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$SL_2(\mathbb{C})$-CHARACTER VARIETY OF A HYPERBOLIC LINK AND REGULATOR

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We analyze a special smooth projective variety Y^h arising from some one-dimensional irreducible slices on the $SL_2(\mathbb{C})$-character variety of a hyperbolic link in S^3. We prove that a natural symbol obtained from these one-dimensional slices is a torsion in $K_2(\mathbb{C}(Y^h))$. By using the regulator map from K_2 to the corresponding Deligne cohomology, we get some variation formulas on some Zariski open subset of Y^h. From this we discuss a possible parametrized volume conjecture for both hyperbolic links and knots.

1. Introduction

This is the sequel to [Li and Wang 2008] on the generalized volume conjecture for a hyperbolic knot in S^3. In this paper, we shall study a hyperbolic link in S^3 and extend several results from the knot case. The main idea is to apply the regulator map in K-theory to the $SL_2(\mathbb{C})$-character varieties of hyperbolic links.

For a link L in S^3, Kashaev [1995] introduced a sequence of complex numbers $\{K_N \mid N \text{ is an odd integer} > 1\}$, which were derived from a matrix version of the quantum dilogarithms. Kashaev's volume conjecture therein predicts that for any hyperbolic link L in S^3, the asymptotic behavior of his invariants $\{K_N\}$ regains the hyperbolic volume of $S^3 \setminus L$. Kashaev verified this for the figure eight knot. The volume conjecture provides an intriguing relationship between the quantum invariants and the hyperbolic volume, but we still do not fully understand it.

For the knot case, Murakami and Murakami [2001] showed that the Kashaev invariants $\{K_N\}$ can be identified with the values of the normalized colored Jones polynomial at the primitive N-th roots of unity. From this, they formulated a new version of volume conjecture, stating that the asymptotic behavior of the colored Jones invariants of any knot equals the Gromov simplicial volume of its complement in S^3. This version of the volume conjecture bridges the quantum invariants of the knot with its classical geometry and topology. However, this formulation does

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not fit well for links since it does not hold for many split links; see [Murakami et al. 2002]. Hence it is a very interesting question to see what is really behind the volume conjecture for links.

Following Witten's SU(2) topological quantum field theory, Gukov [2005] proposed a complex version of Chern–Simons theory and generalized the volume conjecture to a $\mathbb{C}^*$-parametrized version with parameter lying on the zero locus of the A-polynomial of the knot. In [Li and Wang 2008], we constructed a natural torsion element in K_2 of the function field of the curve defined by the A-polynomial. We then showed that the part from the A-polynomial in Gukov's generalized volume conjecture can be interpreted in terms of the regulator map on this torsion element. In particular, this implied the Bohr–Sommerfeld quantization condition posed by Gukov [2005, page 597].

It is natural to ask if there exists a parametrized volume conjecture for links in S^3, as Gukov showed for the knot case. This is the motivation of this paper. Now we have to deal with two problems for links with more than one component. First, its $\mathrm{SL}_2(\mathbb{C})$-character variety has dimension greater than one, and it is not clear how to define an A-polynomial for such a link that will contain geometric information like volume and Chern–Simons as in the knot case. Second, it is not clear how to relate the colored Jones polynomial to its $\mathrm{SL}_2(\mathbb{C})$-character variety. In this paper, we shall focus on the first problem for hyperbolic links. We introduce n curves on the geometric component of the character variety. From these curves, we obtain an n-dimensional smooth projective variety Y^h, where n is the number of the components of the link. We construct a natural torsion element in K_2 of the function field of Y^h. By applying the regulator map on this torsion element, we get the variation formulas (Theorem 3.13) on some Zariski open subset of Y^h. When the link has one component, we recover the results for hyperbolic knots. This suggests that there may exist a parametrized volume conjecture for hyperbolic links and the Y^h may provide a replacement for the zero locus of the A-polynomial of a knot. We do not know how to deal with the second problem, and only give some speculations at the end of Section 4.

On the other hand, Dupont [1987] used the dilogarithm to construct explicitly the Cheeger–Chern–Simons class associated to the second Chern polynomial. This result applied to a closed hyperbolic 3-manifold M gives a number in $\mathbb{C}/\mathbb{Z}$. Dupont also showed that the imaginary part of this number is the hyperbolic volume of M, while the real part is the Chern–Simons invariant of M. In general, for an odd-dimensional hyperbolic manifold of finite volume, Goncharov [1999] constructed an element in Quillen's algebraic K-group of $\mathbb{C}$ and proved that after applying the Borel regulator, we get the volume of the manifold. Here, we use the regulator map for the function field of Y^h; it can be regarded as an analogue of a family version of Dupont and Goncharov's for the $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic link.

The paper is organized as follows. In Section 2, we review the basics of the $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic link. We then study the properties of a smooth projective variety Y^h coming from the one-dimensional slices of the character variety. In Section 3, we recall the definitions and basic properties of K_2 of a commutative ring. We then state and prove our main results. In Section 4, we discuss a parametrized volume conjecture for hyperbolic links.

2. Character variety of a hyperbolic link

2a. Let L be a hyperbolic link in S^3 with n components $K_1, \dots, K_n$. This means that the complement $S^3 \setminus L$ carries a complete hyperbolic structure of finite volume. Let $N(L)$ be an open tubular neighborhood of L in S^3. Then $M_L = S^3 \setminus N(L)$ is a compact 3-manifold with boundary ∂M_L a disjoint union of n tori $T_1, \dots, T_n$, and is called the link exterior. Note that $\pi_1(S^3 \setminus L)$ and $\pi_1(M_L)$ are isomorphic. In the following, we shall identify them.

Let $R(M_L) = \mathrm{Hom}(\pi_1(M_L), \mathrm{SL}_2(\mathbb{C}))$ and $R(T_i) = \mathrm{Hom}(\pi_1(T_i), \mathrm{SL}_2(\mathbb{C}))$ for $i = 1, \dots, n$ be the $\mathrm{SL}_2(\mathbb{C})$-representation spaces. We have the natural action of $\mathrm{SL}_2(\mathbb{C})$ on them by conjugation. According to [Culler and Shalen 1983], they are affine algebraic sets and so are the corresponding character varieties $X(M_L)$ and $X(T_i)$, which are the algebro-geometric quotients of $R(M_L)$ and $R(T_i)$ by $\mathrm{SL}_2(\mathbb{C})$. We then have the canonical surjective morphisms $t : R(M_L) \to X(M_L)$ and $t_i : R(T_i) \to X(T_i)$ that map a representation to its character. The inclusions of $\pi_1(T_i)$ into $\pi_1(M_L)$ induce the restriction map

$$r : X(M_L) \to X(T_1) \times \cdots \times X(T_n).$$

For details on character varieties, see [Culler and Shalen 1983; Culler et al. 1987; Cooper et al. 1994; Shalen 2002].

2b. Let $\rho_0 : \pi_1(M_L) \to \mathrm{SL}_2(\mathbb{C})$ be a representation associated to the complete hyperbolic structure on $S^3 \setminus L$. This representation is irreducible. Denote by χ_0 its character. Fix an irreducible component R_0 of $R(M_L)$ containing ρ_0. Then $X_0 = t(R_0)$ is an affine variety of dimension n [Culler and Shalen 1983; Shalen 2002]. We call X_0 a geometric component of the character variety. We define $Y_0 := \overline{r(X_0)}$, where the bar means the Zariski closure of the image $r(X_0)$ in $X(T_1) \times \cdots \times X(T_n)$.

For $g \in \pi_1(M_L)$, there is a regular function $I_g : X_0 \to \mathbb{C}$ defined by $I_g(\chi) = \chi(g)$ for all $\chi \in X_0$.

