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**REGULARITY OF CANONICAL AND DEFICIENCY MODULES  
FOR MONOMIAL IDEALS**

MANOJ KUMMINI AND SATOSHI MURAI

# REGULARITY OF CANONICAL AND DEFICIENCY MODULES FOR MONOMIAL IDEALS

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**We show that the Castelnuovo–Mumford regularity of the canonical or a deficiency module of the quotient of a polynomial ring by a monomial ideal is bounded by its dimension.**

## 1. Introduction

Let  $R = \mathbb{k}[x_1, \dots, x_n]$  be a standard graded polynomial ring over a field  $\mathbb{k}$ , and let  $\mathfrak{m} = (x_1, \dots, x_n)$  be the homogeneous maximal ideal of  $R$ . We study the Castelnuovo–Mumford regularity of the modules  $\text{Ext}_R^i(R/I, \omega_R)$  when  $I \subset R$  is a monomial ideal; here  $\omega_R = R(-n)$  denotes the canonical module of  $R$ . The modules

$$\text{Ext}_R^i(R/I, \omega_R) \quad \text{for } i > n - \dim R/I$$

are called the *deficiency modules* of  $R/I$ , while

$$\text{Ext}_R^{n-\dim R/I}(R/I, \omega_R)$$

is called the *canonical module* of  $R/I$ .

For any homogeneous ideal  $I \subseteq R$ , the local cohomology modules  $H_{\mathfrak{m}}^i(R/I)$  are important in commutative algebra and algebraic geometry. One is often interested in the vanishing of homogeneous components of  $H_{\mathfrak{m}}^i(R/I)$ . While one cannot expect the vanishing of  $H_{\mathfrak{m}}^i(R/I)$  in negative degrees (unless it has finite length), one can, using the local duality theorem of Grothendieck, obtain some information from  $\text{Ext}_R^{n-i}(R/I, \omega_R)$ . For a finitely generated graded  $R$ -module  $M$ , its (*Castelnuovo–Mumford*) *regularity*  $\text{reg}(M)$  is an invariant that contains information about the stability of homogeneous components in sufficiently large degrees. In light of these, it is desirable to get bounds on  $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$ . Such bounds were studied by L. T. Hoa and E. Hyry [2006] and by M. Chardin, D. T. Ha and Hoa [2009]; see also the references in those papers.

Unfortunately, canonical and deficiency modules can have large regularity. For a finitely generated graded  $R$ -module  $M$ , known bounds for  $\text{reg}(\text{Ext}_R^i(M, \omega_R))$

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are large; see, for example, [Hoa and Hyry 2006, Theorems 9 and 14]. On the other hand, more optimal bounds for  $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$  are known to exist for certain classes of graded ideals  $I$ ; see [Hoa and Hyry 2006, Section 4]. It is an interesting problem to find a class of graded ideals  $I \subset R$  with optimal bounds for  $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$ . In this paper, we focus on monomial ideals. It follows from the theory of square-free modules, introduced by K. Yanagawa [2000], that if  $I$  is a square-free monomial ideal, then  $\text{reg}(\text{Ext}_R^i(R/I, \omega_R)) \leq \dim \text{Ext}_R^i(R/I, \omega_R)$ . This bound is small, since  $\dim \text{Ext}_R^i(R/I, \omega_R) \leq n - i$ ; see [Bruns and Herzog 1993, Corollary 3.5.11].

While one cannot apply the theory of square-free modules to all monomial ideals, there are results that show that  $\text{reg}(\text{Ext}_R^i(R/I, \omega_R))$  is not large when  $I$  is a monomial ideal. For example, we see from [Takayama 2005, Proposition 1, page 333] that if  $\text{Ext}_R^i(R/I, \omega_R)$  has finite length, then its regularity is negative or equal to zero. Again, Hoa and Hyry [2006, Proposition 21] showed that if  $H_m^i(R/I)$  has finite length for  $i = 0, 1, \dots, d-1$ , where  $d = \dim R/I$ , then  $\text{reg}(\text{Ext}_R^{n-d}(R/I, \omega_R)) \leq d$ . We generalize these results in the next theorem:

