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Let $(S^n(r), g_0)$ be the canonical sphere of radius r. Denote by $\widetilde{G}_s$ the Sasaki metric on the unit tangent bundle $T_1S^n(r)$ induced from g_0 and by $\widetilde{\widetilde{G}}_s$ the Sasaki metric on $T_1T_1S^n(r)$ induced from $\widetilde{G}_s$. We resolve here, for $n \geq 7$, a question raised by Boeckx, González–Dávila, and Vanhecke: namely, we prove that the geodesic flow

$$\xi : (T_1S^n(r), \widetilde{G}_s) \to (T_1T_1S^n(r), \widetilde{\widetilde{G}}_s)$$

is an unstable harmonic vector field for any $r > 0$ and $n \geq 7$. In particular, in the case $r = 1$, ξ is an unstable harmonic map. We show that these results are invariant under a four-parameter deformation of the Sasaki metric $\widetilde{\widetilde{G}}_s$.

1. Introduction

Let (M, g) be a compact Riemannian manifold and $\mathfrak{X}^1(M)$ the set of all smooth unit vector fields on (M, g), which we suppose to be nonempty, equivalently, the Euler–Poincaré characteristic of M vanishes. Let $(T_1M, \widetilde{G}_s)$ be the unit tangent sphere bundle equipped with the Sasaki metric $\widetilde{G}_s$. A unit vector field $U \in \mathfrak{X}^1(M)$ determines a map between (M, g) and $(T_1M, \widetilde{G}_s)$ and the energy $E_{\widetilde{G}_s}(U)$ is defined as the energy of the corresponding map

$$U : (M, g) \to (T_1M, \widetilde{G}_s).$$

A unit vector field U is said to be a *harmonic vector field* if it is a critical point for the energy functional $E_{\widetilde{G}_s}$ restricted to $\mathfrak{X}^1(M)$ [Wiegmink 1995; Wood 1997]. Harmonic unit vector fields aren't harmonic maps unless an additional curvature condition is satisfied [Han and Yim 1998; Abbassi et al. 2009a].

For the unit sphere S^{2m+1}, $m > 1$, the Hopf vector fields are unstable harmonic unit vector fields [Wood 1997]. The unit vector fields of minimum energy on the unit sphere S^3 are precisely the Hopf vector fields, equivalently, the unit Killing

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vector fields, and no others [Brito 2000]. Contact metric manifolds which Reeb vector field is harmonic are called H-contact manifolds [Perrone 2004]. In [Perrone 2009a] we studied the stability of the Reeb vector field of a compact H-contact three-manifold. If the unit tangent bundle itself is taken as the source manifold of unit vector fields, then a distinguished unit vector field, namely, the *geodesic flow vector field* ξ, appears in a natural way (it is collinear, with a constant factor, to the Reeb vector field of the standard contact metric structure on T_1M).

Let (M, g) be a Riemannian manifold locally isometric to a two-point homogeneous space, that is, locally flat or locally isometric to a rank-one symmetric space. Boeckx and Vanhecke [2000] proved that $\xi : (T_1M, \widetilde{G}_s) \to (T_1T_1M, \widetilde{\widetilde{G}}_s)$ is a harmonic vector field (and a harmonic map), where $\widetilde{\widetilde{G}}_s$ is the corresponding Sasaki metric on T_1T_1M.

Concerning the stability of the geodesic flow ξ we have few results. Boeckx et al. [2002] studied the stability of ξ as harmonic vector field when such a M is in addition compact (note that, by [Borel 1963], compact quotients always exist) and satisfies some other conditions. More precisely, the authors proved that if $n \geq 3$ and M is of nonpositive curvature with nonzero first Betti number, then the geodesic flow $\xi : (T_1M, \widetilde{G}_s) \to (T_1T_1M, \widetilde{\widetilde{G}}_s)$ is an unstable harmonic vector field. In the positive curvature case they considered a space of constant curvature and proved a similar yet weaker result. Indeed, in such case, they proved that the existence of nonzero Killing vector fields implies the instability of ξ for the energy functional $E_{\tilde{\tilde{G}}}$, in certain ranges of the dimension and the curvature. With these results, the question of stability of ξ remains open, particularly in the case of a compact quotient of a two-point homogeneous space of positive curvature. The most intriguing one, according to Boeckx et al. [2002], concerns the unit spheres $S^n(1)$ for $n > 2$. Their method does not give any answers in this case.

Recently, the papers [Abbassi et al. 2009a; 2009b; 2010a; Perrone 2009b; 2010] examined the question of when a vector field $V : (M, g) \to (TM, G)$ and a unit vector field $U : (M, g) \to (T_1M, \widetilde{G})$ are harmonic vector fields and define harmonic maps, where G is a natural Riemannian metric on TM and $\widetilde{G}$ is its restriction to the unit tangent sphere bundle T_1M. (Natural Riemannian metrics form a very large family, which includes the Sasaki metric, the Cheeger–Gromoll metric, metrics of Cheeger–Gromoll type [Benyounes et al. 2007] and the Kaluza–Klein metrics [Wood 1990].) The restrictions $\widetilde{G}$ of such metrics to T_1M possess a simpler form and globally depend on four real parameters a, b, c, d satisfying some inequalities (the parameters $a = 1,\ b = c = d = 0$ define the Sasaki metric $\widetilde{G}_s$). Suppose that (M, g) is a Riemannian manifold locally isometric to a two-point homogeneous space and T_1M, $T_\rho T_1M$ are equipped with arbitrary natural Riemannian metrics $\widetilde{G}$ and $\widetilde{\widetilde{G}}$ respectively. Then, Abbassi et al. [2010b] proved that the geodesic flow $\xi : (T_1M, \widetilde{G}_s) \to (T_\rho T_1M, \widetilde{\widetilde{G}})$ is always a harmonic vector field, and it also defines a harmonic map under some conditions on the coefficients determining the natural Riemannian metrics.

