neq\beta \text{ and } p\neq q, \\ |\vec{v}_{a\alpha}|^2 + |\vec{v}_{b\beta}|^2 = |\vec{v}_{b\alpha}|^2 + |\vec{v}_{a\beta}|^2 & \text{if } \alpha\neq\beta \text{ and } a\neq b, \\ |\vec{v}_{pa}|^2 + |\vec{v}_{qb}|^2 = |\vec{v}_{qa}|^2 + |\vec{v}_{pb}|^2 & \text{if } a\neq b \text{ and } p\neq q, \end{cases} \tag{3-11}$$

where the dot denotes the standard product in $\mathbb{R}^{m_1}$, $\mathbb{R}^{m_2}$ and $\mathbb{R}^{m_3}$, respectively.

Proof. From (2-10) and (3-8), we have

$$\textstyle\sum_a B_{p\alpha,a}\omega_a = (B_2 - B_3)\omega_{p\alpha}, \tag{3-12}$$

$$\textstyle\sum_p B_{a\alpha,p}\omega_p = (B_1 - B_3)\omega_{a\alpha}, \tag{3-13}$$

$$\textstyle\sum_\alpha B_{pa,\alpha}\omega_\alpha = (B_2 - B_1)\omega_{pa}. \tag{3-14}$$

Differentiating (3-12) and then using (3-6) and (3-7), we get

$$\begin{aligned} \frac{\sum_{a,q,\beta} B_{p\alpha,a}B_{q\beta,a}(B_3-B_2)}{(B_1-B_2)(B_1-B_3)}\omega_q\wedge\omega_\beta \\ + \textstyle\sum_{a,b} B_{p\alpha,a}\omega_{ab}\wedge\omega_b + \sum_a dB_{p\alpha,a}\wedge\omega_a \\ = (B_2-B_3)\bigg(\frac{\sum_{a,q,\beta} B_{p\beta,a}B_{q\alpha,a}}{(B_1-B_2)(B_1-B_3)}\omega_q\wedge\omega_\beta \\ + \textstyle\sum_q \omega_{pq}\wedge\omega_{q\alpha} + \sum_\beta \omega_{p\beta}\wedge\omega_{\beta\alpha} - R_{p\alpha p\alpha}\omega_p\wedge\omega_\alpha\bigg). \end{aligned} \tag{3-15}$$

Comparing the coefficients of $\omega_q\wedge\omega_\beta$ on both sides of (3-15), we obtain

$$\textstyle\sum_a B_{p\alpha,a}B_{q\beta,a} + \sum_a B_{p\beta,a}B_{q\alpha,a} = (B_1-B_2)(B_1-B_3)R_{p\alpha p\alpha}\delta_{pq}\delta_{\alpha\beta}. \tag{3-16}$$

Similarly, by differentiating (3-13) and (3-14), we get

$$\sum_p B_{a\alpha,p}B_{b\beta,p}+\sum_p B_{a\beta,p}B_{b\alpha,p}=(B_2-B_1)(B_2-B_3)R_{a\alpha a\alpha}\delta_{ab}\delta_{\alpha\beta}, \tag{3-17}$$

$$\sum_\alpha B_{pa,\alpha}B_{qb,\alpha}+\sum_\alpha B_{pb,\alpha}B_{qa,\alpha}=(B_3-B_2)(B_3-B_1)R_{papa}\delta_{pq}\delta_{ab}. \tag{3-18}$$

From (3-16), (3-17) and (3-18), the relations in (3-9) and (3-10) immediately follow.

Moreover, from (3-16)–(3-18) and (2-14), we get

$$2|\vec{v}_{p\alpha}|^2=(B_1-B_2)(B_1-B_3)(B_2B_3+A_p+A_\alpha), \tag{3-19}$$

$$2|\vec{v}_{a\alpha}|^2=(B_2-B_1)(B_2-B_3)(B_1B_3+A_a+A_\alpha), \tag{3-20}$$

$$2|\vec{v}_{pa}|^2=(B_3-B_2)(B_3-B_1)(B_1B_2+A_p+A_a). \tag{3-21}$$

Then the relations in (3-11) also immediately follow. □

Lemma 3.2. *If, on some open set, the array $(\vec{v}_{p\alpha})$ contains a zero vector, then all the vectors in either the whole row or in the whole column where the zero vector is located must be zero.*

Proof. For simplicity of notation, in this proof we denote the $m_2\times m_1$ array $(\vec{v}_{p\alpha})$ by $(\vec{v}_{ij})$ for $1\le i\le m_2$ and $1\le j\le m_1$, where $\vec{v}_{ij}\in\mathbb{R}^{m_3}$. By Lemma 3.1, the array has the following properties:

(P1) The vectors of any row form an orthogonal set.

(P2) The vectors of any column form an orthogonal set.

For any 2×2 minor $\begin{pmatrix}\vec{v}_{ik} & \vec{v}_{il}\\ \vec{v}_{jk} & \vec{v}_{jl}\end{pmatrix}$,

(P3) $\vec{v}_{ik}\cdot\vec{v}_{jl}+\vec{v}_{il}\cdot\vec{v}_{jk}=0$, and

(P4) $|\vec{v}_{ik}|^2+|\vec{v}_{jl}|^2=|\vec{v}_{il}|^2+|\vec{v}_{jk}|^2$.

Obviously, all these four properties will remain unchanged if either the rows or the columns of the array are permuted.

Suppose that a vector in the array is zero on an open set $U\subset M^*$. Permuting rows and columns, if necessary, we may assume that $\vec{v}_{11}=0$ on U. Then (P1), (P2) and (P3) imply that at each point of U, the remaining vectors

$$\vec{v}_{12},\dots,\vec{v}_{1m_1}\quad\text{and}\quad\vec{v}_{21},\dots,\vec{v}_{m_21}$$

in the first row and the first column form a mutually orthogonal set of m_1+m_2-2 vectors in $\mathbb{R}^{m_3}$, and at most m_3 vectors of which can be nonzero at any point. Let ξ_0 be a point where a maximal number of these vectors is nonzero. By continuity, the nonzero vectors at ξ_0 will remain nonzero in some open subset $V\subset U$ containing ξ_0. By maximality, the vectors that are zero at ξ_0 must remain zero on V.

By permuting rows and columns if necessary, we may assume that

$$\vec{v}_{11} = \cdots = \vec{v}_{1j} = 0 \quad \text{and} \quad \vec{v}_{11} = \cdots = \vec{v}_{i1} = 0$$

for some $i \in \{1, \ldots, m_2\}$ and $j \in \{1, \ldots, m_1\}$. The remaining vectors of the first column and the first row are all nonzero at each point of V, so the array has first row $(0, 0, \ldots, 0, \vec{v}_{1(j+1)}, \ldots, \vec{v}_{1m_1})$ and first column $(0, \ldots, 0, \vec{v}_{(i+1)1}, \ldots, \vec{v}_{m_2 1})$ and (P4) implies that $\vec{v}_{kl} = 0$ for $1 \le k \le i$ and $1 \le l \le j$. Hence all elements in the upper left $i \times j$ block of the array should be zero vectors on V,

If the first row of the array is zero on V, then we are done. If otherwise, we have $j < m_1$ and $\vec{v}_{1l} \neq 0$ for all $l \ge j+1$. Let us fix an arbitrary $k \in \{i+1, \ldots, m_2\}$ and $l \in \{j+1, \ldots, m_1\}$. Then property (P4) easily implies that

$$|\vec{v}_{k1}| = \cdots = |\vec{v}_{kj}| \quad \text{and} \quad |\vec{v}_{1l}| = \cdots = |\vec{v}_{il}| \neq 0. \tag{3-22}$$

