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**HYPERGEOMETRIC EVALUATION IDENTITIES AND  
SUPERCONGRUENCES**

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## HYPERGEOMETRIC EVALUATION IDENTITIES AND SUPERCONGRUENCES

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**We apply some hypergeometric evaluation identities, including a strange valuation of Gosper, to prove several supercongruences related to special valuations of truncated hypergeometric series. In particular, we prove a conjecture of van Hamme.**

### 1. Introduction

In this article, we use  $p$  to denote an odd prime. Zudilin [2009] proved several Ramanujan-type supercongruences using the Wilf–Zeilberger (WZ) method. One of them, conjectured by van Hamme, says that

$$(1) \quad \sum_{k=0}^{(p-1)/2} (4k+1) \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k \equiv (-1)^{(p-1)/2} p \pmod{p^3},$$

where  $(a)_k = a(a+1) \cdots (a+k-1)$  is the rising factorial for  $a \in \mathbb{C}$  and  $k \in \mathbb{N}$ .

The first proof of (1) was given by Mortenson [2008]. It is said to be of Ramanujan-type because it is a  $p$ -adic version of Ramanujan’s formula

$$\sum_{k=0}^{\infty} (4k+1) \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k = \frac{2}{\pi}.$$

See [Zudilin 2009] for more Ramanujan-type supercongruences.

In this short note, we will present a new proof of (1), which summarizes our strategy in proving similar types of supercongruences.

McCarthy and Osburn [2008] proved van Hamme’s conjecture [1997] that

$$\sum_{k=0}^{(p-1)/2} (4k+1) \left( \frac{(\frac{1}{2})_k}{k!} \right)^5 \equiv \begin{cases} -\frac{p}{\Gamma_p(3/4)^4} \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where  $\Gamma_p(\cdot)$  denotes the  $p$ -adic Gamma function.

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Similarly, van Hamme has conjectured that for any prime  $p > 3$ ,

$$(2) \quad \sum_{k=0}^{(p-1)/2} (6k + 1) \left( \frac{\binom{1}{2}_k}{k!} \right)^3 4^{-k} \equiv (-1)^{(p-1)/2} p \pmod{p^4}.$$

This formula is supported by numerical evidence, but as van Hamme said, “we have no real explanation for our observations”. In our exploration, it will become clear that such supercongruences are a result of extra symmetries, which we are able to interpret using hypergeometric evaluation identities. Of course, they can also be seen from other perspectives, such as the WZ method.

Meanwhile, it is known that some of the truncated hypergeometric series are related to the number of rational points on certain algebraic varieties over finite fields and further to coefficients of modular forms. For instance, based on the result of Ahlgren and Ono [2000], Kilbourn [2006] proved that

$$(3) \quad \sum_{k=0}^{(p-1)/2} \left( \frac{\binom{1}{2}_k}{k!} \right)^4 \equiv a_p \pmod{p^3},$$

where  $a_p$  is the  $p$ -th coefficient of a weight 4 modular form

$$(4) \quad \eta(2z)^4 \eta(4z)^4 := q \prod_{n \geq 1} (1 - q^{2n})^4 (1 - q^{4n})^4, \quad \text{where } q = e^{2\pi iz}.$$

This is one instance of the supercongruences conjectured by Rodriguez-Villegas [2003], which relate special truncated hypergeometric series values and coefficients of Heck eigenforms. McCarthy [2009] proved another supercongruence of this type and his approach provides a general combinatorial framework for all these congruences.

We will establish a few supercongruences mainly via hypergeometric evaluation identities and combinatorics. Since there exist many amazing hypergeometric evaluation identities in the literature, we expect that our approach can be used to prove other interesting congruences.

Here is a summary of our results.

**Theorem 1.1.** *Let  $p > 3$  be a prime and  $r$  be a positive integer. Then*

$$\sum_{k=0}^{(p^r-1)/2} (4k + 1) \left( \frac{\binom{1}{2}_k}{k!} \right)^4 \equiv p^r \pmod{p^{3+r}}.$$

**Theorem 1.2.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} (4k + 1) \left( \frac{\binom{1}{2}_k}{k!} \right)^6 \equiv p \cdot a_p \pmod{p^4}.$$

**Conjecture 1.3.** *Let  $p > 3$  be a prime and  $r$  be a positive integer. Then*

$$\sum_{k=0}^{(p^r-1)/2} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!}\right)^6 \equiv p^r \cdot a_{p^r} \pmod{p^{3+r}},$$

where  $a_{p^r}$  is the  $p^r$ -th coefficient of (4).