The main purpose of this paper is to study the stability of the geodesic flow

$$\xi : (T_1S^n(r), \widetilde{G}_s) \to (T_1T_1S^n(r), \widetilde{\widetilde{G}}),$$

where $S^n(r)$ is the canonical sphere of radius r and $\widetilde{\widetilde{G}}$ is an arbitrary natural Riemannian metric on $T_1T_1S^n(r)$ induced from the Sasaki metric $\widetilde{G}_s$ on $T_1S^n(r)$ (see Theorem 4.2 and Theorem 5.3). In particular, we get that the geodesic flow

$$\xi : (T_1S^n(r), \widetilde{G}_s) \to (T_1T_1S^n(r), \widetilde{\widetilde{G}})$$

is an unstable harmonic vector field (and an unstable harmonic map) for any $r > 0$, $n \geq 7$, and for any natural Riemannian metric $\widetilde{\widetilde{G}}$ on $T_1T_1S^n(r)$ induced from the Sasaki metric $\widetilde{G}_s$. When $\widetilde{\widetilde{G}} = \widetilde{\widetilde{G}}_s$, we resolve the question of posed in [Boeckx et al. 2002, page 202] for any $n \geq 7$. In order to get all these results, we use the Hessian form of the energy functional

$$E_{\widetilde{G}} : \mathfrak{X}^1(M) \to \mathbb{R}, U \mapsto E_{\widetilde{G}}(U) = E\big(U : (M, g) \to (T_1M, \widetilde{G})\big),$$

for an arbitrary natural Riemannian metric $\widetilde{G}$ (see Theorem 3.2). It should be noted that the instability of the Hopf vector fields on S^{2m+1}, $m > 1$, and the stability (instability) results given in [Perrone 2009a] are invariant under a four-parameter deformation of the Sasaki metric $\widetilde{G}_s$ on T_1M (see Corollary 3.4).

2. Natural Riemannian metrics on T_1M

Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. We denote by R the Riemannian curvature tensor of (M, g) with the sign convention $R(X, Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$. Moreover, we denote by Ric the Ricci tensor of type $(0, 2)$, by Q the corresponding endomorphism field and by τ the scalar curvature.

At any point (x, u) of the *tangent bundle* TM, the tangent space of TM splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = \mathscr{H}_{(x,u)} \oplus \mathscr{V}_{(x,u)}.$$

For any vector $X \in M_x$, there exists a unique vector $X^h \in \mathscr{H}_{(x,u)}$ (the *horizontal lift* of X to $(x, u) \in TM$), such that $p_*X^h = X$, where $p : TM \to M$ is the natural projection. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in \mathscr{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all smooth functions f on M. Here we consider 1-forms df on M as smooth functions on TM. The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathscr{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathscr{V}_{(x,u)}$. Each tangent vector

$\widetilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\widetilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors. The *geodesic flow* ξ on TM is a vector field given, in terms of local coordinates, by

$$\xi_{(x,u)} = u^h_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^h_{(x,u)}, \quad \text{where} \quad u = \sum_i u^i (\partial/\partial x^i)_x \in M_x.$$

The *natural Riemannian metrics* form a wide family of Riemannian metrics on TM. These metrics depend on several smooth functions from $\mathbb{R}^+ = [0, +\infty)$ to $\mathbb{R}$ and as their name suggests, they arise from a very "natural" construction starting from a Riemannian metric g over M (see [Abbassi and Sarih 2005; Abbassi et al. 2010a] and the references in [Abbassi 2008]). Given an arbitrary g-natural metric G on the tangent bundle TM of a Riemannian manifold (M, g), there are six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in M_x$, we have

$$\begin{aligned} G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) + (\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) &= \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) &= G_{(x,u)}(X^h, Y^v), \\ G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u), \end{aligned} \tag{2-1}$$

where $r^2 = g_x(u, u)$. Put

$$\begin{aligned} \phi_i(t) &= \alpha_i(t) + t\beta_i(t), \\ \alpha(t) &= \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t), \\ \phi(t) &= \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t), \end{aligned}$$

for all $t \in \mathbb{R}^+$. Then, a g-natural metric G on TM is Riemannian if and only if

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0 \quad \text{for all } t \in \mathbb{R}^+. \tag{2-2}$$

The Sasaki metric G_s, the Cheeger–Gromoll metric, metrics of Cheeger–Gromoll type [Benyounes et al. 2007] and the Kaluza–Klein metrics, as commonly defined on principal bundle [Wood 1990], belong to the subclass of g-natural Riemannian metrics on TM for which horizontal and vertical distribution are mutually orthogonal (i.e., $\alpha_2 = \beta_2 = 0$). More generally, g-natural Riemannian metrics on TM for which horizontal and vertical distribution are mutually orthogonal are called *metrics of Kaluza–Klein type* [Perrone 2010].