Also by using (P4) with the minor $\begin{pmatrix} 0 & \vec{v}_{1l} \\ \vec{v}_{kj} & \vec{v}_{kl} \end{pmatrix}$, we get

$$|\vec{v}_{kl}|^2 = |\vec{v}_{kj}|^2 + |\vec{v}_{1l}|^2 \neq 0. \tag{3-23}$$

On the other hand, the properties (P1), (P2) and (P3) imply that

$$\vec{v}_{k1}, \ldots, \vec{v}_{kj}, \quad \vec{v}_{1l}, \ldots, \vec{v}_{il}, \quad \vec{v}_{kl} \tag{3-24}$$

form an orthogonal set of $i + j + 1$ vectors in $\mathbb{R}^{m_3}$. But, the nonzero vectors in the first column and the first row together form an orthogonal set of $(m_1 - j) + (m_2 - i)$ nonzero vectors. Hence, $m_1 + m_2 - i - j \le m_3$ and thus $i + j + 1 \ge m_1 + m_2 - m_3 + 1 \ge m_1 + 1 > m_3$, so some of the vectors in (3-24) must be zero. By (3-22) and (3-23), it must be the case that $\vec{v}_{k1} = \cdots = \vec{v}_{kj} = 0$. As this is true for $k = i+1, \ldots, m_2$, it follows that the first j columns of the array are all zero on the open set V. □

Lemma 3.3. *If $\nabla \boldsymbol{B} \neq 0$, then for any one of the three arrays $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$, $(\vec{v}_{pa})$, it cannot happen that there exists both a row and a column whose elements are all zero vectors on some open set $U \subset M^*$.*

Proof. Suppose to the contrary that we have such an array $(\vec{v}_{ij})$ for which each element of the $\bar{i}$-th row and the $\bar{j}$-th column is zero on an open set $U \subset M^*$. Then for any $k \neq \bar{i}$ and $l \neq \bar{j}$, the property (P4) gives that

$$|\vec{v}_{kl}|^2 = |\vec{v}_{\bar{i}l}|^2 + |\vec{v}_{k\bar{j}}|^2 - |\vec{v}_{\bar{i}\bar{j}}|^2 = 0.$$

Thus all elements of $(\vec{v}_{ij})$ are zero vectors on U, which contradicts $\nabla \boldsymbol{B} \neq 0$. □

Now we can divide our discussions into two cases:

Case I. $m_1 = m_2 = m_3$.

Case II. $m_1 \ge m_2 \ge m_3$ and $m_1 > m_3$.

Each case corresponds to a main result of this paper:

Theorem 3.1. *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures of multiplicities $m_1 = m_2 = m_3$. If the Möbius second fundamental form is not parallel, then x is locally Möbius equivalent to the Euclidean isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with three distinct principal curvatures.*

Theorem 3.2. *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures of multiplicities m_1, m_2 and m_3 satisfying $m_1 \geq m_2 \geq m_3$ and $m_1 > m_3$. If the Möbius second fundamental form is not parallel, then $m_2 = m_3 := m$ and x is locally Möbius equivalent one of the minimal hypersurfaces as given by part (iii) of the classification theorem.*

The proofs of these two theorems are quite involved and will be given separately in the next two sections.

4. Möbius isoparametric hypersurfaces with $m_1 = m_2 = m_3$

This section is devoted to Case I and giving a proof of Theorem 3.1. Assume that $m_1 = m_2 = m_3 := m$ and $\nabla \boldsymbol{B} \neq 0$.

Proposition 4.1. *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures of the same multiplicity m. If the Möbius second fundamental form $\boldsymbol{B}$ is not parallel, then every vector in each of the three $m \times m$ arrays $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$ and $(\vec{v}_{pa})$ has length equal to $\sqrt{f}/m$, where $f = \sum_{p,a,\alpha} B^2_{pa,\alpha}$ is a constant function.*

To prove the proposition, we first establish two lemmas whose proofs can be given by the crucial algebraic techniques that were essentially discovered by Cecil and Jensen [1998]; we present the proofs here for the reader's convenience.

Lemma 4.1. *There is an open subset $U \subset M^*$ on which every vector is nonzero in each of the three $m \times m$ arrays $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$ and $(\vec{v}_{pa})$.*

Proof. Suppose to the contrary and without loss of generality that $\vec{v}_{(m+1)1} = 0$ on some open set U. Then by Lemma 3.2, one of two cases must occur:

- $\vec{v}_{(m+1)a} = 0$ for $1 \leq a \leq m$, or
- $\vec{v}_{p1} = 0$ for $m + 1 \leq p \leq 2m$.

In the first case, the first component of each vector of $(\vec{v}_{a\alpha})$ is zero. Hence $\vec{v}_{a\alpha}$ can be looked at as if it were in $\mathbb{R}^{m-1}$. By using (P1) and (P2), we see that at least one element both in each row and in each column of the array $(\vec{v}_{a\alpha})$ is zero. Then by using (P4), Lemma 3.2 and Lemma 3.3, we easily get $(\vec{v}_{a\alpha}) = 0$ on U. This contradicts that $\nabla \boldsymbol{B} \neq 0$, so this case does not occur.

In the second case, we can show as above that $(\vec{v}_{p\alpha}) = 0$, also a contradiction. Hence this case cannot occur either. □

Lemma 4.2. *Suppose that every vector in the arrays* $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$ *and* $(\vec{v}_{pa})$ *is nonzero on* $U \subset M^*$. *Then, for each array, all vectors either in each row or in each column have the same length.*

Proof. Consider one of the arrays and denote its first row by $\vec{v}_1, \dots, \vec{v}_m$. By property (P1) and the assumption that none of these vectors is zero, it follows that this is an orthogonal basis of $\mathbb{R}^m$. Thus, there exist linear operators T_j of $\mathbb{R}^m$ for $j = 2, \dots, m$, such that the j-th row of the array is given by $T_j\vec{v}_1, \dots, T_j\vec{v}_m$. For each of these operators, the properties (P1)–(P4) imply also that

(O1) T_j is skew-symmetric for $j = 2, \dots, m$,

(O2) each of the vectors $\vec{v}_1, \dots, \vec{v}_m$ is an eigenvector of T_j^2 for $j = 2, \dots, m$, and

(O3) the relation $|T_j\vec{v}_i|^2 + |\vec{v}_k|^2 = |\vec{v}_i|^2 + |T_j\vec{v}_k|^2$ holds for any $j = 2, \dots, m$ and $i \neq k$, where $1 \leq i, k \leq m$.

In fact, from (P2) we can see that $T_j\vec{v}_i \cdot \vec{v}_i = 0$ holds for all $i = 1, \dots, m$ and $j = 2, \dots, m$. Similarly, $T_j\vec{v}_i \cdot \vec{v}_k + \vec{v}_i \cdot T_j\vec{v}_k = 0$ follows from (P3). Thus, (P2) and (P3) imply (O1). In addition, (P1) implies that $T_j\vec{v}_i \cdot T_j\vec{v}_k = 0$ whenever $i \neq k$, and thus $T_j^2\vec{v}_i \cdot \vec{v}_k = 0$ by (O1). It follows that $\vec{v}_i$ must be an eigenvector of T_j^2. Property (O3) follows immediately from (P4).