**Theorem 1.4.** *Van Hamme’s conjecture (2) is true.*

**Theorem 1.5.** *Let  $p > 3$  be a prime. Then*

$$(5) \quad \sum_{k=0}^{(p-1)/2} (6k+1) \left(\frac{(\frac{1}{2})_k}{k!}\right)^3 \frac{(-1)^k}{8^k} \equiv (-1)^{(p^2-1)/8+(p-1)/2} p \pmod{p^2}.$$

### 2. Preliminaries

**Hypergeometric series.** For any positive integer  $r$ ,

$${}_{r+1}F_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1}; z \\ b_1, \dots, b_r \end{matrix} \right] = \sum_{k \geq 0} \frac{(a_1)_k \cdots (a_{r+1})_k}{k!(b_1)_k \cdots (b_r)_k} z^k,$$

where  $(a)_k$  is the rising factorial and  $z \in \mathbb{C}$ . A hypergeometric series terminates if it is well-defined and at least one of the  $a_i$  is a negative integer. We will make use of this fact to produce various truncated hypergeometric series.

By the definition of the rising factorial,

$$(6) \quad \frac{(\frac{1}{2})_k}{k!} = 2^{-2k} \binom{2k}{k}.$$

**Gamma function.** Let  $\Gamma(x)$  denote the usual Gamma function, which is defined for all  $x \in \mathbb{C}$  except for the nonpositive integers. It satisfies some well known properties, such as  $\Gamma(x+1) = x\Gamma(x)$ . Thus,  $(a)_k = \Gamma(a+k)/\Gamma(a)$  when  $\Gamma(a) \neq 0$  and  $\Gamma(a+k)$  are defined.

Another formula we need is Euler’s reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

**Some combinatorics.** We gather here some results in combinatorics to be used later. It is the author’s pleasure to acknowledge that the approaches used in (7)–(10) are due to Zudilin. Here is a key idea of Zudilin for rising factorials; see also [Chan et al. 2010, Lemma 1]:

$$(7) \quad \begin{aligned} \left(\frac{1}{2} + \varepsilon\right)_k &= \left(\frac{1}{2} + \varepsilon\right)\left(\frac{1}{2} + \varepsilon + 1\right) \cdots \left(\frac{1}{2} + \varepsilon + k - 1\right) \\ &= \left(\frac{1}{2}\right)_k \left(1 + 2\varepsilon \sum_{j=1}^k \frac{1}{2j-1} + 4\varepsilon^2 \sum_{1 \leq i < j \leq k} \frac{1}{(2i-1)(2j-1)} + O(\varepsilon^3)\right). \end{aligned}$$

Hence,  $(\frac{1}{2} + \varepsilon)_k(\frac{1}{2} - \varepsilon)_k$  can be expanded as a power series of  $\varepsilon^2$  as

$$(8) \quad (\frac{1}{2} + \varepsilon)_k(\frac{1}{2} - \varepsilon)_k = (\frac{1}{2})_k^2 \left( 1 - 4\varepsilon^2 \sum_{j=1}^k \frac{1}{(2j-1)^2} + O(\varepsilon^4) \right).$$

Similarly,

$$(9) \quad (1 + \varepsilon)_k(1 - \varepsilon)_k = (1)_k^2 \left( 1 - \varepsilon^2 \sum_{j=1}^k \frac{1}{j^2} + O(\varepsilon^4) \right).$$

Letting  $\varepsilon = -p^r/2$  and  $\varepsilon = p^r/2$  respectively in (7) and taking  $k$  to be an integer between 1 and  $(p^r - 1)/2$ , we obtain

$$(-1)^k \binom{(p^r - 1)/2}{k} \equiv \frac{(\frac{1}{2})_k}{k!} \pmod{p} \quad \text{and} \quad \binom{(p^r - 1)/2 + k}{k} \equiv \frac{(\frac{1}{2})_k}{k!} \pmod{p}.$$

Similarly, letting  $\varepsilon = p^r/2$  in (8) and  $k$  be an integer between 1 and  $(p^r - 1)/2$ , we have

$$(-1)^k \binom{(p^r - 1)/2}{k} \binom{(p^r - 1)/2 + k}{k} \equiv \left( \frac{(\frac{1}{2})_k}{k!} \right)^2 \pmod{p^2}.$$

**Lemma 2.1.** *For any positive integer  $n > 1$ ,*

$$(10) \quad (2n + 1) \sum_{k=0}^n \frac{1}{2k+1} \binom{n}{k} \binom{n+k}{k} (-1)^k = 1.$$

*Proof.* We use the partial fraction decomposition

$$\frac{(t - 1)(t - 2) \cdots (t - n)}{t(t + 1) \cdots (t + n)} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{1}{t+k}.$$

Letting  $t = 1/2$ , this becomes

$$(-1)^n \frac{2}{2n+1} = 2 \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{1}{1+2k},$$

which is equivalent to the claim of the lemma. □

**Lemma 2.2.** *Let  $n$  be an odd positive integer. Then*

$$\frac{(\frac{3}{2} - \frac{1}{4}n)_{(n-1)/2} (1 - \frac{1}{2}n)_{(n-1)/2}}{(2 - \frac{1}{2}n)_{(n-1)/2} (1 - \frac{1}{4}n)_{(n-1)/2}} = (-1)^{(n-1)/2} n.$$

