Pacific Journal of Mathematics

DISCRETE MORSE THEORY AND HOPF BUNDLES

Dmitry N. Kozlov

DISCRETE MORSE THEORY AND HOPF BUNDLES

DMITRY N. KOZLOV

We use Hopf bundles to give an example of a regular CW complex X and an acyclic matching M on the face poset of X, such that there are no critical cells in neighboring dimensions but the complex X is not homotopy equivalent to the corresponding wedge of spheres. The key fact here is that the higher homotopy groups of spheres are nontrivial. We also give a sufficient condition on an acyclic matching M for concluding that X is homotopy equivalent to a wedge of spheres indexed by the critical cells.

1. Introduction

Discrete Morse theory, introduced by Robin Forman [1998], has become quite a useful tool for doing specific computations in combinatorial algebraic topology; see [Kozlov 2008] for the general framework, and [Clark and Ehrenborg ≥ 2011] for an interesting recent application in case of the Frobenius complex.

Let us briefly describe how the computational model provided by discrete Morse theory works. Given a regular CW complex X, let $\mathcal{F}(X)$ denote the poset of all nonempty cells of X. This poset is ranked by the dimensions of the cells. A partial matching on the Hasse diagram of $\mathcal{F}(X)$ is a bijection $M : U \to D$, where U and D are (possibly empty) disjoint sets of elements of $\mathcal{F}(X)$ such that $\dim(\sigma) = \dim(M(\sigma)) + 1$, and $M(\sigma)$ lies on the boundary of σ for all $\sigma \in U$. A partial matching M is called *acyclic* if there do not exist $\sigma_1, \ldots, \sigma_t \in \mathcal{F}(X)$ such that $\sigma_1 \neq \cdots \neq \sigma_t$, and $\sigma_{i+1} > M(\sigma_i)$ for all $i = 1, \ldots, t$, where as usual we set $\sigma_{t+1} := \sigma_1$. We set $C_M := \mathcal{F}(X) \setminus (U \cup D)$ and call the elements of C *critical*. For all i, let $f_i(C_M)$ denote the number of critical cells of dimension i. The main theorem in [Forman 1998] states that whenever M is an acyclic matching, there exists a CW complex $\widetilde{X}$, called the *critical Morse complex*, with $f_i(C_M)$ cells of dimension i, for all i, such that $\widetilde{X}$ is homotopy equivalent to X.

Frequently, the actual goal of applying discrete Morse theory is to prove that X is in fact homotopy equivalent to a wedge of spheres, or at least to compute the

homology groups of X. In a very fortunate situation, one might be able to produce an acyclic matching M such that for all i we have $f_i(C_M) f_{i+1}(C_M) = 0$, that is, there are no critical cells in neighboring dimensions. This would settle the question of computing the homology groups of the space X. However, in order to determine the homotopy type of X, one would want to conclude that the critical cells are somehow independent of each other, and so we have the homotopy equivalence

$$(1\text{-}1) \qquad\qquad X \simeq \bigvee_i \underbrace{S^i \vee \cdots \vee S^i}_{\tilde{f}_i(C_M)},$$

where $\tilde{f}_i(C_M) = f_i(C_M)$ for $i \geq 1$ and $\tilde{f}_0(C_M) = f_0(C_M) - 1$, and we use the convention that the empty wedge is a point.

We will use the fact that higher homotopy groups of spheres are nontrivial to give an example showing that just assuming that there are no critical cells in neighboring dimensions is not enough to conclude that the space is homotopy equivalent to a wedge of spheres.

But first, on the positive side, we give a sufficient condition on acyclic matching that lets us conclude that the space is homotopy equivalent to a wedge of spheres indexed by the critical cells. In fact, for this result we will not need the condition that there are no critical cells in neighboring dimensions; see also [Clark and Ehrenborg ≥ 2011, Theorem 2.5].

2. Acyclic matchings yielding a wedge of spheres

Here we are interested in acyclic matchings that allow us to conclude that the considered complex is in fact homotopy equivalent to a wedge of spheres that are enumerated by the critical cells. First we need some terminology.

Definition 2.1. Let P be a partially ordered set and M a partial matching on P.

(1) A *generalized alternating path* is a sequence $x_1 > x_2 < x_3 > \cdots < x_{2t+1}$ or a sequence $x_1 > x_2 < x_3 > \cdots > x_{2t+2}$, where $t \geq 0$, such that $M(x_{2k+1}) = x_{2k}$ for all $k = 1, \ldots, t$.