Having seen that each $\vec{v}_i$ is an eigenvector of T_j^2, the correspondent eigenvalue is easily seen to be given by

$$T_j^2\vec{v}_i = -\frac{|T_j\vec{v}_i|^2}{|\vec{v}_i|^2}\vec{v}_i. \tag{4-1}$$

This follows from the fact that $a|\vec{v}_i|^2 = a\vec{v}_i \cdot \vec{v}_i = T_j^2\vec{v}_i \cdot \vec{v}_i = -T_j\vec{v}_i \cdot T_j\vec{v}_i$ if $T_j^2\vec{v}_i = a\vec{v}_i$.

Fix any $j \in \{2, \dots, m\}$. Let $T = T_j$ and denote by $a_1, \dots, a_m$ the eigenvalues of T^2. Then property (O3) implies the relation

$$(1 + a_i)|\vec{v}_i|^2 = (1 + a_k)|\vec{v}_k|^2 \quad \text{for all } i,\ k \in \{1, \dots, m\}. \tag{4-2}$$

Consequently, if some eigenvalue a_i is equal to -1, then so are all the others, and thus $T^2 = -I$.

If none of the eigenvalues equals -1, then $a_i = a_k$ if and only if $|\vec{v}_i| = |\vec{v}_k|$.

Suppose that, for some row of the array, the vectors do not have the same length, and suppose likewise for some column. Relabeling if necessary, we may suppose that $\vec{v}_1, \dots, \vec{v}_m$ do not have the same length. Then there must be some vector $\vec{v}_i$ such that $|\vec{v}_i|$ is not equal to $|\vec{v}_k|$ for at least $m - \lfloor m/2 \rfloor$ vectors $\vec{v}_k$, where $\lfloor z \rfloor$

denotes the greatest integer less than or equal to z. Permute the columns so that

$$|\vec{v}_1| \neq |\vec{v}_k| \quad \text{for } \lfloor m/2 \rfloor + 1 \leq k \leq m. \tag{4-3}$$

From (3-19), (3-20) and (3-21), we have

$$\begin{aligned} |\vec{v}_{p\alpha}|^2 - |\vec{v}_{p\beta}|^2 &= \tfrac{1}{2}(B_1 - B_2)(B_1 - B_3)(A_\alpha - A_\beta), \\ |\vec{v}_{p\alpha}|^2 - |\vec{v}_{q\alpha}|^2 &= \tfrac{1}{2}(B_1 - B_2)(B_1 - B_3)(A_p - A_q), \\ |\vec{v}_{a\alpha}|^2 - |\vec{v}_{a\beta}|^2 &= \tfrac{1}{2}(B_2 - B_1)(B_2 - B_3)(A_\alpha - A_\beta), \\ |\vec{v}_{a\alpha}|^2 - |\vec{v}_{b\alpha}|^2 &= \tfrac{1}{2}(B_2 - B_1)(B_2 - B_3)(A_a - A_b), \\ |\vec{v}_{pa}|^2 - |\vec{v}_{pb}|^2 &= \tfrac{1}{2}(B_3 - B_2)(B_3 - B_1)(A_a - A_b), \\ |\vec{v}_{pa}|^2 - |\vec{v}_{qa}|^2 &= \tfrac{1}{2}(B_3 - B_2)(B_3 - B_1)(A_p - A_q). \end{aligned} \tag{4-4}$$

Consequently, if $(\vec{v}_{ij})$ denotes any one of the arrays, then there exist numbers $c_{ij} = -c_{ji}$ and $d_{ij} = -d_{ji}$ such that

$$|\vec{v}_{ij}|^2 - |\vec{v}_{ik}|^2 = c_{jk} \quad \text{for all } i, \tag{4-5}$$

$$|\vec{v}_{ik}|^2 - |\vec{v}_{jk}|^2 = d_{ij} \quad \text{for all } k. \tag{4-6}$$

Now (4-5) implies that (4-3) must hold for every row in our array. Thus (4-3) continues to hold after permuting the rows. We may thus assume that for some i,

$$|\vec{v}_i| \neq |T_j \vec{v}_i| \quad \text{for } \lfloor m/2 \rfloor + 1 \leq j \leq m. \tag{4-7}$$

Then (4-6) implies that (4-7) holds for every column of the array, and in particular for the first column.

In summary, we can conclude that

$$|\vec{v}_1| \neq |\vec{v}_j| \quad \text{and} \quad |\vec{v}_1| \neq |T_j \vec{v}_1| \quad \text{for } \lfloor m/2 \rfloor + 1 \leq j \leq m.$$

Now we fix $j,\ k \in \{\lfloor m/2 \rfloor + 1, \dots, m\}$. Then we claim that $\vec{v}_1$ and $\vec{v}_j$ must be in different eigenspaces of T_k^2. In fact, by (4-1) and (4-4), we see that none of the eigenvalues of T_k^2 is -1. But then by (4-2) and the first part of (4-4), the eigenvalues of T_k^2 associated to the eigenvectors $\vec{v}_1$ and $\vec{v}_j$ must be different.

On the other hand, $\vec{v}_1$ and $T_k \vec{v}_1$ are in the same eigenspace of T_k^2. In fact, if $T_k^2 \vec{v}_1 = a\vec{v}_1$, then $T_k^2 T_k \vec{v}_1 = T_k T_k^2 \vec{v}_1 = a T_k \vec{v}_1$. Thus, $\vec{v}_j$ and $T_k \vec{v}_1$ are in different eigenspaces of T_k^2. Since T_k^2 is symmetric, we have

$$\vec{v}_j \cdot T_k \vec{v}_1 = 0 \quad \text{for } \lfloor m/2 \rfloor + 1 \leq j, k \leq m.$$

By (P1), we also have $\vec{v}_1 \cdot \vec{v}_j = 0$ for $\lfloor m/2 \rfloor + 1 \leq j, k \leq m$. Thus, the $m - \lfloor m/2 \rfloor$ nonzero orthogonal vectors $\vec{v}_{\lfloor m/2 \rfloor + 1}, \dots, \vec{v}_m$ lie in the orthogonal complement of the $(m - \lfloor m/2 \rfloor + 1)$-dimensional space spanned by $\vec{v}_1, T_{\lfloor m/2 \rfloor + 1} \vec{v}_1, \dots, T_m \vec{v}_1$. This is impossible, which implies the impossibility of the assumption above that some row and some column of the array have vectors of unequal length. □

Proof of Proposition 4.1. According to Lemmas 4.1 and 4.2, we may assume that all vectors in each row of array $(\vec{v}_{pa})$ have the same length, that is,

$$|\vec{v}_{p1}|^2 = |\vec{v}_{p2}|^2 = \cdots = |\vec{v}_{pm}|^2 \quad \text{for all } p \in \{m+1, \dots, 2m\}. \tag{4-8}$$

Consider the $m \times m$ matrix

$$F = \begin{pmatrix} B_{p1,2m+1} & B_{p2,2m+1} & \cdots & B_{pm,2m+1} \\ B_{p1,2m+2} & B_{p2,2m+2} & \cdots & B_{pm,2m+2} \\ \vdots & \vdots & \ddots & \vdots \\ B_{p1,n} & B_{p2,n} & \cdots & B_{pm,n} \end{pmatrix},$$

whose i-th row is exactly the components of $\vec{v}_{p(2m+i)}$, and whose j-th column is exactly the components of $\vec{v}_{pj}$, where $1 \le i,\ j \le m$. Using properties (P1) and (P2), we have

$$^tF F = |\vec{v}_{p1}|^2 I_m, \tag{4-9}$$

$$F^t F = \begin{pmatrix} |\vec{v}_{p(2m+1)}|^2 & 0 & \cdots & 0 \\ 0 & |\vec{v}_{p(2m+2)}|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\vec{v}_{pn}|^2 \end{pmatrix}. \tag{4-10}$$

