Pacific Journal of Mathematics

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Volume 249 No. 2 February 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 249, No. 2, 2011

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We apply some hypergeometric evaluation identities, including a strange valuation of Gosper, to prove several supercongruences related to special valuations of truncated hypergeometric series. In particular, we prove a conjecture of van Hamme.

1. Introduction

In this article, we use p to denote an odd prime. Zudilin [2009] proved several Ramanujan-type supercongruences using the Wilf–Zeilberger (WZ) method. One of them, conjectured by van Hamme, says that

$$\sum_{k=0}^{(p-1)/2}(4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3(-1)^k \equiv (-1)^{(p-1)/2}p \mod p^3, \tag{1}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the rising factorial for $a\in\mathbb{C}$ and $k\in\mathbb{N}$.

The first proof of (1) was given by Mortenson [2008]. It is said to be of Ramanujan-type because it is a p-adic version of Ramanujan's formula

$$\sum_{k=0}^{\infty}(4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3(-1)^k = \frac{2}{\pi}.$$

See [Zudilin 2009] for more Ramanujan-type supercongruences.

In this short note, we will present a new proof of (1), which summarizes our strategy in proving similar types of supercongruences.

McCarthy and Osburn [2008] proved van Hamme's conjecture [1997] that

$$\sum_{k=0}^{(p-1)/2}(4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^5 \equiv \begin{cases} -\dfrac{p}{\Gamma_p(3/4)^4} & \mod p^3 \quad \text{if } p\equiv 1 \mod 4,\\ 0 & \mod p^3 \quad \text{if } p\equiv 3 \mod 4,\end{cases}$$

where $\Gamma_p(\,\cdot\,)$ denotes the p-adic Gamma function.

Supported by NSA grant H98230-08-1-0076.
MSC2000: 33C20.
Keywords: Ramanujan supercongruences, hypergeometric identities.

Similarly, van Hamme has conjectured that for any prime $p > 3$,

$$\sum_{k=0}^{(p-1)/2} (6k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 4^{-k} \equiv (-1)^{(p-1)/2} p \mod p^4. \tag{2}$$

This formula is supported by numerical evidence, but as van Hamme said, "we have no real explanation for our observations". In our exploration, it will become clear that such supercongruences are a result of extra symmetries, which we are able to interpret using hypergeometric evaluation identities. Of course, they can also be seen from other perspectives, such as the WZ method.

Meanwhile, it is known that some of the truncated hypergeometric series are related to the number of rational points on certain algebraic varieties over finite fields and further to coefficients of modular forms. For instance, based on the result of Ahlgren and Ono [2000], Kilbourn [2006] proved that

$$\sum_{k=0}^{(p-1)/2} \left(\frac{(\frac{1}{2})_k}{k!}\right)^4 \equiv a_p \mod p^3, \tag{3}$$

where a_p is the p-th coefficient of a weight 4 modular form

$$\eta(2z)^4\eta(4z)^4 := q\prod_{n\geq 1}(1-q^{2n})^4(1-q^{4n})^4, \quad \text{where } q = e^{2\pi i z}. \tag{4}$$

This is one instance of the supercongruences conjectured by Rodriguez-Villegas [2003], which relate special truncated hypergeometric series values and coefficients of Heck eigenforms. McCarthy [2009] proved another supercongruence of this type and his approach provides a general combinatorial framework for all these congruences.

We will establish a few supercongruences mainly via hypergeometric evaluation identities and combinatorics. Since there exist many amazing hypergeometric evaluation identities in the literature, we expect that our approach can be used to prove other interesting congruences.

Here is a summary of our results.

Theorem 1.1. *Let $p > 3$ be a prime and r be a positive integer. Then*

$$\sum_{k=0}^{(p^r-1)/2} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^4 \equiv p^r \mod p^{3+r}.$$

Theorem 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^6 \equiv p\cdot a_p \mod p^4.$$

Conjecture 1.3. *Let $p > 3$ be a prime and r be a positive integer. Then*

$$\sum_{k=0}^{(p^r-1)/2} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^6 \equiv p^r \cdot a_{p^r} \mod p^{3+r},$$

where a_{p^r} is the p^r-th coefficient of (4).

Theorem 1.4. *Van Hamme's conjecture (2) is true.*

Theorem 1.5. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} (6k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 \frac{(-1)^k}{8^k} \equiv (-1)^{(p^2-1)/8+(p-1)/2} p \mod p^2. \tag{5}$$

2. Preliminaries

Hypergeometric series. For any positive integer r,

$${}_{r+1}F_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1}; z \\ b_1, \ldots, b_r \end{matrix}\right] = \sum_{k\geq 0} \frac{(a_1)_k \cdots (a_{r+1})_k}{k!(b_1)_k \cdots (b_r)_k} z^k,$$

where $(a)_k$ is the rising factorial and $z \in \mathbb{C}$. A hypergeometric series terminates if it is well-defined and at least one of the a_i is a negative integer. We will make use of this fact to produce various truncated hypergeometric series.

By the definition of the rising factorial,

$$\frac{(\frac{1}{2})_k}{k!} = 2^{-2k}\binom{2k}{k}. \tag{6}$$

Gamma function. Let $\Gamma(x)$ denote the usual Gamma function, which is defined for all $x \in \mathbb{C}$ except for the nonpositive integers. It satisfies some well known properties, such as $\Gamma(x+1) = x\Gamma(x)$. Thus, $(a)_k = \Gamma(a+k)/\Gamma(a)$ when $\Gamma(a) \neq 0$ and $\Gamma(a+k)$ are defined.