Next, the *tangent sphere bundle of radius r* over a Riemannian manifold (M, g), is the hypersurface $T_r M = \{(x, u) \in TM : g_x(u, u) = r^2\}$. The tangent space of

$T_r M$ at a point $(x,u) \in T_r M$ is given by

$$(T_r M)_{(x,u)} = \{X^h + Y^v : X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{2-3}$$

We call *g-natural metrics on $T_r M$* the restrictions of g-natural metrics of TM to its hypersurface $T_r M$. These metrics possess a simpler form. Precisely, taking in account of (2-1) and (2-3), every natural Riemannian metric $\widetilde{G}$ on $T_r M$ is necessarily induced by a natural Riemannian metric G on TM of the special form (see also [Abbassi 2008; Abbassi et al. 2009a]):

$$\begin{aligned} G_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X,Y) + \beta\, g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) &= G_{(x,u)}(X^v, Y^h) = b\, g_x(X,Y), \\ G_{(x,u)}(X^v, Y^v) &= a\, g_x(X,Y), \end{aligned} \tag{2-4}$$

for three real constants a, b, c and a smooth function $\beta : [0,\infty) \to \mathbb{R}$. It is easily seen that G is obtained by the general expression (2-1) when

$$\alpha_1 = a, \quad \alpha_2 = b, \quad \alpha_3 = c, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 = \beta, \tag{2-5}$$

Such a metric $\widetilde{G}$ on $T_r M$ only depends on the value $d = \beta(r^2)$ of β at r^2. From (2-2) and (2-5) it follows that $\widetilde{G}$ is Riemannian if and only if

$$a > 0, \quad \alpha := a(a+c) - b^2 > 0 \quad \text{and} \quad \phi = a(a+c+r^2 d) - b^2 > 0. \tag{2-6}$$

By (2-4), horizontal and vertical lifts are orthogonal with respect to $\widetilde{G}$ if and only if $b = 0$. Moreover, metrics satisfying $b = 0$ are all and the ones induced by natural Riemannian metrics of Kaluza–Klein type. For this reason, a natural Riemannian metric $\widetilde{G}$ on $T_r M$ will be said to be of *Kaluza–Klein type* if and only if horizontal and vertical lifts are $\widetilde{G}$-orthogonal, that is, $b = 0$ in (2-4). Notice that the Sasaki metric, the Cheeger–Gromoll metric, metrics of Cheeger–Gromoll type and the Kaluza–Klein metrics belong to the subclass of natural Riemannian metrics on $T_1 M$ of Kaluza–Klein type. Moreover, an arbitrary natural Riemannian metric $\widetilde{G}$ on $T_r M$ can be considered as a deformation on four parameters (a,b,c,d) of the Sasaki metric $\widetilde{G}_s$ (which is defined by $a = 1$, $b = c = d = 0$).

When $r = 1$, $T_1 M$ is called *unit tangent sphere bundle*. Now, if $\widetilde{G}$ is an arbitrary g-natural Riemannian metric on $T_1 M$, then by (2-4) it follows that the geodesic flow vector field ξ on $T_1 M$ has constant length $\|\xi\|_{\widetilde{G}} = \sqrt{a+c+d}$ (not necessarily equal to 1). Note that $a+c+d > 0$, since $a > 0$ and $\phi = a(a+c+d) - b^2 > 0$. Hence, ξ defines a map $\xi : T_1 M \to T_\rho T_1 M$ where $\rho := \sqrt{a+c+d}$; if $\widetilde{G} = \widetilde{G}_s$, then $\rho = 1$.

3. The Hessian form for the energy $E_{\widetilde{G}}$

Let (M, g) be a compact Riemannian manifold of dimension n. Every unit vector field U on M defines a map between (M, g) and $(T_1 M, \widetilde{G}_s)$ and we can define $E_{\widetilde{G}_s}(U)$, *the energy of* U, as the energy of the corresponding map:

$$E_{\widetilde{G}_s}(U) = \frac{1}{2}\int_M \|dU\|^2\, v_g = \frac{n}{2}\,\mathrm{vol}(M, g) + \frac{1}{2}\int_M \|\nabla U\|^2\, dv_g.$$

$E(U)$ is equal, up to constants, to $B(U) = \int_M \|\nabla U\|^2\, dv_g$ which is known as the total bending of U [Wiegmink 1995]. Here dv_g denotes the canonical measure on (M, g). U is called a *harmonic vector field* if it is critical for the energy functional

$$E_{\widetilde{G}_s} : \mathfrak{X}^1(M) \to \mathbb{R},\ U \mapsto E_{\widetilde{G}_s}(U) = E\big(U : (M, g) \to (T_1 M, \widetilde{G}_s)\big).$$

The corresponding critical point condition "$\bar{\Delta} V$ is collinear to V" has been determined in [Wiegmink 1995] (see also [Wood 1997]), where $\bar{\Delta} U = -\mathrm{tr}\nabla^2 U$ is the *rough Laplacian* at U. This critical point condition has a tensorial character and may also be considered on non compact manifolds.