Proposition 2.1 [Culler and Shalen 1984, Proposition 2, page 539]. *Let γ_i be a noncontractible simple closed curve in the boundary torus T_i for $1 \le i \le n$. Let $g_i \in \pi_1(M_L)$ be an element whose conjugacy class corresponds to the free homotopy class of γ_i. Let k be an integer with $0 \le k \le n$, and let V be the algebraic subset of X_0 defined by the equations $I^2_{g_i}(\chi) = 4$, with $k < i \le n$. Let V_0 denote an* *irreducible component of V containing χ_{ρ_0}. If χ is a point of V_0, i is an integer with $k < i \le n$, and g is an element of the subgroup* $\operatorname{Im}(\pi_1(T_i) \to \pi_1(M_L))$ (*defined up to conjugacy*), *then we have $I_g(\chi) = \pm 2$. If also $k = 0$, then $V_0 = \{\chi_{\rho_0}\}$.*

The following generalizes the knot case; see [Culler and Shalen 1983; 1984].

Proposition 2.2. *Y_0 is an n-dimensional affine variety.*

Proof. It is clear that Y_0 is an affine variety. We need to show that $\dim Y_0 = n$. Since $\dim X_0 = n$, we have $\dim Y_0 \le n$. Assume that $\dim Y_0 = m < n$. Then for $y \in r(X_0)$, every component of the fiber $r^{-1}(y)$ has dimension $\ge n - m \ge 1$. Take $y = r(\chi_0)$; then there is an irreducible component C of the fiber $r^{-1}(y)$ containing χ_0 and $\dim C \ge 1$. For each boundary torus T_i and a nontrivial $g_i \in \operatorname{Im}(\pi_1(T_i) \to \pi_1(M_L))$, consider the regular function $I_{g_i} : X_0 \to \mathbb{C}$. For all $\chi \in C$, we have $I_{g_i}(\chi) = I_{g_i}(\chi_0)$. Since χ_0 is the character of the complete hyperbolic structure on M_L, we have $I^2_{g_i}(\chi) - 4 = I^2_{g_i}(\chi_0) - 4 = 0$ for all $\chi \in C$ and all $g_i \in \operatorname{Im}(\pi_1(T_i) \to \pi_1(M_L))$ with $1 \le i \le n$. Now we fix n nontrivial $g_i \in \operatorname{Im}(\pi_1(T_i) \to \pi_1(M_L))$ for $1 \le i \le n$. Consider the algebraic subset V of X_0 defined by the equations $I^2_{g_i} - 4 = 0$ for $1 \le i \le n$. By its construction, C is contained in an irreducible component V_0 of V containing χ_0. Hence $\dim V_0 \ge 1$. On the other hand, $V_0 = \{\chi_0\}$ by Proposition 2.1, a contradiction. Therefore, $\dim Y_0 = n$. □

For every boundary torus T_i, we fix a meridian-longitude basis $\{\mu_i, \lambda_i\}$ for $\pi_1(T_i) = H_1(T_i; \mathbb{Z})$. Given $1 \le i \le n$, we define X^i_0 as the subvariety of X_0 defined by the equations $I^2_{\mu_j} - 4 = 0$ for $j \ne i$ and $1 \le j \le n$. Let V_i be an irreducible component of X^i_0 containing χ_0.

Proposition 2.3. *V_i has dimension one for each $i = 1, \ldots, n$.*

Proof. Since X^i_0 is defined by $n-1$ equations and $\dim X_0 = n$, every component of X^i_0 has dimension at least 1. Now assume that $\dim V_i \ge 2$. Let U be the subvariety of V_i defined by the equation $I^2_{\mu_i} - 4 = 0$, and let U_0 be the irreducible component of U containing χ_0. Then $\dim V_i \ge 2$ implies that $\dim U_0 \ge 1$. But this contradicts the last part of Proposition 2.1. Hence, $\dim V_i = 1$. □

Lemma 2.4. *Fix a nontrivial $g_i \in \operatorname{Im}(\pi_1(T_i) \to \pi_1(M_L))$, with $1 \le i \le n$.*

(1) *$I_{g_i} = \pm 2$ is a constant on every V_j with $j \ne i$.*

(2) *I_{g_i} is not a constant on V_i; hence it is not a constant on X_0 either.*

Proof. (1) follows from the definition of V_j and Proposition 2.1.

For (2), suppose I_{g_i} were a constant on V_i. Then $I_{g_i} = I_{g_i}(\chi_0) = \pm 2$. Consider the algebraic subset V of X_0 defined by the n equations $I^2_{\mu_j} = 4$ with $j \ne i$, and $I^2_{g_i} = 4$. Then V_i is contained in some irreducible component V_0 of V that contains χ_{ρ_0}. Hence $\dim V_0 \ge 1$, contradicting Proposition 2.1. □

For each $i = 1, \dots, n$, let p_i be the projection map from $X(T_1) \times \cdots \times X(T_n)$ to the i-th factor $X(T_i)$. Denote by $r_i : X_0 \to X(T_i)$ the composition of r and p_i.

Proposition 2.5. *For every $i = 1, \dots, n$, the Zariski closure W_i of the image $r_i(V_i)$ in $X(T_i)$ has dimension* 1.

Proof. It suffices to consider the case $i = 1$. Since $\dim V_1 = 1$ and r_1 is regular, $\dim W_1 \le 1$. Assume that $\dim W_1 = 0$. This means that $r_1(V_1)$ consists of a single point. Therefore, I_{g_1} is a constant on V_1 for any $g_1 \in \mathrm{Im}(\pi_1(T_1) \to \pi_1(M_L))$. This contradicts Lemma 2.4(2). □

2c. For $1 \le i \le n$, denote by $R_D(T_i)$ the subvariety of $R(T_i)$ that consists of the diagonal representations. For such a representation ρ, it is clear by taking the eigenvalues of $\rho(\mu_i)$ and $\rho(\lambda_i)$ that $R_D(T_i)$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. We denote the coordinates by (l_i, m_i). Let $t_{i|D}$ be the restriction of t_i to $R_D(T_i) = \mathbb{C}^* \times \mathbb{C}^*$. Set $D_i = t_{i|D}^{-1}(W_i)$. By the proof of [Li and Wang 2006, Proposition 3.3], D_i is either irreducible or has two isomorphic irreducible components. Let $y^i \in D_i$ be the point corresponding to the character of the representation of the hyperbolic structure on $S^3 \setminus L$. Let Y_i be an irreducible component of D_i containing y^i. Then Y_i is an algebraic curve. Denote by $\overline{Y}_i$ the smooth projective model of Y_i. Denote by $\mathbb{C}(\overline{Y}_i)$ the function field of $\overline{Y}_i$ that is isomorphic to the function field $\mathbb{C}(Y_i)$ of Y_i. Note that when L is a hyperbolic knot ($n = 1$), Y_1 is the locus of the factor of the A-polynomial corresponding to the geometric component.

We define $Y^h = \overline{Y}_1 \times \overline{Y}_2 \times \cdots \times \overline{Y}_n$. Note that Y^h is an n-dimensional smooth projective variety. Let $\mathbb{C}(Y^h)$ be the function field of Y^h. For each i, we have the injective morphism $j_i : \mathbb{C}(Y_i) = \mathbb{C}(\overline{Y}_i) \to \mathbb{C}(Y^h)$ that is induced by the i-th projection from Y^h to $\overline{Y}_i$. In this way we take the $\mathbb{C}(Y_i)$ as subfields of $\mathbb{C}(Y^h)$. This also induces the map j on the K-groups:

$$j : \bigoplus_{i=1}^{n} K_2(\mathbb{C}(Y_i)) \to K_2(\mathbb{C}(Y^h)).$$

For $f_i, g_i \in \mathbb{C}(Y_i)$ with $i = 1, \dots, n$, we have $j(\sum_{i=1}^{n}\{f_i, g_i\}) = \prod_{i=1}^{n}\{f_i, g_i\}$, where we identify f_i and g_i as rational functions on Y^h via the injection j_i. Note that in this paper we use the multiplication in K_2 instead of addition.