Another formula we need is Euler's reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Some combinatorics. We gather here some results in combinatorics to be used later. It is the author's pleasure to acknowledge that the approaches used in (7)–(10) are due to Zudilin. Here is a key idea of Zudilin for rising factorials; see also [Chan et al. 2010, Lemma 1]:

$$\begin{aligned}\left(\tfrac{1}{2}+\varepsilon\right)_k &= (\tfrac{1}{2}+\varepsilon)(\tfrac{1}{2}+\varepsilon+1)\cdots(\tfrac{1}{2}+\varepsilon+k-1) \\ &= \left(\tfrac{1}{2}\right)_k\left(1+2\varepsilon\sum_{j=1}^{k}\frac{1}{2j-1}+4\varepsilon^2\sum_{1\leq i<j\leq k}\frac{1}{(2i-1)(2j-1)}+O(\varepsilon^3)\right).\end{aligned} \tag{7}$$

Hence, $\left(\frac{1}{2}+\varepsilon\right)_k\left(\frac{1}{2}-\varepsilon\right)_k$ can be expanded as a power series of ε^2 as

$$\left(\tfrac{1}{2}+\varepsilon\right)_k\left(\tfrac{1}{2}-\varepsilon\right)_k=\left(\tfrac{1}{2}\right)_k^2\Big(1-4\varepsilon^2\sum_{j=1}^{k}\frac{1}{(2j-1)^2}+O(\varepsilon^4)\Big). \tag{8}$$

Similarly,

$$(1+\varepsilon)_k(1-\varepsilon)_k=(1)_k^2\Big(1-\varepsilon^2\sum_{j=1}^{k}\frac{1}{j^2}+O(\varepsilon^4)\Big). \tag{9}$$

Letting $\varepsilon=-p^r/2$ and $\varepsilon=p^r/2$ respectively in (7) and taking k to be an integer between 1 and $(p^r-1)/2$, we obtain

$$(-1)^k\binom{(p^r-1)/2}{k}\equiv\frac{(\frac{1}{2})_k}{k!}\mod p\quad\text{and}\quad\binom{(p^r-1)/2+k}{k}\equiv\frac{(\frac{1}{2})_k}{k!}\mod p.$$

Similarly, letting $\varepsilon=p^r/2$ in (8) and k be an integer between 1 and $(p^r-1)/2$, we have

$$(-1)^k\binom{(p^r-1)/2}{k}\binom{(p^r-1)/2+k}{k}\equiv\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\mod p^2.$$

Lemma 2.1. *For any positive integer $n>1$,*

$$(2n+1)\sum_{k=0}^{n}\frac{1}{2k+1}\binom{n}{k}\binom{n+k}{k}(-1)^k=1. \tag{10}$$

Proof. We use the partial fraction decomposition

$$\frac{(t-1)(t-2)\cdots(t-n)}{t(t+1)\cdots(t+n)}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}\frac{1}{t+k}.$$

Letting $t=1/2$, this becomes

$$(-1)^n\frac{2}{2n+1}=2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}\frac{1}{1+2k},$$

which is equivalent to the claim of the lemma. □

Lemma 2.2. *Let n be an odd positive integer. Then*

$$\frac{(\frac{3}{2}-\frac{1}{4}n)_{(n-1)/2}(1-\frac{1}{2}n)_{(n-1)/2}}{(2-\frac{1}{2}n)_{(n-1)/2}(1-\frac{1}{4}n)_{(n-1)/2}}=(-1)^{(n-1)/2}n.$$

Proof. Using $(a)_k = \Gamma(a+k)/\Gamma(a)$, we have

$$\begin{aligned}
&\frac{(\frac{3}{2}-\frac{1}{4}n)_{(n-1)/2}(1-\frac{1}{2}n)_{(n-1)/2}}{(2-\frac{1}{2}n)_{(n-1)/2}(1-\frac{1}{4}n)_{(n-1)/2}} \\
&\qquad = \frac{\Gamma(\frac{3}{2}-\frac{1}{4}n+\frac{1}{2}(n-1))\Gamma(\frac{1}{2})\Gamma(2-\frac{1}{2}n)\Gamma(1-\frac{1}{4}n)}{\Gamma(\frac{3}{2}-\frac{1}{4}n)\Gamma(1-\frac{1}{2}n)\Gamma(\frac{3}{2})\Gamma(1-\frac{1}{4}n+\frac{1}{2}(n-1))} \\
&\qquad = \frac{(1-\frac{1}{2}n)}{\frac{1}{2}} \frac{\frac{1}{4}n\cdot\Gamma(\frac{1}{4}n)\Gamma(1-\frac{1}{4}n)}{(\frac{1}{2}-\frac{1}{4}n)\cdot\Gamma(\frac{1}{2}+\frac{n}{4})\Gamma(\frac{1}{2}-\frac{1}{4}n)} \\
&\qquad = n\cdot\frac{\sin(\pi/2-\pi n/4)}{\sin(\pi n/4)} = n\cdot\cot(\pi n/4) = (-1)^{(n-1)/2}n. \qquad \square
\end{aligned}$$

