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NONCONVENTIONAL ERGODIC AVERAGES AND MULTIPLE RECURRENCE FOR VON NEUMANN DYNAMICAL SYSTEMS

TIM AUSTIN, TANJA EISNER AND TERENCE TAO

The Furstenberg recurrence theorem (or equivalently Szemerédi's theorem) can be formulated in the language of von Neumann algebras as follows: given an integer $k \geq 2$, an abelian finite von Neumann algebra $(\mathcal{M}, \tau)$ with an automorphism $\alpha : \mathcal{M} \to \mathcal{M}$, and a nonnegative $a \in \mathcal{M}$ with $\tau(a) > 0$, one has $\liminf_{N\to\infty} N^{-1} \sum_{n=1}^{N} \operatorname{Re} \tau(a\alpha^n(a) \cdots \alpha^{(k-1)n}(a)) > 0$; a later result of Host and Kra shows this limit exists. In particular, $\operatorname{Re} \tau(a\alpha^n(a) \cdots \alpha^{(k-1)n}(a))$ is positive for all n in a set of positive density.

From the von Neumann algebra perspective, it is natural to ask to what remains of these results when the abelian hypothesis is dropped. All three claims hold for $k = 2$, and we show that all three claims hold for all k when the von Neumann algebra is asymptotically abelian, and that the last two claims hold for $k = 3$ when the von Neumann algebra is ergodic. However, we show that the first claim can fail for $k = 3$ even with ergodicity, the second claim can fail for $k \geq 4$ even when assuming ergodicity, and the third claim can fail for $k = 3$ without ergodicity, or $k \geq 5$ and odd assuming ergodicity. The second claim remains open for nonergodic systems with $k = 3$, and the third claim remains open for ergodic systems with $k = 4$.

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1. Introduction

1a. ***Multiple recurrence.*** Let $(X, \mathscr{X}, \mu)$ be a probability space, and let $T : X \to X$ be a measure-preserving invertible transformation on X (that is, T and T^{-1} are both measurable, and $\mu(T(A)) = \mu(A)$ for all measurable A). From the mean ergodic theorem we know that for any $f \in L^\infty(X)$, the averages $N^{-1}\sum_{n=1}^N f \circ T^{-n}$ converge in (say) $L^2(X)$ norm,[1] which implies in particular that the averages $N^{-1}\sum_{n=1}^N \int_X f_1(f_2 \circ T^{-n})\, d\mu$ converge for all $f_1, f_2 \in L^\infty(X)$. Furthermore, if $f_1 = f_2 = f$ is nonnegative with positive mean $\int_X f\, d\mu > 0$, then the Poincaré recurrence theorem implies that this latter limit is strictly positive. In particular, this implies that the mean $\int_X f(f \circ T^{-n})\, d\mu$ is positive for all natural numbers n in a set $E \subset \mathbb{N}$ of positive (lower) density (that is, the set E is a set such that $\liminf_{N\to\infty} N^{-1}\#\{1 \le n \le N : n \in E\} > 0$).

Thanks to a long effort starting with Furstenberg's ground breaking new proof [1977] of Szemerédi's theorem on arithmetic progressions [1975], it is now known that all of these single recurrence results extend to multiple recurrence:

Theorem 1.1 (abelian multiple recurrence). *Let $(X, \mathscr{X}, \mu)$ be a probability space, let $k \ge 2$ be an integer, and let $T : X \to X$ be a measure-preserving invertible transformation.*

- (Convergence in norm.) *For any $f_1, \dots, f_{k-1} \in L^\infty(X)$, the averages*

$$\frac{1}{N}\sum_{n=1}^N (f_1 \circ T^{-n}) \cdots (f_{k-1} \circ T^{-(k-1)n})$$

  converge in $L^2(X)$ norm as $N \to \infty$.

- (Weak convergence.) *For any $f_0, f_1, \dots, f_{k-1} \in L^\infty(X)$, the averages*

$$\frac{1}{N}\sum_{n=1}^N \int_X f_0(f_1 \circ T^{-n}) \cdots (f_{k-1} \circ T^{-(k-1)n})\, d\mu$$

  converge as $N \to \infty$.

- (Recurrence on average.) *For any nonnegative $f \in L^\infty(X)$ with $\int_X f\, d\mu > 0$, one has*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \int_X f(f \circ T^{-n}) \cdots (f \circ T^{-(k-1)n})\, d\mu > 0. \tag{1}$$

[1] The minus sign here is not of particular significance (other than to conform to some minor notational conventions) and can be ignored in the sequel if desired.

- (Recurrence on a dense set.) *For any nonnegative* $f \in L^\infty(X)$ *such that* $\int_X f\, d\mu > 0$, *one has*

$$\int_X f(f \circ T^{-n}) \cdots (f \circ T^{-(k-1)n})\, d\mu > c > 0 \tag{2}$$

for some $c > 0$ *and all* n *in a set of natural numbers of positive lower density.*

We have called this result the "abelian" multiple recurrence theorem in order to emphasise the abelian nature of the algebra $L^\infty(X)$.

Remarks 1.2. Clearly, convergence in norm implies weak convergence; also, because the averages (2) are bounded and nonnegative, recurrence on average implies recurrence on a dense set. Using the weak convergence result, the limit inferior in (1) can be replaced with a limit, but we have retained the limit inferior in order to keep the two claims logically independent of each other.

As mentioned earlier, the $k = 2$ cases of Theorem 1.1 follow from classical ergodic theorems. Furstenberg [1977] established recurrence on average (and hence recurrence on a dense set) for all k, and observed that this result was equivalent (by what is now known as the *Furstenberg correspondence principle*) to Szemerédi's famous theorem [1975] on arithmetic progressions, thus providing an important new proof of that theorem. Convergence in norm (and hence in mean) was established for $k = 3$ by Furstenberg [1977], for $k = 4$ by Conze and Lesigne [1984; 1988a; 1988b] assuming total ergodicity and by Host and Kra [2001] in general, for $k = 5$ in some cases by Ziegler [2005], and for all k by Host and Kra [2005] and subsequently also by Ziegler [2007]. See [Kra 2006] for a survey of these results, and their relation to other topics such as dynamics of nilsequences, and arithmetic progressions in number-theoretic sets such as the primes.

There is also a multidimensional generalisation of the results above to multiple commuting shifts:

Theorem 1.3 (abelian multidimensional multiple recurrence). *Let* $(X, \mathcal{X}, \mu)$ *be a probability space, let* $k \geq 2$ *be an integer, and let* $T_0, \dots, T_{k-1} : X \to X$ *be a commuting system of measure-preserving invertible transformations.*

- (Convergence in norm.) *For any* $f_1, \dots, f_{k-1} \in L^\infty(X)$, *the averages*

$$\frac{1}{N} \sum_{n=1}^{N} T_0^n((f_1 \circ T_1^{-n}) \cdots (f_{k-1} \circ T_{k-1}^{-n}))$$

converge in $L^2(X)$ *norm.*

- (Weak convergence.) *For any* $f_0, f_1, \dots, f_{k-1} \in L^\infty(X)$, *the averages*

$$\frac{1}{N}\sum_{n=1}^{N}\int_X (f_0 \circ T_0^{-n})(f_1 \circ T_1^{-n}) \cdots (f_{k-1} \circ T_{k-1}^{-n})\, d\mu$$

converge.