**Theorem 1.1.** *Let  $I \subseteq R$  be a monomial ideal. Then*

$$\text{reg}(\text{Ext}_R^i(R/I, \omega_R)) \leq \dim \text{Ext}_R^i(R/I, \omega_R) \quad \text{for all } 0 \leq i \leq n.$$

Since  $\dim \text{Ext}_R^i(R/I, \omega_R) \leq n - i$ , we immediately get this:

**Corollary 1.2.** *Let  $I \subseteq R$  be a monomial ideal. Then*

$$\text{reg}(\text{Ext}_R^i(R/I, \omega_R)) \leq n - i \quad \text{for all } 0 \leq i \leq n.$$

In general, this conclusion need not hold without the assumption that  $I$  is a monomial ideal; see [Chardin and D’Cruz 2003, Example 3.5].

Our approach to bounding the regularity of canonical and deficiency modules differs from that of Hoa and Hyry. We show that if  $I$  is a monomial ideal, then  $\text{Ext}_R^i(R/I, \omega_R)$  has a multigraded filtration, called the *Stanley filtration* and introduced by D. Maclagan and G. G. Smith [2005]; the bound on regularity follows from this filtration.

In the next section, we discuss some preliminaries on Stanley filtrations and local cohomology. In Section 3, we prove our main result.

## 2. Preliminaries

Hereafter we take  $R$ -modules to be graded by  $\mathbb{Z}^n$ , giving  $\deg x_i = e_i$ , the  $i$ -th unit vector of  $\mathbb{Z}^n$ . We call this the *multigrading* of  $R$  and  $R$ -modules.

**Notation 2.1.** Let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ . Write

$$\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^n x_i^{a_i} \in \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

We say that  $\mathbf{a}$  is the *degree* of  $\mathbf{x}^{\mathbf{a}}$  and write  $\deg \mathbf{x}^{\mathbf{a}} = \mathbf{a}$ . Define  $\text{Supp}(\mathbf{a}) = \{i : a_i \neq 0\}$ , and define  $\mathbf{a}^+, \mathbf{a}^- \in \mathbb{N}^n$  by the conditions

$$\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^- \quad \text{and} \quad \text{Supp}(\mathbf{a}^+) \cap \text{Supp}(\mathbf{a}^-) = \emptyset.$$

Write  $\|\mathbf{a}\|$  for  $\sum_{i=1}^n a_i$ , the *total degree* of  $\mathbf{a}$  (and of the monomial  $\mathbf{x}^{\mathbf{a}}$ ). We will say that  $\mathbf{a}$  (or equivalently  $\mathbf{x}^{\mathbf{a}}$ ) is *square-free* if  $a_i \in \{0, 1\}$  for all  $i$ . Let  $[n] = \{1, \dots, n\}$ . For  $\Lambda \subseteq [n]$ , we set  $\mathbf{e}_{\Lambda} = \sum_{i \in \Lambda} \mathbf{e}_i$  and abbreviate the (square-free) monomial  $\mathbf{x}^{\mathbf{e}_{\Lambda}}$  as  $x_{\Lambda}$ . The canonical module of  $R$  is  $\omega_R = R(-\mathbf{e}_{[n]})$ .

Let  $M$  be a finitely generated multigraded  $R$ -module. Let  $m \in M$  be a homogeneous element, and let  $G \subset \{x_1, \dots, x_n\}$  be a subset such that  $um \neq 0$  for all monomials  $u \in \mathbb{k}[G]$ . The  $\mathbb{k}$ -subspace  $\mathbb{k}[G]m$  of  $M$  generated by all the  $um$ , where  $u$  is a monomial in  $\mathbb{k}[G]$ , is called a *Stanley space*. A *Stanley decomposition* of  $M$  is a finite set  $\mathcal{S}$  of pairs  $(m, G)$  of homogeneous elements  $m \in M$  and  $G \subseteq \{x_1, \dots, x_n\}$  such that  $\mathbb{k}[G]m$  is a Stanley space for all  $(m, G) \in \mathcal{S}$  and

$$(1) \quad M =_{\mathbb{k}} \bigoplus_{(m, G) \in \mathcal{S}} \mathbb{k}[G]m.$$