(2) Let x be an element of P. We set $F(x)$ to be the set of the endpoints of all generalized alternating paths starting at x, and call $F(x)$ the *feasibility domain* of x.

Note that in a generalized alternating path, we require that x_{2k+1} covers x_{2k} for all $k = 1, \ldots, t$, but we obviously do not require that x_{2k-1} covers x_{2k} for all such k.

It is easy to see that $F(x)$ shall always contain a critical cell of dimension 0. Let A denote the set of 0-dimensional cells in $F(x)$. If none of them is critical, then there exists the set of 1-dimensional cells $B \subset F(x)$ such that $M : B \to A$ is a bijection. Since every $y \in B$ covers two elements, the graph with the vertex set

$A \cup B$ and the covering relations as edges cannot be a forest, so it contains cycles, which contradicts the assumption that the matching M is acyclic.

The following theorem gives a sufficient condition on an acyclic matching for the critical Morse complex to be homotopy equivalent to a wedge of spheres enumerated by critical cells.

Theorem 2.2. *Let X be a connected regular CW complex, and let M be an acyclic partial matching on $\mathcal{F}(X)$. Assume that for every critical cell c of dimension larger than 0, its feasibility domain $F(c)$ contains precisely two critical cells: c itself and one critical cell of dimension 0. Then X is homotopy equivalent to a wedge of spheres enumerated by critical cells, that is, (1-1) is true.*

Proof. For this argument, we adopt the point of view of [Kozlov 2008] and follow the proof of its Theorem 11.13(b). There the main theorem of discrete Morse theory for CW complexes is proved by a stepwise attachment of either a critical cell or of a pair of cells matched by M, with a parallel explicit construction of a Morse homotopy map. This stepwise attachment is done along a certain linear extension of the face poset of X, which we denote by l. When a pair of matched cells is attached, we simply have a strong deformation retraction of the obtained complex to what we have had before that attachment, so we just need to understand the case of attaching a critical cell.

Assume that a critical cell c of dimension at least 1 is being attached. The cells in $F(c) \setminus \{c\}$ form a subcomplex C of X. The assumption of the theorem implies that C is collapsible along the matching M. It means that prior to the attachment of c, the Morse homotopy has already shrunk the complex C to a point a, where a is the critical 0-dimensional cell of $F(c) \setminus \{c\}$. Since the image of the attaching map of the cell c lies inside C, we conclude that in the critical Morse complex, the attaching map of c will simply map everything to the point a. Thus we can conclude that all the attaching maps in the critical Morse complex are trivial.

Finally, we need to see that the matching M has exactly one 0-dimensional critical cell, which will imply that all the critical cells will be attached to the same vertex. Assume we have another critical 0-dimensional cell b, and assume that b occurs after a in the linear extension l. Then, when b is added, it will form a new connected component. So, since the total complex X is connected, at some point in the inductive process of adding critical cells and pairs of matched cells we will have to connect that connected component to the connected component containing a. This can only be achieved by adding a critical 1-dimensional cell, which we call e. The set $F(e)$ cannot contain any critical 0-dimensional cells other than b. Let v_1 denote the vertex of e that does not lie in the same connected component as b. The vertex v_1 is not critical, and we set $e_1 := M^{-1}(v_1)$. Both v_1 and e_1 were added before e. We now proceed, starting with $k = 1$ be letting
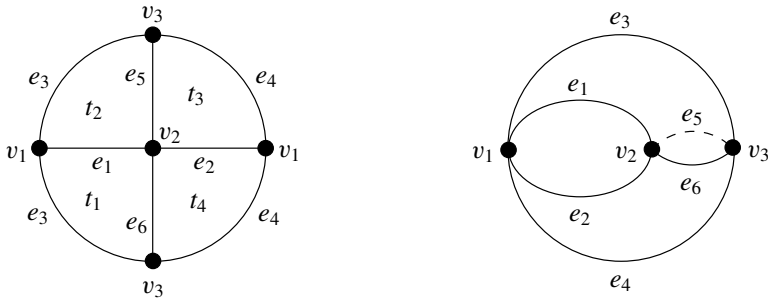
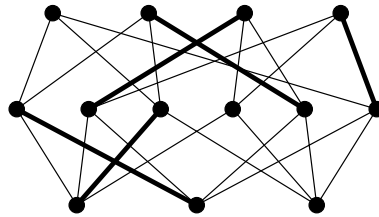**Figure 2.1.** Two presentations of the simplicial complex L.