Lemma 2.3. *Let n be an odd integer. Then*

$$\frac{(\frac{3}{2}-\frac{1}{4}n)_{(n-1)/2}}{(2-\frac{1}{2}n)_{(n-1)/2}}2^{(n-1)/2} = (-1)^{(n^2-1)/8+(n-1)/2}n.$$

Proof. We have

$$\frac{(\frac{3}{2}-\frac{1}{4}n)_{(n-1)/2}}{(2-\frac{1}{2}n)_{(n-1)/2}}2^{(n-1)/2} = \frac{(3-\frac{1}{2}n)(5-\frac{1}{2}n)\cdots\frac{n}{2}}{(2-\frac{1}{2}n)(3-\frac{1}{2}n)\cdots\frac{1}{2}} = \mathrm{sgn}\cdot n,$$

where $\mathrm{sgn} = (-1)^{\#}$ and # is the number of negative terms appearing in the fraction above. It is easy to see that

$$\# = \lfloor\tfrac{1}{2}(\tfrac{1}{2}n+1)\rfloor + \lfloor\tfrac{1}{2}n\rfloor - 2 \equiv \tfrac{1}{8}(n^2-1)+\tfrac{1}{2}(n-1) \mod 2. \qquad \square$$

Lemma 2.4 [Cai 2002]. *For any prime $p>3$ and positive integer r,*

$$(-1)^{(p^r-1)/2}\binom{p^r-1}{\frac{1}{2}(p^r-1)} \equiv \left(\frac{(\frac{1}{2})_{(p^r-1)/2}}{(\frac{1}{2}(p^r-1))!}\right)^2 \mod p^3. \tag{11}$$

Using (6), the congruence (11) is equivalent to

$$\binom{p^r-1}{\frac{1}{2}(p^r-1)} \equiv (-1)^{(p^r-1)/2}2^{2(p^r-1)} \mod p^3.$$

When $r=1$, this was proved in [Morley 1895].

A generalized harmonic sum. Let $H_k^{(2)} := \sum_{j=1}^{k}\frac{1}{j^2}$.

Lemma 2.5 [Morley 1895]. *Let $p>3$ be a prime. We have*

$$H^{(2)}_{(p-1)/2} \equiv 0 \mod p \quad \textit{and} \quad \sum_{j=1}^{(p-1)/2}\frac{1}{(2j-1)^2} \equiv 0 \mod p.$$

Using arguments in [Morley 1895] or elementary congruence, it is easy to see the following lemma holds.

Lemma 2.6. *Let $p>3$ be a prime. Then for every integer k between* 1 *and* $p-2$,

$$H_k^{(2)}+H_{p-1-k}^{(2)}\equiv 0 \mod p.$$

Lemma 2.7. *Let $p>3$ be a prime and s be a positive integer. Then*

$$\sum_{k=0}^{(p-1)/2}\left(\frac{(\frac{1}{2})_k}{k!}\right)^{2s}\cdot H_{2k}^{(2)}\equiv 0 \mod p.$$

Proof. Using the fact that

$$(-1)^k\binom{\frac{1}{2}(p-1)}{k}\equiv\frac{(\frac{1}{2})_k}{k!} \mod p,$$

we have

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\left(\frac{(\frac{1}{2})_k}{k!}\right)^{2s}H_{2k}^{(2)}&\equiv\sum_{k=0}^{(p-1)/2}\binom{\frac{1}{2}(p-1)}{k}^{2s}H_{2k}^{(2)} \mod p\\&=\frac{1}{2}\left(\sum_{k=0}^{(p-1)/2}\binom{\frac{1}{2}(p-1)}{k}^{2s}H_{2k}^{(2)}+\sum_{k=0}^{(p-1)/2}\binom{\frac{1}{2}(p-1)}{\frac{1}{2}(p-1)-k}^{2s}H_{p-1-2k}^{(2)}\right)\\&=\frac{1}{2}\left(\sum_{k=0}^{(p-1)/2}\binom{\frac{1}{2}(p-1)}{k}^{2s}(H_{2k}^{(2)}+H_{p-1-2k}^{(2)})\right)\\&\equiv 0 \quad \mod p.\end{aligned}$$

□

2.1. ***An elementary p-adic analysis.*** Let $F(x_1,\dots,x_t;z)$ be a $(t+1)$-variable formal power series. For instance, it could be a scalar multiple of a terminating hypergeometric series as follows:

$$C\cdot{}_{r+1}F_r\left[\begin{matrix}a_1,a_2,\dots, & a_r, & -n;z\\ b_1,\dots,b_{r-1}, & b_r\end{matrix}\right].$$

Assume that by specifying values $x_i=a_i$ for $i=1,\dots,t$ and $z=z_0$, we have

$$F(a_1,\dots,a_t;z_0)\in\mathbb{Z}_p.$$

Now we fix z_0 and deform the parameters a_i into polynomials $a_i(x)\in\mathbb{Z}_p[x]$ such that $a_i(0)=a_i$ for all $1\le i\le t$, and assume that the resulting function $F(a_1(x),\dots,a_t(x);z_0)$ is a formal power series in x^2 with coefficients in $\mathbb{Z}_p$, that is, $F(a_1(x),\dots,a_t(x);z_0)=A_0+A_2x^2+A_4x^4+\cdots$ for $A_i\in\mathbb{Z}_p$, where $A_0=F(a_1,\dots,a_t;z_0)$.