From (4-9), we see that $F^t F = {}^tF F$. Then we compare (4-9) with (4-10) to obtain

$$|\vec{v}_{p(2m+1)}|^2 = \cdots = |\vec{v}_{pn}|^2 = |\vec{v}_{p1}|^2 \quad \text{for all } p \in \{m+1, \dots, 2m\}. \tag{4-11}$$

Now, from (3-21) and (4-8), we get $A_a = A_b$ for all $a \neq b$. Similarly, from (3-19) and (4-11) we get $A_\alpha = A_\beta$ for all $\alpha \neq \beta$. These facts together with (3-20) give

$$|\vec{v}_{a\alpha}|^2 = \frac{1}{m^2} \sum_{b,\beta,p} B^2_{b\beta,p} = \frac{1}{m^2} f \quad \text{for all } a, \alpha.$$

Proceeding as in the proof of (4-11), we get

$$|\vec{v}_{(m+1)a}|^2 = \cdots = |\vec{v}_{(2m)a}|^2 = |\vec{v}_{a(2m+1)}|^2 = \cdots = |\vec{v}_{an}|^2 \quad \text{for all } a \in \{1, \dots, m\}, \tag{4-12}$$

$$|\vec{v}_{(m+1)\alpha}|^2 = \cdots = |\vec{v}_{(2m)\alpha}|^2 = |\vec{v}_{1\alpha}|^2 = \cdots = |\vec{v}_{m\alpha}|^2 \quad \text{for all } \alpha \in \{2m+1, \dots, n\}. \tag{4-13}$$

Then (4-11)–(4-13) imply that every vector in each of the three arrays $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$ and $(\vec{v}_{pa})$ has length equal to $\sqrt{f}/m$.

Next, we will show that f is constant. Using (2-17), (3-6) and (3-8), we get

$$\sum_i B_{ab,pi}\omega_i=\sum_\alpha B_{a\alpha,p}\omega_{\alpha b}+\sum_\alpha B_{\alpha b,p}\omega_{\alpha a}=\sum_{\alpha,q} B_{a\alpha,p}\Gamma^\alpha_{qb}\omega_q+\sum_{\alpha,q} B_{\alpha b,p}\Gamma^\alpha_{qa}\omega_q.$$

Comparing two sides of this, we obtain $B_{ab,p\alpha}=0$. A similar argument gives $B_{pq,a\alpha}=0$ and $B_{\alpha\beta,ap}=0$. By (2-14), (2-18), (3-1) and (3-4), we easily see that the four indices in $B_{pa,\alpha i}$ for $1\le i\le n$ are totally symmetric. Hence we get

$$0=\sum_i B_{pa,\alpha i}\omega_i=dB_{pa,\alpha}+\sum_b B_{pb,\alpha}\omega_{ba}+\sum_q B_{qa,\alpha}\omega_{qp}+\sum_\beta B_{pa,\beta}\omega_{\beta\alpha}.$$

Multiplying this equation by $B_{pa,\alpha}$ and summing, we get

$$0=\sum_{p,a,\alpha} B_{pa,\alpha}dB_{pa,\alpha}+\sum_{p,a,b,\alpha} B_{pa,\alpha}B_{pb,\alpha}\omega_{ba}+\sum_{p,a,q,\alpha} B_{pa,\alpha}B_{qa,\alpha}\omega_{qp}+\sum_{p,a,\alpha,\beta} B_{pa,\alpha}B_{pa,\beta}\omega_{\beta\alpha},$$

or, equivalently,

$$0=\tfrac{1}{2}df+\sum_{p,a,b}(\vec{v}_{pa}\cdot\vec{v}_{pb})\,\omega_{ba}+\sum_{p,q,a}(\vec{v}_{pa}\cdot\vec{v}_{qa})\,\omega_{qp}+\sum_{p,\alpha,\beta}(\vec{v}_{p\alpha}\cdot\vec{v}_{p\beta})\,\omega_{\beta\alpha}. \tag{4-14}$$

Lemma 3.1 and (4-14) imply that $df=0$, showing that f is constant. □

Lemma 4.3. *The eigenvalues of the Blaschke tensor* $\mathbf{A}$ *are all constant on* M.

Proof. By (2-14) and (3-19)–(3-21), we get

$$R_{apap}=\frac{2|\vec{v}_{pa}|^2}{(B_3-B_1)(B_3-B_2)}=B_1B_2+A_a+A_p, \tag{4-15}$$

$$R_{a\alpha a\alpha}=\frac{2|\vec{v}_{a\alpha}|^2}{(B_2-B_1)(B_2-B_3)}=B_1B_3+A_a+A_\alpha, \tag{4-16}$$

$$R_{p\alpha p\alpha}=\frac{2|\vec{v}_{p\alpha}|^2}{(B_1-B_2)(B_1-B_3)}=B_2B_3+A_p+A_\alpha. \tag{4-17}$$

Using Proposition 4.1 and adding (4-15), (4-16) and (4-17), we have

$$B_1B_2+B_1B_3+B_2B_3+2(A_a+A_p+A_\alpha)=0. \tag{4-18}$$

From (4-15) up to (4-18) we get

$$\begin{aligned}
A_a&=\tfrac{1}{2}(B_2B_3-B_1B_2-B_1B_3)-\frac{2f}{m^2(B_1-B_2)(B_1-B_3)},\\
A_p&=\tfrac{1}{2}(B_1B_3-B_1B_2-B_2B_3)-\frac{2f}{m^2(B_2-B_1)(B_2-B_3)},\\
A_\alpha&=\tfrac{1}{2}(B_1B_2-B_1B_3-B_2B_3)-\frac{2f}{m^2(B_3-B_1)(B_3-B_2)}.
\end{aligned} \tag{4-19}$$

Therefore all the eigenvalues of $\boldsymbol{A}$ are constant on M^*. On the other hand, the well-defined continuous functions $A_1, A_2, \ldots, A_n$ satisfy (3-5). Thus we can indeed choose a frame field $\{E_i\}$ around each point of M so that (3-1) and (3-4) hold identically. This fact and the argument above show that the open set M^* is also closed in M. By connectedness, we know that $M^* = M$. □

Remark 4.1. Now that the Blaschke eigenvalues $A_1, A_2, \ldots, A_n$ are constant, we can find everywhere local frame fields $\{E_i\}$ such that (3-1) and (3-4) hold at the same time.