*Proof.* Using  $(a)_k = \Gamma(a+k)/\Gamma(a)$ , we have

$$\begin{aligned} & \frac{(\frac{3}{2} - \frac{1}{4}n)_{(n-1)/2} (1 - \frac{1}{2}n)_{(n-1)/2}}{(2 - \frac{1}{2}n)_{(n-1)/2} (1 - \frac{1}{4}n)_{(n-1)/2}} \\ &= \frac{\Gamma(\frac{3}{2} - \frac{1}{4}n + \frac{1}{2}(n-1))\Gamma(\frac{1}{2})\Gamma(2 - \frac{1}{2}n)\Gamma(1 - \frac{1}{4}n)}{\Gamma(\frac{3}{2} - \frac{1}{4}n)\Gamma(1 - \frac{1}{2}n)\Gamma(\frac{3}{2})\Gamma(1 - \frac{1}{4}n + \frac{1}{2}(n-1))} \\ &= \frac{(1 - \frac{1}{2}n)}{\frac{1}{2}} \frac{\frac{1}{4}n \cdot \Gamma(\frac{1}{4}n)\Gamma(1 - \frac{1}{4}n)}{(\frac{1}{2} - \frac{1}{4}n) \cdot \Gamma(\frac{1}{2} + \frac{n}{4})\Gamma(\frac{1}{2} - \frac{1}{4}n)} \\ &= n \cdot \frac{\sin(\pi/2 - \pi n/4)}{\sin(\pi n/4)} = n \cdot \cot(\pi n/4) = (-1)^{(n-1)/2} n. \quad \square \end{aligned}$$

**Lemma 2.3.** *Let  $n$  be an odd integer. Then*

$$\frac{(\frac{3}{2} - \frac{1}{4}n)_{(n-1)/2} 2^{(n-1)/2}}{(2 - \frac{1}{2}n)_{(n-1)/2}} = (-1)^{(n^2-1)/8+(n-1)/2} n.$$

*Proof.* We have

$$\frac{(\frac{3}{2} - \frac{1}{4}n)_{(n-1)/2} 2^{(n-1)/2}}{(2 - \frac{1}{2}n)_{(n-1)/2}} = \frac{(3 - \frac{1}{2}n)(5 - \frac{1}{2}n) \cdots \frac{n}{2}}{(2 - \frac{1}{2}n)(3 - \frac{1}{2}n) \cdots \frac{1}{2}} = \text{sgn} \cdot n,$$

where  $\text{sgn} = (-1)^\#$  and  $\#$  is the number of negative terms appearing in the fraction above. It is easy to see that

$$\# = \lfloor \frac{1}{2}(\frac{1}{2}n + 1) \rfloor + \lfloor \frac{1}{2}n \rfloor - 2 \equiv \frac{1}{8}(n^2 - 1) + \frac{1}{2}(n - 1) \pmod{2}. \quad \square$$

**Lemma 2.4** [Cai 2002]. *For any prime  $p > 3$  and positive integer  $r$ ,*

$$(11) \quad (-1)^{(p^r-1)/2} \binom{p^r - 1}{\frac{1}{2}(p^r - 1)} \equiv \left( \frac{(\frac{1}{2})^{(p^r-1)/2}}{(\frac{1}{2}(p^r - 1))!} \right)^2 \pmod{p^3}.$$

Using (6), the congruence (11) is equivalent to

$$\binom{p^r - 1}{\frac{1}{2}(p^r - 1)} \equiv (-1)^{(p^r-1)/2} 2^{2(p^r-1)} \pmod{p^3}.$$

When  $r = 1$ , this was proved in [Morley 1895].

**A generalized harmonic sum.** Let  $H_k^{(2)} := \sum_{j=1}^k \frac{1}{j^2}$ .

**Lemma 2.5** [Morley 1895]. *Let  $p > 3$  be a prime. We have*

$$H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{j=1}^{(p-1)/2} \frac{1}{(2j-1)^2} \equiv 0 \pmod{p}.$$

Using arguments in [Morley 1895] or elementary congruence, it is easy to see the following lemma holds.

**Lemma 2.6.** *Let  $p > 3$  be a prime. Then for every integer  $k$  between 1 and  $p - 2$ ,*

$$H_k^{(2)} + H_{p-1-k}^{(2)} \equiv 0 \pmod{p}.$$

**Lemma 2.7.** *Let  $p > 3$  be a prime and  $s$  be a positive integer. Then*

$$\sum_{k=0}^{(p-1)/2} \left( \frac{\binom{1}{2}k}{k!} \right)^{2s} \cdot H_{2k}^{(2)} \equiv 0 \pmod{p}.$$

*Proof.* Using the fact that

$$(-1)^k \binom{\frac{1}{2}(p-1)}{k} \equiv \frac{\binom{1}{2}k}{k!} \pmod{p},$$

we have

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \left( \frac{\binom{1}{2}k}{k!} \right)^{2s} H_{2k}^{(2)} &\equiv \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{k}^{2s} H_{2k}^{(2)} \pmod{p} \\ &= \frac{1}{2} \left( \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{k}^{2s} H_{2k}^{(2)} + \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{\frac{1}{2}(p-1)-k}^{2s} H_{p-1-2k}^{(2)} \right) \\ &= \frac{1}{2} \left( \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{k}^{2s} (H_{2k}^{(2)} + H_{p-1-2k}^{(2)}) \right) \\ &\equiv 0 \pmod{p}. \end{aligned} \quad \square$$