Lemma 2.8. *Under the setting above, if $p^s \mid A_2$ for $s = 1, 2$, then*

$$F(a_1(p), \ldots, a_t(p); z_0) \equiv A_0 \mod p^{2+s}.$$

3. A new proof of (1)

We briefly outline our method for proving the next few supercongruences; we are motivated by [McCarthy and Osburn 2008] and [Mortenson 2008]. To each congruence, we first identify a corresponding hypergeometric evaluation identity, which with specified parameters is congruent to a target truncated hypergeometric series evaluation up to some power of p. Usually the power of p so obtained is weaker than the conjectural exponent. In our cases, we reduce the optimal congruences to some congruence combinatorial identities, which are established using additional hypergeometric evaluation identities or combinatorics.

Our strategy can be best implemented in the following new proof of (1). An identity of Whipple [1926, (5.1)] says

$${}_4F_3\left[\begin{matrix} a, 1+a/2, & c, & d; & -1 \\ a/2, & 1+a-c, & 1+a-d & \end{matrix}\right] = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}.$$

Letting $a = \frac{1}{2}$, $c = \frac{1}{2} + \frac{1}{2}p$ and $d = \frac{1}{2} - \frac{1}{2}p$, we conclude immediately that

$$\sum_{k=0}^{(p-1)/2} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 (-1)^k \equiv \frac{\Gamma(1-\frac{1}{2}p)\Gamma(1+\frac{1}{2}p)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = (-1)^{(p-1)/2} p \mod p^2.$$

To achieve the congruence modulo p^3, we consider the expansion of the terminating hypergeometric series (it terminates since $(1-p)/2$ is a negative integer)

$$(12)\quad {}_4F_3\left[\begin{matrix} \frac{1}{2}(1-p), \frac{5}{4}, \frac{1}{2}(1-x), \frac{1}{2}(1+x); -1 \\ \frac{1}{4}, \ 1+\frac{1}{2}x, \ \ 1-\frac{1}{2}x \end{matrix}\right]$$
$$= \sum_{k=0}^{(p-1)/2} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 (-1)^k + A_2 x^2 + \cdots \quad \text{for some } A_2 \in \mathbb{Z}_p.$$

By Lemma 2.8, if $p \mid A_2$, we are done. Now we follow Mortenson [2008] by using another hypergeometric evaluation identity, which is a specialization of Whipple's ${}_7F_6$ formula (see [Bailey 1935, page 28]):

$${}_6F_5\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d, & e; & -1 \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix}\right]$$
$$= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\, {}_3F_2\left[\begin{matrix} 1+a-b-c, & d, & e; & 1 \\ & 1+a-b, & 1+a-c & \end{matrix}\right].$$

Letting $a=\frac{1}{2},\ b=\frac{1-x}{2},\ c=\frac{1}{2}(1+x),\ e=\frac{1}{2}(1-p)$ and $d=1$, we have

$$(13)\quad {}_6F_5\left[\begin{matrix}\frac{1}{2}, \frac{5}{4}, \frac{1}{2}(1-x), \frac{1}{2}(1+x), \frac{1}{2}(1-p), & 1; & -1\\ \frac{1}{4},\ 1+\frac{1}{2}x,\ 1-\frac{1}{2}x,\ \frac{1}{2}, & 1+\frac{1}{2}p & \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+\frac{1}{2}p)}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}p)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, & 1, & \frac{1}{2}-\frac{1}{2}p;1\\ & 1+\frac{1}{2}x, & 1-\frac{1}{2}x\end{matrix}\right].$$

Since $\Gamma(\frac{1}{2})\Gamma(1+\frac{1}{2}p)/((\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}p))=p$, every x-coefficient above is in $p\mathbb{Z}_p$. Moreover, modulo p the left side of (12) is congruent to that of (13). So when we expand the left side of (12) in terms of x, the coefficients are all in $p\mathbb{Z}_p$. In particular, $p\,|\,A_2$ and this concludes the proof of (1).

4. Proofs of Theorems 1.1, 1.2, 1.4, and 1.5

Whipple [1926, (7.7)] proved that

$$(14)\quad {}_7F_6\left[\begin{matrix}a, 1+\frac{1}{2}a, & c, & d, & e, & f, & g; & 1\\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f & 1+a-g; & \end{matrix}\right]$$
$$= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-f)} \times {}_4F_3\left[\begin{matrix}1+a-c-d, & e, & f, & g; & 1\\ & e+f+g-a, & 1+a-c, & 1+a-d & \end{matrix}\right],$$

provided the ${}_4F_3$ is a terminating series.