Proof of Theorem 3.1. From Proposition 4.1 and (4-19), we get

$$A_1 = \cdots = A_m, \quad A_{m+1} = \cdots = A_{2m}, \quad A_{2m+1} = \cdots = A_n. \tag{4-20}$$

From Lemma 4.1 we know that $\vec{v}_{pa} \neq 0$; thus there exist α such that $B_{pa,\alpha} \neq 0$. From (3-6), (3-7), (2-9) and that both $A_{ij,k}$ and $B_{ij,k}$ are totally symmetric, we get

$$A_{pa,\alpha} = (A_p - A_a)\Gamma^p_{\alpha a} = (A_a - A_\alpha)\Gamma^a_{p\alpha} = (A_\alpha - A_p)\Gamma^\alpha_{ap}, \tag{4-21}$$

$$B_{pa,\alpha} = (B_2 - B_1)\Gamma^p_{\alpha a} = (B_1 - B_3)\Gamma^a_{p\alpha} = (B_3 - B_2)\Gamma^\alpha_{ap}. \tag{4-22}$$

From (4-21) and (4-22), we derive

$$\frac{A_{pa,\alpha}}{B_{pa,\alpha}} = \frac{A_p - A_a}{B_2 - B_1} = \frac{A_a - A_\alpha}{B_1 - B_3} = \frac{A_\alpha - A_p}{B_3 - B_2},$$

which together with (4-20) implies the existence of constant functions λ and μ with the property

$$\begin{aligned} A_1 + \lambda B_1 = \cdots = A_m + \lambda B_1 = A_{m+1} + \lambda B_2 = \cdots = A_{2m} + \lambda B_2 \\ = A_{2m+1} + \lambda B_3 = \cdots = A_n + \lambda B_3 = \mu. \end{aligned}$$

Hence we have $\boldsymbol{A} + \lambda \boldsymbol{B} - \mu g = 0$, and by it we can apply the result of Li and Wang [2003] to conclude that $x : M \to \mathbb{S}^{n+1}$ is locally Möbius equivalent to one of the following hypersurfaces:

- a hypersurface $\tilde{x} : \tilde{M} \to \mathbb{S}^{n+1}$ with constant mean curvature and constant scalar curvature;
- the image under σ of a hypersurface $\tilde{x} : \tilde{M} \to \mathbb{R}^{n+1}$ with constant mean curvature and constant scalar curvature;
- the image under τ of a hypersurface $\tilde{x} : \tilde{M} \to \mathbb{H}^{n+1}$ with constant mean curvature and constant scalar curvature. Here, we recall that we have defined the conformal diffeomorphism $\tau : \mathbb{H}^{n+1} \to \mathbb{S}^{n+1}_+$, $y \mapsto (1, y')/y_0$, where

$$\begin{aligned} \mathbb{H}^{n+1} &= \{(y_0, y_1, \ldots, y_{n+1}) \in \mathbb{L}^{n+2} \mid \langle y, y\rangle_1 = -1, y_0 \geq 1)\}, \\ \mathbb{S}^{n+1}_+ &= \{(x_1, \ldots, x_{n+2}) \in \mathbb{S}^{n+1} \mid x_1 > 0)\}, \end{aligned}$$

and $y' = (y_1, \dots, y_{n+1})$.

For each of these possibilities, from [Hu et al. 2007, Propositions 3.1 and 3.2], and because the B_i are all constant, we see that $\tilde{x} : \tilde{M} \to \mathbb{S}^{n+1}$, or $\tilde{x} : \tilde{M} \to \mathbb{R}^{n+1}$, or $\tilde{x} : \tilde{M} \to \mathbb{H}^{n+1}$, respectively, are all Euclidean isoparametric hypersurfaces with three distinct principal curvatures. From the classical result that isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ can have at most two distinct principal curvatures, we finally see that x is Möbius equivalent to an open part of some isoparametric hypersurface in $\mathbb{S}^{n+1}$ with three distinct principal curvatures. □

5. Möbius isoparametric hypersurfaces with $m_1 > m_3$

This section is devoted to Case II and proving Theorem 3.2. Assume that

(5-1) $$\nabla \boldsymbol{B} \neq 0 \quad \text{and} \quad m_1 \geq m_2 \geq m_3 \text{ such that } m_1 > m_3.$$

To add to the index conventions (3-3), we introduce the notation

$$\begin{aligned} \mathcal{I}_1 &= \{1, 2, \dots, m_1\}, \\ \mathcal{I}_2 &= \{m_1+1, m_1+2, \dots, m_1+m_2\}, \\ \mathcal{I}_3 &= \{m_1+m_2+1, m_1+m_2+2, \dots, n\}. \end{aligned}$$

In follows, we will concentrate on the $m_2 \times m_1$ array $(\vec{v}_{pa})$ of vectors in $\mathbb{R}^{m_3}$.

Lemma 5.1. *There exists an integer m_1', where $0 < m_1 - m_3 \leq m_1' < m_1$, such that exactly m_1' columns of the $m_2 \times m_1$ array $(\vec{v}_{pa})$ are identically zero on an open set $U \subset M^*$. Explicitly, there exists a subset $\mathcal{D}_0 \subset \mathcal{I}_1$ of m_1' elements, with complement $\mathcal{D}_1$ in $\mathcal{I}_1$, such that*

(5-2) $$\vec{v}_{pa} = 0 \quad \textit{for all } a \in \mathcal{D}_0 \textit{ and } p \in \mathcal{I}_2,$$

(5-3) $$\vec{v}_{pc} \neq 0 \quad \textit{for all } c \in \mathcal{D}_1 \textit{ and } p \in \mathcal{I}_2.$$

Proof. By Lemma 3.1, for each $\bar{p} \in \mathcal{I}_2$, the vectors in row $\bar{p}$ of the array $(\vec{v}_{pa})$ constitute a set of m_1 mutually orthogonal vectors in $\mathbb{R}^{m_3}$. Thus, at least $m_1 - m_3$ vectors in row $\bar{p}$ must be zero at any point of M^*. On the other hand, by Lemmas 3.2 and 3.3 we know that it is impossible that a whole row is zero in the array $(\vec{v}_{pa})$. Permute the columns of $(\vec{v}_{pa})$, so that row $\bar{p}$ has all its nonzero vectors occurring first (left to right). Let $\vec{v}_{\bar{p}\tilde{m}_1}$ denote the last nonzero vector in this row. Then $1 < \tilde{m}_1 \leq m_3 < m_1$. Thus we have

$$\vec{v}_{\bar{p}c} \neq 0 \quad \text{if } 1 \leq c \leq \tilde{m}_1 \quad \text{and} \quad \vec{v}_{\bar{p}a} = 0 \quad \text{if } \tilde{m}_1 + 1 \leq a \leq m_1.$$

Since at least one vector is nonzero in row $\bar{p}$, by Lemma 3.2 the last $m_1 - \tilde{m}_1$ columns of array $(\vec{v}_{pa})$ are all zero on an open set $U \subset M^*$. That is,

$$\text{if } \tilde{m}_1 + 1 \leq a \leq m_1, \quad \text{then } \vec{v}_{pa} = 0 \quad \text{for all } p \in \mathcal{I}_2.$$

Now we apply property (P4) to the minor

$$\begin{pmatrix} \vec{v}_{\bar{p}c} & \vec{v}_{\bar{p}a} \\ \vec{v}_{pc} & \vec{v}_{pa} \end{pmatrix} \quad \text{with } 1 \le c \le \tilde{m}_1,\ \tilde{m}_1 + 1 \le a \le m_1 \text{ and any } p \in \mathcal{I}_2,$$

to obtain

$$|\vec{v}_{(m_1+1)c}| = \cdots = |\vec{v}_{(m_1+m_2)c}| = |\vec{v}_{\bar{p}c}| \ne 0 \quad \text{for all } 1 \le c \le \tilde{m}_1. \tag{5-4}$$