**2.1. An elementary  $p$ -adic analysis.** Let  $F(x_1, \dots, x_t; z)$  be a  $(t + 1)$ -variable formal power series. For instance, it could be a scalar multiple of a terminating hypergeometric series as follows:

$$C \cdot {}_{r+1}F_r \left[ \begin{matrix} a_1, a_2, \dots, a_r, -n; z \\ b_1, \dots, b_{r-1}, b_r \end{matrix} \right].$$

Assume that by specifying values  $x_i = a_i$  for  $i = 1, \dots, t$  and  $z = z_0$ , we have

$$F(a_1, \dots, a_t; z_0) \in \mathbb{Z}_p.$$

Now we fix  $z_0$  and deform the parameters  $a_i$  into polynomials  $a_i(x) \in \mathbb{Z}_p[x]$  such that  $a_i(0) = a_i$  for all  $1 \leq i \leq t$ , and assume that the resulting function  $F(a_1(x), \dots, a_t(x); z_0)$  is a formal power series in  $x^2$  with coefficients in  $\mathbb{Z}_p$ , that is,  $F(a_1(x), \dots, a_t(x); z_0) = A_0 + A_2x^2 + A_4x^4 + \dots$  for  $A_i \in \mathbb{Z}_p$ , where  $A_0 = F(a_1, \dots, a_t; z_0)$ .

**Lemma 2.8.** *Under the setting above, if  $p^s \mid A_2$  for  $s = 1, 2$ , then*

$$F(a_1(p), \dots, a_t(p); z_0) \equiv A_0 \pmod{p^{2+s}}.$$

### 3. A new proof of (1)

We briefly outline our method for proving the next few supercongruences; we are motivated by [McCarthy and Osburn 2008] and [Mortenson 2008]. To each congruence, we first identify a corresponding hypergeometric evaluation identity, which with specified parameters is congruent to a target truncated hypergeometric series evaluation up to some power of  $p$ . Usually the power of  $p$  so obtained is weaker than the conjectural exponent. In our cases, we reduce the optimal congruences to some congruence combinatorial identities, which are established using additional hypergeometric evaluation identities or combinatorics.

Our strategy can be best implemented in the following new proof of (1). An identity of Whipple [1926, (5.1)] says

$${}_4F_3 \left[ \begin{matrix} a, 1+a/2, & c, & d; & -1 \\ a/2, & 1+a-c, & 1+a-d \end{matrix} \right] = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}.$$

Letting  $a = \frac{1}{2}$ ,  $c = \frac{1}{2} + \frac{1}{2}p$  and  $d = \frac{1}{2} - \frac{1}{2}p$ , we conclude immediately that

$$\sum_{k=0}^{(p-1)/2} (4k+1) \left( \frac{\binom{1}{2}k}{k!} \right)^3 (-1)^k \equiv \frac{\Gamma(1-\frac{1}{2}p)\Gamma(1+\frac{1}{2}p)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = (-1)^{(p-1)/2} p \pmod{p^2}.$$

To achieve the congruence modulo  $p^3$ , we consider the expansion of the terminating hypergeometric series (it terminates since  $(1-p)/2$  is a negative integer)

$$\begin{aligned} (12) \quad & {}_4F_3 \left[ \begin{matrix} \frac{1}{2}(1-p), \frac{5}{4}, \frac{1}{2}(1-x), \frac{1}{2}(1+x); -1 \\ \frac{1}{4}, 1+\frac{1}{2}x, 1-\frac{1}{2}x \end{matrix} \right] \\ &= \sum_{k=0}^{(p-1)/2} (4k+1) \left( \frac{\binom{1}{2}k}{k!} \right)^3 (-1)^k + A_2x^2 + \dots \quad \text{for some } A_2 \in \mathbb{Z}_p. \end{aligned}$$

By Lemma 2.8, if  $p \mid A_2$ , we are done. Now we follow Mortenson [2008] by using another hypergeometric evaluation identity, which is a specialization of Whipple’s  ${}_7F_6$  formula (see [Bailey 1935, page 28]):

$$\begin{aligned} & {}_6F_5 \left[ \begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d, & e; & -1 \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} \right] \\ &= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)} {}_3F_2 \left[ \begin{matrix} 1+a-b-c, & d, & e; & 1 \\ & 1+a-b, & 1+a-c \end{matrix} \right]. \end{aligned}$$



Letting  $a = \frac{1}{2}$ ,  $b = \frac{1-x}{2}$ ,  $c = \frac{1}{2}(1+x)$ ,  $e = \frac{1}{2}(1-p)$  and  $d = 1$ , we have

$$(13) \quad {}_6F_5 \left[ \begin{matrix} \frac{1}{2}, \frac{5}{4}, \frac{1}{2}(1-x), \frac{1}{2}(1+x), \frac{1}{2}(1-p), & 1; & -1 \\ \frac{1}{4}, 1 + \frac{1}{2}x, 1 - \frac{1}{2}x, & \frac{1}{2}, & 1 + \frac{1}{2}p \end{matrix} \right] \\ = \frac{\Gamma(\frac{1}{2})\Gamma(1 + \frac{1}{2}p)}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}p)} {}_3F_2 \left[ \begin{matrix} \frac{1}{2}, & 1, & \frac{1}{2} - \frac{1}{2}p; 1 \\ & 1 + \frac{1}{2}x, & 1 - \frac{1}{2}x \end{matrix} \right].$$

