

*Pacific
Journal of
Mathematics*

**DUAL PAIRS AND CONTRAGREDIENTS
OF IRREDUCIBLE REPRESENTATIONS**

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Let G be one of the classical groups $GL(n)$, $U(n)$, $O(n)$ or $Sp(2n)$, over a nonarchimedean local field of characteristic zero. It is well known that the contragredient of an irreducible admissible smooth representation of G is isomorphic to a twist of it by an automorphism of G . We prove that similar results hold for double covers of G that occur in the study of local theta correspondences.

1. Introduction and the results

Fix a nonarchimedean local field \mathbb{k} of characteristic zero. We introduce the notation in order to treat the four classes of classical groups $GL(n)$, $U(n)$, $O(n)$ and $Sp(2n)$ simultaneously. Let A be a \mathbb{k} -algebra and τ be a \mathbb{k} -algebra involution of A such that

$$(A, \tau) = \begin{cases} (\mathbb{k} \times \mathbb{k}, \text{the nontrivial automorphism}), \\ \text{(a quadratic field extension of } \mathbb{k}, \text{ the nontrivial automorphism), or} \\ (\mathbb{k}, \text{the trivial automorphism}). \end{cases}$$

Let $\epsilon \in \pm 1$ and let E be an ϵ -Hermitian A -module; namely, E is a free A -module of finite rank equipped with a nondegenerate \mathbb{k} -bilinear map

$$\langle \cdot, \cdot \rangle_E : E \times E \rightarrow A$$

satisfying $\langle u, v \rangle_E = \epsilon \langle v, u \rangle_E^\tau$ and $\langle au, v \rangle_E = a \langle u, v \rangle_E$ for $a \in A$ and $u, v \in E$. Denote by $U(E)$ the group of all A -module automorphisms of E that preserve the form $\langle \cdot, \cdot \rangle_E$. Depending on the choice of A and ϵ , it is either a general linear group, a unitary group, an orthogonal group or a symplectic group.

Following Mœglin, Vigneras and Waldspurger [1987, Proposition 4.I.2], we extend $U(E)$ to a larger group $\check{U}(E)$ consisting of pairs $(g, \delta) \in GL_{\mathbb{k}}(E) \times \{\pm 1\}$ such that either

$$\delta = 1 \quad \text{and} \quad g \in U(E),$$

Supported by NSFC grants number 10801126 and 10931006.

MSC2000: 22E35, 22E46.

Keywords: contragredient representation, dual pair, irreducible representation.

or

$$\begin{aligned} \delta &= -1, \\ g(au) &= a^\tau g(u) \quad \text{for } a \in A, u \in E, \quad \text{and} \\ \langle gu, gv \rangle_E &= \langle v, u \rangle_E \quad \text{for } u, v \in E. \end{aligned}$$

Clearly $\check{U}(E)$ contains $U(E)$ as a subgroup of index two.

In general, if π is a representation of a group H and g is an element of a group that acts on H as automorphisms, we define the twist π^g to be the representation of H that has the same underlying space as that of π , and whose action is given by $\pi^g(h) := \pi(gh)$ for $h \in H$. If \check{H} is a group containing H as a subgroup of index two, we always let it act on H by conjugation:

$$\text{Ad} : \check{H} \times H \rightarrow H, \quad (\check{g}, x) \mapsto \text{Ad}_{\check{g}}(x) := \check{g}x\check{g}^{-1}.$$

It is a classical result in linear algebra that

$$(1) \quad \check{g}x\check{g}^{-1} \text{ is conjugate to } x^{-1} \text{ inside } U(E)$$

for all $\check{g} \in \check{U}(E) \setminus U(E)$ and all $x \in U(E)$. For example, when $U(E)$ is a general linear group, this amounts to saying that every square matrix is conjugate to its transpose. For orthogonal groups, this says that every element of an orthogonal group is conjugate to its inverse. The following considerations (which lead to [Theorem 1.1](#) below) appear in [[Mœglin et al. 1987](#)]. By the localization principle of Bernšteĭn and Zelevinskĭĭ [[1976](#), Theorem 6.9 and Theorem 6.15.A], result (1) implies that

$$(2) \quad f(\check{g}x\check{g}^{-1}) = f(x^{-1}) \quad (\text{as generalized functions on } U(E))$$

for all Ad-invariant generalized functions f on $U(E)$ and all $\check{g} \in \check{U}(E) \setminus U(E)$. For the usual notion of generalized functions, see [[Sun 2009](#), Section 2]. We get the following well known result by (2) and by considering characters of irreducible admissible smooth representations (which are conjugation invariant generalized functions).