Let $m_1' = m_1 - \tilde{m}_1$. Then $0 < m_1 - m_3 \le m_1' < m_1$ and the assertion follows by setting

$$\mathcal{D}_0 = \{\tilde{m}_1 + 1, \tilde{m}_1 + 2, \dots, m_1\} \quad \text{and} \quad \mathcal{D}_1 = \{1, 2, \dots, \tilde{m}_1\}. \qquad \square$$

Lemma 5.2. *Assume that $\nabla \boldsymbol{B} \ne 0$ and $m_1 \ge m_2 \ge m_3$. If $m_1 > m_3$, then $m_2 = m_3$.*

Proof. By (5-3) and Lemma 3.1, for each $c \in \mathcal{D}_1$ the vectors in column c of the array constitute a set of m_2 mutually orthogonal nonzero vectors in $\mathbb{R}^{m_3}$; hence we have $m_2 \le m_3$. By the assumption $m_2 \ge m_3$, we get $m_2 = m_3$. $\square$

Lemma 5.3. *For all $a, b \in \mathcal{D}_0,\ c \in \mathcal{D}_1,\ p, q \in \mathcal{I}_2$ and $\alpha, \beta \in \mathcal{I}_3$, we have*

$$A_a = A_b \ne A_c, \quad A_p = A_q, \quad A_\alpha = A_\beta.$$

Proof. From (5-2) and (5-3), we get that, for all $a, b \in \mathcal{D}_0,\ c \in \mathcal{D}_1$ and $p, q \in \mathcal{I}_2$,

$$|\vec{v}_{pa}| = |\vec{v}_{pb}| = |\vec{v}_{qa}| = 0 \ne |\vec{v}_{pc}|.$$

This combined with (3-21) gives $A_a = A_b \ne A_c$ and $A_p = A_q$.

From (5-2) we have

$$B_{pa,\alpha} = 0 \quad \text{for all } a \in \mathcal{D}_0,\ p \in \mathcal{I}_2,\ \alpha \in \mathcal{I}_3. \tag{5-5}$$

The fact that $B_{ij,k}$ is totally symmetric and (5-5) implies that $\vec{v}_{a\alpha} = 0$ for all $a \in \mathcal{D}_0$ and $\alpha \in \mathcal{I}_3$. Combining this with (3-20), we get $A_\alpha = A_\beta$. $\square$

Lemma 5.4. $\tilde{m}_1 = m_3 = m_2$.

Proof. By Lemma 5.3, we get $A_p = A_q$ and $A_\alpha = A_\beta$. Combining (3-19) with (5-1), we obtain

$$|\vec{v}_{p\alpha}|^2 = \frac{1}{m_2^2} \sum_{q,\beta,c} B_{q\beta,c}^2 = \frac{1}{m_2^2} f \ne 0 \quad \text{for all } p, \alpha. \tag{5-6}$$

From (5-5) we know that the last $m_1 - \tilde{m}_1$ components of each vector $\vec{v}_{p\alpha}$ are zero on the open set U as we stated in Lemma 5.1; thus $\vec{v}_{p\alpha}$ can be regarded as an element of $\mathbb{R}^{\tilde{m}_1}$. By Lemma 3.1, for each p the vectors in row p of the array $(\vec{v}_{p\alpha})$ constitute a set of m_3 mutually orthogonal nonzero vectors in $\mathbb{R}^{\tilde{m}_1}$. Hence $m_3 \le \tilde{m}_1$, while Lemma 5.1 tells that $\tilde{m}_1 \le m_3$. Hence $\tilde{m}_1 = m_3 = m_2$. $\square$

Next, by using (5-4), (5-6) and Lemma 3.1, we get the following by adapting the proof of Proposition 4.1.

Proposition 5.1. *All the nonzero vectors of the arrays* $(\vec{v}_{p\alpha})$, $(\vec{v}_{a\alpha})$ *and* $(\vec{v}_{pa})$ *have constant length equal to* $\sqrt{f}/m_2$. *That is, we have*

$$|\vec{v}_{cp}|^2 = |\vec{v}_{d\alpha}|^2 = |\vec{v}_{q\beta}|^2 = f/m_2^2 := L^2 = \text{const} \tag{5-7}$$

for any $c, d \in \mathfrak{D}_1,\ p, q \in \mathcal{I}_2$ *and* $\alpha, \beta \in \mathcal{I}_3$.

Now, we are ready to prove one of the main results in this section.

Proposition 5.2. *Let* $x : M^n \to \mathbb{S}^{n+1}$ *be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures of multiplicities* $m_1 \geq m_2 \geq m_3$ *and* $m_1 > m_3$. *If the Möbius second fundamental form is not parallel, then it must be the case that* $m_2 = m_3 := m$ *and that the Möbius principal curvatures satisfy* $B_1 = 0$ *and* $B_2 = -B_3 = \pm\sqrt{(n-1)/(2mn)}$.

Proof. By Lemma 5.2 we may assume that $m_2 = m_3 := m$. Let us take $a \in \mathfrak{D}_0$, $c \in \mathfrak{D}_1,\ p \in \mathcal{I}_2$ and $\alpha \in \mathcal{I}_3$. Then by the proof of Lemma 5.3, we have $\vec{v}_{a\alpha} = 0$. By using (2-14), (3-19)–(3-21) and Lemma 5.1, we obtain

$$R_{apap} = B_1B_2 + A_a + A_p = \frac{2|\vec{v}_{pa}|^2}{(B_3 - B_1)(B_3 - B_2)} = 0, \tag{5-8}$$

$$R_{a\alpha a\alpha} = B_1B_3 + A_a + A_\alpha = \frac{2|\vec{v}_{a\alpha}|^2}{(B_2 - B_1)(B_2 - B_3)} = 0, \tag{5-9}$$

$$R_{cpcp} = B_1B_2 + A_c + A_p = \frac{2|\vec{v}_{pc}|^2}{(B_3 - B_1)(B_3 - B_2)}, \tag{5-10}$$

$$R_{c\alpha c\alpha} = B_1B_3 + A_c + A_\alpha = \frac{2|\vec{v}_{c\alpha}|^2}{(B_2 - B_1)(B_2 - B_3)}, \tag{5-11}$$

$$R_{p\alpha p\alpha} = B_2B_3 + A_p + A_\alpha = \frac{2|\vec{v}_{p\alpha}|^2}{(B_1 - B_2)(B_1 - B_3)}. \tag{5-12}$$

With the summation (5-9) + (5-10) − (5-8) − (5-11), we get

$$\frac{2|\vec{v}_{pc}|^2}{(B_3 - B_1)(B_3 - B_2)} - \frac{2|\vec{v}_{c\alpha}|^2}{(B_2 - B_1)(B_2 - B_3)} = 0.$$

This equation and (5-7) imply that $B_2 + B_3 - 2B_1 = 0$. Combining this with (3-2), we obtain $B_1 = 0$ and $B_2 = -B_3 = \pm\sqrt{(n-1)/(2mn)}$. □

Without loss of generality, in what follows we may assume that

$$B_1 = 0, \quad B_2 = \sqrt{\frac{n-1}{2mn}}, \quad B_3 = -\sqrt{\frac{n-1}{2mn}}. \tag{5-13}$$