Proposition 2.6. *There exists a finite field extension F of $\mathbb{C}(Y^h)$ with the property that for every $i = 1, \dots, n$, there is a representation $P_i : \pi_1(M_L) \to \mathrm{SL}_2(F)$ such that for $1 \le j \le n$, if $j \ne i$, the traces of $P_i(\lambda_j)$ and $P_i(\mu_j)$ are either* 2 *or* -2. *If $j = i$, then*

$$P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \quad \textit{and} \quad P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.$$

Proof. By definition, W_i for each i is the Zariski closure of $r_i(V_i)$ in $X(T_i)$ and Y_i is mapped dominatingly to W_i. The canonical morphism $t : R_0 \to X_0$ is surjective, so we can choose a curve $E_i \subset R_0$ such that $t(E_i)$ is dense in V_i. Hence the composition $r_i \circ t : E_i \to W_i$ is dominating. Then the function fields $\mathbb{C}(E_i)$ and $\mathbb{C}(Y_i)$ are finite extensions of $\mathbb{C}(W_i)$. By [Culler and Shalen 1983, page 115], there is a tautological representation $p_i : \pi_1(M_L) \to \mathrm{SL}_2(\mathbb{C}(E_i))$, and the trace of $p_i(g)$ equals I_g for any $g \in \pi_1(M_L)$. The composite field F_i of $\mathbb{C}(E_i)$ and $\mathbb{C}(Y_i)$ is finite over both $\mathbb{C}(E_i)$ and $\mathbb{C}(Y_i)$. We shall view p_i as a representation in $\mathrm{SL}_2(F_i)$. Since $t(E_i)$ is dense in V_i, by Lemma 2.4 we have that the traces of $p_i(\lambda_j)$ and $p_i(\mu_j)$ are ± 2 if $j \neq i$, and the traces of $p_i(\lambda_i)$ and $p_i(\mu_i)$ are nonconstant functions on E_i if $j = i$. Since $p_i(\lambda_i)$ and $p_i(\mu_i)$ are commuting and their eigenvalues l_i and m_i are in F_i, the representation p_i is conjugate in $\mathrm{GL}_2(F_i)$ to a representation

$$P_i : \pi_1(M_L) \to \mathrm{SL}_2(F_i)$$

such that if $j \neq i$, the traces of $P_i(\lambda_j)$ and $P_i(\mu_j)$ are either 2 or -2. If $j = i$, then

$$P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \quad \text{and} \quad P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.$$

Fix an algebraic closure $\overline{\mathbb{C}(Y^h)}$ of $\mathbb{C}(Y^h)$. As above, by viewing $\mathbb{C}(Y_i)$ as a subfield of $\overline{\mathbb{C}(Y^h)}$, we can identify the finite field extension F_i as a subfield of $\overline{\mathbb{C}(Y^h)}$. In $\overline{\mathbb{C}(Y^h)}$, take the composition K_i of F_i and $\mathbb{C}(Y^h)$ over $\mathbb{C}(Y_i)$. Then $F_i \subset K_i$, and K_i is a finite extension of $\mathbb{C}(Y^h)$ because the extension $F_i/\mathbb{C}(Y_i)$ is finite. Now let F be the composition of the fields $K_1, \ldots, K_n$ in $\overline{\mathbb{C}(Y^h)}$. Then F is a finite extension of $\mathbb{C}(Y^h)$ since each K_i is. Now compose each P_i with the embedding $\mathrm{SL}_2(F_i) \hookrightarrow \mathrm{SL}_2(F)$; the proposition follows. □

3. K-theory and Deligne cohomology

First we recall the definitions of K_2 of a commutative ring A; see [Milnor 1971]. Let $\mathrm{GL}(A)$ be the direct limit of the groups $\mathrm{GL}_n(A)$, and let $E(A)$ be the direct limit of the groups $E_n(A)$ generated by all $n \times n$ elementary matrices.

Definition 3.1. For $n \geq 3$, the *Steinberg group* $\mathrm{St}(n, A)$ is the group defined by generators x_{ij}^{λ} for $1 \leq i \neq j \leq n$, with $\lambda \in A$, subject to the relations

(i) $x_{ij}^{\lambda} \cdot x_{ij}^{\mu} = x_{ij}^{\lambda+\mu}$,

(ii) $[x_{ij}^{\lambda}, x_{jl}^{\mu}] = x_{il}^{\lambda\mu}$ for $i \neq l$, and

(iii) $[x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1$ for $j \neq k$ and $i \neq l$.

We have the canonical homomorphism $\phi_n : \mathrm{St}(n, A) \to \mathrm{GL}_n(A)$ by $\phi(x_{ij}^{\lambda}) = e_{ij}^{\lambda}$, where $e_{ij}^{\lambda} \in \mathrm{GL}_n(A)$ is the elementary matrix with entry λ in the (i, j) place. Taking the direct limit as $n \to \infty$, we get $\phi : \mathrm{St}(A) \to \mathrm{GL}(A)$. Its image $\phi(\mathrm{St}(A))$ is equal to $E(A)$, the commutator subgroup of $\mathrm{GL}(A)$.

Definition 3.2. $K_2(A) = \operatorname{Ker}\phi$.

It is well known that $K_2(A)$ is the center of the Steinberg group $\mathrm{St}(A)$ and there is a canonical isomorphism $\alpha : H_2(E(A); \mathbb{Z}) \to K_2(A)$; see [Milnor 1971, Theorems 5.1 and 5.10], respectively.

3a. ***The symbol.*** Let U and V be two commuting elements of $E(A)$. Choose $u, v \in \mathrm{St}(A)$ such that $U = \phi(u)$ and $V = \phi(v)$. Then the commutator $[u, v] = uvu^{-1}v^{-1}$ is in the kernel of ϕ. Hence $[u, v] \in K_2(A)$. We can check that $[u, v]$ is independent of the choices of u and v, and we denote it by $U \star V$.

Lemma 3.3. (1) *The product is skew-symmetric*: $U \star V = (V \star U)^{-1}$.

(2) *It is bimultiplicative*: $(U_1 \cdot U_2) \star V = (U_1 \star V) \cdot (U_2 \star V)$.

(3) *It is conjugation invariant*: $(PUP^{-1}) \star (PVP^{-1}) = U \star V$ *for* $P \in \mathrm{GL}(A)$.

Proof. This is [Milnor 1971, Lemma 8.1]. For (3), we remark that since $E(A)$ is a normal subgroup of $\mathrm{GL}(A)$, the left side of the formula makes sense. If P, U and V are in $\mathrm{GL}(n, A)$, then choose $p \in \mathrm{St}(A)$ such that

$$\phi(p) = \begin{bmatrix} P & 0 \\ 0 & P^{-1} \end{bmatrix} \in E(A).$$

Now we have $\phi(pup^{-1}) = PUP^{-1}$ and $\phi(pvp^{-1}) = PVP^{-1}$. Hence

$$[pup^{-1}, pvp^{-1}] = p[u, v]p^{-1} = [u, v]. \qquad \square$$

Given two units f and g of A, consider the matrices

$$D_f = \begin{bmatrix} f & 0 & 0 \\ 0 & f^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D'_g = \begin{bmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{bmatrix}.$$

They are in $E(A)$ and commute. Define the symbol $\{f, g\} := D_f \star D'_g$.

Lemma 3.4 [Milnor 1971, Lemmas 8.2 and 8.3]. (1) *The symbol* $\{f, g\}$ *is skew-symmetric*: $\{f, g\} = \{g, f\}^{-1}$.