Now, consider on T_1M an arbitrary g-natural Riemannian metric $\widetilde{G}$. Then a unit vector field U defines a mapping from (M, g) to $(T_1M, \widetilde{G})$ and we can consider the energy functional

$$E_{\widetilde{G}} : \mathfrak{X}^1(M) \to \mathbb{R},\ U \mapsto E_{\widetilde{G}}(U) = E\big(U : (M, g) \to (T_1M, \widetilde{G})\big) = \int_M e(U)\, dv_g,$$

where $e(U)$ is the energy density of $U : (M, g) \to (T_1M, \widetilde{G})$ and is given by [Abbassi et al. 2009a]

$$2e(U) = n(a + c) + d + a\, \|\nabla U\|^2 + 2b\, \mathrm{div}\, U, \tag{3-1}$$

and so, integrating over M we get

$$E_{\widetilde{G}}(U) = \frac{1}{2}[n(a + c) + d]\, \mathrm{vol}(M, g) + \frac{a}{2}\int_M \|\nabla U\|^2 dv_g. \tag{3-2}$$

In [Abbassi et al. 2009a] we proved that the critical point condition for the energy $E_{\widetilde{G}_s}$ is invariant under a four-parameter deformation of the Sasaki metric $\widetilde{G}_s$. More precisely:

Theorem 3.1 [Abbassi et al. 2009a]. *Let* (M,g) *be a compact Riemannian manifold of dimension n. Then, a unit vector field* $U \in \mathfrak{X}^1(M)$ *is a harmonic vector field for the energy* $E_{\widetilde{G}}$ *if and only if* U *is a harmonic vector field for the energy* $E_{\widetilde{G}_s}$, *that is,* $\Delta U = \|\nabla U\|^2 U$. *Moreover,* $U : (M, g) \to (T_1M, \widetilde{G})$ *is a harmonic map if and only if U is a harmonic vector field and*

$$b\, QU + a\ \mathrm{tr}[R(\nabla.U, U)\cdot] = (b\, \|\nabla V\|^2 - d\, \mathrm{div}\, U)U + d\nabla_U U. \tag{3-3}$$

In the case of the Sasaki metric $\widetilde{G}_s$, (3-3) gives a result of [Han and Yim 1998].

Wiegmink [1995] obtained the second variation formula for the energy $E_{\widetilde{G}_s}$. The second variation formula for the energy $E_{\widetilde{G}}$ could be deduced directly from (3-1) by using Theorem 3.1. In the sequel, we include the proof for completeness. Let U be a harmonic vector field for the energy $E_{\widetilde{G}}$, and $U(t)$ a variation of U in $\mathfrak{X}^1(M)$. Then, by (3-1) we have

$$2e(t) := 2e(U(t)) = n(a+c) + d + a\,\|\nabla U(t)\|^2 + 2b\,\operatorname{div} U(t),$$

and integrating over M, we find

$$E_{\widetilde{G}}(t) := E_{\widetilde{G}}(U(t)) = \frac{n(a+c)+d}{2}\operatorname{vol}(M,g) + \frac{a}{2}\int_M \|\nabla U(t)\|^2 dv_g. \tag{3-4}$$

Differentiating (3-4) we obtain

$$E'_{\widetilde{G}}(t) = a\int_M g\big(\nabla U(t), \nabla U'(t)\big)\,dv_g,$$

and hence

$$E''_{\widetilde{G}}(t) = a\int_M g\big(\nabla U'(t), \nabla U'(t)\big)\,dv_g + a\int_M g\big(\nabla U(t), \nabla U''(t)\big)\,dv_g.$$

Therefore

$$E''_{\widetilde{G}}(0) = a\int_M \|\nabla W\|^2 dv_g + a\int_M g(\nabla U, \nabla A)\,dv_g,$$

where $W = U'(0)$ is orthogonal to U and $A = U''(0)$. On the other hand, for any $X, Y \in \mathfrak{X}(M)$, by a direct calculation, one gets the Bochner-type formula (see [Poor 1981, page 158] for $X = Y$):

$$\Delta g(X,Y) = g(\bar{\Delta}X, Y) + g(X, \bar{\Delta}Y) - 2g(\nabla X, \nabla Y), \tag{3-5}$$

where Δ is the Laplacian acting on functions. This formula implies

$$\int_M g(\bar{\Delta}U, A)\,dv_g = \int_M g(\nabla U, \nabla A) dv_g,$$

where, using Theorem 3.1, $\bar{\Delta}U = \|\nabla U\|^2 U$. Then

$$E''_{\widetilde{G}}(0) = a\int_M (\|\nabla W\|^2 + \|\nabla U\|^2 g(U,A))\,dv_g.$$

Moreover, $\|\nabla U\|^2 = 1$ implies

$$\|W\|^2 = g\big(U'(0), U'(0)\big) = -g\big(U(0), U''(0)\big) = -g(U,A).$$

Thus, we get:

Theorem 3.2. *Let (M,g) be a compact Riemannian manifold. If $U \in \mathfrak{X}^1(M)$ is a critical point of the energy functional $E_{\tilde{G}}$. Then*

$$(\text{Hess} E_{\tilde{G}})_U(W) = a \int_M \left(\|\nabla W\|^2 - \|\nabla U\|^2 \|W\|^2\right) dv_g \tag{3-6}$$

for any $W \in U^{\perp}$.

When T_1M is equipped with the Sasaki metric $\tilde{G}_s$, we get the Hessian form given in [Wiegmink 1995].