Lemma 5.5. *For all $a\in\mathscr{D}_0$, $c\in\mathscr{D}_1$, $p\in\mathscr{I}_2$ and $\alpha\in\mathscr{I}_3$, we have*

$$\omega_{ac}=\omega_{ap}=\omega_{a\alpha}=0, \qquad \omega_{cp}=\frac{1}{B_1-B_2}\sum\nolimits_\alpha B_{cp,\alpha}\omega_\alpha,$$
$$\omega_{c\alpha}=\frac{1}{B_1-B_3}\sum\nolimits_p B_{cp,\alpha}\omega_p, \qquad \omega_{p\alpha}=\frac{1}{B_2-B_3}\sum\nolimits_c B_{cp,\alpha}\omega_c,$$
$$R_{apap}=R_{a\alpha a\alpha}=0, \qquad R_{cpcp}=\frac{2}{(B_3-B_1)(B_3-B_2)}|\vec{v}_{cp}|^2,$$
$$R_{c\alpha c\alpha}=\frac{2}{(B_2-B_1)(B_2-B_3)}|\vec{v}_{c\alpha}|^2, \quad R_{p\alpha p\alpha}=\frac{2}{(B_1-B_2)(B_1-B_3)}|\vec{v}_{p\alpha}|^2.$$

Proof. The formulas follow directly from (2-14), (3-6)–(3-8) and (3-19)–(3-21). First of all, from (5-5) we get $\omega_{ap}=\omega_{a\alpha}=0$. The remaining formulas in Lemma 5.5 except $\omega_{ac}=0$ can be easily obtained.

To show that $\omega_{ac}=0$ holds for any $a\in\mathscr{D}_0$ and $c\in\mathscr{D}_1$, we use the following two equations for any $p\in\mathscr{I}_2$ and $\alpha\in\mathscr{I}_3$:

$$0=-R_{apap}\omega_a\wedge\omega_p=d\omega_{ap}-\sum_i\omega_{ai}\wedge\omega_{ip}=-\sum_{\beta\in\mathscr{I}_3,\,c\in\mathscr{D}_1}\Gamma^c_{\beta p}\omega_{ac}\wedge\omega_\beta, \tag{5-14}$$

$$0=-R_{a\alpha a\alpha}\omega_a\wedge\omega_\alpha=d\omega_{a\alpha}-\sum_i\omega_{ai}\wedge\omega_{i\alpha}=-\sum_{q\in\mathscr{I}_2,\,c\in\mathscr{D}_1}\Gamma^c_{q\alpha}\omega_{ac}\wedge\omega_q. \tag{5-15}$$

Let us write

$$\omega_{ac}=\sum_{b\in\mathscr{D}_0}\Gamma^a_{bc}\omega_b+\sum_{d\in\mathscr{D}_1}\Gamma^a_{dc}\omega_d+\sum_{q\in\mathscr{I}_2}\Gamma^a_{qc}\omega_q+\sum_{\beta\in\mathscr{I}_3}\Gamma^a_{\beta c}\omega_\beta.$$

Then the two equations above give that

$$\sum_{c\in\mathscr{D}_1}\Gamma^a_{bc}\Gamma^c_{\alpha p}=0\quad\text{for all } a,b\in\mathscr{D}_0,\ p\in\mathscr{I}_2,\ \alpha\in\mathscr{I}_3, \tag{5-16}$$

$$\sum_{c\in\mathscr{D}_1}\Gamma^a_{dc}\Gamma^c_{\alpha p}=0\quad\text{for all } a\in\mathscr{D}_0,\ d\in\mathscr{D}_1,\ p\in\mathscr{I}_2,\ \alpha\in\mathscr{I}_3, \tag{5-17}$$

$$\sum_{c\in\mathscr{D}_1}\Gamma^a_{qc}\Gamma^c_{\alpha p}=0\quad\text{for all } a\in\mathscr{D}_0,\ p,q\in\mathscr{I}_2,\ \alpha\in\mathscr{I}_3, \tag{5-18}$$

$$\sum_{c\in\mathscr{D}_1}\Gamma^a_{\beta c}\Gamma^c_{\alpha p}=0\quad\text{for all } a\in\mathscr{D}_0,\ p\in\mathscr{I}_2,\ \alpha,\beta\in\mathscr{I}_3. \tag{5-19}$$

From (5-16), we get for any $b\in\mathscr{D}_0$ a linear system of equations on $\{\Gamma^a_{bc}\}_{1\le c\le m}$:

$$\begin{cases} B_{p(m_1+m+1),1}\Gamma^a_{b1}+B_{p(m_1+m+1),2}\Gamma^a_{b2}+\cdots+B_{p(m_1+m+1),m}\Gamma^a_{bm}=0,\\ B_{p(m_1+m+2),1}\Gamma^a_{b1}+B_{p(m_1+m+2),2}\Gamma^a_{b2}+\cdots+B_{p(m_1+m+2),m}\Gamma^a_{bm}=0,\\ \quad\vdots\\ B_{pn,1}\Gamma^a_{b1}\ \ +B_{pn,2}\Gamma^a_{b2}\ \ +\cdots+B_{pn,m}\Gamma^a_{bm}=0.\end{cases} \tag{5-20}$$

By using (P1), (P2) and Proposition 5.1, we see that the coefficient matrix F of (5-20) satisfies ${}^tFF = \operatorname{diag}(|\vec{v}_{p1}|^2, |\vec{v}_{p2}|^2, \dots, |\vec{v}_{pm}|^2) = |\vec{v}_{p1}|^2 I_m$. Hence we have $|F| \neq 0$, and then (5-20) implies that $\Gamma^a_{b1} = \Gamma^a_{b2} = \cdots = \Gamma^a_{bm} = 0$ for all $b \in \mathfrak{D}_0$, that is,

$$\Gamma^a_{bc} = 0 \quad \text{for all } b \in \mathfrak{D}_0.$$

Analogously, from (5-17), (5-18) and (5-19), respectively, we can show that

$$\Gamma^a_{dc} = \Gamma^a_{qc} = \Gamma^a_{\beta c} = 0 \quad \text{for all } d \in \mathfrak{D}_1,\ q \in \mathfrak{I}_2 \text{ and } \beta \in \mathfrak{I}_3.$$

Hence $\Gamma^a_{ic} = 0$ for all i, and $\omega_{ac} = 0$ follows. □

Lemma 5.6. *For all $p \in \mathfrak{I}_2$, $\alpha \in \mathfrak{I}_3$ and $a \in \mathfrak{D}_0$, $c \in \mathfrak{D}_1$,*

$$A_a = -A_c = -A_p = -A_\alpha = -\frac{n-1}{12mn}.$$

Proof. Lemma 5.5 and (2-16) imply that $R_{acij} = 0$ and thus we have $R_{acac} = 0$. On the other hand, (2-14) gives that $R_{acac} = B_1^2 + A_a + A_c$. It follows that $A_a = -A_c$. From (5-8), (5-9) and (5-13), we further get $A_a = -A_p = -A_\alpha$ and hence

$$A_a = -A_c = -A_p = -A_\alpha. \tag{5-21}$$

These together with (5-10), (5-12) and (5-13) give that

$$A_c = \frac{L^2}{2{B_2}^2} = A_p = \frac{B_2^2}{2} - \frac{L^2}{B_2^2}.$$

It follows that $L^2 = \frac{1}{3}B_2^4$ and $A_c = \frac{1}{6}B_2^2$. Then our conclusions follow immediately from (5-13) and (5-21). □

Remark 5.1. Because all the Blaschke eigenvalues $A_1, A_2, \dots, A_n$ are constant on M^*, the reasoning of the proof of Lemma 4.3 shows that $M = M^*$. Hence we can find everywhere local frame fields $\{E_i\}$, such that (3-1) and (3-4) hold simultaneously in Case II.