(2) *It is bimultiplicative*: $\{f_1 f_2, g\} = \{f_1, g\}\{f_2, g\}$.

(3) *Denote by* $\operatorname{diag}(f_1, \dots, f_n)$ *a diagonal matrix with diagonal entries the* f_i. *If* $f_1 \cdots f_n = g_1 \cdots g_n = 1$, *then*

$$\operatorname{diag}(f_1, \dots, f_n) \star \operatorname{diag}(g_1, \dots, g_n) = \{f_1, g_1\}\{f_2, g_2\} \cdots \{f_n, g_n\}.$$

where the right side means the product of the symbols $\{f_i, g_i\}$ *for* $1 \le i \le n$.

Let F be a field. Let $\mathrm{SL}(F)$ be the direct limit of the groups $\mathrm{SL}_n(F)$. We know that $\mathrm{SL}(F) = E(F)$ and any element of $\mathrm{SL}_n(F)$ is also naturally an element of $E(F)$.

Lemma 3.5. *Let $u, t \in F$.*

(1) $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \star \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = 1.$

(2) $\begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} \star \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \star \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}$ *and* $\begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} \star \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}$ *are* 2*-torsion in* $K_2(F)$.

(3) *If U and V are two commuting matrices in* $\mathrm{SL}_2(F)$ *and their traces are* 2 *or* -2, *then $U \star V$ is* 2*-torsion in* $K_2(F)$. *In particular, if both have trace* 2, *then $U \star V = 1$.*

Proof. For $s \in F$, let

$$M(1, s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M(-1, s) = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}.$$

In particular, $M(1, 0)$ is the 2×2 identity matrix and $M(-1, 0)$ is the 2×2 diagonal matrix with diagonal entries -1.

For (1), $M(1, t) \star M(1, u) = [x_{12}^t, x_{12}^u] = 1$ by the definition of $\mathrm{St}(A)$.

For (2), notice that by the definition, $M(1, 0) \star A = 1$ and $A \star A = 1$ for any $A \in E(F)$. By Lemma 3.3,

$$1 = (M(-1, 0) \cdot M(-1, 0)) \star M(1, s) = (M(-1, 0) \star M(1, s))^2,$$

so $M(-1, 0) \star M(1, s)$ is a 2-torsion in $K_2(F)$. Since

$$M(-1, t) = M(-1, 0) \cdot M(1, -t) \quad \text{and} \quad M(-1, u) = M(-1, 0) \cdot M(1, -u),$$

by Lemma 3.3 and the first part, we have

$$\begin{aligned} M(-1, t) \star M(1, u) &= (M(-1, 0) \star M(1, u))(M(1, -t) \star M(1, u)) \\ &= M(-1, 0) \star M(1, u), \\ M(-1, t) \star M(-1, u) &= (M(-1, 0) \star M(1, -u))(M(1, -t) \star M(-1, 0)); \end{aligned}$$

hence they are 2-torsion.

For (3), we can find $P \in \mathrm{GL}_2(F)$ such that

$$PUP^{-1} = \begin{bmatrix} \pm 1 & t \\ 0 & \pm 1 \end{bmatrix} \quad \text{and} \quad PVP^{-1} = \begin{bmatrix} \pm 1 & u \\ 0 & \pm 1 \end{bmatrix}.$$

Then it follows from the first two parts and Lemma 3.3(3). $\square$

The following proposition slightly generalizes [Cooper et al. 1994, Lemma 4.1]. The proof is the same.

Proposition 3.6. *Let π be a free abelian group of rank two with $\{e_1, e_2\}$ its basis. Let $f : \pi \to E(A)$ be a group homomorphism defined by $f(e_1) = U$ and $f(e_2) = V$. Then there is a generator t of $H_2(\pi; \mathbb{Z})$ such that $\alpha(f_*(t)) = U \star V$. Here $\alpha : H_2(E(A); \mathbb{Z}) \to K_2(A)$ is the canonical isomorphism and $f_* : H_2(\pi; \mathbb{Z}) \to H_2(E(A); \mathbb{Z})$ is the homomorphism induced by f.*

Proof. Since π is abelian, U and V commute. $U \star V$ is well-defined. Let F be the free group on $\{e_1, e_2\}$. The homomorphism f gives rise to a commutative diagram of short exact sequences of groups:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & {[F,F]} \arrow[r] \arrow[d, "f_2"'] & F \arrow[r] \arrow[d, "f_1"'] & \pi \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ 0 \arrow[r] & K_2(A) \arrow[r] & \mathrm{St}(A) \arrow[r, "\phi"] & E(A) \arrow[r] & 0, \end{tikzcd}$$

where $f_2([e_1, e_2]) = U \star V$. Applying the homology spectral sequence to this diagram, we obtain the diagram

$$\begin{tikzcd} H_2(\pi; \mathbb{Z}) \arrow[r] \arrow[d, "f_*"'] & H_0(\pi; H_1([F,F]; \mathbb{Z})) \arrow[d, "g"'] \\ H_2(E(A); \mathbb{Z}) \arrow[r, "\alpha"] & K_2(A). \end{tikzcd}$$

The top horizontal arrow is an isomorphism. The class of $[e_1, e_2]$ is the generator of $H_0(\pi; H_1([F, F]; \mathbb{Z}))$. It is mapped to $U \star V$ by g, which is induced by f_2. Let t be the generator of $H_2(\pi; \mathbb{Z})$ mapped to the class of $[e_1, e_2]$. Then we have $\alpha(f_*(t)) = U \star V$ by the commutative diagram. □

Corollary 3.7. (1) *If $U = \mathrm{diag}(u, u^{-1})$ and $V = \mathrm{diag}(v, v^{-1})$, where u, v are units of A, then there is a generator t of $H_2(\pi; \mathbb{Z})$ such that $\alpha(f_*(t)) = \{u, v\}^2$.*

(2) *Suppose A is a field. If U and V are two commuting matrices in $\mathrm{SL}_2(A)$ and their traces are 2 or -2, then the image of any generator of $H_2(\pi; \mathbb{Z})$ is 2-torsion in $K_2(A)$.*

Proof. For (1), we have $U \star V = \{u, v\}\{u^{-1}, v^{-1}\} = \{u, v\}^2$ by Lemma 3.4.

For (2), $U \star V$ is 2-torsion in $K_2(F)$ by Lemma 3.5(3). □

Theorem 3.8. *For each $i = 1, \dots, n$, there is an integer $\epsilon(i) = 1$ or -1 such that the symbol $\prod_{i=1}^{n} \{l_i, m_i\}^{\epsilon(i)}$ is a torsion element in $K_2(\mathbb{C}(Y^h))$.*

Proof. First, by Proposition 2.6, for each $i = 1, \dots, n$ there exist a finite extension F of $\mathbb{C}(Y^h)$ and a representation $P_i : \pi_1(M_L) \to \mathrm{SL}_2(F)$ such that for $1 \le j \le n$, the traces of $P_i(\lambda_j)$ and $P_i(\mu_j)$ are either 2 or -2 if $j \neq i$ and, if $j = i$,

$$P_i(\lambda_i) = \begin{bmatrix} l_i & 0 \\ 0 & l_i^{-1} \end{bmatrix} \quad \text{and} \quad P_i(\mu_i) = \begin{bmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{bmatrix}.$$

The inclusions of $\pi_1(T_i)$ into $\pi_1(M_L)$ induce homomorphisms $\pi_1(T_i) \to E(F)$ by composition with P_i. This gives rise to homomorphisms

$$\bigoplus_{i=1}^{n} H_2(\pi_1(T_i); \mathbb{Z}) \xrightarrow{\alpha} H_2(\pi_1(M_L); \mathbb{Z}) \xrightarrow{\beta} H_2(E(F); \mathbb{Z}) = K_2(F) \tag{3-1}$$

in group homology, where $\alpha = j_{1*} + \cdots + j_{n*}$, $\beta = P_{1*} + \cdots + P_{n*}$, the j_{i*} are the morphisms on the group homology induced by the inclusions $j_i : \pi_1(T_i) \hookrightarrow \pi_1(M_L)$, and the P_{i*} are those induced by the P_i.