- (Recurrence on average.) *For any nonnegative* $f \in L^\infty(X)$ *with* $\int_X f\, d\mu > 0$, *one has*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_X (f \circ T_0^{-n})(f \circ T_1^{-n}) \cdots (f \circ T_{k-1}^{-n})\, d\mu > 0. \tag{3}$$

- (Recurrence on a dense set.) *For any nonnegative* $f \in L^\infty(X)$ *such that* $\int_X f\, d\mu > 0$, *one has*

$$\int_X (f \circ T_0^{-n})(f \circ T_1^{-n}) \cdots (f \circ T_{k-1}^{-n})\, d\mu > c > 0 \tag{4}$$

for some $c > 0$ *and all* n *in a set of natural numbers of positive lower density.*

Of course, Theorem 1.1 is the special case of Theorem 1.3 when $T_i := T^i$. It is often customary to normalise T_0 to be the identity transformation (by replacing each of the T_i with $T_0^{-1}T_i$).

Remarks 1.4. The $k = 2$ case is again classical. Recurrence on average (and hence on a dense set) in this theorem was established for all k by Furstenberg and Katznelson [1978], which by the Furstenberg correspondence principle implies a multidimensional version of Szemerédi's theorem, a combinatorial proof of which in full generality has only been obtained relatively recently in [Nagle et al. 2006] and [Gowers 2006]. Convergence in norm (and weak convergence) was established for $k = 3$ in [Conze and Lesigne 1984], for some special cases of $k = 4$ in [Zhang 1996], for all k assuming total ergodicity in [Frantzikinakis and Kra 2005], and for all k unconditionally in [Tao 2008], with subsequent proofs in [Towsner 2007; Austin 2010; Host 2009]. The results can fail if the shifts $T_0, \dots, T_{k-1}$ do not commute [Bergelson and Leibman 2004]. Note that noncommutativity of the shifts should *not* be confused with the noncommutativity of the underlying algebra, which is the focus of this paper.

1b. ***Noncommutative analogues.*** From the perspective of the theory of von Neumann algebras, the space $L^\infty(X)$ appearing in these theorems can be interpreted as an abelian von Neumann algebra, with a finite trace $\tau(f) := \int_X f\, d\mu$ and with an automorphism $T : L^\infty(X) \to L^\infty(X)$ defined by $Tf := f \circ T^{-1}$. It is then natural to ask whether the results can be extended to nonabelian settings. More precisely, we recall the following definitions.

Definition 1.5 (noncommutative systems). A *finite von Neumann algebra* is a pair $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a von Neumann algebra (that is, an algebra of bounded operators on a separable[2] complex Hilbert space that contains the identity 1, is closed under adjoints, and is closed in the weak operator topology), and $\tau : \mathcal{M} \to \mathbb{C}$ is a finite faithful trace (that is, a linear map with $\tau(a^*) = \overline{\tau(a)}$, $\tau(ab) = \tau(ba)$, and $\tau(a^*a) \geq 0$ for all $a, b \in \mathcal{M}$, with $\tau(a^*a) = 0$ if and only if $a = 0$ and $\tau(1) = 1$). The operator norm of an element $a \in \mathcal{M}$ is denoted $\|a\|$. We say that an element $a \in \mathcal{M}$ is *nonnegative* if one has $a = b^*b$ for some $b \in \mathcal{M}$. An element $a \in \mathcal{M}$ is *central* if one has $ab = ba$ for all $b \in \mathcal{M}$. The set of all central elements is denoted $\mathcal{Z}(\mathcal{M})$ and referred to as the *centre* of $\mathcal{M}$; the algebra $\mathcal{M}$ is *abelian* if $\mathcal{Z}(\mathcal{M}) = \mathcal{M}$.

A *shift* α on a finite von Neumann algebra $(\mathcal{M}, \tau)$ is trace-preserving $*$-automorphism, that is, α is an algebra isomorphism such that $\alpha(a^*) = \alpha(a)^*$ and $\tau(\alpha(a)) = \tau(a)$ for all $a \in \mathcal{M}$. We say that the shift is *ergodic* if the invariant algebra $\{a \in \mathcal{M} : \alpha(a) = a\}$ consists only of the constants $\mathbb{C}1$. We refer to the triple $(\mathcal{M}, \tau, \alpha)$ as a *von Neumann $\mathbb{Z}$-system*, or a *von Neumann dynamical system*. More generally, if $\alpha_0, \dots, \alpha_{k-1}$ are k commuting shifts on M, we refer to $(\mathcal{M}, \tau, \alpha_0, \dots, \alpha_{k-1})$ as a *von Neumann $\mathbb{Z}^k$-system*.

It is easy to verify that if $(X, \mathcal{X}, \mu)$ is a (classical) probability space with a shift $T : X \to X$, then $(L^\infty(X), \int_X \cdot \, d\mu, \circ T^{-1})$ is an (abelian example of a) von Neumann dynamical system, and more generally if $T_0, \dots, T_{k-1} : X \to X$ are commuting shifts, then $(L^\infty(X), \int_X \cdot \, d\mu, \circ T_0^{-1}, \dots, \circ T_{k-1}^{-1})$ is an abelian example of a von Neumann $\mathbb{Z}^k$-system. In fact, all abelian von Neumann dynamical systems arise (up to isomorphism of the algebras) as such examples; see [Kadison and Ringrose 1997, Chapter 5].

A finite von Neumann algebra $(\mathcal{M}, \tau)$ gives rise to an inner product $\langle a, b \rangle := \tau(a^*b)$ on $\mathcal{M}$; the properties of the trace ensure that this inner product is positive definite. (We use the convention for a scalar product to be conjugate linear in the first coordinate.) The Hilbert space completion of $\mathcal{M}$ with respect to this inner product will be referred to as $L^2(\tau)$. Note that α extends to a unitary transformation on $L^2(\tau)$. In the abelian case when $\mathcal{M} = L^\infty(X, \mathcal{X}, \mu)$, the space $L^2(\tau)$ can be canonically identified with $L^2(X, \mathcal{X}, \mu)$.

Inspired by Theorems 1.1 and 1.3, we now make the following definitions:

Definition 1.6 (noncommutative recurrence and convergence). Let $k \geq 2$ be an integer, $(\mathcal{M}, \tau, \alpha)$ be a von Neumann dynamical system, and $(\mathcal{M}, \tau, \alpha_0, \dots, \alpha_{k-1})$ be a von Neumann $\mathbb{Z}^k$-system.

[2] In our applications, the hypothesis of separability can be omitted since one can always pass to the separable subalgebra generated by a finite collection $a_0, \dots, a_{k-1}$ of elements and their shifts if desired.

- We say $(\mathcal{M}, \tau, \alpha)$ enjoys *order k convergence in norm* if for any $a_1, \ldots, a_{k-1}$ in $\mathcal{M}$, the averages

$$\frac{1}{N}\sum_{n=1}^{N}(\alpha^n(a_1))(\alpha^{2n}(a_2))\cdots(\alpha^{(k-1)n}(a_{k-1}))$$

converge in $L^2(\tau)$ as $N \to \infty$.

- We say $(\mathcal{M}, \tau, \alpha)$ enjoys *order k weak convergence* if for any $a_0, a_1, \ldots, a_{k-1}$ in $\mathcal{M}$, the averages

$$\frac{1}{N}\sum_{n=1}^{N}\tau(a_0(\alpha^n(a_1))(\alpha^{2n}(a_2))\cdots(\alpha^{(k-1)n}(a_{k-1})))$$

converge as $N \to \infty$.