Theorem 1.1 [[Mœglin et al. 1987](#), Theorem 4.II.1]. *Let $\check{g} \in \check{U}(E) \setminus U(E)$, and let π be an irreducible admissible smooth representation of $U(E)$. Then π^\vee is isomorphic to $\pi^{\check{g}}$.*

Here and as usual, we use “ \vee ” to indicate the contragredient of an admissible smooth representation of a totally disconnected locally compact group.

If E is a symplectic space, that is, if $\epsilon = -1$ and $A = \mathbb{k}$, then $\check{\text{Sp}}(E) := \check{U}(E)$ is equal to the subgroup of $\text{GSp}(E)$ with similitudes ± 1 . Denote by

$$(3) \quad 1 \rightarrow \{\pm 1\} \rightarrow \check{\text{Sp}}(E) \rightarrow \text{Sp}(E) \rightarrow 1$$

the metaplectic cover of the symplectic group $\mathrm{Sp}(E)$. It is shown in [Mœglin et al. 1987, page 36] that there is a unique continuous action

$$(4) \quad \widetilde{\mathrm{Ad}} : \check{\mathrm{Sp}}(E) \times \widetilde{\mathrm{Sp}}(E) \rightarrow \widetilde{\mathrm{Sp}}(E)$$

of $\check{\mathrm{Sp}}(E)$ on $\widetilde{\mathrm{Sp}}(E)$ as group automorphisms that lifts the adjoint action

$$\mathrm{Ad} : \check{\mathrm{Sp}}(E) \times \mathrm{Sp}(E) \rightarrow \mathrm{Sp}(E)$$

and leaves the central element $-1 \in \widetilde{\mathrm{Sp}}(E)$ fixed.

We first extend [Theorem 1.1](#) to the case of metaplectic groups:

Theorem 1.2. *Assume that E is a symplectic space. Let $\check{g} \in \check{\mathrm{Sp}}(E) \setminus \mathrm{Sp}(E)$, and let π be a genuine irreducible admissible smooth representation of $\check{\mathrm{Sp}}(E)$. Then π^\vee is isomorphic to $\pi^{\check{g}}$.*

Here and henceforth, “genuine” means that the central element $-1 \in \widetilde{\mathrm{Sp}}(E)$ acts via the scalar multiplication by -1 .

Remark. In the case that the character of π is a locally integrable function, [Theorem 1.2](#) is proved in [Mœglin et al. 1987, Theorem 4.II.2].

Harish-Chandra [1999] proved locally integrability of irreducible characters for p -adic linear reductive groups, but he did not treat metaplectic groups.

The proofs of [Theorem 1.1](#) in [Mœglin et al. 1987] and [Theorem 1.2](#) in [Section 2](#) do not depend on locally integrability of irreducible characters.

Now we consider dual pairs. Write $\epsilon' := -\epsilon$, and let $(E', \langle \cdot, \cdot \rangle_{E'})$ be an ϵ' -Hermitian A -module. Then $\mathbf{E} := E \otimes_A E'$ is a skew-Hermitian A -module under the form $\langle u \otimes u', v \otimes v' \rangle_{\mathbf{E}} := \langle u, v \rangle_E \langle u', v' \rangle_{E'}$. Write $\mathbf{E}_{\mathbb{k}} := \mathbf{E}$, viewed as a \mathbb{k} -symplectic space under the form $\langle u, v \rangle_{\mathbf{E}_{\mathbb{k}}} := \mathrm{tr}_{A/\mathbb{k}}(\langle u, v \rangle_{\mathbf{E}})$. Put

$$G := \mathrm{U}(E), \quad \check{G} := \check{\mathrm{U}}(E), \quad G' := \mathrm{U}(E'), \quad \check{G}' := \check{\mathrm{U}}(E').$$

The group G obviously maps to the symplectic group $\mathrm{Sp}(\mathbf{E}_{\mathbb{k}})$. Define the fiber product $\widetilde{G} := \widetilde{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}}) \times_{\mathrm{Sp}(\mathbf{E}_{\mathbb{k}})} G$. This is a double cover of G that depends on both E and E' .