The orientation of M_L induces an orientation on each boundary torus T_i. Let $[T_i]$ be the orientation class of $H_2(T_i; \mathbb{Z}) = \mathbb{Z}$. By Corollary 3.7(1), for each i there is a generator ξ_i of $H_2(\pi_1(T_i))$ such that $P_{i*}(j_{i*}(\xi_i)) = \{l_i, m_i\}^2$. Since T_i is a $K(\pi_1(T_i), 1)$ space, $H_2(\pi_1(T_i); \mathbb{Z}) = H_2(T_i; \mathbb{Z})$. If $\xi_i = [T_i]$, define $\epsilon(i) = 1$; if $\xi_i = -[T_i]$, then define $\epsilon(i) = -1$.

Since L is a hyperbolic link, M_L is a $K(\pi_1(M_L), 1)$ space. Hence we have $H_2(\pi_1(M_L); \mathbb{Z}) = H_2(M_L; \mathbb{Z})$. Under this identification, we have

$$\alpha(\epsilon(1)\xi_1, \ldots, \epsilon(n)\xi_n) = \sum_{i=1}^{n} [T_i] = [\partial M_L] = 0 \quad \text{in } H_2(M_L; \mathbb{Z}).$$

Therefore,

$$\beta(\alpha(\epsilon(1)\xi_1, \ldots, \epsilon(n)\xi_n)) = 1 \quad \text{in } K_2(F). \tag{3-2}$$

On the other hand, we have

$$\begin{aligned} \beta(\alpha(\epsilon(1)\xi_1, \ldots, \epsilon(n)\xi_n)) &= \beta\Big(\sum_{i=1}^{n} j_{i*}(\epsilon(i)\xi_i)\Big) \\ &= \sum_{k=1}^{n} P_{k*}\Big(\sum_{i=1}^{n} j_{i*}(\epsilon(i)\xi_i)\Big) \\ &= \sum_{i=1}^{n} P_{i*}(j_{i*}(\epsilon(i)\xi_i)) + \sum_{1 \le i \neq k \le n} P_{k*}(j_{i*}(\epsilon(i)\xi_i)) \\ &= \prod_{i=1}^{n} \{l_i, m_i\}^{2\epsilon(i)} \cdot \prod_{1 \le i \neq k \le n} P_k(\mu_i) \star P_k(\lambda_i), \end{aligned}$$

where the last step follows from Proposition 3.6 and Corollary 3.7. Note also that we use multiplication in $K_2(F)$.

Now $\prod_{1\le i\ne k\le n} P_k(\mu_i) \star P_k(\lambda_i)$ is 2-torsion by Corollary 3.7(2). Comparing with (3-2), we see that $\prod_{i=1}^n \{l_i, m_i\}^{2\epsilon(i)}$ is 2-torsion in $K_2(F)$. By the argument of [Li and Wang 2008, Proposition 3.2], $\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)}$ is torsion in $K_2(\mathbb{C}(Y^h))$. $\square$

Remark 3.1. This theorem is a natural generalization of [Li and Wang 2008, Proposition 3.2], which concerned the hyperbolic knot case.

Remark 3.2. The proof of Theorem 3.8 uses the condition that the geometric component contains the character χ_0 of the complete hyperbolic structure. For a nongeometric component of the character variety, it is not clear whether we can still have the analogous torsion property on it.

3b. ***Deligne cohomology.*** Here we recall the definition of Deligne cohomology, give the construction of the regulator map, and apply it to our situation.

Let X be a nonsingular variety over $\mathbb{C}$. First recall the definition of the (holomorphic) Deligne cohomology groups of X. For more details, see [Beĭlinson 1984; Brylinski 2008; Esnault and Viehweg 1988]. We define the complex $\mathbb{Z}(p)_{\mathscr{D}}$ of sheaves on X by

$$\mathbb{Z}(p)_{\mathscr{D}} : \mathbb{Z}(p) \longrightarrow \mathbb{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X, \tag{3-3}$$

where $\mathbb{Z}(p)$ is the constant sheaf $(2\pi\sqrt{-1})^p\mathbb{Z}$ and sits in degree zero, $\mathbb{O}_X$ is the sheaf of holomorphic functions on X, and Ω^i_X is the sheaf of holomorphic i-forms on X. The first map in (3-3) is the inclusion and d is the exterior differential. The Deligne cohomology groups of X are defined as the hypercohomology of the complex $\mathbb{Z}(p)_{\mathscr{D}}$:

$$H^q_{\mathscr{D}}(X; \mathbb{Z}(p)) := \mathbb{H}^q(X; \mathbb{Z}(p)_{\mathscr{D}}).$$

For example, the exponential exact sequence of sheaves on X

$$0 \to \mathbb{Z}(1) \to \mathbb{O}_X \to \mathbb{O}^*_X \to 0$$

gives rise to a quasiisomorphism between $\mathbb{Z}(1)_{\mathscr{D}}$ and $\mathbb{O}^*_X[-1]$, where $\mathbb{O}^*_X$ is the sheaf of nonvanishing holomorphic functions on X. Moreover there is a quasi-isomorphism between $\mathbb{Z}(2)_{\mathscr{D}}$ and the complex [Esnault and Viehweg 1988, page 46]

$$(\mathbb{O}^*_X \xrightarrow{d\log} \Omega^1_X)[-1].$$

Therefore, we have for any integer q

$$H^q_{\mathscr{D}}(X; \mathbb{Z}(1)) = H^{q-1}(X; \mathbb{O}^*_X) \quad \text{and} \quad H^q_{\mathscr{D}}(X; \mathbb{Z}(2)) = \mathbb{H}^{q-1}(X; \mathbb{O}^*_X \to \Omega^1_X).$$

On the other hand, Deligne [1991] interprets $\mathbb{H}^1(X; \mathbb{O}^*_X \to \Omega^1_X) = H^2_{\mathscr{D}}(X; \mathbb{Z}(2))$ as the group of holomorphic line bundles with (holomorphic) connections over X. For details, see [Brylinski 2008, Theorem 2.2.20].

Let $\mathbb{C}(X)$ be the function field of X. Given two functions $f, g \in \mathbb{C}(X)$, let $D(f, g)$ be the divisors of the zeros and poles of f and g, and let $|D(f, g)|$ denote its support. Then we have the morphism

$$(f, g) : X - |D(f, g)| \to \mathbb{C}^* \times \mathbb{C}^*,$$

given by $(f, g)(x) = (f(x), g(x))$.