Corollary 3.3. *Let (M,g) be a compact Riemannian manifold and U a unit vector field on M. Then the property of $U : (M,g) \to (T_1M, \tilde{G}_s)$ being a stable (or unstable) harmonic vector field is invariant under a four-parameter deformation of the Sasaki metric $\tilde{G}_s$ on T_1M.*

Wood [1997] showed that for the unit sphere S^{2m+1}, $m > 1$, the Hopf vector fields are unstable for the energy $E_{\tilde{G}_s}$. Contact metric manifolds which Reeb vector field is harmonic are called H-contact manifolds [Perrone 2004]. Recently, in [Perrone 2009a] we studied the stability of the Reeb vector field of a compact H-contact three manifold for the energy $E_{\tilde{G}_s}$. From Corollary 3.3 we get:

Corollary 3.4. *The instability of the Hopf vector fields on S^{2m+1}, $m > 1$, and the stability (or instability) results given in* [Perrone 2009a] *are invariant under a four-parameter deformation of the Sasaki metric $\tilde{G}_s$ on T_1M.*

4. Instability of the geodesic flow

Let (M,g) be a Riemannian manifold locally isometric to a two-point homogeneous space, that is, locally flat or locally isometric to a rank-one symmetric space. We denote by $\tilde{G}_s$ the Sasaki metric on T_1M, by $\tilde{\tilde{G}}_s$ the corresponding Sasaki metric on T_1T_1M and by $\tilde{\tilde{G}}$ an arbitrary natural Riemannian metric on T_1T_1M constructed from $\tilde{G}_s$. Boeckx and Vanhecke [2000] proved that $\xi : (T_1M, \tilde{G}_s) \to (T_1T_1M, \tilde{\tilde{G}}_s)$ is a harmonic map, in particular ξ is a harmonic vector field for the energy $E_{\tilde{\tilde{G}}}$. About the stability of ξ, we have:

Theorem 4.1 [Boeckx et al. 2002]. *Let (M,g) be a compact quotient of a two-point homogeneous space of nonpositive curvature and with first Betti number $b_1(M) \neq 0$, $\dim M = n \geq 3$. Then the geodesic flow ξ on T_1M is unstable for the energy $E_{\tilde{\tilde{G}}}$.*

In the positive curvature case they proved a similar yet weaker result. Indeed, in such case, the existence of nonzero Killing vector fields implies the instability of ξ for the energy functional $E_{\tilde{\tilde{G}}_s}$, in certain ranges of the dimension n and of curvature. With these results, the question of stability of ξ remains open. The most intriguing one (according to [Boeckx et al. 2002, page 202]) concerns the unit spheres $S^n(1)$ for $n > 2$. Their method does not give any answers in this case.

Now, we consider on T_1M the Sasaki metric $\widetilde{G}_s$ while on T_1T_1M consider an arbitrary natural Riemannian metric $\tilde{\widetilde{G}}$ constructed from $\widetilde{G}_s$, where (M, g) is a compact quotient of a two-point homogeneous space of dimension n. Abbassi et al. [2010b, Theorem 5] proved that $\xi : (T_1M, \widetilde{G}_s) \to (T_1T_1M, \tilde{\widetilde{G}})$ is a harmonic vector field for the energy $E_{\tilde{\tilde{G}}}$. From Theorem 3.2 we have that the geodesic flow ξ is stable (or unstable) with respect to $E_{\tilde{\tilde{G}}}$ if and only if it has the same property with respect to $E_{\tilde{\tilde{G}}_s}$, that is, when $\xi : (T_1M, \widetilde{G}_s) \to (T_1T_1M, \tilde{\widetilde{G}}_s)$. So we consider Hess $E_{\tilde{\tilde{G}}_s}$; from the general expression (3-6), we have

$$(\text{Hess } E_{\tilde{\tilde{G}}_s})_\xi(W) = \int_{T_1M} \big(\|\widetilde{\nabla} W\|^2 - \|\widetilde{\nabla}\xi\|^2 \|W\|^2\big)\, dv_{\widetilde{G}_s} \tag{4-1}$$

for any vector field W on T_1M such that $\widetilde{G}_s(\xi, W) = 0$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $(T_1M, \widetilde{G}_s)$. If X is an arbitrary vector field on M, the tangential lift $X^t_z = X^v_z - g_x(X_x, u)u^v$, $z = (x, u)$, is a vector field on T_1M orthogonal to ξ, but the horizontal lift X^h in general is not. For that reason, we define the modified horizontal $\bar{X}^h_z = X^h_z - g(X_p, u)\xi_z$, $z = (p, u)$. This vector field on T_1M is orthogonal to ξ and tangent to T_1M. Moreover, we have, from [Boeckx et al. 2002, Lemma 1, page 206],

$$\int_{T_1M} \big(\|\widetilde{\nabla} X^t\|^2 - \|\widetilde{\nabla}\xi\|^2 \|X^t\|^2\big)\, dv_{\widetilde{G}_s} = a_{n-1} \int_M \big(\|\nabla X\|^2 + A_t \|X\|^2\big)\, dv_g, \tag{4-2}$$

$$\int_{T_1M} \big(\|\widetilde{\nabla} \bar{X}^h\|^2 - \|\widetilde{\nabla}\xi\|^2 \|\bar{X}^h\|^2\big)\, dv_{\widetilde{G}_s} = a_{n-1} \int_M \big(\|\nabla X\|^2 + A_h \|X\|^2\big)\, dv_g, \tag{4-3}$$

where $\dfrac{n}{n-1}\, a_{n-1}$ is the volume of the unit sphere S^{n-1}, and

$$A_t = \frac{5-2n}{4n(n-1)(n+2)}\|R\|^2 - \frac{\tau^2}{2n^2(n+2)} + \frac{\tau}{n} - n + 2,$$

$$A_h = \frac{4-n}{4n(n-1)(n+2)}\|R\|^2 - \frac{\tau^2}{2n(n-1)(n+2)} + \frac{(n-2)\tau}{n(n-1)} - n + 3.$$