In what follows, we define an action

$$(5) \quad \widetilde{\mathrm{Ad}} : \check{G} \times \widetilde{G} \rightarrow \widetilde{G}$$

that lifts the adjoint action $\mathrm{Ad} : \check{G} \times G \rightarrow G$ and fixes the central element $-1 \in \widetilde{G}$. Let $\check{g} = (g, \delta) \in \check{G}$. Choose an arbitrary element $(g', \delta) \in \check{G}'$. Then

$$\check{g} := (g \otimes g', \delta) \in \check{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}}),$$

and the automorphism

$$(6) \quad \widetilde{\mathrm{Ad}}_{\check{g}} \times \mathrm{Ad}_{\check{g}} : \widetilde{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}}) \times G \rightarrow \widetilde{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}}) \times G$$

leaves the subgroup \tilde{G} stable. It restricts to an automorphism

$$(7) \quad \tilde{\text{Ad}}_{\check{g}} : \tilde{G} \rightarrow \tilde{G}$$

that is independent of the choice of g' . We obtain (5) by gluing (7) for all $\check{g} \in \check{G}$.

The following is a generalization of [Theorem 1.2](#) in the setting of dual pairs.

Theorem 1.3. *Let $\check{g} \in \check{G} \setminus G$, and let π be a genuine irreducible admissible smooth representation of \tilde{G} . Then π^\vee is isomorphic to $\pi^{\check{g}}$.*

Remark. When $E' = A = \mathbb{k}$ and $\epsilon = -1$, [Theorem 1.3](#) specializes to [Theorem 1.2](#). The statement for the general case reduces essentially to those of [Theorem 1.1](#) and [Theorem 1.2](#). [Theorem 1.3](#) is proved in [Section 3](#).

[Theorem 1.3](#) has the following consequence, which is known to experts (up to a proof of [Theorem 1.2](#)). As far as the author knows, no proof of it in full generality is found in the literature.

Theorem 1.4. *Denote by ω_ψ the smooth oscillator representation of $\tilde{\text{Sp}}(E_{\mathbb{k}})$ corresponding to a nontrivial character ψ of \mathbb{k} . Then for all genuine irreducible admissible smooth representation π of \tilde{G} and π' of \tilde{G}' , we have*

$$\dim \text{Hom}_{G \times G'}(\omega_\psi \otimes \pi \otimes \pi', \mathbb{C}) = \dim \text{Hom}_{G \times G'}(\omega_\psi^\vee \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}).$$

Here $\tilde{G}' := \tilde{\text{Sp}}(E_{\mathbb{k}}) \times_{\text{Sp}(E_{\mathbb{k}})} G'$ is a double cover of G' . Note that both $\omega_\psi \otimes \pi \otimes \pi'$ and $\omega_\psi^\vee \otimes \pi^\vee \otimes \pi'^\vee$, which are originally representations of $\tilde{G} \times \tilde{G}'$, descend to representations of $G \times G'$.

Remark. In a follow-up paper [[Li et al. 2009](#)], [Theorem 1.4](#) is used to prove multiplicity preservations in theta correspondences (for all residue characteristics), that is, the dimension in [Theorem 1.4](#) is at most one. This is the main reason for providing a detailed proof of [Theorem 1.4](#) here.

In the archimedean case, the analog of [Theorem 1.4](#) is proved by T. Przebinda [[1988](#), [Theorem 5.5](#)], while the analog of [Theorem 1.3](#) is a consequence of [[1988](#), [Theorem 2.6](#)]. His method is different from ours in that he uses the Langlands classification.