(We used “ $=_{\mathbb{k}}$ ” to emphasize that the decomposition is only as vector spaces.) Properties of such decompositions have been widely studied; we follow the approach of [Maclagan and Smith 2005, Section 3], where Stanley decompositions were used to get bounds for multigraded regularity. Following [Maclagan and Smith 2005, Definition 3.7], we define a *Stanley filtration* to be a Stanley decomposition with an ordering of pairs  $\{(m_i, G_i) : 1 \leq i \leq p\}$  such that

$$\left( \sum_{i=1}^j Rm_i \right) / \left( \sum_{i=1}^{j-1} Rm_i \right) = \mathbb{k}[G_j](-\deg m_j) \quad \text{for } j = 1, 2, \dots, p$$

as  $R$ -modules. Note, in this case, that

$$0 \subseteq Rm_1 \subseteq \dots \subseteq \sum_{i=1}^j Rm_i \subseteq \dots \subseteq \sum_{i=1}^p Rm_i = M$$

is a prime filtration of  $M$ , as in [Eisenbud 1995, Proposition 3.7, page 93].

**Proposition 2.2.** *Let  $M$  be a multigraded  $R$ -module with a Stanley decomposition  $\mathcal{S}$  such that  $(\deg m)^+$  is square-free and  $G = \text{Supp}((\deg m)^+)$  for all  $(m, G) \in \mathcal{S}$ . Then,  $\mathcal{S}$  gives a Stanley filtration. Moreover,  $\text{reg } M \leq \max\{\|\deg m\| : (m, G) \in \mathcal{S}\}$ .*

*Proof.* We order  $\mathcal{G} = \{(m_1, G_1), \dots, (m_p, G_p)\}$  so that  $\|\deg m_1\| \geq \dots \geq \|\deg m_p\|$ . It follows from our hypothesis that

$$(2) \quad \text{span}_{\mathbb{k}}\{m_1, \dots, m_p\} = \text{span}_{\mathbb{k}}\{m \in M : \text{Supp}((\deg m)^+) \text{ is square-free}\},$$

where  $\text{span}_{\mathbb{k}}(V)$  denotes the  $\mathbb{k}$ -vector space spanned by the elements in  $V$ . We write  $M^{(j)}$  for  $\sum_{i=1}^j Rm_i$ . We will now show, inductively on  $j$ , that

- (a)  $M^{(j-1)} :_R m_j = (x_k; x_k \notin G_j)$ , and
- (b) the set  $\bigcup_{i=1}^j \{um_i : u \text{ is a monomial in } \mathbb{k}[G_i]\}$  is a  $\mathbb{k}$ -basis for  $M^{(j)}$ .

These imply that  $\mathcal{G}$  is a Stanley filtration of  $M$ .

Let  $j = 1$ . We will show that  $(0 :_R m_1) = (x_k; x_k \notin G_1)$ . We have  $um_1 \neq 0$  for all monomials  $u \in \mathbb{k}[G_1]$  from the definition of the decomposition. Therefore we must show that  $x_l m_1 = 0$  for any  $x_l \notin G_1$ . Let  $x_l \notin G_1$ . Then  $(\deg x_l m_1)^+$  is square-free, and  $x_l m_1 \in \text{span}_{\mathbb{k}}\{m_1, \dots, m_p\}$  by (2). However, from the choice of  $m_1$ , we see that  $x_l m_1 = 0$ . Therefore  $(0 :_R m_1) = (x_k; k \notin G_1)$ , proving (a). Then (b) follows immediately.

Now, assume that  $j > 1$  and that the assertion is known for all  $i < j$ . We first show (a). Let  $u$  be a monomial in  $\mathbb{k}[G_j]$ . By statement (b) for  $j - 1$ , the set  $\bigcup_{i=1}^{j-1} \{vm_i : v \text{ is a monomial in } \mathbb{k}[G_i]\}$  is a  $\mathbb{k}$ -basis for  $M^{(j-1)}$ . Since  $um_j$  is an element of the basis of  $M$  coming from the Stanley decomposition,  $um_j$  is not in the  $\mathbb{k}$ -linear span of  $\bigcup_{i=1}^{j-1} \{vm_i : v \text{ is a monomial in } \mathbb{k}[G_i]\}$ , that is,  $um_j \notin M^{(j-1)}$ . It remains to prove that  $x_l m_j \in M^{(j-1)}$  for any  $x_l \notin G_j$ . Let  $x_l \notin G_j$ . Since  $(\deg x_l m_j)^+$  is square-free, it follows from (2) and the ordering of the  $(m_i, G_i)$  that