Figure 2.2. The face poset of L. The vertices, edges, and triangles
are shown with the index increasing from left to right. The bold
edges indicate an acyclic matching.

v_{k+1} be the vertex of e_k other than v_k. Since $v_{k+1} \in F(e)$, we see that v_{k+1} is
not critical, and set $e_{k+1} := M^{-1}(v_{k+1})$. Both v_{k+1} and e_{k+1} were added before
e. Eventually we will have to conclude that for some $k \geq 1$ the vertex v_{k+1} lies in
the same connected component as b. But this means that b was connected to the
vertex v_1 even before adding e, yielding a contradiction to the choice of e. □

The condition of Theorem 2.2 is not necessary for getting a wedge of spheres
enumerated by critical cells. For example, let L be the simplicial complex shown
on Figure 2.1. A direct examination yields that the matching shown on Figure 2.2 is
acyclic with one critical cell in each of the dimensions 0, 1, and 2. The condition
of Theorem 2.2 is not satisfied, but the space L is homotopy equivalent to $S^1 \vee S^2$.

3. Hopf fiber bundles

We now give an example that simply having an acyclic matching with no critical
cells in neighboring dimensions is not sufficient to conclude that the space X is
homotopy equivalent to a wedge of spheres. Our example exploits the fact that the
higher homotopy groups (unlike the homology groups) of spheres are nontrivial.
The first such nontrivial group is $\pi_3(S^2) = \mathbb{Z}$, and it is this one which we use for
our construction.

Consider the set $A := \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}$, and let the multiplicative group $G = \{z \mid |z| = 1\} \subseteq \mathbb{C}$ act on A diagonally by multiplication: $z : (z_1, z_2) \mapsto (zz_1, zz_2)$. The quotient A/G can be viewed as a complex projective line $\mathbb{CP}^1$, with the quotient map $q : A \to A/G$ simply being $q : (z_1, z_2) \mapsto (z_1 : z_2)$. Note that topologically $A \cong S^3$, $G \cong S^1$, and $A/G \cong S^2$. The obtained fiber bundle $S^1 \to S^3 \to S^2$ is the first example of a Hopf bundle; see [Hatcher 2002, Example 4.45].

Consider the CW structure on A obtained by intersecting with the real coordinate hyperplanes $\operatorname{Re} z_1 = 0$, $\operatorname{Im} z_1 = 0$, $\operatorname{Re} z_2 = 0$, and $\operatorname{Im} z_2 = 0$. Then A is a regular CW complex with face vector $(8, 24, 32, 16)$. Furthermore, consider the CW structure on A/G consisting of the two vertices $v_1 = (1 : 0)$ and $v_2 = (0 : 1)$, four edges $e_1 = \{(1 : r) \mid r > 0\}$, $e_2 = \{(1 : ir) \mid r > 0\}$, $e_3 = \{(1 : -r) \mid r > 0\}$, and $e_4 = \{(1 : -ir) \mid r > 0\}$, and four 2-cells denoted s_1, s_2, s_3, s_4, where s_i is bound by e_i and e_{i+1} for $i = 1, 2, 3$, and s_4 is bound by e_1 and e_4. Again A/G is a regular CW complex with the face vector $(2, 4, 4)$, and one sees that q is a cellular map.

Set $C := ((A \times [0, 1]) \coprod (A/G))/\sim$ to be the mapping cylinder of q, that is $\sim$ is given by $(a, 1) \sim q(a)$, for all $a \in A$. We choose a CW structure on C by taking all the cells of $A/G \subseteq C$, subdividing $A \times \{0\}$, the top copy of A, as described above, and taking the open cells $\tilde{\sigma} := \operatorname{int} \sigma \times (0, 1)$ for all cells σ of A. Here we write $\operatorname{int} \sigma = \sigma \setminus \partial \sigma$ to denote the relative interiors of cells. Finally, let X be the regular CW complex obtained from C by attaching a 4-cell k along $A \times \{0\} \cong S^3$.