- We say $(\mathcal{M}, \tau, \alpha)$ enjoys *order k recurrence on average* if for any nonnegative $a \in \mathcal{M}$ with $\tau(a) > 0$, one has

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\operatorname{Re}\tau(a(\alpha^n(a))(\alpha^{2n}(a))\cdots(\alpha^{(k-1)n}(a))) > 0. \tag{5}$$

- We say that $(\mathcal{M}, \tau, \alpha)$ enjoys *order k recurrence on a dense set* if for any nonnegative $a \in \mathcal{M}$ with $\tau(a) > 0$, one has

$$\operatorname{Re}\tau(a(\alpha^n(a))(\alpha^{2n}(a))\cdots(\alpha^{(k-1)n}(a))) > c > 0 \tag{6}$$

for some $c > 0$ and all n in a set of natural numbers of positive lower density.

- We say $(\mathcal{M}, \tau, \alpha_0, \ldots, \alpha_{k-1})$ *converges in norm* if for any $a_1, \ldots, a_{k-1} \in \mathcal{M}$, the averages

$$\frac{1}{N}\sum_{n=1}^{N}\alpha_0^{-n}((\alpha_1^n(a_1))(\alpha_2^n(a_2))\cdots(\alpha_{k-1}^n(a_{k-1})))$$

converge in $L^2(\tau)$ as $N \to \infty$.

- We say $(\mathcal{M}, \tau, \alpha_0, \ldots, \alpha_{k-1})$ *converges weakly* if for any $a_0, a_1, \ldots, a_{k-1} \in \mathcal{M}$, the averages

$$\frac{1}{N}\sum_{n=1}^{N}\tau((\alpha_0^n(a_0))(\alpha_1^n(a_1))(\alpha_2^n(a_2))\cdots(\alpha_{k-1}^n(a_{k-1})))$$

converge as $N \to \infty$.

- We say that $(\mathcal{M}, \tau, \alpha_0, \ldots, \alpha_{k-1})$ enjoys *recurrence on average* if for any nonnegative $a \in \mathcal{M}$ with $\tau(a) > 0$, one has

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \operatorname{Re} \tau((\alpha_0^n(a))(\alpha_1^n(a)) \cdots (\alpha_{k-1}^n(a))) > 0. \tag{7}$$

- We say that $(\mathcal{M}, \tau, \alpha)$ enjoys *order k recurrence on a dense set* if for any nonnegative $a \in \mathcal{M}$ with $\tau(a) > 0$, one has

$$\operatorname{Re} \tau((\alpha_0^n(a))(\alpha_1^n(a)) \cdots (\alpha_{k-1}^n(a))) > c > 0. \tag{8}$$

  for some $c > 0$ and all n in a set of natural numbers of positive lower density.

Remark 1.7. As before, we may normalise α_0 to be the identity. Of course, the first four properties here are nothing more than the specialisations of the last four to the case $\alpha_i = \alpha^i$ for $0 \le i \le k-1$. The real part is needed in (5), (6), (7) and (8) because there is no necessity for the traces here to be real-valued (the difficulty being that the product of two nonnegative elements of a nonabelian von Neumann algebra need not remain nonnegative). In the case of (5), one can omit the real part by taking averages from $-N$ to N, since one has the symmetry

$$\begin{aligned}
\overline{\tau(a(\alpha^n(a))(\alpha^{2n}(a)) \cdots (\alpha^{(k-1)n}(a)))} &= \tau((a(\alpha^n(a))(\alpha^{2n}(a)) \cdots (\alpha^{(k-1)n}(a)))^*) \\
&= \tau((\alpha^{(k-1)n}(a)) \cdots (\alpha^{2n}(a))(\alpha^n(a))a) \\
&= \tau(a(\alpha^{-n}(a)) \cdots (\alpha^{-(k-1)n}(a)))
\end{aligned}$$

for any self-adjoint a.

Note however that it is quite possible for the expressions (6) or (8) to be negative even when a is nonnegative. Because of this, while recurrence on average still implies recurrence on a dense set, the converse is not true; one can have recurrence on a dense set but end up with a zero or even negative average due to the presence of large negative values of (6) or (8). We will see examples of this later.

Remark 1.8. As we said earlier, the Furstenberg correspondence principle equates recurrence results with combinatorial statements (such as Szemerédi's theorem) that can be formulated in a purely finitary fashion. However, we do not know whether the same is true for noncommutative recurrence results. Formulating a finitary statement that would imply recurrence results for some nonabelian von Neumann dynamical system probably requires some quite strong approximate embeddability of the system into finite-dimensional matrix algebras with approximate shifts, together with a recurrence assertion for such finite-dimensional systems in which the various parameters may all be chosen independent of the dimension. Since many of the results we prove below in the infinitary setting are negative anyway, we will not pursue this issue here.

These properties (and related topics) for von Neumann dynamical systems have been studied by Niculescu, Ströh and Zsidó [2003], Duvenhage [2009], Beyers, Duvenhage and Ströh [2010], and Fidaleo [2009]. A variant of these questions, in which one averages over a higher-dimensional range of shifts, was also studied in [Fidaleo 2007]. In this paper we shall develop further positive and negative results regarding these properties, which we now present.

1c. ***Positive results.*** When $k = 2$, all systems enjoy norm and weak convergence, as well as recurrence on average and on a dense set, thanks to the ergodic theorem for von Neumann algebras; see for example [Krengel 1985, Section 9.1]. Indeed, from that theorem, we know that for any von Neumann dynamical system $(\mathcal{M}, \tau, \alpha)$ and $a \in \mathcal{M}$, the averages $N^{-1} \sum_{n=1}^{N} \alpha^n(a)$ converge in $L^2(\tau)$ to the orthogonal projection of a to the invariant space $L^2(\tau)^\alpha := \{f \in L^2(\tau) : \alpha(f) = f\}$, giving the convergence results. If a is nonnegative and nonzero, this projection can be verified to have a positive inner product with a, giving the recurrence results.

Now we consider the cases $k \geq 3$. We have already seen from Theorems 1.1 and 1.3 that we have convergence and recurrence in those abelian systems arising from ergodic theory, and have recalled above that in fact these include all examples (up to isomorphism).

Proposition 1.9. *Let $k \geq 2$. If $(\mathcal{M}, \tau, \alpha)$ is an abelian von Neumann dynamical system, then $(\mathcal{M}, \tau, \alpha)$ enjoys weak convergence and convergence in norm, and recurrence on average and on a dense set.*

More generally, an abelian von Neumann $\mathbb{Z}^k$-system $(\mathcal{M}, \tau, \alpha_0, \dots, \alpha_{k-1})$ enjoys weak convergence and convergence in norm, and recurrence on average and on a dense set.

We now generalise these results to the wider class of *asymptotically abelian* systems.

Definition 1.10 (asymptotic abelianness). A von Neumann dynamical system $(\mathcal{M}, \tau, \alpha)$ is *asymptotically abelian* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \|[\alpha^n(a), b]\|_{L^2(\tau)} = 0 \quad \text{for all } a, b \in \mathcal{M},$$

where $[a, b] := ab - ba$ is the commutator.

Remark 1.11. In previous literature such as [Beyers et al. 2010], a stronger version of asymptotic abelianness is assumed, in which the $L^2(\tau)$ norm is replaced by the operator norm. Variants of this type of "topological asymptotic abelianness", and their relationship with noncommutative topological weak mixing have also been considered in [Kerr and Li 2007].

Our work also singles out this case as special, since the assumption of asymptotic abelianness seems to be essential for the correct working of some the chief technical tools taken from the commutative setting (particularly the van der Corput estimate). In [Niculescu et al. 2003; Beyers et al. 2010; Duvenhage 2009], convergence and recurrence were shown for all orders k for asymptotically abelian systems under some additional assumptions such as weak mixing or compactness. Our first main result shows that in fact all asymptotically abelian systems enjoy convergence and recurrence.