As shown in [[Przebinda 1988](#)], [Theorem 1.4](#) together with the Howe duality conjecture implies that theta lifting maps Hermitian representations to Hermitian representations.

2. [Theorem 1.2](#) and its analog

Throughout this section, we assume that $\epsilon = -1$.

2.1. Skew Hermitian modules and Jacobi groups. As in the last section, E is an ϵ -Hermitian A -module, and $E_{\mathbb{k}} := E$ is a symplectic space under the form

$$\langle u, v \rangle_{E_{\mathbb{k}}} := \operatorname{tr}_{A/\mathbb{k}}(\langle u, v \rangle_E).$$

Denote by $H(E) := E_{\mathbb{k}} \times \mathbb{k}$ the Heisenberg group associated to $E_{\mathbb{k}}$, whose multiplication is given by $(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_{E_{\mathbb{k}}})$. The group $\check{U}(E)$ acts on $H(E)$ as group automorphisms by

$$(8) \quad (g, \delta)(u, t) := (gu, \delta t).$$

It defines a semidirect product $\check{J}(E) := \check{U}(E) \ltimes H(E)$, which contains $J(E) := U(E) \ltimes H(E)$ as a subgroup of index two.

The results of this note depend heavily on the following.

Lemma 2.1 [Sun 2009, Theorem D]. *Let f be a generalized function on $J(E)$. If f is invariant under conjugations by $U(E)$, that is,*

$$f(gxg^{-1}) = f(x) \quad \text{for all } g \in U(E),$$

then

$$f(\check{g}x\check{g}^{-1}) = f(x^{-1}) \quad \text{for all } \check{g} \in \check{U}(E) \setminus U(E).$$

Actually, we only need the following lemma, which is much weaker.

Lemma 2.2. *Let f be a conjugation-invariant generalized function on $J(E)$. Then*

$$f(\check{g}x\check{g}^{-1}) = f(x^{-1}) \quad \text{for all } \check{g} \in \check{J}(E) \setminus J(E).$$

A consequence of Lemma 2.2 is this:

Proposition 2.3. *Let $\check{g} \in \check{J}(E) \setminus J(E)$, and let π be an irreducible admissible smooth representation of $J(E)$. Then $\pi^{\check{g}}$ is isomorphic to $\pi^{\check{g}}$.*

Proof. Denote by f the character of π , which is thus a conjugation-invariant generalized function on $J(E)$. Therefore

$$(9) \quad f(\check{g}x\check{g}^{-1}) = f(x^{-1})$$

by Lemma 2.2. The left side of (9) is the character of $\pi^{\check{g}}$, and the right side is the character of $\pi^{\check{g}}$. Therefore $\pi^{\check{g}}$ and $\pi^{\check{g}}$ have the same character, and they are thus isomorphic to each other. \square

2.2. Proof of Theorem 1.2 and its analog. We reuse the notation of Section 2.1. Denote by

$$\tilde{U}(E) := \tilde{\operatorname{Sp}}(E_{\mathbb{k}}) \times_{\operatorname{Sp}(E_{\mathbb{k}})} U(E)$$

the double cover of $U(E)$ induced by the metaplectic cover

$$(10) \quad 1 \rightarrow \{\pm 1\} \rightarrow \tilde{\operatorname{Sp}}(E_{\mathbb{k}}) \rightarrow \operatorname{Sp}(E_{\mathbb{k}}) \rightarrow 1.$$

As in (5), we have an action

$$(11) \quad \widetilde{\text{Ad}} : \check{\text{U}}(E) \times \widetilde{\text{U}}(E) \rightarrow \widetilde{\text{U}}(E).$$

The following theorem reduces to [Theorem 1.2](#) when $A = \mathbb{k}$.