$$x_l m_j \in \text{span}_{\mathbb{k}}\{m_i : 1 \leq i \leq p, \deg m_i > \deg m_j\} \subseteq \text{span}_{\mathbb{k}}\{m_1, \dots, m_{j-1}\}.$$

Therefore  $x_l m_j \in M^{(j-1)}$ , proving the statement (a) for  $j$ .

From (a), we see that the sequence

$$(3) \quad 0 \rightarrow M^{(j-1)} \rightarrow M^{(j)} \rightarrow \mathbb{k}[G_j]m_j \rightarrow 0$$

is exact. Now, statement (b) for  $j$  follows from the induction hypothesis.

Theorem 4.1 of [Maclagan and Smith 2005] essentially gives the assertion about regularity, but we give a quick proof here by showing that

$$\text{reg } M^{(j)} \leq \max\{\|\deg m_i\| : 1 \leq i \leq j\} \quad \text{for all } 1 \leq j \leq p.$$

It holds for  $j = 1$ . For  $j > 1$ , it follows from [Eisenbud 1995, Corollary 20.19] and the exact sequence (3) that

$$\text{reg } M^{(j)} \leq \max\{\text{reg } M^{(j-1)}, \|\deg m_j\|\}.$$

Then induction completes the proof.  $\square$

Finally, we recall some basics of local cohomology. We follow [Bruns and Herzog 1993, Sections 3.5 and 3.6]. Let  $\check{C}^\bullet$  be the Čech complex on  $x_1, \dots, x_n$ ; the term at the  $i$ -th cohomological degree is

$$\check{C}^i = \bigoplus_{\Lambda \subseteq [n], |\Lambda|=i} R_{x_\Lambda},$$

where  $R_{x_\Lambda}$  denotes inverting the monomial  $x_\Lambda$ . Note that  $\check{C}^\bullet$  is a complex of  $\mathbb{Z}^n$ -graded  $R$ -modules, with differentials of degree 0. For a finitely generated  $R$ -module  $M$ , we set  $\check{C}^\bullet(M) = \check{C}^\bullet \otimes_R (M)$ . Then  $H_m^i(M) = H^i(\check{C}^\bullet(M))$ .

**Definition 2.3.** Let  $F \subseteq [n]$ . We define  $\check{C}_F^\bullet$  to be the subcomplex of  $\check{C}^\bullet$  obtained by setting

$$\check{C}_F^i = \begin{cases} 0 & \text{if } i < |F|, \\ \bigoplus_{\substack{F \subseteq \Lambda \subseteq [n] \\ |\Lambda|=i}} R_{x_\Lambda} & \text{otherwise.} \end{cases}$$

**Lemma 2.4.** Let  $I$  be a monomial ideal and  $F \subseteq [n]$ . If  $\mathbf{a} \in \mathbb{Z}^n$  is such that  $\text{Supp}(\mathbf{a}^-) = F$ , then  $H_m^i(R/I)_{\mathbf{a}} = H^i(\check{C}_F^\bullet \otimes_R (R/I))_{\mathbf{a}}$ .

*Proof.* The proof of [Takayama 2005, Theorem 1] uses this argument implicitly. Since  $H_m^i(R/I)_{\mathbf{a}} = H^i((\check{C}^\bullet(R/I))_{\mathbf{a}})$ , it suffices to show that

$$(\check{C}^\bullet(R/I))_{\mathbf{a}} = (\check{C}_F^\bullet \otimes_R (R/I))_{\mathbf{a}}.$$

This, in turn, stems from the fact that  $\check{C}_F^j \otimes_R (R/I)$  consists precisely of the direct summands of  $\check{C}^j(R/I)$  that are nonzero in multidegree  $\mathbf{a}$  for all  $1 \leq j \leq n$ .  $\square$

### 3. Proof of the main theorem

**Lemma 3.1.** Let  $I \subset R$  be a monomial ideal. Let  $\mathbf{a} \in \mathbb{Z}^n$  and  $j \in \text{Supp}(\mathbf{a}^+)$ . The multiplication map

$$x_j : \text{Ext}_R^i(R/I, \omega_R)_{\mathbf{a}} \rightarrow \text{Ext}_R^i(R/I, \omega_R)_{\mathbf{a}+e_j}$$

is bijective.