Since  $\Gamma(\frac{1}{2})\Gamma(1 + \frac{1}{2}p)/(\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}p)) = p$ , every  $x$ -coefficient above is in  $p\mathbb{Z}_p$ . Moreover, modulo  $p$  the left side of (12) is congruent to that of (13). So when we expand the left side of (12) in terms of  $x$ , the coefficients are all in  $p\mathbb{Z}_p$ . In particular,  $p|A_2$  and this concludes the proof of (1).

**4. Proofs of Theorems 1.1, 1.2, 1.4, and 1.5**

Whipple [1926, (7.7)] proved that

$$(14) \quad {}_7F_6 \left[ \begin{matrix} a, 1 + \frac{1}{2}a, & c, & d, & e, & f, & g; & 1 \\ \frac{1}{2}a, & 1 + a - c, & 1 + a - d, & 1 + a - e, & 1 + a - f & 1 + a - g; \end{matrix} \right] \\ = \frac{\Gamma(1 + a - e)\Gamma(1 + a - f)\Gamma(1 + a - g)\Gamma(1 + a - e - f - g)}{\Gamma(1 + a)\Gamma(1 + a - f - g)\Gamma(1 + a - e - f)\Gamma(1 + a - e - f)} \\ \times {}_4F_3 \left[ \begin{matrix} 1 + a - c - d, & e, & f, & g; & 1 \\ & e + f + g - a, & 1 + a - c, & 1 + a - d \end{matrix} \right],$$

provided the  ${}_4F_3$  is a terminating series.

*Proof of Theorem 1.1.* Let  $r$  be a positive integer and  $p > 3$  a prime. In (14), we let

$$a = \frac{1}{2}, \quad c = \frac{1}{2} + i\frac{1}{2}p^r, \quad d = \frac{1}{2} - i\frac{1}{2}p^r, \quad e = \frac{1}{2} + \frac{1}{2}p^r, \quad f = \frac{1}{2} - \frac{1}{2}p^r, \quad g = 1,$$

where  $i = \sqrt{-1}$ . Then following McCarthy and Osburn’s argument, we know the left side of (14) is congruent to

$$\sum_{k=0}^{(p^r-1)/2} (4k + 1) \left( \frac{\binom{1}{2}k}{k!} \right)^4 \pmod{p^{4r}}$$

and the right side of (14) equals

$$\frac{\Gamma(1 - \frac{1}{2}p^r)\Gamma(1 + \frac{1}{2}p^r)\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}p^r)\Gamma(\frac{1}{2}p^r)} {}_4F_3 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} + \frac{1}{2}p^r, \frac{1}{2} - \frac{1}{2}p^r, & 1; & 1 \\ \frac{3}{2}, & 1 - i\frac{1}{2}p^r, & 1 + i\frac{1}{2}p^r \end{matrix} \right].$$

Since

$$\frac{\Gamma(1 - \frac{1}{2}p^r)\Gamma(1 + \frac{1}{2}p^r)\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}p^r)\Gamma(\frac{p^r}{2})} = p^{2r},$$

it suffices to prove

$$p^r \cdot \sum_{k=0}^{(p^r-1)/2} \frac{1}{2k+1} \left( \frac{\binom{1}{2}k}{k!} \right)^2 \equiv 1 \pmod{p^3} \quad \text{for } p > 3.$$

Recall that Lemma 2.1 says for any odd integer  $n > 1$ ,

$$(2n+1) \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} \binom{n+k}{k} = 1.$$

Therefore, combining this identity, congruence (8), and Lemma 2.4, we have

$$\begin{aligned} p^r \cdot \sum_{k=0}^{(p^r-1)/2} \frac{1}{2k+1} \left( \frac{\binom{1}{2}k}{k!} \right)^2 &= p^r \cdot \sum_{k=0}^{(p^r-1)/2-1} \frac{1}{2k+1} \left( \frac{\binom{1}{2}k}{k!} \right)^2 + \left( \frac{\binom{1}{2}^{(p^r-1)/2}}{\left(\frac{1}{2}(p^r-1)\right)!} \right)^2 \\ &\equiv p^r \cdot \sum_{k=0}^{(p^r-1)/2-1} \frac{(-1)^k}{2k+1} \binom{\frac{1}{2}(p^r-1)}{k} \binom{\frac{1}{2}(p^r-1)+k}{k} \\ &\quad + (-1)^{(p^r-1)/2} \binom{p^r-1}{\frac{1}{2}(p^r-1)} \pmod{p^3} \\ &\equiv 1 \pmod{p^3}. \end{aligned} \quad \square$$