Denote by Δ_1 the Laplacian acting on 1-forms. Recall that Δ_1 also acts on vector fields via duality and it is related to the rough Laplacian $\bar{\Delta}$ and the Ricci operator Q by the well-known Weitzenböck formula [Poor 1981, page 168]:

$$\Delta_1 = \bar{\Delta} + Q. \tag{4-4}$$

Moreover, for any $X \in \mathfrak{X}(M)$, from (3-5) we have

$$-\tfrac{1}{2}\Delta\|X\|^2 = \|\nabla X\|^2 - g(\bar{\Delta}X, X). \tag{4-5}$$

Then (4-4) and (4-5) imply that

$$-\tfrac{1}{2}\Delta\|X\|^2 = \|\nabla X\|^2 - g(\Delta_1 X, X) + \mathrm{Ric}(X, X).$$

As M is locally isometric to a two-point homogeneous space, it is Einstein, that is, $\mathrm{Ric} = (\tau/n)g$, the above equation gives

$$\int_M \|\nabla X\|^2 dv_g = \int_M (g(\Delta_1 X, X) - \frac{\tau}{n}\|X\|^2)\, dv_g. \tag{4-6}$$

Then, (4-1), (4-2), and (4-6) imply

$$(\mathrm{Hess}\, E_{\tilde{\tilde{G}}_s})_\xi(X^t) = a_{n-1}\int_M \Big(g(\Delta_1 X, X) + \Big(A_t - \frac{\tau}{n}\Big)\|X\|^2\Big)\, dv_g, \tag{4-7}$$

$$(\mathrm{Hess}\, E_{\tilde{\tilde{G}}_s})_\xi(\bar{X}^h) = a_{n-1}\int_M \Big(g(\Delta_1 X, X) + \Big(A_h - \frac{\tau}{n}\Big)\|X\|^2\Big)\, dv_g. \tag{4-8}$$

Let λ_1 the first eigenvalue of the Laplacian Δ acting on functions. Consider an eigenfunction f related to the eigenvalue λ_1. Set $\omega = df$, so that

$$\Delta_1\omega = (d\delta + \delta d)\, df = d\delta df = d\Delta f = \lambda_1 df = \lambda_1\omega.$$

Hence, if X_0 is the vector field defined by $g(X_0, \cdot) = \omega$, we obtain

$$\Delta_1 X_0 = \lambda_1 X_0.$$

Consequently, $(\mathrm{Hess}\, E_{\tilde{\tilde{G}}_s})_\xi(X_0^t) < 0$ if and only if λ_1 satisfies

$$\lambda_1 < \frac{\tau}{n} - A_t = \frac{2n-5}{4n(n-1)(n+2)}\|R\|^2 + \frac{\tau^2}{2n^2(n+2)} + n - 2, \tag{4-9}$$

and $(\mathrm{Hess}\, E_{\tilde{\tilde{G}}_s})_\xi(\bar{X}_0^h) < 0$ if and only if λ_1 satisfies

$$\begin{aligned}\lambda_1 &< \frac{\tau}{n} - A_h \\ &= \frac{n-4}{4n(n-1)(n+2)}\|R\|^2 + \frac{\tau^2}{2n(n-1)(n+2)} + \frac{\tau}{n(n-1)} + n - 3.\end{aligned} \tag{4-10}$$

Now, suppose that (M, g) is a space of constant curvature $\kappa > 0$. Then,

$$\tau = n(n-1)\kappa, \quad \|R\|^2 = 2n(n-1)\kappa^2 = \frac{2\tau^2}{n(n-1)} \quad \text{and}$$

$$A_t - \frac{\tau}{n} = \frac{(5-2n)2n(n-1)\kappa^2}{4n(n-1)(n+2)} - \frac{n^2(n-1)^2\kappa^2}{2n^2(n+2)} - (n-2),$$

that is,

$$\frac{\tau}{n} - A_t = (n-2)\Big(\frac{\kappa^2}{2} + 1\Big) > 0 \quad \text{for any } n > 2.$$

Moreover,

$$A_h - \frac{\tau}{n} = \frac{(4-n)2n(n-1)\kappa^2}{4n(n-1)(n+2)} - \frac{n^2(n-1)^2\kappa^2}{2n(n-1)(n+2)} + \frac{(n-2)n(n-1)\kappa}{n(n-1)} - (n-3) - \frac{\tau}{n}$$
$$= \frac{(2-n)}{2}\kappa^2 - \kappa - (n-3),$$

that is,

$$\frac{\tau}{n} - A_h = \frac{(n-2)}{2}\kappa^2 + \kappa + n - 3.$$

Therefore, by (4-9), $(\mathrm{Hess}\, E_{\tilde{\tilde{G}}_s})_\xi(X_0^t) < 0$ if and only if λ_1 satisfies

(4-11) $$\lambda_1 < \frac{\tau}{n} - A_t = (n-2)\Big(\frac{\kappa^2}{2} + 1\Big)$$

and, by (4-10), $(\mathrm{Hess}\, E_{\tilde{\tilde{G}}_s})_\xi(\bar{X}_0^h) < 0$ if and only if λ_1 satisfies

(4-12) $$\lambda_1 < \frac{\tau}{n} - A_h = \frac{(n-2)}{2}\kappa^2 + \kappa + n - 3.$$

Now, for a space of constant sectional curvature $\kappa > 0$, a result of Lichnerowicz and Obata [Berger et al. 1971, pages 179–180] states that the eigenvalue λ_1 satisfies $\lambda_1 \geq n\kappa$, where the equality holds if and only if M is isometric to the canonical sphere of radius $r = \sqrt{1/\kappa}$. So, for the sphere $S^n(r)$ of radius $r > 0$, that is of constant sectional curvature $\kappa = 1/r^2$, the conditions (4-11), (4-12) become