Let $\mathscr{H}$ be the Heisenberg line bundle with connection on $\mathbb{C}^* \times \mathbb{C}^*$. For its construction, see [Bloch 1981] and [Ramakrishnan 1989, Section 4]. Pull back $\mathscr{H}$ along (f, g) to obtain a line bundle $r(f, g)$ with connection on $X - |D(f, g)|$. Hence $r(f, g) \in \mathbb{H}^1(V; \mathbb{O}_V^* \to \Omega_V^1) = H^2_{\mathscr{D}}(V; \mathbb{Z}(2))$, where $V = X - |D(f, g)|$. Moreover we can represent $r(f, g)$ in terms of Čech cocycles for $\mathbb{H}^1(V; \mathbb{O}_V^* \to \Omega_V^1)$. Indeed, choose an open covering $(U_i)_{i\in I}$ of V such that the logarithm $\log_i f$ of f is well-defined on every U_i. Then $r(f, g)$ is represented by the cocyle (c_{ij}, ω_i), with

$$c_{ij} = g^{(\log_j f - \log_i f)/(2\pi\sqrt{-1})} \qquad \text{on } U_i \cap U_j, \tag{3-4}$$

$$\omega_i = \frac{1}{2\pi\sqrt{-1}} \log_i f \; \frac{dg}{g} \qquad \text{on } U_i. \tag{3-5}$$

Its curvature is

$$R = \frac{1}{2\pi\sqrt{-1}} \frac{df}{f} \wedge \frac{dg}{g}. \tag{3-6}$$

Remark 3.3. There is a cup product on the Deligne cohomology groups [Beĭlinson 1984; Esnault and Viehweg 1988]. For $f, g \in H^0(X; \mathbb{O}_X^*) = H^1_{\mathscr{D}}(X; \mathbb{Z}(1))$ as above, the cup product $f \cup g$ is exactly the line bundle $r(f, g) \in H^2_{\mathscr{D}}(X; \mathbb{Z}(2))$.

Furthermore, we have the following properties of $r(f, g)$:

Proposition 3.9. *$r(f_1 f_2, g) = r(f_1, g) \otimes r(f_2, g)$, $r(f, g) = r(g, f)^{-1}$, and the Steinberg relation $r(f, 1 - f) = 1$ holds if $f \neq 0$ and $f \neq 1$.*

Proof. See [Bloch 1981; Esnault and Viehweg 1988] and [Ramakrishnan 1989, Section 4]. The proofs there assume that X is a curve. But they are valid for arbitrary X without change. To prove the Steinberg relation, we need the ubiquitous dilogarithm function. □

Corollary 3.10. *We have the regulator map*

$$r : K_2(\mathbb{C}(X)) \to \varinjlim_{U\subset X:\text{Zariski open}} H^2_{\mathscr{D}}(U; \mathbb{Z}(2)),$$

which maps the symbol $\{f, g\}$ to the line bundle $r(f, g)$.

This follows from the definition of K_2 and Proposition 3.9.

When $\dim X = 1$, the line bundle $r(f, g)$ is always flat, but $r(f, g)$ is not necessarily flat if $\dim X > 1$. Nevertheless:

Proposition 3.11. *If $x \in K_2(\mathbb{C}(X))$ is torsion, the corresponding line bundle $r(x)$ is flat.*

Proof. Let U be the Zariski open subset over which the line bundle $r(x)$ is defined. Since x is torsion in $K_2(\mathbb{C}(X))$, $r(x)$ is torsion in $\mathbb{H}^1(U; \mathbb{O}_U^* \to \Omega_U^1)$. Choose a suitable open covering $(U_i)_{i\in I}$ of U such that $r(x)$ is represented by a Čech cocyle (c_{ij}, ω_i) with $c_{ij} \in \mathbb{O}^*(U_i \cap U_j)$ and $\omega_i \in \Omega^1(U_i)$. Then there exists an integer $n > 0$ such that the class represented by the cocycle $((c_{ij})^n, n\omega_i)$ is zero. Hence, there exists $t_i \in \mathbb{O}_X^*(U_i)$ (or by a refinement covering of $\{U_i\}$) , such that

$$c_{ij}^n = \frac{t_j}{t_i} \quad \text{and} \quad \omega_i = \frac{1}{n}\frac{dt_i}{t_i}.$$

Therefore, $d\omega_i = 0$ for all i and the curvature is 0. □

Let $|D|$ be the support of the divisors of zeros and poles of the rational functions m_i and l_i on Y^h for $1 \le i \le n$. Define $Y_0^h = Y^h - |D|$. The line bundle $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ is well-defined over Y_0^h.

Corollary 3.12. *The line bundle $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ over Y_0^h is flat; therefore it is an element of $H^1(Y_0^h; \mathbb{C}^*)$.*

Proof. This follows from Theorem 3.8 and Proposition 3.11. □

Using the Čech cocycle for $r(f, g)$ given in (3-4) and (3-5), we can represent $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ as follows. Choose an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of Y_0^h such that on every U_α, the logarithms of l_i are well-defined and denoted by $\log_\alpha l_i$. Then $r(\prod_{i=1}^n \{l_i, m_i\}^{\epsilon(i)})$ is represented by the cocyle $(c_{\alpha\beta}, \omega_\alpha)$:

$$c_{\alpha\beta} = \prod_{i=1}^n m_i^{\epsilon(i)((\log_\beta l_i - \log_\alpha l_i))/(2\pi\sqrt{-1})} \qquad \text{on } U_\alpha \cap U_\beta, \tag{3-7}$$

$$\omega_\alpha = \sum_{i=1}^n \frac{\epsilon(i)}{2\pi\sqrt{-1}}(\log_\alpha l_i)\,\frac{dm_i}{m_i} \qquad \text{on } U_\alpha. \tag{3-8}$$

Let $t_0 = (l_1^0, m_1^0, \ldots, l_n^0, m_n^0) \in Y_0^h$ be a point corresponding to the hyperbolic structure of the link complement $S^3 \backslash L$. Then the monodromy of the flat line bundle $r(\prod_{i=1}^n \{m_i, l_i\}^{\epsilon(i)})$ give rises to the representation $M : \pi_1(Y_0^h, t_0) \to \mathbb{C}^*$. With its explicit descriptions (3-7) and (3-8), we have the following formula for M. Let γ be a loop based at t_0. Let $\log l_i$ be a branch of logarithm of l_i over $\gamma - \{t_0\}$, then by a direct calculation we have

$$M(\gamma) = \exp\biggl(\sum_{i=1}^n \Bigl(-\frac{\epsilon(i)}{2\pi\sqrt{-1}}\Bigr)\Bigl(\int_\gamma \log l_i \frac{dm_i}{m_i} - \log m_i(t_0)\int_\gamma \frac{dl_i}{l_i}\Bigr)\biggr); \tag{3-9}$$

see [Deligne 1991, (2.7.2)].

Now we have the main theorem:

Theorem 3.13. (i) *The real* 1*-form*

$$\eta=\sum_{i=1}^{n}\epsilon(i)(\log|l_i|\,d\arg m_i-\log|m_i|\,d\arg l_i)$$

is exact on Y_0^h. *Hence there exists a smooth function* $V:Y_0^h\to\mathbb{R}$ *such that*

$$dV=\sum_{i=1}^{n}\epsilon(i)(\log|l_i|\,d\arg m_i-\log|m_i|\,d\arg l_i).$$

(ii) *Suppose* $m_i^0=1$ *for* $1\le i\le n$. *For a loop* γ *with initial point* t_0 *in* Y_0^h

$$\frac{1}{4\pi^2}\sum_{i=1}^{n}\epsilon(i)\int_\gamma(\log|m_i|d\log|l_i|+\arg l_i\;d\arg m_i)=\frac{p}{q},$$

where q *is the order of the symbol* $\prod_{i=1}^{n}\{l_i,m_i\}^{\epsilon(i)}$ *in* $K_2(\mathbb{C}(Y^h))$, *and* p *is some integer depending on the loop* $\gamma\in\pi_1(Y_0^h,t_0)$ *and the branches of* $\arg l_i$ *for* $1\le i\le n$.

Proof. First, by (3-8), the curvature of the flat line bundle is

$$R=\sum_{i=1}^{n}\frac{\epsilon(i)}{2\pi\sqrt{-1}}\Big(\frac{dl_i}{l_i}\wedge\frac{dm_i}{m_i}\Big)=0.$$

On the other hand, we have $d\eta=\operatorname{Im}(\sum_{i=1}^{n}\epsilon(i)(dl_i/l_i\wedge dm_i/m_i))$; hence η is a real closed 1-form.