Theorem 1.12. *Let $k \geq 2$. If $(\mathcal{M}, \tau, \alpha)$ is an asymptotically abelian von Neumann dynamical system, then $(\mathcal{M}, \tau, \alpha)$ enjoys weak convergence and convergence in norm, and recurrence on average and on a dense set.*

More generally, if $(\mathcal{M}, \tau, \alpha_0, \dots, \alpha_{k-1})$ is a von Neumann $\mathbb{Z}^k$-system, and the $\alpha_i\alpha_j^{-1}$ for $i \neq j$ are each individually asymptotically abelian, then this $\mathbb{Z}^k$-system enjoys weak convergence and convergence in norm, and recurrence on average and on a dense set.

Theorem 1.12 can be deduced from the genuinely abelian case (Proposition 1.9) by using two results. The first one is essentially from [Beyers et al. 2010] or [Duvenhage 2009], which considered the model case $\alpha_i = \alpha^i$; for the sake of completeness, we present a proof in Appendix A.

Theorem 1.13 (multiple ergodic averages for relatively weakly mixing extensions). *Let $(\mathcal{M}, \tau, \alpha_0, \dots, \alpha_{k-1})$ be a von Neumann $\mathbb{Z}^k$-system, and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ that is invariant under all of the α_i. If for any distinct $0 \leq i, j \leq k-1$ the shift $\alpha_i\alpha_j^{-1}$ is asymptotically abelian and weakly mixing relative to $\mathcal{N}$, then the associated multiple ergodic averages satisfy*

$$\left\| \frac{1}{N} \sum_{n=1}^{N} \alpha_0^{-n} \prod_{i=1}^{k-1} \alpha_i^n(a_i) - \frac{1}{N} \sum_{n=1}^{N} \alpha_0^{-n} \prod_{i=1}^{k-1} \alpha_i^n(E_{\mathcal{N}}(a_i)) \right\|_{L^2(\tau)} \to 0$$

as $N \to \infty$, where $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ is the conditional expectation constructed from τ, and the products are from left to right.

We will recall the notions of relative weak mixing and conditional expectation in Section 3.

The second result, which is new and may have other applications elsewhere, can be viewed as a partial analogue for asymptotically abelian systems of the Furstenberg–Zimmer structure theorem [Furstenberg et al. 1982].

Theorem 1.14 (structure theorem for asymptotically abelian systems). *If $(\mathcal{M}, \tau, \alpha)$ is an asymptotically abelian von Neumann dynamical system, then α is weakly mixing relative to the centre $\mathfrak{Z}(\mathcal{M}) \subset \mathcal{M}$.*

Remark 1.15. In the case when $\mathcal{M}$ is a factor (that is, when the centre is trivial), results of this nature (with a slightly different notion of mixing and of asymptotic abelianness) were established in [Bratteli and Robinson 1987, Example 4.3.24].

These results quickly imply Theorem 1.12. Indeed, when studying (for instance) convergence in norm for a $\mathbb{Z}^k$-system, one can use Theorem 1.14 followed by Theorem 1.13 to replace each of the $a_0, \dots, a_{k-1}$ by their conditional expectations $E_{\mathfrak{Z}(\mathcal{M})}(a_0), \dots, E_{\mathfrak{Z}(\mathcal{M})}(a_{k-1})$ without any affect on the convergence, at which point one can apply Proposition 1.9. (Note that the centre $\mathfrak{Z}(\mathcal{M})$ does not depend on what shift $\alpha_i^{-1}\alpha_j$ one is analysing.) The other claims are similar (using Lemma 3.1 to ensure that if a is nonnegative with positive trace, then so is the conditional expectation $E_{\mathfrak{Z}(\mathcal{M})}(a)$).

Remark 1.16. The arguments above in fact show a more quantitative statement: if a is nonnegative with $\|a\| \le 1$ and $\tau(a) \ge \delta$ for some $0 \le \delta \le 1$, then one has the same lower bound $c(k, \delta) \ge 0$ for (6) as is given by Szemerédi's theorem for (1) for nonnegative functions f with $\|f\|_{L^\infty(X)} \le 1$ and $\int_X f\, d\mu \ge \delta$; in particular, one could insert the bound of Gowers [2001]. Similar remarks apply to multiple commuting shifts. We leave the details to the reader.

The proof of Theorem 1.14, given in Section 3 below, rests on noncommutative versions of several of the steps on the way to the Furstenberg–Zimmer structure theorem in the commutative world of ergodic theory [Furstenberg 1977; Zimmer 1976b; 1976a]. In particular, it rests on a version of the dichotomy between relatively weakly mixing inclusions and those containing a relatively isometric subinclusion, well known in ergodic theory from the cited work of Furstenberg and Zimmer and already generalised to the noncommutative world by Popa [2007], for applications to the study of superrigidity phenomena.

If $(\mathcal{M}, \tau, \alpha)$ is not asymptotically abelian, matters are rather more complicated, with positive results only obtaining under additional restrictions. For $k = 3$ and for ergodic shifts, we have a positive result, established in Section 5:

Theorem 1.17. *If $k = 3$ and $(\mathcal{M}, \tau, \alpha)$ is an ergodic von Neumann dynamical system, one has weak convergence and convergence in norm, as well as recurrence on a dense set.*

The weak convergence result was previously established in [Fidaleo 2009].

1d. *Negative results.* Recurrence on average cannot be included in Theorem 1.17.

Theorem 1.18. *Let $k = 3$. Then there exists an ergodic von Neumann dynamical system $(\mathcal{M}, \tau, \alpha)$ for which recurrence on average fails. (In fact one can make the average (5) strictly negative.)*

We establish this in Section 2b. The main tool is a sophisticated version of the Behrend set construction, combined with the crossed product construction.

Without the ergodicity assumption,[3] one also loses recurrence on a dense set:

Theorem 1.19. *Let $k = 3$. There exists a von Neumann dynamical system $(\mathcal{M}, \tau, \alpha)$ for which recurrence on a dense set fails.* (*In fact one can make the means* (6) *equal to a negative constant for all nonzero n.*)

This result, also proved in Section 2b, is simpler to prove than Theorem 1.18, and uses the original Behrend set construction and crossed product constructions.

One loses recurrence on a dense set for larger k even when ergodicity is assumed:

Theorem 1.20. *Let $k \geq 5$ be odd. There exists an ergodic von Neumann dynamical system $(\mathcal{M}, \tau, \alpha)$ for which recurrence on a dense set fails.* (*In fact one can make the means* (6) *equal to a negative constant for all nonzero n.*)

We establish this in Section 2c. This result uses a counterexample of Bergelson, Host, Kra, and Ruzsa [Bergelson et al. 2005] combined with a group theoretic construction. The restriction to odd k is mostly technical and can almost certainly be removed; however, we are unable to decide whether Theorem 1.20 can be extended to the $k = 4$ case because it was shown in [Bergelson et al. 2005] that the $k = 5$ counterexample in that paper cannot be replicated for $k = 4$.

For convergence, we have counterexamples for $k \geq 4$ even when we assume ergodicity:

Theorem 1.21. *Let $k \geq 4$. There exists an ergodic von Neumann dynamical system $(\mathcal{M}, \tau, \alpha)$ for which weak convergence and convergence in norm fail.*

We establish this in Section 2a. The main tool is a group theoretic construction.