Theorem 2.4. *Assume that $\epsilon = -1$. Let $\check{g} \in \check{\text{U}}(E) \setminus \text{U}(E)$, and let π be a genuine irreducible admissible smooth representation of $\widetilde{\text{U}}(E)$. Then π^\vee is isomorphic to $\pi^{\check{g}}$.*

Proof. Denote by ω_ψ the smooth oscillator representation of $\widetilde{\text{Sp}}(E_{\mathbb{k}}) \times \text{H}(E)$ that corresponds to a nontrivial character ψ of \mathbb{k} . Up to isomorphism, this is the only genuine smooth representation that, as a representation of $\text{H}(E)$, is irreducible and has central character ψ .

Both ω_ψ and π are viewed as smooth representations of $\check{\text{J}}(E) := \widetilde{\text{U}}(E) \times \text{H}(E)$, via the restriction and the inflation, respectively. The tensor product $\omega_\psi \otimes \pi$ descends to an irreducible admissible smooth representation of $\text{J}(E)$ [[Sun 2009](#), Lemma 5.3].

The actions of $\check{\text{U}}(E)$ on $\widetilde{\text{U}}(E)$, $\text{U}(E)$ and $\text{H}(E)$ induce its actions on the semi-direct products $\check{\text{J}}(E)$ and $\text{J}(E)$. By [Proposition 2.3](#),

$$(\omega_\psi \otimes \pi)^{\check{g}} \cong (\omega_\psi \otimes \pi)^\vee$$

as irreducible admissible smooth representations of $\text{J}(E)$, or equivalently

$$\omega_\psi^{\check{g}} \otimes \pi^{\check{g}} \cong \omega_\psi^\vee \otimes \pi^\vee.$$

Note that $\omega_\psi^{\check{g}} \cong \omega_\psi^\vee$ as smooth representations of $\check{\text{J}}(E)$. (This is a special case of [Lemma 3.3](#).) Therefore

$$(12) \quad \omega_\psi^\vee \otimes \pi^{\check{g}} \cong \omega_\psi^\vee \otimes \pi^\vee.$$

As in the proof of [[Sun 2009](#), Lemma 5.3], we have

$$(13) \quad \pi^{\check{g}} \cong \text{Hom}_{\text{H}(E)}(\omega_\psi^\vee, \omega_\psi^\vee \otimes \pi^{\check{g}}).$$

Here the right side carries the action of $\widetilde{\text{U}}(E)$ given by $(\check{g}\phi)(v) := g(\phi(\check{g}^{-1}v))$, where

$$\check{g} \in \widetilde{\text{U}}(E), \quad \phi \in \text{Hom}_{\text{H}}(\omega_\psi^\vee, \omega_\psi^\vee \otimes \pi^{\check{g}}), \quad v \in \omega_\psi^\vee,$$

and g is the image of \check{g} under the covering map $\widetilde{\text{U}}(E) \rightarrow \text{U}(E)$. Similarly,

$$(14) \quad \pi^\vee \cong \text{Hom}_{\text{H}}(\omega_\psi^\vee, \omega_\psi^\vee \otimes \pi^\vee).$$

We finish the proof by combining (12), (13) and (14). □

3. Proofs of Theorem 1.3 and Theorem 1.4

3.1. Proof of Theorem 1.3 for symplectic groups. Now we return to the notation of Section 1. First assume that $A = \mathbb{k}$ and $\epsilon = -1$. Then G is a symplectic group and is thus perfect, that is, G equals its own commutator group. Consequently, there is only one action of \check{G} on \check{G} that lifts the adjoint action and fixes the central element $-1 \in \check{G}$. There are two cases.

Case 1. The covering map $\check{G} \rightarrow G$ splits. Then $\check{G} = G \times \{\pm 1\}$, and Theorem 1.3 is one case of Theorem 1.1.

Case 2. The covering map $\check{G} \rightarrow G$ does not split. Then $\check{G} = \check{S}p(E)$ [Moore 1968, Theorem 10.4], and Theorem 1.3 is one case of Theorem 1.2.

3.2. Proof of Theorem 1.3 when $A \neq \mathbb{k}$. Assume that $A \neq \mathbb{k}$. Then $U(E)$ is a general linear group or a unitary group.