Since the symbol $\prod_{i=1}^{n}\{l_i,m_i\}^{\epsilon(i)}$ has order q in $K_2(\mathbb{C}(Y^h))$, by (3-9) we have for a loop $\gamma\in\pi_1(Y_0^h,t_0)$ that

$$1=M(\gamma)^q=\bigg(\exp\Big(\sum_{i=1}^{n}\Big(-\frac{\epsilon(i)}{2\pi\sqrt{-1}}\Big)\Big(\int_\gamma\log l_i\,\frac{dm_i}{m_i}-\log m_i(t_0)\int_\gamma\frac{dl_i}{l_i}\Big)\Big)\bigg)^q.$$

Decompose part of this into real and imaginary parts as

$$\sum_{i=1}^{n}\epsilon(i)\Big(\int_\gamma\log l_i\,\frac{dm_i}{m_i}-\log m_i(t_0)\int_\gamma\frac{dl_i}{l_i}\Big)=\operatorname{Re}+i\operatorname{Im},$$

Then we have $\exp\,(q\cdot\operatorname{Im}/(2\pi)+q\cdot\operatorname{Re}/(2\pi\sqrt{-1}))=1$. Therefore, $\operatorname{Im}=0$ and $q\cdot\operatorname{Re}/(2\pi\sqrt{-1})=2\pi\sqrt{-1}p$ for some integer p. A straightforward calculation or [Li and Wang 2008, Lemma 3.4] shows that

$$\begin{aligned} &\mathrm{Im} = \int_\gamma \eta, \\ &\mathrm{Re} = -\sum_{i=1}^{n} \epsilon(i) \int_\gamma (\log|m_i|\, d\log|l_i| + \arg l_i \;\, d\arg m_i) = \int_\gamma \xi. \end{aligned} \tag{3-10}$$

These immediately imply both parts of the theorem. □

Remark 3.4. When $n = 1$, our V is (up to sign) the volume function of the representation of the knot complement [Dunfield 1999]. For $n \geq 2$, up to some constant and signs related to the orientations on each boundary component of the hyperbolic link exterior, the function V should be closely related to the volume function given in [Hodgson 1986, Theorem 5.5].

Remark 3.5. From the proof of Theorem 3.8, the signs $\epsilon(i)$ for $1 \leq i \leq n$ are determined by the orientation of M_L on its n boundary tori. For knots, the sign can be neglected since there is only one term in the 1-form η. For links (where $n \geq 2$), if they are not the same, they could have quite contributions different from those in the knot case. On the other hand, it is not clear what are the exact geometric meanings of these signs for the link L.

Remark 3.6. If there exists any representation $\rho : \pi_1(Y^h) \to \mathrm{GL}_n(\mathbb{C})$ with $n \geq 2$, then Reznikov [1995, Theorem 1.1] proved that for all $i \geq 2$, the Chern classes $c_i \in H^{2i}_{\mathscr{D}}(Y^h; \mathbb{Z}(i))$ in the Deligne cohomology groups are torsion.

3c. ***On the Bohr–Sommerfeld quantization condition for hyperbolic links.*** We now discuss the Theorem 3.13(ii) from a symplectic point of view. When $n = 1$, this is the Bohr–Sommerfeld quantization condition proposed by Gukov for knots in [Gukov 2005, page 597], and is proved in [Li and Wang 2008, Theorem 3.3(2)].

Let Σ be a closed surface with fundamental group π. Its $\mathrm{SL}_2(\mathbb{C})$-character variety is the space of equivalence classes of representations from π into $\mathrm{SL}_2(\mathbb{C})$. This variety carries a natural complex-symplectic structure, where a complex-symplectic structure is a nondegenerate closed holomorphic exterior 2-form; see [Goldman 1984; 2004].

A homomorphism $\rho : \pi \to \mathrm{SL}_2(\mathbb{C})$ is irreducible if it has no proper linear invariant subspace of $\mathbb{C}^2$, and irreducible representations are stable points, denoted by $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))^s$. Now $\mathrm{SL}_2(\mathbb{C})$ acts freely and properly on $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))^s$, and the quotient $X^s(\Sigma) = \mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))^s / \mathrm{SL}_2(\mathbb{C})$ is an embedding onto an open subset in the geometric quotient $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C})) \mathbin{/\!/} \mathrm{SL}_2(\mathbb{C})$. Thus $X^s(\Sigma)$ is a smooth irreducible complex quasiaffine variety that is dense in the geometric quotient [Goldman 2004, Section 1]. Note that ρ is a nonsingular point if and only if $\dim Z(\rho)/Z(\mathrm{SL}_2(\mathbb{C})) = 0$, and this corresponds to the top stratum $X^s(\Sigma)$, where

$Z(u)$ is the centralizer of u in $\mathrm{SL}_2(\mathbb{C})$. If $\rho \in \mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))$ is a singular point (that is, $\dim Z(\rho)/Z(\mathrm{SL}_2(\mathbb{C})) > 0$), then all points of $\sigma \in \mathrm{Hom}(\pi, Z(Z(\rho)))^s$ with $\mathrm{stab}(\sigma) = Z(\sigma) = Z(\rho)$ have the same orbit type and form a stratification of the $\mathrm{SL}_2(\mathbb{C})$-character variety [Goldman 1984, Section 1].

We have the $\mathrm{SL}_2(\mathbb{C})$-character variety $X(T^2)$ of the torus T^2 as a surface in $\mathbb{C}^3$ given by

$$x^2 + y^2 + z^2 - xyz - 4 = 0.$$

See [Li and Wang 2006, Proposition 3.2]. There is a natural symplectic structure on the smooth top stratum $X^s(T^2)$ of $X(T^2)$, and there exists a symplectic structure ω on the character variety $X^s(\partial M_L) = \prod_{i=1}^n X^s(T_i^2)$ such that $X(M_L) \cap X^s(\partial M_L)$ (a subset of $X(M_L)$) is a Lagrangian subvariety of $X^s(\partial M_L)$, where $X^s(\partial M_L)$ is a smooth irreducible variety that is open and dense in $X(\partial M_L)$.

The inclusion $\partial M_L \to M_L$ indeed induces a degree one map on the irreducible components. Thus $r(X_0)^s$ (the smooth part of the image $r(X_0)$) is a Lagrangian submanifold of the symplectic manifold $X^s(\partial M_L)$. Note that the pullback of the symplectic 2-form on the double covering of $X^s(T_i^2)$ is again skew-symmetric and nondegenerate. The symplectic form $\tilde{\omega}_i$ induced by the map $t_i : r(X_0) \to X(T_i^2)$ gives the Lagrangian property for the corresponding pullback of the Lagrangian part $r(X_0^i)^s$. Hence we have the product Lagrangian smooth part of the pullback of $\prod_{i=1}^n r(X_0^i)^s$. Then we need to see that the smooth projective model preserves the Lagrangian and symplectic property.

Let $\tilde{X}(T_i^2)$ be the symplectic blowup of the double covering of $X(T_i^2)$ as in [McDuff and Salamon 1998]. The blowup in the complex category carries a natural symplectic structure on $\tilde{X}(T_i^2)$; see [McDuff and Salamon 1998, Section 7.1]. On the other hand, the corresponding part $\overline{Y}_i$ of Y_i (the irreducible component of D_i containing y_i) lies in the symplectic manifold $\tilde{X}(T_i^2)$.