Proof of Theorem 1.1. Let r be a positive integer and $p>3$ a prime. In (14), we let

$$a=\tfrac{1}{2},\quad c=\tfrac{1}{2}+i\tfrac{1}{2}p^r,\quad d=\tfrac{1}{2}-i\tfrac{1}{2}p^r,\quad e=\tfrac{1}{2}+\tfrac{1}{2}p^r,\quad f=\tfrac{1}{2}-\tfrac{1}{2}p^r,\quad g=1,$$

where $i=\sqrt{-1}$. Then following McCarthy and Osburn's argument, we know the left side of (14) is congruent to

$$\sum_{k=0}^{(p^r-1)/2}(4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^4 \mod p^{4r}$$

and the right side of (14) equals

$$\frac{\Gamma(1-\frac{1}{2}p^r)\Gamma(1+\frac{1}{2}p^r)\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}p^r)\Gamma(\frac{1}{2}p^r)}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}+\frac{1}{2}p^r, & \frac{1}{2}-\frac{1}{2}p^r, & 1; & 1\\ \frac{3}{2}, & 1-i\frac{1}{2}p^r, & 1+i\frac{1}{2}p^r & \end{matrix}\right].$$

Since

$$\frac{\Gamma(1-\frac{1}{2}p^r)\Gamma(1+\frac{1}{2}p^r)\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}p^r)\Gamma(\frac{p^r}{2})}=p^{2r},$$

it suffices to prove

$$p^r \cdot \sum_{k=0}^{(p^r-1)/2} \frac{1}{2k+1}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2 \equiv 1 \mod p^3 \quad \text{for } p > 3.$$

Recall that Lemma 2.1 says for any odd integer $n > 1$,

$$(2n+1)\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k}\binom{n+k}{k} = 1.$$

Therefore, combining this identity, congruence (8), and Lemma 2.4, we have

$$\begin{aligned}
p^r \cdot \sum_{k=0}^{(p^r-1)/2} \frac{1}{2k+1}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2 &= p^r \cdot \sum_{k=0}^{(p^r-1)/2-1} \frac{1}{2k+1}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2 + \left(\frac{(\frac{1}{2})_{(p^r-1)/2}}{(\frac{1}{2}(p^r-1))!}\right)^2 \\
&\equiv p^r \cdot \sum_{k=0}^{(p^r-1)/2-1} \frac{(-1)^k}{2k+1}\binom{\frac{1}{2}(p^r-1)}{k}\binom{\frac{1}{2}(p^r-1)+k}{k} \\
&\qquad + (-1)^{(p^r-1)/2}\binom{p^r-1}{\frac{1}{2}(p^r-1)} \mod p^3 \\
&\equiv 1 \mod p^3. \qquad \square
\end{aligned}$$

Proof of Theorem 1.2. In (14), take

$$a = \tfrac{1}{2}, \quad c = \tfrac{1}{2} + i\tfrac{1}{2}p, \quad d = \tfrac{1}{2} - i\tfrac{1}{2}p, \quad e = \tfrac{1}{2} - \tfrac{1}{2}p, \quad f = \tfrac{1}{2} + \tfrac{1}{2}p, \quad g = \tfrac{1}{2} - p^4.$$

Then the left side of (14) is congruent to

$$\sum_{k=0}^{(p-1)/2} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^6 \mod p^4.$$

Meanwhile, the right side of (14) is congruent to

$$\begin{aligned}
&\frac{\Gamma(1-\frac{1}{2}p)\Gamma(1+\frac{1}{2}p)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \frac{\Gamma(1+p^4)\Gamma(p^4)}{\Gamma(\frac{1}{2}+\frac{1}{2}p+p^4)\Gamma(\frac{1}{2}-\frac{1}{2}p+p^4)} \\
&\qquad \times \sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^2(\frac{1}{2}+\frac{1}{2}p)_k(\frac{1}{2}-\frac{1}{2}p)_k}{k!^2(1-i\frac{1}{2}p)_k(1+i\frac{1}{2}p)_k} \mod p^4,
\end{aligned}$$

where

$$\frac{\Gamma(1-\frac{1}{2}p)\Gamma(1+\frac{1}{2}p)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = (-1)^{(p-1)/2}p$$

and

$$\frac{\Gamma(1+p^4)\Gamma(p^4)}{\Gamma(\frac{1}{2}+\frac{1}{2}p+p^4)\Gamma(\frac{1}{2}-\frac{1}{2}p+p^4)}=\frac{(p^4-\frac{1}{2}(p-1))_{(p-1)/2}}{(1+p^4)_{(p-1)/2}}$$
$$\equiv\frac{(-\frac{1}{2}(p-1))(-\frac{1}{2}(p-1)+1)\cdots(-1)}{1\cdot 2\cdots(\frac{1}{2}(p-1))} \mod p=(-1)^{(p-1)/2}.$$

Therefore, Theorem 1.2 follows from the result of Kilbourn (see (3)) and the next lemma. □

Lemma 4.1. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^2(\frac{1}{2}+\frac{1}{2}p)_k(\frac{1}{2}-\frac{1}{2}p)_k}{k!^2(1-i\frac{1}{2}p)_k(1+i\frac{1}{2}p)_k}\equiv\sum_{k=0}^{(p-1)/2}\left(\frac{(\frac{1}{2})_k}{k!}\right)^4 \mod p^3.$$

Proof. Expand

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^2(\frac{1}{2}+\frac{1}{2}x)_k(\frac{1}{2}-\frac{1}{2}x)_k}{k!^2(1-i\frac{1}{2}x)_k(1+i\frac{1}{2}x)_k}=\sum_{k=0}^{(p-1)/2}\left(\frac{(\frac{1}{2})_k}{k!}\right)^4(1+b_{2,k}x^2+b_{4,k}x^4+\cdots).$$

Using (8) and (9), we have

$$b_{2,k}=-\sum_{j=1}^{k}\frac{1}{(2j-1)^2}-\frac{1}{4}\sum_{j=1}^{k}\frac{1}{j^2}=-\sum_{j=1}^{2k}\frac{1}{j^2}.$$

The claim is verified by using Lemma 2.8 and taking $s=2$ in Lemma 2.7. □

Proof of Theorem 1.4. We start with the following combinatorial identity.