*Proof of Theorem 1.2.* In (14), take

$$a = \frac{1}{2}, \quad c = \frac{1}{2} + i\frac{1}{2}p, \quad d = \frac{1}{2} - i\frac{1}{2}p, \quad e = \frac{1}{2} - \frac{1}{2}p, \quad f = \frac{1}{2} + \frac{1}{2}p, \quad g = \frac{1}{2} - p^4.$$

Then the left side of (14) is congruent to

$$\sum_{k=0}^{(p-1)/2} (4k+1) \left( \frac{\binom{1}{2}k}{k!} \right)^6 \pmod{p^4}.$$

Meanwhile, the right side of (14) is congruent to

$$\begin{aligned} \frac{\Gamma(1 - \frac{1}{2}p)\Gamma(1 + \frac{1}{2}p)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \frac{\Gamma(1 + p^4)\Gamma(p^4)}{\Gamma(\frac{1}{2} + \frac{1}{2}p + p^4)\Gamma(\frac{1}{2} - \frac{1}{2}p + p^4)} \\ \times \sum_{k=0}^{(p-1)/2} \frac{\binom{1}{2}k^2 \binom{1}{2}k \binom{1}{2}p \binom{1}{2}k \binom{1}{2}p}{k!^2 (1 - i\frac{1}{2}p)_k (1 + i\frac{1}{2}p)_k} \pmod{p^4}, \end{aligned}$$

where

$$\frac{\Gamma(1 - \frac{1}{2}p)\Gamma(1 + \frac{1}{2}p)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = (-1)^{(p-1)/2} p$$

and

$$\begin{aligned} \frac{\Gamma(1+p^4)\Gamma(p^4)}{\Gamma(\frac{1}{2}+\frac{1}{2}p+p^4)\Gamma(\frac{1}{2}-\frac{1}{2}p+p^4)} &= \frac{(p^4-\frac{1}{2}(p-1))_{(p-1)/2}}{(1+p^4)_{(p-1)/2}} \\ &\equiv \frac{(-\frac{1}{2}(p-1))(-\frac{1}{2}(p-1)+1)\cdots(-1)}{1\cdot 2\cdots(\frac{1}{2}(p-1))} \pmod p = (-1)^{(p-1)/2}. \end{aligned}$$

Therefore, Theorem 1.2 follows from the result of Kilbourn (see (3)) and the next lemma. □

**Lemma 4.1.** *Let  $p > 3$  be a prime, then*

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^2(\frac{1}{2}+\frac{1}{2}p)_k(\frac{1}{2}-\frac{1}{2}p)_k}{k!^2(1-i\frac{1}{2}p)_k(1+i\frac{1}{2}p)_k} \equiv \sum_{k=0}^{(p-1)/2} \left(\frac{(\frac{1}{2})_k}{k!}\right)^4 \pmod{p^3}.$$

*Proof.* Expand

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^2(\frac{1}{2}+\frac{1}{2}x)_k(\frac{1}{2}-\frac{1}{2}x)_k}{k!^2(1-i\frac{1}{2}x)_k(1+i\frac{1}{2}x)_k} = \sum_{k=0}^{(p-1)/2} \left(\frac{(\frac{1}{2})_k}{k!}\right)^4 (1+b_{2,k}x^2+b_{4,k}x^4+\cdots).$$

Using (8) and (9), we have

$$b_{2,k} = -\sum_{j=1}^k \frac{1}{(2j-1)^2} - \frac{1}{4} \sum_{j=1}^k \frac{1}{j^2} = -\sum_{j=1}^{2k} \frac{1}{j^2}.$$

The claim is verified by using Lemma 2.8 and taking  $s = 2$  in Lemma 2.7. □

*Proof of Theorem 1.4.* We start with the following combinatorial identity.

**Lemma 4.2.** 
$$\sum_{k=0}^{(p-1)/2} (6k+1) \frac{(\frac{1}{2})_k(\frac{1}{2}-\frac{1}{2}p)_k(\frac{1}{2}+\frac{1}{2}p)_k}{(1)_k(1+\frac{1}{4}p)_k(1-\frac{1}{4}p)_k} \frac{1}{4^k} = (-1)^{(p-1)/2} p.$$

*Proof.* Recall that [Gessel 1995, (31.1)] says

$$\begin{aligned} {}_5F_4 \left[ \begin{matrix} \frac{1}{2}+a-c, & -n, & n+1, & 2-2c+n, & \frac{5}{3}-\frac{2}{3}c+\frac{1}{3}n; & \frac{1}{4} \\ & 2-c+n, & \frac{2}{3}-\frac{2}{3}c+\frac{1}{3}n, & n-2a+2, & \frac{3}{2}-c \end{matrix} \right] \\ = \frac{(2-c)_n(2-2a)_n}{(3-2c)_n(\frac{3}{2}-a)_n}. \end{aligned}$$