(4-13) $$(\kappa^2 - 2\kappa + 2)\Big(n - \frac{2(\kappa^2+2)}{\kappa^2 - 2\kappa + 2}\Big) > 0,$$

(4-14) $$(\kappa^2 - 2\kappa + 2)\Big(n - \frac{2\kappa^2 - 2\kappa + 6}{\kappa^2 - 2\kappa + 2}\Big) > 0.$$

Examining these expressions, we conclude:

If n and κ satisfy one of the following conditions, then (4-11) is satisfied:

- $\kappa > 0$ and $n \geq 7$,
- $\kappa \in \,]0, 1[\, \cup \,]2, +\infty[$ and $n \geq 6$,
- $\kappa \in \,]0, \frac{1}{3}(5-\sqrt{7})[\, \cup \,]\frac{1}{3}(5+\sqrt{7}), +\infty[$ and $n \geq 5$,
- $\kappa \in \,]0, 2-\sqrt{2}[\, \cup \,]2+\sqrt{2}, +\infty[$ and $n \geq 4$,
- $\kappa \in \,]0, 3-\sqrt{7}[\, \cup \,]3+\sqrt{7}, +\infty[$ and $n \geq 3$.

If n and κ satisfy one of the following conditions, then (4-12) is satisfied:

- $\kappa > 0$ and $n \geq 7$,
- $\kappa \in \,]0, 1[\, \cup \,]\frac{3}{2}, +\infty[$ and $n \geq 6$,
- $\kappa \in \,]0, \frac{2}{3}[\, \cup \,]2, +\infty[$ and $n \geq 5$,
- $\kappa \in \,]0, 3-2\sqrt{2}[\, \cup \,]3+2\sqrt{2}, +\infty[$ and $n \geq 4$,
- $\kappa \in \,]4, +\infty[$ and $n \geq 3$.

Summarizing:

Theorem 4.2. *Let $S^n(r)$ be the canonical sphere of radius r, and let $\kappa = 1/r^2$. If one of the following conditions holds, then the geodesic flow ξ on $T_1 S^n(r)$ is unstable for the energy $E_{\tilde{\tilde{G}}}$:*

- $\kappa > 0$ *and* $n \geq 7$,
- $\kappa \in]0, 1[\cup]\frac{3}{2}, +\infty[$ *and* $n \geq 6$,
- $\kappa \in]0, \frac{2}{3}[\cup]2, +\infty[$ *and* $n \geq 5$,
- $\kappa \in]0, 2-\sqrt{2}[\cup]2+\sqrt{2}, +\infty[$ *and* $n \geq 4$,
- $\kappa \in]0, 3-\sqrt{7}[\cup]4, +\infty[$ *and* $n \geq 3$.

Corollary 4.3. *The geodesic flow ξ on $T_1 S^n(1)$ is unstable for the energy $E_{\tilde{\tilde{G}}}$, for $n \geq 7$.*

The two-dimensional case. Let (M, g) be a compact Riemannian surface of constant curvature $\kappa > 0$. If $\kappa < 1$, Theorem 7 of [Boeckx et al. 2002] gives that the geodesic flow ξ on $T_1 M$ is an unstable harmonic vector field for the energy $E_{\tilde{\tilde{G}}_s}$. If $\kappa = 1$, $(T_1 M, G_s)$ is a compact Riemannian three-manifold of constant curvature $c = \frac{1}{4}$ and ξ is a unit Killing vector field. Brito [2000] proved that the unit vector fields of minimum energy on the unit sphere S^3 are precisely the unit Killing vector fields, and no others. Recently, we proved an analogue of Brito's theorem for a compact Sasakian three-manifold [Perrone 2008, page 20]. A consequence of its proof gives: *the unit vector fields of minimum energy on a compact Riemannian three-manifold of constant sectional curvature $c \geq 0$ are precisely the unit Killing vector fields, and no others.*

Other positively curved two-point homogeneous spaces. There are known analogues of Theorem 4.2 for other compact positively curved two-point homogeneous spaces, though with different conditions. We mention:

– For the real projective space $\mathbb{RP}^n$ of constant sectional curvature $\kappa > 0$, we know from [Gallot 1980, page 38] that $\lambda_1 = 2(n+1)\kappa$. The conditions (4-11) and (4-12) become

$$n(\kappa^2 - 4\kappa + 2) - 2(\kappa^2 + 2\kappa + 2) > 0, \quad n(\kappa^2 - 4\kappa + 2) - 2(\kappa^2 + \kappa + 3) > 0.$$

Examining this inequality we find that if $n \geq 3$ and $\kappa \in]0, 8 - \sqrt{62}[\cup]14, +\infty[$, the geodesic flow ξ on $T_1 \mathbb{RP}^n$ is unstable for the energy $E_{\tilde{\tilde{G}}}$.

– For the complex projective space $\mathbb{CP}^m$, $n = 2m$, of constant holomorphic sectional curvature $\mu > 0$, we have, from [Gray and Vanhecke 1979, page 177] and [Gallot 1980, page 38],

$$\tau = m(m+1)\mu, \quad \|R\|^2 = 2m(m+1)\mu^2, \quad \lambda_1 = (m+1)\mu. \tag{4-15}$$

Using this, we obtain conditions, like Theorem 4.2, which imply the instability of the geodesic flow on the unit tangent sphere bundle of the corresponding space. For $m > 1$, the condition $\lambda_1 + A_t - \tau/n < 0$ becomes

$$(m-1)(2m+11)\mu^2 - 16(m+1)(2m-1)\mu + 32(m-1)(2m-1) > 0.$$

The other condition, $\lambda_1 + A_h - \tau/n < 0$, becomes

$$(m-1)(m+4)\mu^2 - 4(m+1)(4m-3)\mu + 8(2m-3)(2m-1) > 0.$$

A similar remark applies to the next two examples. The references are also the same.