Lemma 4.2.
$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{2}-\frac{1}{2}p)_k(\frac{1}{2}+\frac{1}{2}p)_k}{(1)_k(1+\frac{1}{4}p)_k(1-\frac{1}{4}p)_k}\frac{1}{4^k}=(-1)^{(p-1)/2}p.$$

Proof. Recall that [Gessel 1995, (31.1)] says

$${}_5F_4\left[\begin{matrix}\frac{1}{2}+a-c, & -n, & n+1, & 2-2c+n, & \frac{5}{3}-\frac{2}{3}c+\frac{1}{3}n;\frac{1}{4}\\ & 2-c+n, & \frac{2}{3}-\frac{2}{3}c+\frac{1}{3}n, & n-2a+2, & \frac{3}{2}-c\end{matrix}\right]$$
$$=\frac{(2-c)_n(2-2a)_n}{(3-2c)_n(\frac{3}{2}-a)_n}.$$

Letting $a=\frac{1}{2}+\frac{1}{4}p$, $c=\frac{1}{2}+\frac{1}{4}p$, and $n=\frac{1}{2}(p-1)$ and using Lemma 2.2, we have

$${}_5F_4\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{7}{6},\frac{1}{2}-\frac{1}{2}p,\ \frac{1}{2}+\frac{1}{2}p;\frac{1}{4}\\ \frac{1}{2},\frac{1}{6},1-\frac{1}{4}p,\ 1+\frac{1}{4}p\end{matrix}\right]=\frac{(\frac{3}{2}-\frac{1}{4}p)_{(p-1)/2}(1-\frac{1}{2}p)_{(p-1)/2}}{(2-\frac{1}{2}p)_{(p-1)/2}(1-\frac{1}{4}p)_{(p-1)/2}}$$
$$=(-1)^{(p-1)/2}p.$$ □

Lemma 4.3. *The function*

$$\left(\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_k}{(1)_k(1+\frac{1}{4}x)_k(1-\frac{1}{4}x)_k}\frac{1}{4^k}\right)\Bigg/\left(\sum_{k=0}^{(p-1)/2}\frac{6k+1}{4^k}\left(\frac{(\frac{1}{2})_k}{k!}\right)^3\right)$$

is a formal power series in x^2 with coefficients in $\mathbb{Z}_p$. Its x^2 coefficient is zero modulo p.

Proof. We use the *strange* valuation of Gosper:

$${}_5F_4\left[\begin{matrix}2a, & 2b, & 1-2b, & 1+\frac{2}{3}a, & -n; & \frac{1}{4}\\ & a+b-1, & a+b+\frac{1}{2}, & \frac{2}{3}a, & 1+2a+2n & \end{matrix}\right]=\frac{(a+\frac{1}{2})_n(a+1)_n}{(a+b+\frac{1}{2})_n(a-b+1)_n}.$$

See [Gessel and Stanton 1982, (1.2)]. Let $a=\frac{1}{4}$, $b=\frac{1}{4}-\frac{1}{4}x$ and $n=\frac{1}{2}(p-1)$. Then the left side of the above equals

(15)
$${}_5F_4\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}-\frac{1}{2}x, & \frac{1}{2}+\frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2}-\frac{1}{2}p; & \frac{1}{4}\\ & \frac{1}{2}+p, & \frac{1}{6}, & 1-\frac{1}{4}x, & 1+\frac{1}{4}x & \end{matrix}\right]=\frac{(\frac{3}{4})_{(p-1)/2}(\frac{5}{4})_{(p-1)/2}}{(1-\frac{1}{4}x)_{(p-1)/2}(1+\frac{1}{4}x)_{(p-1)/2}}.$$

We remark that

$$(16)\quad {}_5F_4\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}-\frac{1}{2}x, & \frac{1}{2}+\frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2}-\frac{1}{2}p; & \frac{1}{4}\\ & \frac{1}{2}+p, & \frac{1}{6}, & 1-\frac{1}{4}x, & 1+\frac{1}{4}x & \end{matrix}\right] \equiv \sum_{k=0}^{(p-1)/2}\frac{6k+1}{4^k}\frac{(\frac{1}{2})_k(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_k}{(1)_k(1+\frac{1}{4}x)_k(1-\frac{1}{4}x)_k} \mod p.$$