Letting  $a = \frac{1}{2} + \frac{1}{4}p$ ,  $c = \frac{1}{2} + \frac{1}{4}p$ , and  $n = \frac{1}{2}(p-1)$  and using Lemma 2.2, we have

$$\begin{aligned} {}_5F_4 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{7}{6}, \frac{1}{2}-\frac{1}{2}p, \frac{1}{2}+\frac{1}{2}p; & \frac{1}{4} \\ & \frac{1}{2}, \frac{1}{6}, 1-\frac{1}{4}p, 1+\frac{1}{4}p \end{matrix} \right] &= \frac{(\frac{3}{2}-\frac{1}{4}p)_{(p-1)/2}(1-\frac{1}{2}p)_{(p-1)/2}}{(2-\frac{1}{2}p)_{(p-1)/2}(1-\frac{1}{4}p)_{(p-1)/2}} \\ &= (-1)^{(p-1)/2} p. \end{aligned} \quad \square$$

**Lemma 4.3.** *The function*

$$\left( \sum_{k=0}^{(p-1)/2} (6k+1) \frac{(\frac{1}{2})_k (\frac{1}{2} - \frac{1}{2}x)_k (\frac{1}{2} + \frac{1}{2}x)_k}{(1)_k (1 + \frac{1}{4}x)_k (1 - \frac{1}{4}x)_k} \frac{1}{4^k} \right) / \left( \sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k} \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 \right)$$

is a formal power series in  $x^2$  with coefficients in  $\mathbb{Z}_p$ . Its  $x^2$  coefficient is zero modulo  $p$ .

*Proof.* We use the *strange* valuation of Gosper:

$${}_5F_4 \left[ \begin{matrix} 2a, & 2b, & 1-2b, & 1+\frac{2}{3}a, & -n; & \frac{1}{4} \\ a+b-1, & a+b+\frac{1}{2}, & \frac{2}{3}a, & 1+2a+2n \end{matrix} \right] = \frac{(a+\frac{1}{2})_n (a+1)_n}{(a+b+\frac{1}{2})_n (a-b+1)_n}.$$

See [Gessel and Stanton 1982, (1.2)]. Let  $a = \frac{1}{4}$ ,  $b = \frac{1}{4} - \frac{1}{4}x$  and  $n = \frac{1}{2}(p-1)$ . Then the left side of the above equals

$$(15) \quad {}_5F_4 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2} - \frac{1}{2}p; & \frac{1}{4} \\ \frac{1}{2} + p, & \frac{1}{6}, & 1 - \frac{1}{4}x, & 1 + \frac{1}{4}x \end{matrix} \right] = \frac{(\frac{3}{4})_{(p-1)/2} (\frac{5}{4})_{(p-1)/2}}{(1 - \frac{1}{4}x)_{(p-1)/2} (1 + \frac{1}{4}x)_{(p-1)/2}}.$$

We remark that

$$(16) \quad {}_5F_4 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2} - \frac{1}{2}p; & \frac{1}{4} \\ \frac{1}{2} + p, & \frac{1}{6}, & 1 - \frac{1}{4}x, & 1 + \frac{1}{4}x \end{matrix} \right] \equiv \sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k} \frac{(\frac{1}{2})_k (\frac{1}{2} - \frac{1}{2}x)_k (\frac{1}{2} + \frac{1}{2}x)_k}{(1)_k (1 + \frac{1}{4}x)_k (1 - \frac{1}{4}x)_k} \pmod{p}.$$

When  $x = 0$ , the right hand side of (15) equals  $(\frac{3}{4})_{(p-1)/2} (\frac{5}{4})_{(p-1)/2} / (1)_{(p-1)/2}^2$ , which is in  $p\mathbb{Z}_p$ . In fact, if  $p \equiv 1 \pmod{4}$  then  $\frac{5}{4} + \frac{1}{4}(p-1) - 1 = \frac{1}{4}p$ , and if  $p \equiv 3 \pmod{4}$ , then  $\frac{3}{4} + \frac{1}{4}(p-3) = \frac{1}{4}p$ , while  $(1)_{(p-1)/2}$  is a  $p$ -adic unit. It is not difficult to see that  $p$  divides  $((3/4)_{(p-1)/2} (\frac{5}{4})_{(p-1)/2} / (1)_{(p-1)/2}^2)$  exactly. Consequently, if we expand

$${}_5F_4 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2} - \frac{1}{2}p; & \frac{1}{4} \\ \frac{1}{2} + p, & \frac{1}{6}, & 1 - \frac{1}{4}x, & 1 + \frac{1}{4}x \end{matrix} \right]$$

in terms of formal power series of  $x$  (in fact,  $x^2$ ), each coefficient is in  $p\mathbb{Z}_p$ . Thus the coefficients of the right side of (16), including the coefficient of  $x^2$ , are all divisible by  $p$ . By Lemmas 2.8 and 4.2,

$$\sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k} \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 \equiv (-1)^{(p-1)/2} p \pmod{p^3}.$$

Namely,

$$\sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k} \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 = (-1)^{(p-1)/2} p + ap^3 \quad \text{for some } a \in \mathbb{Z}_p.$$

The statement of Theorem 1.4 is equivalent to  $a \in p\mathbb{Z}_p$ .