Consider the following acyclic matching: $M(\tilde{\sigma}) = \sigma$ whenever σ is a cell of $A \times \{0\}$, $M(s_i) = e_i$ for $i = 1, 2, 3$, and $M(e_4) = v_2$. The partial matching M has three critical cells: v_1, k, and s_4, in dimensions 0, 2, and 4. It is easily verified directly that all the matched pairs are regular in the sense of [Cohen 1973] and [Kozlov 2008, Definition 11.12]; in particular the main theorem of discrete Morse theory (see [Forman 1998]) can be applied and we can conclude that X is homotopy equivalent to a CW complex with one cell in each of the dimensions 0, 2, and 4.

However, the space X is not homotopy equivalent to $S^2 \vee S^4$. For example, these two spaces have different π_3 groups.[1] Namely $\pi_3(X) = 0$, while $\pi_3(S^2 \vee S^4) = \mathbb{Z}$. Both of these statements can be seen using the long exact sequence for relative homotopy; see [Hatcher 2002, page 344]. Indeed, when a space $\widetilde{Y}$ is obtained from a space $Y \cong S^2$ by attaching a 4-cell along some continuous map $\varphi : S^3 \to S^2$, the relevant part of the long exact sequence for homotopy of the pair $(\widetilde{Y}, Y)$ is

$$
(3\text{-}1) \qquad \cdots \to \pi_4(\widetilde{Y}, Y, y) \xrightarrow{\partial} \pi_3(Y, y) \xrightarrow{i_*} \pi_3(\widetilde{Y}, y) \to \pi_3(\widetilde{Y}, Y, y) \to \cdots,
$$

---

[1] Serge Ochanine pointed out to the author in 2009 that the constructed space X is actually the complex projective plane. We use the long exact sequence of the homotopy of a pair to show how we came up with the example.

where $y \in Y$ is a base point, the map ∂ comes from restricting maps $(D^n, S^{n-1}, s) \to$ $(\widetilde{Y}, Y, y)$ to S^{n-1}, and i_* is the map between homotopy groups induced by the inclusion map $i : Y \hookrightarrow \widetilde{Y}$. Since $(Y, y) \cong (S^2, x_2)$, and $(\widetilde{Y}, Y, y) \simeq (D^4, S^3, x_3)$, where $x_2 \in S^2$ and $x_3 \in S^3$ are corresponding base points, the sequence (3-1) translates to

$$(3\text{-}2) \qquad \cdots \to \pi_3(S^3, x_3) \xrightarrow{\varphi_*} \pi_3(S^2, x_2) \to \pi_3(\widetilde{Y}, y) \to 0 \to \cdots .$$

For the space $S^2 \vee S^4$ the map φ_* is trivial, and hence $\pi_3(\widetilde{Y}, y) = \pi_3(S^2, x_2) = \mathbb{Z}$; for the case of the Hopf bundle above, the map φ_* is surjective, and so we get $\pi_3(\widetilde{Y}, y) = 0$. Clearly, this technique can be used to produce further examples that might be needed to test various hypothesis.

Taking the barycentric subdivision of X will yield a simplicial complex with the same property: It has an acyclic matching with one critical cell of dimensions 0, 2, and 4 each, but is of course homeomorphic to the regular CW version; in particular, it is not homotopy equivalent to $S^2 \vee S^4$. We leave finding such an acyclic matching to the reader.

Acknowledgments

References

[Clark and Ehrenborg ≥ 2011] E. Clark and R. Ehrenborg, "The Frobenius complex", preprint. To appear in *Ann. Comb.*

[Cohen 1973] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics **10**, Springer, New York, 1973. MR 50 #14762 Zbl 0261.57009

[Forman 1998] R. Forman, "Morse theory for cell complexes", *Adv. Math.* **134**:1 (1998), 90–145. MR 99b:57050 Zbl 0896.57023

[Hatcher 2002] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 2002k:55001 Zbl 1044.55001

[Kozlov 2008] D. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics **21**, Springer, Berlin, 2008. MR 2008j:55001 Zbl 1130.55001

DMITRY N. KOZLOV
DEPARTMENT OF MATHEMATICS
BREMEN UNIVERSITY
BIBLIOTHEKSSTRASSE 1
D-28334 BREMEN
GERMANY
dfk@math.uni-bremen.de
http://www.math.uni-bremen.de/~dfk/

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

PACIFIC JOURNAL OF MATHEMATICS