– For the quaternionic projective space, $n = 4m$, of constant quaternionic sectional curvature $\nu > 0$, we have

$$\tau = 4m(m+2)\nu, \|R\|^2 = 4m(5m+1)\nu^2, \lambda_1 = 2(m+1)\nu. \tag{4-16}$$

– For the Cayley projective plane, $n = 16$, of maximum sectional curvature $\zeta > 0$,

$$\tau = 144\zeta, \|R\|^2 = 576\zeta^2, \lambda_1 = 48\zeta. \tag{4-17}$$

5. Instability of harmonic maps defined by the geodesic flow

In the theory of harmonic maps, a fundamental question concerns the existence of harmonic maps between two given Riemannian manifolds (M, g) and (M', g'). If (M, g) is compact and (M', g') is of nonpositive sectional curvature, there exists a harmonic map $f : (M, g) \to (M', g')$ in each homotopy class [Eells and Sampson 1964]. However, there is no general existence result when (M', g') does not satisfy this condition. This fact makes it interesting to find examples of harmonic maps having such a target manifold. Since the standard existence theory for harmonic maps does not apply, examples have to be constructed ad hoc.

Now, let $\widetilde{G}$ be an arbitrary Riemannian g-natural metric on T_1M. By (2-4), the geodesic flow vector field ξ on T_1M has constant length $\|\xi\|_{\widetilde{G}} = \rho = \sqrt{a+c+d}$ (not necessarily equal to 1). Hence, we can study the harmonicity of the geodesic flow as a map $\xi : T_1M \to T_\rho T_1M$. We equip $T_\rho T_1M$ with an arbitrary g-natural Riemannian metric $\widetilde{\widetilde{G}}$ coming from $\widetilde{G}$. By (2-6), $\widetilde{\widetilde{G}}$ will depend on four constants a', b', c', d', satisfying

$$a' > 0, \quad a'(a'+c') - (b')^2 > 0, \quad a'(a'+c'+\rho^2 d') - (b')^2 > 0.$$

The following result shows that in many cases, the geodesic flow also defines a harmonic map.

Theorem 5.1 [Abbassi et al. 2010b]. *Let (M, g) be a two-point homogeneous space. The map $\xi : (T_1M, \widetilde{G}) \to (T_\rho T_1M, \widetilde{\widetilde{G}})$ is a harmonic map if and only if*

$$\text{(5-1)} \qquad na\alpha b' \sum_{i=1}^{n-1} \lambda_i^2 = \big[a'b^3d + 2b'\alpha(\alpha - b^2)\big]\tau - n(n-1)b'\alpha(a+c)^2,$$

where $\alpha = a(a+c) - b^2$ and the λ_i are the eigenvalues of the Jacobi operator $R_u = R(\cdot\,, u)u$.

In particular, if $\widetilde{G} = \widetilde{G}_s$ (i.e., $a = 1, b = c = d = 0$) and M has constant sectional curvature κ, then $\lambda_i = \kappa$, $\tau = n(n-1)\kappa$ and (5-1) becomes $n(n-1)b'(\kappa-1)^2 = 0$. Thus we get:

Theorem 5.2. *Let (M, g) be a space of constant sectional curvature κ.*

(i) *If $\kappa = 1$, the geodesic flow determines a harmonic map*

$$\xi : (T_1M, \widetilde{G}_s) \to (T_1T_1M, \widetilde{\widetilde{G}})$$

for any natural Riemannian metric $\widetilde{\widetilde{G}}$ on T_1T_1M induced from $\widetilde{G}_s$.

(ii) *If $\kappa \neq 1$, the geodesic flow determines a harmonic map*

$$\xi : (T_1M, \widetilde{G}_s) \to (T_\rho T_1M, \widetilde{\widetilde{G}})$$

if and only if $\widetilde{\widetilde{G}}$ is of Kaluza–Klein type, that is, $b' = 0$.

Since instability for the energy restricted to $\mathfrak{X}^1(T_1M)$ clearly implies instability in the large sense, combining Theorem 4.2 and Theorem 5.2 we get:

Theorem 5.3. (i) *The geodesic flow vector field on $T_1S^n(1)$, $n > 6$, determines an unstable harmonic map $\xi : (T_1S^n(1), \widetilde{G}_s) \to (T_1T_1S^n(1), \widetilde{\widetilde{G}})$ for any natural Riemannian metric $\widetilde{\widetilde{G}}$ on $T_1T_1S^n(1)$ induced from $\widetilde{G}_s$.*

(ii) *Let $S^n(\kappa)$ be the canonical sphere of constant curvature κ, where*

$$\kappa \in \,]0, 3 - \sqrt{7}[\, \cup \,]4, +\infty[,$$

and let $n \geq 3$. Then the geodesic flow on $T_1S^n(\kappa)$ determines an unstable harmonic map

$$\xi : (T_1S^n(\kappa), \widetilde{G}_s) \to (T_1T_1S^n(\kappa), \widetilde{\widetilde{G}})$$

for any metric of Kaluza–Klein type $\widetilde{\widetilde{G}}$ on $T_1T_1S^n(\kappa)$ induced from $\widetilde{G}_s$.

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