The quotient

$$(17) \quad \left( \sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k} \frac{(\frac{1}{2})_k (\frac{1}{2} - \frac{1}{2}x)_k (\frac{1}{2} + \frac{1}{2}x)_k}{(1)_k (1 + \frac{1}{4}x)_k (1 - \frac{1}{4}x)_k} \right) / \left( \sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k} \frac{(\frac{1}{2})_k}{(1)_k} \right)$$

is a formal power series in  $x^2$  with  $p$ -integral coefficients, since the denominators are divisible by  $p$  exactly. The same conclusion applies to

$$\begin{aligned} & {}_5F_4 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x, \frac{7}{6}, \frac{1}{2} - \frac{1}{2}p; \frac{1}{4} \end{matrix} \middle/ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6}, \frac{1}{2} - \frac{1}{2}p; \frac{1}{4} \end{matrix} \right] \\ &= {}_5F_4 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x, \frac{7}{6}, \frac{1}{2} - \frac{1}{2}p; \frac{1}{4} \end{matrix} \middle/ \begin{matrix} \frac{1}{2} + p, \frac{1}{6}, 1 - \frac{1}{4}x, 1 + \frac{1}{4}x \end{matrix} \right] \\ &= \frac{(1)_{(p-1)/2}^2}{(1 - \frac{1}{4}x)_{(p-1)/2} (1 + \frac{1}{4}x)_{(p-1)/2}}. \end{aligned}$$

On the other hand, by (9), the  $x^2$  coefficient of

$$\frac{(1)_{(p-1)/2}^2}{(1 - \frac{1}{4}x)_{(p-1)/2} (1 + \frac{1}{4}x)_{(p-1)/2}}$$

is a scalar multiple of  $H_{(p-1)/2}^{(2)}$ , which is in  $p\mathbb{Z}_p$  by Lemma 2.5; so is the  $x^2$  coefficient of (17). □

By Lemma 2.8 and the analysis above,

$$\frac{(-1)^{(p-1)/2} p}{(-1)^{(p-1)/2} p + ap^3} = \frac{(-1)^{(p-1)/2}}{(-1)^{(p-1)/2} + ap^2} \equiv 1 \pmod{p^3};$$

hence  $a \in p\mathbb{Z}_p$ , which concludes the proof of Theorem 1.4. □

**Lemma 4.4.**

$$\sum_{k=0}^{(p-1)/2} (6k+1) \frac{(\frac{1}{2})_k (\frac{1}{2} - \frac{1}{2}p)_k (\frac{1}{2} + \frac{1}{2}p)_k (-1)^k}{(1)_k (1 + \frac{1}{4}p)_k (1 - \frac{1}{4}p)_k 8^k} = (-1)^{(p^2-1)/8 + (p-1)/2} p.$$

*Proof.* This time, we use [Gessel 1995, last identity of page 544]

$${}_4F_3 \left[ \begin{matrix} 2a + n + 1, & n + 1, & \frac{2}{3}a + \frac{1}{3}n + \frac{4}{3}, & -n; & -\frac{1}{8} \\ & a + \frac{3}{2} + n, & \frac{2}{3}a + \frac{1}{3}n + \frac{1}{3}, & 1 + a \end{matrix} \right] = \frac{(a + \frac{3}{2})_n}{(2a + 2)_n} 2^n.$$

Letting  $a = -\frac{1}{4}p$  and  $n = \frac{1}{2}(p - 1)$  and using Lemma 2.3, we have

$$\begin{aligned} {}_4F_3 \left[ \begin{matrix} \frac{1}{2}, \frac{7}{6}, \frac{1}{2} + \frac{1}{2}p, & \frac{1}{2} - \frac{1}{2}p; & -\frac{1}{8} \\ & \frac{1}{6}, 1 - \frac{1}{4}p, & 1 + \frac{1}{4}p \end{matrix} \right] &= \frac{(\frac{3}{2} - \frac{1}{4}p)_{(p-1)/2}}{(2 - \frac{1}{2}p)_{(p-1)/2}} 2^{(p-1)/2} \\ &= (-1)^{(p^2-1)/8+(p-1)/2} p. \end{aligned} \quad \square$$

*Proof of Theorem 1.5.* Equation (5) is a consequence of Lemma 4.4. □

**Remark 1.** Van Hamme’s conjecture that

$$\sum_{k=0}^{(p-1)/2} (6k + 1) \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 \frac{(-1)^k}{8^k} \equiv (-1)^{(p^2-1)/8+(p-1)/2} p \pmod{p^3}$$

holds if

$$\sum_{k=0}^{(p-1)/2} (6k + 1) \left( \frac{(\frac{1}{2})_k}{k!} \right)^3 \left( \sum_{j=1}^k \frac{1}{(2j-1)^2} - \frac{1}{16} \sum_{j=1}^k \frac{1}{j^2} \right) \frac{(-1)^k}{8^k} \equiv 0 \pmod{p}.$$

The proof of the latter is left to the interested reader.

**Remark 2.** In [2009], Zudilin proved the congruence (2) modulo  $p^2$  and the congruence (5) modulo  $p$ .

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