Lemma 5.6 shows that the Blaschke tensor has exactly two distinct constant eigenvalues. Then applying Theorem 2.4 we immediately get the following result.

Theorem 5.1. *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with nonparallel Möbius second fundamental form and three distinct Möbius principal curvatures whose multiplicities are not equal. Then there is an $\tilde{n}$ with $2 \le \tilde{n} \le n-1$, and locally x is Möbius equivalent to one of the following two families of hypersurfaces in $\mathbb{S}^{n+1}$:*

($\mathfrak{C}1$) *Minimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^{\tilde{n}} \times \mathbb{H}^{n-\tilde{n}}(-r^{-2}) \to \mathbb{S}^{n+1},$$

with $r > 0$ *and*

$$\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0, \quad y_0 \in \mathbb{R}^+, \quad y_1 \in \mathbb{R}^{\tilde{n}+2}, \quad y_2 \in \mathbb{R}^{n-\tilde{n}},$$

where $y_1 : N^{\tilde{n}} \to \mathbb{S}^{\tilde{n}+1}(r) \hookrightarrow \mathbb{R}^{\tilde{n}+2}$ *is an umbilic-free minimal hypersurface immersed into the* $(\tilde{n}+1)$*-dimensional sphere of radius* r *and constant scalar curvature*

$$\tilde{R}_1 = \frac{n\tilde{n}(\tilde{n}-1) - (n-1)r^2}{nr^2},$$

and $(y_0, y_2) : \mathbb{H}^{n-\tilde{n}}(-r^{-2}) \to \mathbb{L}^{n-\tilde{n}+1}$ *is the standard embedding of hyperbolic space of sectional curvature* $-r^{-2}$ *into the* $(n-\tilde{n}+1)$*-dimensional Lorentz space with* $-y_0^2 + y_2^2 = -r^2$.

($\mathfrak{C}2$) *Nonminimal hypersurfaces defined by*

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \tilde{M}^n = N^{\tilde{n}} \times \mathbb{S}^{n-\tilde{n}}(r) \to \mathbb{S}^{n+1},$$

with $r > 0$ *and*

$$\tilde{x}_1 = y_1/y_0, \quad \tilde{x}_2 = y_2/y_0, \quad y_0 \in \mathbb{R}^+, \quad y_1 \in \mathbb{R}^{\tilde{n}+1}, \quad y_2 \in \mathbb{R}^{n-\tilde{n}+1},$$

where $(y_0, y_1) : N^{\tilde{n}} \to \mathbb{H}^{\tilde{n}+1}(-r^{-2}) \hookrightarrow \mathbb{L}^{\tilde{n}+2}$, *with* $-y_0^2 + y_1^2 = -r^2$, *is an umbilic-free minimal hypersurface immersed into* $(\tilde{n}+1)$*-dimensional hyperbolic space of sectional curvature* $-r^{-2}$ *and constant scalar curvature*

$$\tilde{R}_1 = -\frac{n\tilde{n}(\tilde{n}-1) + (n-1)r^2}{nr^2},$$

and $y_2 : \mathbb{S}^{n-\tilde{n}}(r) \to \mathbb{R}^{n-\tilde{n}+1}$ *is the standard embedding of the* $(n-\tilde{n})$*-sphere of radius* r.

Determining which of the hypersurfaces ($\mathfrak{C}1$) and ($\mathfrak{C}2$) is Möbius isoparametric requires knowing their Möbius invariants — but this was done in [Hu and Li 2007, Section 4]. For simplicity we will not repeat this calculation here. With the omitted calculations and Lemma 5.6, we immediately get the following results.

Proposition 5.3. *A hypersurface* $\tilde{x}$ *in* ($\mathfrak{C}_1$) *is Möbius isoparametric if and only if it satisfies*

(1) $\tilde{n} = 3m$;

(2) $r = \sqrt{6mn/(n-1)}$;

(3) $y_1 : N^{3m} \to \mathbb{S}^{3m+1}(\sqrt{6mn/(n-1)})$ *is a minimal isoparametric hypersurface with constant scalar curvature* $\tilde{R}_1 = 3(m-1)(n-1)/(2n)$; *moreover, it has three distinct principal curvatures with values given by* (1-1), *each of them with the same multiplicity* m.

Remark 5.2. Cartan [1939] proved that minimal isoparametric hypersurfaces in $\mathbb{S}^{3m+1}(\sqrt{6mn/(n-1)})$ with three distinct principal curvatures do exist and are unique with principal curvatures having the same multiplicities $m \in \{1, 2, 4, 8\}$. More precisely, it is the tube of constant radius over a standard Veronese embedding of a projective plane $\mathbb{F}P^2$ into $\mathbb{S}^{3m+1}(\sqrt{6mn/(n-1)})$ with principal curvatures of (1-1) where $m = 1, 2, 4$ or 8, and $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) or $\mathbb{O}$ (Cayley numbers), respectively.

Proposition 5.4. *If a hypersurface $\tilde{x}$ in $(\mathfrak{C}_2)$ is Möbius isoparametric, then it must satisfy the following three conditions:*

(1) $\tilde{n} = n - 3m$*;*

(2) $r = \sqrt{6mn/(n-1)}$*;*

(3) $\tilde{y} = (y_0, y_1) : N^{n-3m} \to \mathbb{H}^{n-3m+1}(-(n-1)/(6mn))$ *is a minimal isoparametric hypersurface with the principal curvatures of* (1-1).

On the other hand, by Cartan's theorem [1938], an isoparametric hypersurface M^n in the hyperbolic space $\mathbb{H}^{n+1}$ can have at most two distinct principal curvatures, which can only be either totally umbilic or else an open subset of a standard product $\mathbb{S}^k \times \mathbb{H}^{n-k}$ in $\mathbb{H}^{n+1}$. Moreover, the latter must be nonminimal. From this fact and Proposition 5.4, we immediately get the following:

Proposition 5.5. *There is no Möbius isoparametric hypersurface in* ($\mathfrak{C}$2) *that has three distinct Möbius principal curvatures.*

Proof of Theorem 3.2. This is an immediate consequence of the Theorem 5.1, Remark 5.1 and Propositions 5.3 and 5.5. □

6. Completion of the proof of the classification theorem

Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct Möbius principal curvatures whose multiplicities satisfy $m_1 \geq m_2 \geq m_3$.

If x has parallel Möbius second fundamental form, then we apply Theorem 2.3 to obtain that it is locally Möbius equivalent to a hypersurface in part (i) of the classification theorem.

If x has nonparallel Möbius second fundamental form, then we have exactly two cases as we stated in section three:

For Case I, we apply Theorem 3.1 and Cartan's theorem to obtain that it is locally Möbius equivalent to a hypersurface in (ii). For Case II, we can apply Theorem 3.2 and Cartan's theorem to conclude that it is locally Möbius equivalent to the hypersurface in (iii). □

Final remarks. For the general theory (see [Wang 1998]) of Möbius submanifolds in $\mathbb{S}^{n+p}$, the Möbius form Φ is an important invariant. Closely related to Möbius isoparametric hypersurfaces is the concept of Blaschke isoparametric hypersurfaces in spheres. It is interesting to mention a conjecture by X. X. Li [Li and Zhang 2009; Li and Peng 2010]: *A Blaschke isoparametric hypersurfaces with more than two distinct Blaschke eigenvalues is Möbius isoparametric*. For definitions and some recent progress on Blaschke isoparametric hypersurfaces, see [Li and Peng 2010; Li and Zhang 2006; 2007; 2009].

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