Define a compatible Lagrangian blowup with respect to the complex blowup as following. Define a real submanifold $\tilde{\mathbb{R}}^n$ of $\mathbb{R}^n \times \mathbb{R}P^{n-1}$ (a subset of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$) as a subspace of pairs (x, l) with $x = \mathrm{Re}(z) \in l$, where $l \in \mathbb{R}P^{n-1}$ is a real line in $\mathbb{R}^n$. If $I_{\mathbb{C}}$ is complex conjugation on $\mathbb{C}^n$ and $J_{\mathbb{C}P^{n-1}}$ is the complex involution on $\mathbb{C}P^{n-1}$ given by complex conjugation on each component, then

$$\begin{aligned}\tilde{\mathbb{R}}^n &= \mathrm{Fix}(I_{\mathbb{C}} \times J_{\mathbb{C}P^{n-1}}|_{\tilde{\mathbb{C}}^n}) \subset \tilde{\mathbb{C}}^n \\ &= \{(z_1, \dots, z_n; [w_1 : \cdots : w_n]) \mid w_j z_k = w_k z_j,\ 1 \le j, k \le n\}.\end{aligned}$$

It is clear that $\tilde{\mathbb{R}}^n$ is Lagrangian in $\tilde{\mathbb{C}}^n$. Hence the real Lagrangian blowup $\tilde{\overline{Y}}_i$ is Lagrangian in $\tilde{X}(T_i^2)$, and the Lagrangian submanifold $\tilde{Y}^h$ is Lagrangian in the symplectic manifold $\prod_{i=1}^n \tilde{X}(T_i^2)$. In this way, the symplectic and Lagrangian properties are preserved under the blowup, and we can treat the Lagrangian blowup in a real blowup by looking at the complex one.

Now we have a Lagrangian submanifold $\tilde{Y}_0^h$ in a symplectic manifold. Suppose $m_i^0 = 2$ for $1 \le i \le n$. For a loop γ with initial point t_0 in $\tilde{Y}_0^h$, Theorem 3.13(ii) gives

$$\frac{1}{4\pi^2}\sum_{i=1}^{n}\epsilon(i)\int_\gamma(\log|m_i|\,d\log|l_i| + \arg l_i\,d\arg m_i) = \frac{p}{q},$$

where p is some integer and q is the order of the symbol $\prod_{i=1}^n\{l_i, m_i\}^{\epsilon(i)}$ in $K_2(\mathbb{C}(Y^h))$. We shall call this result the Bohr–Sommerfeld quantization condition for hyperbolic links. It would be interesting to give an interpretation from mathematical physics, as what Gukov did for hyperbolic knots.

4. On a possible unified volume conjecture for both knots and links

By Corollary 3.12, the class $r(\prod_{i=1}^n\{l_i, m_i)^{\varepsilon_i})$ corresponds to a flat line bundle over Y_0^h; therefore the curvature of the holomorphic connection is zero. Formally this can be expressed as $d(\xi + \sqrt{-1}\eta) = 0$, where ξ and η are defined in (3-10). Hence, $(\xi + \sqrt{-1}\eta)/(2\pi\sqrt{-1})$ can be viewed as the Chern–Simons 1-*form* of the line bundle $r(\prod_{i=1}^n\{l_i, m_i)^{\varepsilon_i})$.

Given a point $p \in Y_0^h$, choose a path $\gamma : [0, 1] \to Y_0^h$ with $\gamma(1) = p$ and $\gamma(0) = t_0$ a point corresponding to the complete hyperbolic structure. Write

$$\gamma(t) = (l(t), m(t)) = (l_1(t), m_1(t), \ldots, l_n(t), m_n(t)).$$

Recall that q is the order of the symbol $\prod_{i=1}^n\{l_i, m_i\}^{\varepsilon_i}$ in $K_2(\mathbb{C}(Y^h))$. Let $\mathrm{Vol}(L)$ and $\mathrm{CS}(L)$ be the volume and usual Chern–Simons invariant of the complete hyperbolic structure on $S^3 \setminus L$, respectively. Now we define

$$V(p) = \mathrm{Vol}(L) + 2\cdot\sum_{i=1}^{n}\epsilon(i)\int_\gamma(\log|l_i|\,d\arg m_i - \log|m_i|\,d\arg l_i). \tag{4-1}$$

$$U(p) = 4\pi^2\,\mathrm{CS}(L) + q\cdot\sum_{i=1}^{n}\epsilon(i)\int_\gamma(\log|m_i|\,d\log|l_i| + \arg l_i\,d\arg m_i). \tag{4-2}$$

According to Theorem 3.13, $R(p) = (2\pi)^{-1}(V(p) + \sqrt{-1}(2\pi)^{-1}U(p))$ is independent of the choices of the path γ and takes values in $\mathbb{C}/\mathbb{Z}$. We call

$$\frac{1}{4\pi^2}U(p)$$

the *special Chern–Simons invariant* of the hyperbolic link L at p. When $p = t_0$, it equals $\mathrm{CS}(L)$.

Remark 4.1. For $p \ne t_0$, $U(p)/(4\pi^2)$ is different from the usual Chern–Simons invariant for a 3-dimensional manifold. The latter comes from the transgressive 3-form of the second Chern class of the 3-dimensional manifold.

In order to formulate a parametrized conjecture parallel to the knot case as in [Li and Wang 2008, Conjecture 3.9], we have to find a way to relate the quantum invariants to the n-dimensional variety Y_0^h that comes from the $SL_2(\mathbb{C})$ character variety. By the work of Kashaev [1995] and Baseilhac and Benedetti [2004], there exists an $SL_2(\mathbb{C})$ quantum hyperbolic invariant for a hyperbolic link in S^3, which is conjectured to give the information of the volume and Chern–Simons at the point for the complete hyperbolic structure.

Here is a conjectural description. Given a point $p \in Y_0^h$ corresponding to an $SL_2(\mathbb{C})$ representation of $\pi_1(M_L)$, let's assume that we can define certain quantum invariants $K_N(L, p)$. Then we formulate the following:

Conjecture 4.1 (a possibly unified parametrized volume conjecture).

$$\lim_{N\to\infty} \frac{\log K_N(L, p)}{N} = \frac{1}{2\pi}\Big(V(p) + \frac{\sqrt{-1}}{2\pi} U(p)\Big).$$

Remark 4.2. When L is a hyperbolic knot (that is, $n = 1$), Y^h is the smooth projective model of an irreducible component of the locus of the A-polynomial that contains the complete hyperbolic structure. Fix a number a. For $p = (l, m) \in Y_0^h$ with $m = -\exp(\sqrt{-1}\pi a)$, we take $K_N(L, p) = J_N(L, e^{2\pi\sqrt{-1}a/N})$, the values of the colored Jones polynomial of L evaluated at $e^{2\pi\sqrt{-1}\,a/N}$. Then Conjecture 4.1 reduces to the reformulated generalized volume conjecture (3.9) of [Li and Wang 2008] for hyperbolic knots. When γ is the constant path at t_0, or equivalently $p = t_0$, it reduces to the complexification of Kashaev's conjecture for hyperbolic knots; see [Murakami et al. 2002, Conjeture 1.2].

Remark 4.3. When $n \geq 2$, we can take $K_N(L, t_0)$ to be the Kashaev and Baseilhac–Benedetti invariant that is based on the triangulations of the manifold and is conjectured to give the information of the volume and Chern–Simons at the complete hyperbolic structure t_0. See [Baseilhac and Benedetti 2004, Section 5]. For a general $p \in Y_0^h$, we do not have a rigorous definition, although we expect that there is a way of deforming $K_N(L, t_0)$ to get $K_N(L, p)$.

Remark 4.4. If the point corresponding to the hyperbolic structure in Y_i is not smooth, then the point t_0 in the definition of (4-1) and (4-2) is not unique. If we make different choices of t_0, then $V(p)$ and $U(p)$ will differ by a constant, corresponding to choice made in the integrals in (4-1) and (4-2). We can modify the left side of the Conjecture 4.1 by this constant accordingly. So the choice of t_0 is not essential, and it seems that there is no canonical choice.

Remark 4.5. From the regulator point of view developed in this paper, we expect there exists a parametrized version of the volume conjecture for both hyperbolic links and knots.

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