When $x=0$, the right hand side of (15) equals $(\frac{3}{4})_{(p-1)/2}(\frac{5}{4})_{(p-1)/2}/(1)^2_{(p-1)/2}$, which is in $p\mathbb{Z}_p$. In fact, if $p\equiv 1 \bmod 4$ then $\frac{5}{4}+\frac{1}{4}(p-1)-1=\frac{1}{4}p$, and if $p\equiv 3 \bmod 4$, then $\frac{3}{4}+\frac{1}{4}(p-3)=\frac{1}{4}p$, while $(1)_{(p-1)/2}$ is a p-adic unit. It is not difficult to see that p divides $((3)/4)_{(p-1)/2}(\frac{5}{4})_{(p-1)/2}/(1)^2_{(p-1)/2}$ exactly. Consequently, if we expand

$${}_5F_4\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}-\frac{1}{2}x, & \frac{1}{2}+\frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2}-\frac{1}{2}p; & \frac{1}{4}\\ & \frac{1}{2}+p, & \frac{1}{6}, & 1-\frac{1}{4}x, & 1+\frac{1}{4}x & \end{matrix}\right]$$

in terms of formal power series of x (in fact, x^2), each coefficient is in $p\mathbb{Z}_p$. Thus the coefficients of the right side of (16), including the coefficient of x^2, are all divisible by p. By Lemmas 2.8 and 4.2,

$$\sum_{k=0}^{(p-1)/2}\frac{6k+1}{4^k}\left(\frac{(\frac{1}{2})_k}{k!}\right)^3\equiv(-1)^{(p-1)/2}p \mod p^3.$$

Namely,

$$\sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k}\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 = (-1)^{(p-1)/2}p + ap^3 \quad \text{for some } a \in \mathbb{Z}_p.$$

The statement of Theorem 1.4 is equivalent to $a \in p\mathbb{Z}_p$.

The quotient

$$\left(\sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k}\frac{(\frac{1}{2})_k(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_k}{(1)_k(1+\frac{1}{4}x)_k(1-\frac{1}{4}x)_k}\right) \Big/ \left(\sum_{k=0}^{(p-1)/2} \frac{6k+1}{4^k}\frac{(\frac{1}{2})_k}{(1)_k}\right) \tag{17}$$

is a formal power series in x^2 with p-integral coefficients, since the denominators are divisible by p exactly. The same conclusion applies to

$$\begin{aligned}
&{}_5F_4\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}x, \frac{1}{2}+\frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2}-\frac{1}{2}p; \frac{1}{4}\\ \frac{1}{2}+p, & \frac{1}{6}, & 1-\frac{1}{4}x,\ 1+\frac{1}{4}x\end{matrix}\right] \Big/ {}_5F_4\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{7}{6}, & \frac{1}{2}-\frac{1}{2}p; \frac{1}{4}\\ \frac{1}{2}+p, & \frac{1}{6}, & 1-\frac{1}{4}x, & 1+\frac{1}{4}x\end{matrix}\right]\\
&= {}_5F_4\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}x, \frac{1}{2}+\frac{1}{2}x, & \frac{7}{6}, & \frac{1}{2}-\frac{1}{2}p; \frac{1}{4}\\ \frac{1}{2}+p, & \frac{1}{6}, & 1-\frac{1}{4}x,\ 1+\frac{1}{4}x\end{matrix}\right] \Big/ \left(\frac{(\frac{3}{4})_{(p-1)/2}(\frac{5}{4})_{(p-1)/2}}{(1)^2_{(p-1)/2}}\right)\\
&= \frac{(1)^2_{(p-1)/2}}{(1-\frac{1}{4}x)_{(p-1)/2}(1+\frac{1}{4}x)_{(p-1)/2}}.
\end{aligned}$$

On the other hand, by (9), the x^2 coefficient of

$$\frac{(1)^2_{(p-1)/2}}{(1-\frac{1}{4}x)_{(p-1)/2}(1+\frac{1}{4}x)_{(p-1)/2}}$$

is a scalar multiple of $H^{(2)}_{(p-1)/2}$, which is in $p\mathbb{Z}_p$ by Lemma 2.5; so is the x^2 coefficient of (17). $\square$

By Lemma 2.8 and the analysis above,

$$\frac{(-1)^{(p-1)/2}p}{(-1)^{(p-1)/2}p+ap^3} = \frac{(-1)^{(p-1)/2}}{(-1)^{(p-1)/2}+ap^2} \equiv 1 \mod p^3;$$

hence $a \in p\mathbb{Z}_p$, which concludes the proof of Theorem 1.4. $\square$

Lemma 4.4.

$$\sum_{k=0}^{(p-1)/2} (6k+1)\frac{(\frac{1}{2})_k(\frac{1}{2}-\frac{1}{2}p)_k(\frac{1}{2}+\frac{1}{2}p)_k}{(1)_k(1+\frac{1}{4}p)_k(1-\frac{1}{4}p)_k}\frac{(-1)^k}{8^k} = (-1)^{(p^2-1)/8+(p-1)/2}p.$$

Proof. This time, we use [Gessel 1995, last identity of page 544]

$$ {}_4F_3\left[\begin{matrix} 2a+n+1, & n+1, & \frac{2}{3}a+\frac{1}{3}n+\frac{4}{3}, & -n; -\frac{1}{8} \\ & a+\frac{3}{2}+n, & \frac{2}{3}a+\frac{1}{3}n+\frac{1}{3}, & 1+a \end{matrix}\right] = \frac{(a+\frac{3}{2})_n}{(2a+2)_n}2^n. $$