The counterexamples above were for the single shift case, but of course they are also counterexamples to the more general situation of multiple commuting shifts. Table 1 summarises the positive and negative results (in the single shift case).

We note in particular that the following questions remain open:

Question 1.22. If $k = 3$, does weak or norm convergence hold for nonergodic von Neumann dynamical systems $(\mathcal{M}, \tau, \alpha)$?

Question 1.23. If $k = 3$, does weak or norm convergence hold for von Neumann $\mathbb{Z}^3$-systems $(\mathcal{M}, \tau, \alpha_0, \alpha_1, \alpha_2)$, (possibly after imposing suitable ergodicity hypotheses)?

Question 1.24. If $k = 4$ (or if $k \geq 6$ is even), does recurrence on a dense set hold for ergodic von Neumann dynamical systems $(\mathcal{M}, \tau, \alpha)$?

[3] In the commutative case, an easy application of the ergodic decomposition allows one to recover the nonergodic case of the recurrence and convergence results from the ergodic case. Unfortunately, in the noncommutative case, the ergodic decomposition is only available when the invariant factor $\mathcal{M}^\tau$ is central, which is the case in the asymptotically abelian case, but not in general.

	Conv. norm?	Conv. mean?	Recur. avg.?	Recur. dense?
$k = 2$	Yes	Yes	Yes	Yes
$k = 3$, erg.	Yes	Yes	No	Yes
$k = 3$, nonerg.	???	???	No	No
$k \geq 4$, even, erg.	No	No	No?	???
$k \geq 4$, even, nonerg.	No	No	No?	No?
$k \geq 5$, odd, erg.	No	No	No	No
$k \geq 5$, odd, nonerg.	No	No	No	No

Table 1. Positive and negative results for noncommutative convergence and recurrence of a single shift for various values of k, and for various assumptions of ergodicity. The entries marked "No?" would be expected to have a negative answer if one adopts the principle that recurrence results which fail for one value of k, should also fail for higher values of k.

We present some remarks on the first two problems in Section 6.

Notational remark. Unfortunately this paper stands between two quite unrelated uses of the word "factor", one from operator algebras and one from ergodic theory. In the hope that it may be of interest to operator algebraists, we have deferred to their usage (even though the true notion of a factor due to Murray and von Neumann is actually not essential to our work), and will refer throughout to inclusions of von Neumann algebras, even in the commutative setting where these can be identified with ergodic-theoretic "factors".

2. Counterexamples

In this section we construct various counterexamples of von Neumann systems $(\mathcal{M}, \tau, \alpha)$ that will demonstrate the negative results in Theorems 1.18-1.21. The material in this section is independent of the positive results in the rest of the paper, but may provide some cautionary intuition to keep in mind when reading the proofs of those results.

2a. ***Nonconvergence for $k \geq 4$.*** We first show that convergence results fail for $k \geq 4$, even if one assumes ergodicity. In fact the divergence is so bad that it is essentially arbitrary:

Theorem 2.1 (no convergence for $k \geq 4$). *Let $k \geq 4$ be an integer, and let $A \subset \mathbb{Z}$ be a set. Then there exist an ergodic von Neumann system $(\mathcal{M}, \tau, \alpha)$ and elements $a_0, \dots, a_{k-1} \in \mathcal{M}$ such that*

$$\tau(a_0 \alpha^n(a_1) \cdots \alpha^{(k-1)n}(a_{k-1})) = 1_A(n) \quad \textit{for all integers } n.$$

It is clear that this implies Theorem 1.21 by choosing A appropriately (and noting that failure of weak convergence implies failure of convergence in norm, by Cauchy–Schwarz applied in the contrapositive).

Proof. It will suffice to verify the $k = 4$ case, as the higher cases follow by setting $a_j = 1$ for $j \geq 4$. We will need a group G with four distinguished elements e_0, e_1, e_2, e_3 and an automorphism $T : G \to G$ such that T^k has no fixed points other than the identity for all $k \neq 0$ and such that

$$e_0(T^r e_1)(T^{2r} e_2)(T^{3r} e_3) = \mathrm{id}$$

holds for all $r \in A$ and fails for all $r \in \mathbb{Z} \backslash A$. Constructing such a group is somewhat nontrivial and is deferred to Appendix B, and in particular to Proposition B.11.

The group algebra $\mathbb{C}G$ of formal finite linear combinations of group elements of G acts (on the left) on the Hilbert space $\ell^2(G)$ in the obvious way (arising from convolution on G) and can thus be viewed as a subspace of the von Neumann algebra $B(\ell^2(G))$; note that all the elements of G become unitary in this perspective. We can place a finite faithful trace τ on $\mathbb{C}G$ by declaring the identity element to have trace 1, and all other elements of G to have trace zero. If we then define $\mathcal{M}$ to be the closure of $\mathbb{C}G$ in the weak operator topology of $B(\ell^2(G))$, we obtain a finite von Neumann algebra, known as the *group von Neumann algebra LG* of G. The shift T leads to an algebra isomorphism α of $\mathbb{C}G$, which then easily extends to a shift α on $\mathcal{M} = LG$. Because none of the powers of T have any nontrivial fixed points, the orbit of any nonzero group element contains no repetitions, and so one can easily establish that $\alpha^n f$ converges weakly to $\tau(f)$ as $n \to \infty$ for every $f \in \mathbb{C}G$ and hence by approximation that the unitary operator on $\ell^2(G)$ associated to α has no fixed points outside $\mathbb{C}\delta_{\mathrm{id}}$. This implies that $(\mathcal{M}, \tau, \alpha)$ is ergodic, since given $a \in \mathcal{M}$ for which $\alpha(a) = a$ and $\tau(a) = 0$ it follows that $a(\delta_{\mathrm{id}}) \in \ell^2(G)$ is a fixed point for the action of T on $\ell^2(G)$, which must therefore equal $\tau(a)\delta_{\mathrm{id}} = 0$, and hence $\tau(a^*a) = \|a(\delta_{\mathrm{id}})\|_2^2 = 0$ and so $a = 0$, by the faithfulness of τ. If we now set $a_j = e_j$ for $j = 0, 1, 2, 3$, we obtain the claim. □

Remark 2.2. An inspection of the proofs of Theorem 2.1 and Proposition B.11 shows that the expression $a_0\alpha^n(a_1)\alpha^{2n}(a_2)\alpha^{3n}(a_3)$ can more generally be replaced by $\alpha^{c_0 n}(a_0)\alpha^{c_1 n}(a_1)\alpha^{c_2 n}(a_2)\alpha^{c_3 n}(a_3)$ whenever c_0, c_1, c_2, c_3 are integers such that $c_i \neq c_{i+1}$ for all $i = 0, 1, 2, 3$ (with the cyclic convention $c_{i+4} = c_i$). Thus for instance one can construct von Neumann systems for which

$$\tau(a_0(\alpha^n(a_1))a_2\alpha^n(a_3)) = 1_A(n)$$

for an arbitrary set A. We omit the details.

Remark 2.3. The examples of nonconvergence given above are not self-adjoint or positive, and the a_i are not equal to each other. However, it is not hard to modify the examples to give an example of a positive $a_i = a$ for which the averages $N^{-1}\sum_{n=1}^N \tau(a\alpha^n(a)\alpha^{2n}(a)\alpha^{3n}(a))$ do not converge. Indeed, one can repeat the above construction with

$$a := \mathrm{id} + \frac{1}{100}\sum_{i=0}^{3}(e_i + e_i^*);$$

this is easily seen to be positive and self-adjoint, and a modification of the above computations then shows that

$$\tau(a\alpha^n(a)\alpha^{2n}(a)\alpha^{3n}(a)) = 1 + \frac{2}{100^4}1_A(n) \quad \text{for all } n,$$

which is enough to ensure divergence by choosing A appropriately. We leave the details to the reader.