Lemma 3.1. *There exists a genuine character on $\check{U}(E)$.*

Proof. It is well known that the exact sequence

$$1 \rightarrow \mathbb{C}^\times \rightarrow (\check{S}p(\mathbf{E}_{\mathbb{k}}) \times \mathbb{C}^\times) / \text{diag}(\{\pm 1\}) \rightarrow Sp(\mathbf{E}_{\mathbb{k}}) \rightarrow 1$$

splits continuously over $U(E)$ (this is trivial for general linear groups, and for unitary groups, see [Kudla 1994, Proposition 4.1] or [Harris et al. 1996, Section 1]). Write ι for such a splitting and write $p : \check{U}(E) \rightarrow U(E)$ for the covering map. Then $x \in \check{U}(E) \mapsto x^{-1} \iota(p(x)) \in \mathbb{C}^\times$ is a genuine character. \square

Lemma 3.2. *There exists a genuine character χ of \check{G} such that $\chi^{\check{g}} = \chi^{-1}$ for all $\check{g} \in \check{G} \setminus G$.*

Proof. As in Section 1, let $\check{g} = (g, -1) \in \check{G} \setminus G$ and $(g', -1) \in \check{G}' \setminus G'$, and write $\check{g} := (g \otimes g', -1) \in \check{U}(E) \setminus U(E)$. It is obvious that the diagram

$$(15) \quad \begin{array}{ccc} \check{U}(E) & \xrightarrow{\text{Ad}_{\check{g}}} & \check{U}(E) \\ \uparrow & & \uparrow \\ \check{G} & \xrightarrow{\text{Ad}_{\check{g}}} & \check{G} \end{array}$$

commutes.

Take a character χ_E as in Lemma 3.1, and denote by χ its restriction to \check{G} . Then

$$\begin{aligned} \chi^{\check{g}} &= (\chi_E|_{\check{G}})^{\check{g}} \\ &= (\chi_E^{\check{g}})|_{\check{G}} && \text{by commutativity of (15)} \\ &= (\chi_E^{-1})|_{\check{G}} && \text{by Theorem 2.4} \\ &= \chi^{-1}. \end{aligned} \quad \square$$

Fix χ as in Lemma 3.2. Let $\check{g} \in \check{G} \setminus G$, and let π be a genuine irreducible admissible smooth representation of \check{G} . Then $\pi \otimes \chi$ descends to an irreducible admissible smooth representation of G . By Theorem 1.1, $(\pi \otimes \chi)^{\check{g}} \cong (\pi \otimes \chi)^\vee$, or equivalently, $\pi^{\check{g}} \otimes \chi^{\check{g}} \cong \pi^\vee \otimes \chi^{-1}$. Therefore, $\pi^{\check{g}} \cong \pi^\vee$ since $\chi^{\check{g}} = \chi^{-1}$. This proves Theorem 1.3 when $A \neq \mathbb{k}$.

3.3. Proof of Theorem 1.3 for orthogonal groups. Assume that $A = \mathbb{k}$ and $\epsilon = 1$, that is, G is an orthogonal group. In what follows, we show that Lemma 3.2 still holds in this case. Fix a complete polarization $E' = E'_+ \oplus E'_-$ of the symplectic space E' . Then $E = E_+ \oplus E_-$ is a complete polarization of the symplectic space E , where $E_\pm := E \otimes E'_\pm$. Depending on this polarization, we define a skew-Hermitian $\mathbb{k} \times \mathbb{k}$ -module E' as follows. As an abelian group, $E' = E$. The scalar multiplication is given by

$$(ae_1 + be_2)(u + v) := au + bv \quad \text{for } a, b \in \mathbb{k}, u \in E_+, v \in E_-,$$

where $e_1 := (1, 0)$ and $e_2 := (0, 1)$ are the two idempotent elements of $\mathbb{k} \times \mathbb{k}$. The skew-Hermitian form is given by

$$\langle u_+ + u_-, v_+ + v_- \rangle_{E'} := \langle u_+, v_- \rangle_E e_1 + \langle u_-, v_+ \rangle_E e_2,$$

where $u_+, v_+ \in E_+, u_-, v_- \in E_-$.