Letting $a=-\frac{1}{4}p$ and $n=\frac{1}{2}(p-1)$ and using Lemma 2.3, we have

$$ \begin{aligned} {}_4F_3\left[\begin{matrix} \frac{1}{2}, \frac{7}{6}, \frac{1}{2}+\frac{1}{2}p, \frac{1}{2}-\frac{1}{2}p; -\frac{1}{8} \\ \frac{1}{6}, 1-\frac{1}{4}p, \ 1+\frac{1}{4}p \end{matrix}\right] &= \frac{(\frac{3}{2}-\frac{1}{4}p)_{(p-1)/2}}{(2-\frac{1}{2}p)_{(p-1)/2}}2^{(p-1)/2} \\ &= (-1)^{(p^2-1)/8+(p-1)/2}p. \end{aligned} $$

□

Proof of Theorem 1.5. Equation (5) is a consequence of Lemma 4.4. □

Remark 1. Van Hamme's conjecture that

$$ \sum_{k=0}^{(p-1)/2}(6k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3\frac{(-1)^k}{8^k} \equiv (-1)^{(p^2-1)/8+(p-1)/2}p \mod p^3 $$

holds if

$$ \sum_{k=0}^{(p-1)/2}(6k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3\left(\sum_{j=1}^{k}\frac{1}{(2j-1)^2}-\frac{1}{16}\sum_{j=1}^{k}\frac{1}{j^2}\right)\frac{(-1)^k}{8^k} \equiv 0 \mod p. $$

The proof of the latter is left to the interested reader.

Remark 2. In [2009], Zudilin proved the congruence (2) modulo p^2 and the congruence (5) modulo p.

Acknowledgments

I would like to thank Heng Huat Chan and Wadim Zudilin for their encouragement, enlightening discussions and valuable comments. In particular, Zudilin pointed me to a few useful ideas and the reference [Gessel and Stanton 1982]. I further thank the anonymous referees for their detailed comments, including pointing out the reference [Cai 2002], on an earlier version of this article.

References

[Ahlgren and Ono 2000] S. Ahlgren and K. Ono, "A Gaussian hypergeometric series evaluation and Apéry number congruences", *J. Reine Angew. Math.* **518** (2000), 187–212. MR 2001c:11057 Zbl 0940.33002

[Bailey 1935] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics **32**, Cambridge University Press, London, 1935. MR 32 #2625 Zbl 0011.02303

[Cai 2002] T. Cai, "A congruence involving the quotients of Euler and its applications, I", *Acta Arith.* **103**:4 (2002), 313–320. MR 2003d:11007 Zbl 1008.11001

[Chan et al. 2010] H. H. Chan, L. Long, and W. Zudilin, "A supercongruence motivated by the Legendre family of elliptic curves", *Mat. Zametki* **88**:4 (2010), 621–625. In Russian; translated in *Mathematical Notes* **88**:4 (2010), 599–602.

[Gessel 1995] I. M. Gessel, "Finding identities with the WZ method", *J. Symbolic Comput.* **20**:5-6 (1995), 537–566. MR 97j:05010 Zbl 0908.33004

[Gessel and Stanton 1982] I. Gessel and D. Stanton, "Strange evaluations of hypergeometric series", *SIAM J. Math. Anal.* **13**:2 (1982), 295–308. MR 83c:33002 Zbl 0486.33003

[van Hamme 1997] L. van Hamme, "Some conjectures concerning partial sums of generalized hypergeometric series", pp. 223–236 in *p-adic functional analysis* (Nijmegen, 1996), edited by W. H. Schikhof et al., Lecture Notes in Pure and Appl. Math. **192**, Dekker, New York, 1997. MR 98k:33011 Zbl 0895.11051

[Kilbourn 2006] T. Kilbourn, "An extension of the Apéry number supercongruence", *Acta Arith.* **123**:4 (2006), 335–348. MR 2007e:11049 Zbl 1170.11008

[McCarthy 2009] D. McCarthy, "Supercongruence conjectures of Rodriguez–Villegas", preprint, 2009. arXiv 0907.5089

[McCarthy and Osburn 2008] D. McCarthy and R. Osburn, "A p-adic analogue of a formula of Ramanujan", *Arch. Math.* (*Basel*) **91**:6 (2008), 492–504. MR 2009k:11191 Zbl 1175.33004

[Morley 1895] F. Morley, "Note on the congruence $2^{4n} \equiv (-1)^n (2n)!/(n!)^2$, where $2n+1$ is prime", *Annals of Math.* **9** (1895), 168–170.

[Mortenson 2008] E. Mortenson, "A p-adic supercongruence conjecture of van Hamme", *Proc. Amer. Math. Soc.* **136**:12 (2008), 4321–4328. MR 2010g:11203 Zbl 1171.11061

[Rodriguez-Villegas 2003] F. Rodriguez-Villegas, "Hypergeometric families of Calabi–Yau manifolds", pp. 223–231 in *Calabi–Yau varieties and mirror symmetry* (Toronto, ON, 2001), edited by N. Yui and J. D. Lewis, Fields Inst. Commun. **38**, Amer. Math. Soc., Providence, RI, 2003. MR 2005b:11086 Zbl 1062.11038

[Whipple 1926] F. J. W. Whipple, "On well-posed series, generalised hypergeometric series having parameters in pairs, each pair with the same sum", *Proc. London Math. Soc.* **24**:1 (1926), 247–263.

[Zudilin 2009] W. Zudilin, "Ramanujan-type supercongruences", *J. Number Theory* **129**:8 (2009), 1848–1857. MR 2522708 Zbl 05569026

Received December 1, 2009. Revised September 27, 2010.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 249 No. 2 February 2011