Remark 2.4. The group G constructed here can easily be shown to have infinite conjugacy classes (by the same methods used to prove Proposition B.11). This implies that the group algebra LG is a factor. See [Kadison and Ringrose 1997, Theorem 6.7.5] for details.

2b. ***Negative averages for k = 3.*** We now show the negativity of various triple averages. The main tool is the following Behrend-type construction of a set that avoids progressions of length three, but contains many "hexagons":

Lemma 2.5 (Behrend-type example). *Let $\varepsilon > 0$. Then for all sufficiently large d, there exists a subset F of $\mathbb{Z}/d\mathbb{Z}$ such that $|F| \geq d^{1-\varepsilon}$, but F contains no nontrivial arithmetic progressions of length three*; *thus $n, n+r, n+2r \in F$ can only occur if $r = 0$. On the other hand*, *the set*

$$\{(x, h, k) \in \mathbb{Z}/d\mathbb{Z} : x, x+h, x+k, x+k+2h, x+2k+h, x+2k+2h \in F\}$$

of "hexagons" in F has cardinality at least $d^{3-\varepsilon}$.

The first part of the lemma already follows directly from [Behrend 1946] or the earlier [Salem and Spencer 1942]. The claim about hexagons will be needed in the proof of Theorem 2.11 below, but is not needed for the simpler results in Corollary 2.7 or Theorem 2.10.

Proof. Let R be a large multiple of 400 (depending on ε). We claim that for n a large enough multiple of 4 (depending on R), the set $\{-R, \dots, R\}^n \subset \mathbb{Z}^n$ contains a subset E of cardinality $|E| \geq e^{-O(n)}R^n$ (where the implied constant in the O notation is absolute), and which contains $\geq e^{-O(n)}R^{3n}$ hexagons

$$\{x, x+h, x+k, x+k+2h, x+2k+h, x+2k+2h\}$$

but contains no arithmetic progressions of length three. Choosing d sufficiently large, letting n be the largest integer such that $(10R)^n \leq d$, and then embedding

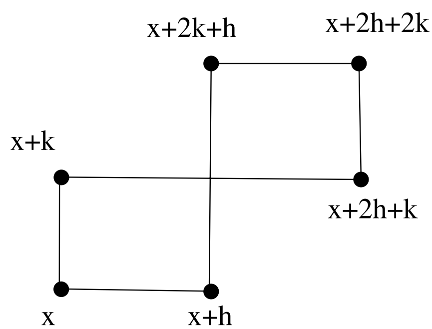

Figure 1. A hexagon. Note the absence of arithmetic progressions of length three.

$\{-R, \dots, R\}^n$ in $\mathbb{Z}/d\mathbb{Z}$ using base $10R$ (say), as in the work of Behrend or Salem and Spencer, this claim will imply the lemma (after choosing R sufficiently large depending on ε).

The claim itself remains. From the classical results on the Waring problem (see for example [Vaughan 1997]), we know that every large integer N has $\sim N^{(k-2)/2}$ representations as the sum of k squares for k large enough (one can for instance take $k = 5$, but for our purposes any fixed k will suffice). Using this, we see that for any fixed $\delta \in (0, \frac{1}{10})$, every integer r such that $\delta R^2 n \leq r \leq \frac{1}{10} R^2 n$ (say) will have $\geq (c_\delta R)^{n-C_\delta}$ representations as the sum of n squares of integers less than R, where $c_\delta, C_\delta > 0$ depend only on δ. In other words, the sphere

$$E_r := \{x \in \{-R, \dots, R\}^n : |x|^2 = r\}$$

has cardinality at least $(c_\delta R)^{n-C_\delta}$. On the other hand, such spheres have no nontrivial progressions of length three. Thus it will suffice (for n large enough) by the pigeonhole principle to show that there are at least $e^{-O(n)} R^{3n}$ hexagons

$$\{x, x+h, x+k, x+k+2h, x+2k+h, x+2k+2h\} \quad \text{in } \{-R, \dots, R\}^n$$

such that

$$|x|^2 = |x+h|^2 = |x+k|^2 = |x+k+2h|^2 = |x+2k+h|^2 = |x+2k+2h|^2 \leq \tfrac{1}{10} R^2 n \tag{9}$$

(note that the case when $|x|^2 \leq \delta R^2 n$ for sufficiently small δ can be eliminated by crude estimates).

To count the solutions to (9), we perform some elementary changes of variable to replace the constraints in (9) with simpler constraints. We begin by observing that if $a, b, c \in \{-R/100, \dots, R/100\}^n$ are such that

$$a \cdot b = b \cdot c = c \cdot a = 0 \quad \text{and} \quad c \cdot c = 3b \cdot b, \tag{10}$$

then $x := a - 2b$, $h := b + c$, $k := b - c$ can be verified to be a solution to (9), with the map $(a, b, c) \to (x, h, k)$ being injective, so it suffices to show that there are at least $e^{-O(n)} R^{3n}$ triples (a, b, c) with the properties above.

For reasons that will become clearer later, we will initially work in dimension $n/4$ rather than n. Using the Waring problem results as before, we can find at least $e^{-O(n)} R^{3n/4}$ triples $a, b, c \in \{-R/400, \dots, R/400\}^{n/4}$ such that

$$c \cdot c = 3b \cdot b.$$

This is one of the four constraints required for (10). To obtain the remaining ones, we use a pigeonholing trick followed by a tensor power trick. First, observe that if $a, b, c \in \{-R/400, \dots, R/400\}^{n/4}$, then $a \cdot b$, $b \cdot c$, $c \cdot a$ are of order $O(R^2 n) \le e^{O(n)}$. Applying the pigeonhole principle, one can thus find $h_1, h_2, h_3 = O(R^2 n)$ such that there are $e^{-O(n)} R^{3n/4}$ triples

$$a, b, c \in \{-R/400, \dots, R/400\}^{n/4} \tag{11}$$

with

$$a \cdot b = h_1, \quad b \cdot c = h_2, \quad c \cdot a = h_3, \quad c \cdot c = 3b \cdot b. \tag{12}$$

This is an inhomogeneous version of (10) (at dimension $n/4$ rather than n), with the zero coefficients replaced by more general coefficients h_1, h_2, h_3. To eliminate these coefficients we use a tensor power trick. Let S be the set of all triples (a, b, c) obeying (11) and (12). We then observe that if $(a_i, b_i, c_i) \in S$ for $i = 1, 2, 3, 4$, then the vectors $a, b, c \in \mathbb{Z}^n$ defined by

$$a := (a_1, a_2, a_3, a_4); \quad b := (b_1, b_2, -b_3, -b_4); \quad c := (c_1, -c_2, c_3, -c_4)$$

solve (10). The map from the (a_i, b_i, c_i) to (a, b, c) is an injection from S^4 to the solution set of (10), and so we obtain at least $|S|^4 \ge e^{-O(n)} R^{3n}$ solutions to (10) as required. $\square$

This leads to a useful matrix counterexample:

Lemma 2.6 (restricted third moment can be negative). *There exists a positive semi-definite Hermitian matrix $(A(j, k))_{1 \le j,k \le d}$ for which the quantity*

$$\sum_{n,r \in \mathbb{Z}/d\mathbb{Z}} A(n, n + r) A(n + r, n + 2r) A(n + 2r, n) \tag{13}$$

is negative, where we extend $A(i, j)$ periodically in both variables by d.