Let $\check{g} = (g, -1) \in \check{G} \setminus G$. Choose an element $(g', -1) \in \check{G}' \setminus G'$ such that $g'(E'_+) = E'_-$ and $g'(E'_-) = E'_+$. Then

$$\check{g} := (g \otimes g', -1) \in \check{U}(E') \setminus U(E'),$$

and we have a commutative diagram

$$\begin{array}{ccc} \check{U}(E') & \xrightarrow{\text{Ad}_{\check{g}}} & \check{U}(E') \\ \uparrow & & \uparrow \\ \check{G} & \xrightarrow{\text{Ad}_{\check{g}}} & \check{G}. \end{array}$$

Take a genuine character $\chi_{E'}$ of $\check{U}(E')$ as in Lemma 3.1, and denote by χ its restriction to \check{G} . Then as in the proof of Lemma 3.2, we show that χ fulfills the requirement of Lemma 3.2. Now we argue as in the end of the last subsection, and prove Theorem 1.3 for orthogonal groups.

3.4. Proof of Theorem 1.4. The group

$$\check{G} := \check{G} \times_{\{\pm 1\}} \check{G}' = \{(g, g', \delta) \mid (g, \delta) \in \check{G}, (g', \delta) \in \check{G}'\}$$

contains $G := G \times G'$ as a subgroup of index two. Define a homomorphism

$$\xi : \check{G} \rightarrow \check{S}p(E_{\mathbb{k}}), \quad (g, g', \delta) \mapsto (g \otimes g', \delta).$$

By using the covering map $\tilde{G} \times \tilde{G}' \rightarrow \mathbf{G} = G \times G'$ and the map $\xi|_{\mathbf{G}} : \mathbf{G} \rightarrow \mathrm{Sp}(\mathbf{E}_{\mathbb{k}})$, we form the semidirect product $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$ as in [Section 2.1](#). Let $\check{\mathbf{G}}$ act on $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$ as group automorphisms by

$$(16) \quad \check{\mathbf{g}}(x, y, z) := (\widetilde{\mathrm{Ad}}_{\check{\mathbf{g}}}(x), \widetilde{\mathrm{Ad}}_{\check{\mathbf{g}}}(y), \xi(\check{\mathbf{g}})z),$$

where

$$\check{\mathbf{g}} = (g, g', \delta), \quad \check{\mathbf{g}} = (g, \delta), \quad \check{\mathbf{g}}' = (g', \delta),$$

and the last term of the right hand side of (16) is defined as in (8).

Let ω_{ψ} , π and π' be as in [Theorem 1.4](#).

Lemma 3.3. *View ω_{ψ} as an admissible smooth representation of $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$ (via the restriction). Then for every $\check{\mathbf{g}} \in \check{\mathbf{G}} \setminus \mathbf{G}$, we have*

$$\omega_{\psi}^{\vee} \cong \omega_{\psi}^{\check{\mathbf{g}}}.$$

Proof. Recall that the group $\check{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}})$ acts on $\check{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}}) \ltimes \mathrm{H}(\mathbf{E})$ diagonally through its action on the two factors. We have

$$(17) \quad \omega_{\psi}^{\vee} \cong \omega_{\psi}^{\xi(\check{\mathbf{g}})}$$

as smooth oscillator representations of $\check{\mathrm{Sp}}(\mathbf{E}_{\mathbb{k}}) \ltimes \mathrm{H}(\mathbf{E})$, since both correspond to the character ψ^{-1} . We prove the lemma by restricting both sides of (17) to the group $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$. \square

Lemma 3.4. *Via the inflations, view π and π' as admissible smooth representations of $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$. Then for every $\check{\mathbf{g}} \in \check{\mathbf{G}} \setminus \mathbf{G}$, we have*

$$(18) \quad \pi^{\vee} \cong \pi^{\check{\mathbf{g}}} \quad \text{and} \quad \pi'^{\vee} \cong \pi'^{\check{\mathbf{g}}}.$$