*Proof.* We first claim that the multiplication map

$$x_j : H_m^{n-i}(R/I)_{-\mathbf{a}-e_j} \rightarrow H_m^{n-i}(R/I)_{-\mathbf{a}}$$

is bijective. By local duality [Bruns and Herzog 1993, Theorem 3.6.19], this map is the Matlis dual of the multiplication by  $x_j$  on  $\text{Ext}_R^i(R/I, \omega_R)_{\mathbf{a}}$ ; hence, it suffices to prove the claim above.

Set  $F = \text{Supp}(\mathbf{a}^+)$ . Note that  $\text{Supp}(\mathbf{a}^+ + e_j) = F$ . For all  $i$ , the map  $x_j$  acts as a unit on  $\check{C}_F^i$ . Therefore the homomorphism of complexes

$$\check{C}_F^\bullet \otimes_R (R/I) \rightarrow \check{C}_F^\bullet \otimes_R (R/I)$$

induced by the multiplication map  $x_j : \check{C}_F^i \otimes_R (R/I) \rightarrow \check{C}_F^i \otimes_R (R/I)$  is an isomorphism. The claim now follows from [Lemma 2.4](#), which implies that

$$\begin{aligned} H_m^i(R/I)_{-a-e_j} &= H^i(\check{C}_F^\bullet \otimes_R (R/I))_{-a-e_j}, \\ H_m^i(R/I)_{-a} &= H^i(\check{C}_F^\bullet \otimes_R (R/I))_{-a}. \end{aligned} \quad \square$$

The previous lemma says that, if  $I$  is a monomial ideal, then  $\text{Ext}_R^i(R/I, \omega_R)$  is a  $(1, 1, \dots, 1)$ -determined module in the sense of [\[Miller 2000, Definition 2.1\]](#).

*Proof of Theorem 1.1.* For  $F \subseteq [n]$ , let  $\mathcal{M}_F^i$  be a multigraded  $\mathbb{k}$ -basis for

$$\bigoplus_{a \in \mathbb{N}^n, \text{Supp}(a) \cap F = \emptyset} \text{Ext}_R^i(R/I, \omega_R)_{e_F - a}.$$

Let  $\mathcal{S}_i = \{(m, F) : F \subseteq [n] \text{ and } m \in \mathcal{M}_F^i\}$ . It follows from [Lemma 3.1](#) that  $\mathcal{S}_i$  is a Stanley decomposition of  $\text{Ext}_R^i(R/I, \omega_R)$ . In particular,

$$\dim \text{Ext}_R^i(R/I, \omega_R) = \max\{|F| : \mathcal{M}_F^i \neq \emptyset\}.$$

By the construction of  $\mathcal{M}_F^i$ , this Stanley decomposition satisfies the assumption of [Proposition 2.2](#). Therefore

$$\begin{aligned} \text{reg}(\text{Ext}_R^i(R/I, \omega_R)) &\leq \max_{F \subseteq [n]} \{\max\{\|\deg m\| : m \in \mathcal{M}_F^i\}\} \\ &\leq \max_{F \subseteq [n]} \{|F| : \mathcal{M}_F^i \neq \emptyset\} \\ &= \dim \text{Ext}_R^i(R/I, \omega_R), \end{aligned}$$

as desired. (The second inequality follows since  $\|\deg u\| = |F| - \|(\deg u)^-\|$  for any  $u \in \mathcal{M}_F^i$ .) □

We remark that, by using [\[Takayama 2005, Theorem 1\]](#) and local duality, one can determine whether  $\mathcal{M}_F^i \neq \emptyset$  from certain subcomplexes of the Stanley–Reisner complex of the radical  $\sqrt{I}$  of  $I$ .

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