Proof. We will take d to be a multiple of 3, and $A(j, k)$ to take the form

$$A(j, k) := 1_E(j) 1_E(k) + 1_E(j) \omega^{-j} 1_E(k) \omega^k,$$

where $E \subset \mathbb{Z}/d\mathbb{Z}$ is a set to be determined later and $\omega := e^{2\pi i/3}$ is a cube root of unity. The matrix $(A(j,k))_{1\leq j,k\leq d}$ is then the sum of two rank one projections and is thus positive semidefinite and Hermitian. The expression (13) can be expanded as

$$\sum_{\substack{n,r\in\mathbb{Z}/d\mathbb{Z}:\\ n,n+r,n+2r\in E}} (1+\omega^r)(1+\omega^r)(1+\omega^{-2r}).$$

The summand can be computed to equal 8 when r is divisible by 3, and -1 otherwise. Thus, to establish the claim, it suffices to find a set E such that the set

$$\{(n,r)\in\mathbb{Z}/d\mathbb{Z} : n, n+r, n+2r \in E,\ r \neq 0 \bmod 3\}$$

is more than eight times larger than the set

$$\{(n,r)\in\mathbb{Z}/d\mathbb{Z} : n, n+r, n+2r \in E,\ r = 0 \bmod 3\};$$

thus the length three arithmetic progressions in E with spacing not divisible by 3 need to overwhelm the length three progressions with spacing divisible by 3.

To do this, we use Lemma 2.5 to get a subset $F \subset \{1,\ldots,[d/10]\}$ of cardinality $|F| \geq d^{0.99}$ that contains no arithmetic progressions of length three. We then pick three random shifts $h_0, h_1, h_2 \in \{1,\ldots,d/3\}$ uniformly at random, and consider the set

$$E := \{3(f+h_i)+i : i = 0,1,2,\ f\in F\}$$

consisting of three randomly shifted, dilated copies of F.

By construction, the only length three progressions in E with spacing divisible by 3 are the trivial progressions n,n,n with $r=0$, so the total number of such progressions is at most d. On the other hand, for any fixed $f_0, f_1, f_2 \in F$, the numbers $3(f_i+h_i)+i$ for $i=0,1,2$ have a probability $3/d$ of forming an arithmetic progression with spacing not divisible by 3, due to the random nature of the h_i. Thus the expected value of the total number of such progressions is at least $(d^{0.99})^3 \times 3/d = 3d^{1.97}$. For d large enough, this gives the claim. □

This gives a simple example of negative averages for nonergodic systems:

Corollary 2.7 (negative average for nonergodic system). *There exists a finite von Neumann algebra $(\mathcal{M},\tau)$ with a shift α and a nonnegative element $a\in\mathcal{M}$, such that $(2N+1)^{-1}\sum_{n=-N}^{N}\tau(a\alpha^n(a)\alpha^{2n}(a))$ converges to a negative number.*

Proof. Let $a = (A(j,k))_{1\leq j,k\leq d}$ be as in Lemma 2.6. We let $\mathcal{M}$ be the von Neumann algebra of complex $d\times d$ matrices with the normalised trace τ and with the shift

$$\alpha(B(j,k))_{1\leq j,k\leq d} := (e^{2\pi i(j-k)/d}B(j,k))_{1\leq j,k\leq d}.$$

This is easily verified to be a shift. We see that

$$\tau(a\alpha^n(a)\alpha^{2n}(a)) = \frac{1}{d} \sum_{j,k,l \in \mathbb{Z}/d\mathbb{Z}} e^{2\pi i n(k+l-2j)/d} A(j,k)A(k,l)A(l,j).$$

This expression is periodic in n with period d and has average

$$\frac{1}{d} \sum_{l,r \in \mathbb{Z}/d\mathbb{Z}} A(l,l+r)A(l+r,l+2r)A(l+2r,l)$$

and the claim then follows from Lemma 2.6. □

This shows that recurrence on average for $k = 3$ can fail for nonergodic systems. However, this is not yet enough to establish either Theorem 1.18 or Theorem 1.19. To obtain these stronger results we must introduce the *crossed product construction* in von Neumann algebras. For a comprehensive introduction to this concept, see [Kadison and Ringrose 1997, Chapter 13]. We shall just recall the key properties of this construction we need here.

Suppose we have a finite von Neumann algebra $(\mathcal{M}, \tau)$, and an action U of a (discrete) group G on $\mathcal{M}$; thus for each $g \in G$ we have a shift $U(g) : \mathcal{M} \to \mathcal{M}$ such that $U(g)U(h) = U(gh)$ for all $g, h \in G$, with $U(\mathrm{id})$ being the identity. Then there exists a crossed product $(\mathcal{M} \rtimes_U G, \tau)$ that contains both the original space $(\mathcal{M}, \tau)$ and the group algebra $\mathbb{C}G$ as subalgebras. Furthermore, in this crossed product we have

$$U(g)a = gag^{-1} \tag{14}$$

for all $a \in \mathcal{M}$ and $g \in G$, and

$$\tau(ga) = \tau(ag) = 0$$

for all $a \in \mathcal{M}$ and $g \in G$ with g not equal to the identity. Finally, the span of the elements ag for $a \in \mathcal{M}$ and $g \in G$ is dense in $\mathcal{M} \rtimes_U G$.

Remark 2.8. The exact construction of the crossed product is not relevant for our applications, but for the convenience of the reader we sketch one such construction here. We first form the Hilbert space

$$\mathfrak{h} := \ell^2(G, L^2(\tau)) = \bigoplus_{g \in G} L^2(\tau)$$

consisting of tuples $(x_g)_{g \in G}$ in $L^2(\tau)$. This space has an action of $\mathcal{M}$ defined by

$$a(x_g)_{g \in G} := ((U(g^{-1})a)x_g)_{g \in G}$$

for $a \in \mathcal{M}$, and an action of G (and hence $\mathbb{C}G$) defined by

$$h(x_g)_{g \in G} := (x_{h^{-1}g})_{g \in G}.$$

One can verify that these actions combine to an action of the twisted convolution algebra $\ell^1(G, \mathcal{M})$ on $\mathfrak{h}$, defined as the space of formal sums $\sum_{h\in G} ha_h$ with $\sum_{h\in G} \|a_h\| < \infty$, and subject to the relations (14). We define a trace on such sums by the formula $\tau(\sum_{h\in G} ha_h) := \tau(a_{\mathrm{id}})$. One can then show that one can extend this to a finite trace on the weak operator topology closure of $\ell^1(G, \mathcal{M})$, viewed as a subset of $B(\mathfrak{h})$; this closure can then be denoted $\mathcal{M} \rtimes_U G$. In other words, $\mathcal{M} \rtimes_U G$ is constructed as the von Neumann algebra generated by the action of $\mathcal{M}$ and G on $\mathfrak{h}$.

Example 2.9. The group von Neumann algebra LG can be viewed as $\mathbb{C} \rtimes G$, where G acts trivially on the one-dimensional von Neumann algebra $\mathbb{C}$.