Proof. Write $\check{\mathbf{g}} = (g, g', -1)$ and $\check{\mathbf{g}} = (g, -1)$. By [Theorem 1.3](#), we have $\pi^{\vee} \cong \pi^{\check{\mathbf{g}}}$ as irreducible admissible smooth representations of \tilde{G} . By pulling back this isomorphism to the group $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$, we obtain the first isomorphism of (18). The second isomorphism follows similarly. \square

Lemma 3.5. *For every $\check{\mathbf{g}} \in \check{\mathbf{G}} \setminus \mathbf{G}$, we have*

$$(19) \quad \omega_{\psi}^{\vee} \otimes \pi^{\vee} \otimes \pi'^{\vee} \cong (\omega_{\psi} \otimes \pi \otimes \pi')^{\check{\mathbf{g}}}$$

as smooth representations of $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$.

Proof. This is a combination of [Lemma 3.3](#) and [Lemma 3.4](#). \square

Fix an element $\check{\mathbf{g}} \in \check{\mathbf{G}} \setminus \mathbf{G}$. Since the action of $\check{\mathbf{g}}$ stabilizes the subgroup $\tilde{G} \times \tilde{G}'$ of $(\tilde{G} \times \tilde{G}') \ltimes \mathrm{H}(\mathbf{E})$, we have

$$(20) \quad \mathrm{Hom}_{\tilde{G} \times \tilde{G}'}(\omega_{\psi} \otimes \pi \otimes \pi', \mathbb{C}) = \mathrm{Hom}_{\tilde{G} \times \tilde{G}'}((\omega_{\psi} \otimes \pi \otimes \pi')^{\check{\mathbf{g}}}, \mathbb{C}).$$

Now [Theorem 1.4](#) is a consequence of (19) and (20).

Acknowledgments

I thank the referee for many detailed suggestions.

References

- [Bernšteĭn and Zelevinskiĭ 1976] I. N. Bernšteĭn and A. V. Zelevinskiĭ, “Representations of the group $GL(n, F)$, where F is a local non-Archimedean field”, *Uspehi Mat. Nauk* **31**:3 (1976), 5–70. In Russian; translated in *Russian Mathematical Surveys* **31**:3 (1976), 1–68. [MR 54 #12988](#)
- [Harish-Chandra 1999] Harish-Chandra, *Admissible invariant distributions on reductive p -adic groups*, University Lecture Series **16**, American Mathematical Society, Providence, RI, 1999. [MR 2001b:22015](#) [Zbl 0928.22017](#)
- [Harris et al. 1996] M. Harris, S. S. Kudla, and W. J. Sweet, “Theta dichotomy for unitary groups”, *J. Amer. Math. Soc.* **9**:4 (1996), 941–1004. [MR 96m:11041](#) [Zbl 0870.11026](#)
- [Kudla 1994] S. S. Kudla, “Splitting metaplectic covers of dual reductive pairs”, *Israel J. Math.* **87**:1-3 (1994), 361–401. [MR 95h:22019](#) [Zbl 0840.22029](#)
- [Li et al. 2009] J.-S. Li, B. Sun, and Y. Tian, “The multiplicity one conjecture for local theta correspondences”, preprint, 2009. To appear in *Invent. Math.* [arXiv 0903.1419](#)
- [Mœglin et al. 1987] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p -adique*, Lecture Notes in Mathematics **1291**, Springer, Berlin, 1987. [MR 91f:11040](#) [Zbl 0642.22002](#)
- [Moore 1968] C. C. Moore, “Group extensions of p -adic and adelic linear groups”, *Inst. Hautes Études Sci. Publ. Math.* **35** (1968), 157–222. [MR 39 #5575](#) [Zbl 0159.03203](#)
- [Przebinda 1988] T. Przebinda, “On Howe’s duality theorem”, *J. Funct. Anal.* **81**:1 (1988), 160–183. [MR 89j:22031](#) [Zbl 0678.22007](#)
- [Sun 2009] B. Sun, “Multiplicity one theorems for Fourier–Jacobi models”, preprint, 2009. [arXiv 0903.1417](#)

Received December 7, 2009. Revised April 21, 2010.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

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Typeset in L^AT_EX

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PACIFIC JOURNAL OF MATHEMATICS

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