We can now get a stronger version of Corollary 2.7:

Theorem 2.10 (negative trace for nonergodic system). *There exists a von Neumann dynamical system* $(\mathcal{M}, \tau, \alpha)$ *and a nonnegative element* $a \in \mathcal{M}$, *such that* $\tau(a\alpha^n(a)\alpha^{2n}(a))$ *is negative* (*and independent of* n) *for all nonzero* n. *In particular*, *Theorem 1.19 holds.*

Proof. Let $(\mathcal{M}', \tau, \beta)$ be a von Neumann dynamical system to be chosen later. Using the crossed product construction, we can build an extension $\mathcal{M} := \mathcal{M}' \rtimes_U \mathbb{Z}^2$ of $\mathcal{M}'$ generated by $\mathcal{M}'$ and two commuting unitary elements u and m, such that

$$mam^{-1} = \beta(a) \tag{15}$$

and $uau^{-1} = a$ for all $a \in \mathcal{M}'$. In particular, the element u is central. It is then easy to see that we can build[4] a shift α on $\mathcal{M}$ for which

$$\alpha(a) = a, \quad \alpha(u) = u, \quad \alpha(m) = mu$$

for all $a \in \mathcal{M}'$, since the action of the group $\mathbb{Z}^2$ generated by m and u on $\mathcal{M}'$ is unchanged when one replaces m by mu.

Now let $a \in \mathcal{M}$ be an element of the form

$$a = \Big(\sum_{i\in\mathbb{Z}} f_i m^i\Big)\Big(\sum_{i\in\mathbb{Z}} f_i m^i\Big)^*,$$

where $f_i \in \mathcal{M}'$ and only finitely many of the f_i are nonzero. This is clearly nonnegative, and can be simplified by (15) to the power series

$$a = \sum_{h\in\mathbb{Z}} g_h m^h,$$

[4]To build α explicitly, we can view $\mathcal{M}$ as an algebra of operators on the Hilbert space $\mathfrak{h} := \bigoplus_{(j,k)\in\mathbb{Z}^2} L^2(\tau)$ as per Remark 2.8, and let α be the conjugation $a \mapsto WaW^*$ by the unitary operator $W : \mathfrak{h} \to \mathfrak{h}$ defined by $W(x_{(j,k)})_{(j,k)\in\mathbb{Z}^2} := (x_{(j,k-j)})_{(j,k)\in\mathbb{Z}^2}$.

where the $g_h \in \mathcal{M}'$ are the twisted autocorrelations of the f_j, given by

$$g_h = \sum_{j \in \mathbb{Z}} f_{j+h} \beta^h(f_j^*).$$

Let n be nonzero. The expression $\tau(a\alpha^n(a)\alpha^{2n}(a))$ can be expanded as

$$\sum_{h_1, h_2, h_3 \in \mathbb{Z}} \tau(g_{h_1} m^{h_1} g_{h_2} (mu^n)^{h_2} g_{h_3} (mu^{2n})^{h_3}).$$

The net power of the central element u here is $n(h_2 + 2h_3)$, and the net power of m is $h_1 + h_2 + h_3$. Thus we see that the trace vanishes unless $h_2 + 2h_3 = h_1 + h_2 + h_3 = 0$, or equivalently if $(h_1, h_2, h_3) = (h, -2h, h)$ for some h. Performing this substitution and using (15), we simplify this expression to

$$\sum_{h \in \mathbb{Z}} \tau(g_h \beta^h(g_{-2h}) \beta^{-h}(g_h)). \tag{16}$$

In particular, this expression is now manifestly independent of $n \neq 0$.

We now select $\mathcal{M}'$ to be the commutative von Neumann system $L^\infty(\mathbb{Z}/d\mathbb{Z})$ with the shift $\beta(f(x)) := f(x+1)$ and the normalised trace. Thus the g_h and f_h are now complex-valued functions on $\mathbb{Z}/d\mathbb{Z}$, and the expression above can be expanded explicitly as

$$\frac{1}{d} \sum_{x \in \mathbb{Z}/d\mathbb{Z}} \sum_{h \in \mathbb{Z}} g_h(x) g_{-2h}(x+h) g_h(x-h).$$

Meanwhile, the $g_h(x)$ by definition can be written as

$$g_h(x) = \sum_{j \in \mathbb{Z}} f_{j+h}(x) \overline{f_j(x+h)}.$$

We pick a large number N to be chosen later, and set

$$f_j(x) := b(x, x+j) 1_{1 \leq j \leq Nd},$$

where $b : \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}$ is a function periodic in two variables of period d to be chosen later. Then we can compute

$$g_h(x) = \left(1 - \frac{|h|}{dN}\right)_+ N A(x, x+h) + O(1),$$

where

$$A(x, y) := \sum_{z \in \mathbb{Z}/d\mathbb{Z}} b(x, z) \overline{b(y, z)} \tag{17}$$

and $O(1)$ denotes a quantity that can depend on d (and b) but is uniformly bounded in N. The expression (16) can then be computed to be

$$C\frac{N^4}{d}\sum_{x,h\in\mathbb{Z}/d\mathbb{Z}} A(x,x+h)A(x+h,x-h)A(x-h,x)+O(N^3),$$

where $C>0$ is the explicit constant $C:=\int_{\mathbb{R}}(1-|h|)_+^2(1-|2h|)_+\,dh$. By the substitutions $x=m+r$ and $h=r$, we can reexpress this as

$$C\frac{N^4}{d}\sum_{m,r\in\mathbb{Z}/d\mathbb{Z}} A(m,m+r)A(m+r,m+2r)A(m+2r,m)+O(N^3). \tag{18}$$

Now, let d and $A(j,k)$ be as in Lemma 2.6. By the spectral theorem (which in particular allows one to construct self-adjoint square roots of positive definite matrices), we can find $b(x,y)$ such that (17) holds. The summand in (18) is then negative, and the claim follows by choosing N large enough depending on all other parameters. $\square$

Of course, by Theorem 1.17, one cannot have such a result when the underlying shift α is ergodic. On the other hand, one can extend Corollary 2.7 to the ergodic case:

Theorem 2.11. *There exists an* ergodic *von Neumann system* $(\mathcal{M},\tau,\alpha)$ *and a non-negative element* $a\in\mathcal{M}$, *such that* $(2N+1)^{-1}\sum_{n=-N}^{N}\tau(a\alpha^n(a)\alpha^{2n}(a))$ *converges to a negative number. In particular*, *Theorem 1.18 holds.*

Proof. Let d be a large odd number, and let $u:=e^{2\pi i/d}$ be a primitive d-th root of unity. We will let $\mathcal{M}$ be a completion of the *noncommutative torus*. This is obtained by first forming the C*-algebra generated by two unitary generators e_1 and e_2 obeying the commutation relation

$$e_2e_1=ue_1e_2$$

and with all of the expressions $e_1^je_2^k$ having zero trace unless $j=k=0$, in which case the trace is 1, and then completing in the weak operator topology resulting from the Gel'fand–Naimark–Segal representation on $L^2(\tau)$. One can represent this finite von Neumann algebra more explicitly by letting e_1 and e_2 act on $L^2((\mathbb{R}/\mathbb{Z})^2)$ by the maps $e_1f(x,y):=e^{2\pi ix}f(x,y)$ and $e_2f(x,y):=e^{2\pi iy}f(x+1/d,y)$, with the trace τ given by $\tau(a)=\langle\Omega,a\Omega\rangle_{L^2((\mathbb{R}/\mathbb{Z})^2)}$, where $\Omega\equiv 1$ is the identity function on $(\mathbb{R}/\mathbb{Z})^2$.

We let $\theta_1,\theta_2\in S^1$ be generic unit phases, and then define the shift α on $\mathcal{M}$ by setting

$$\alpha(e_1):=\theta_1e_1 \quad\text{and}\quad \alpha(e_2):=\theta_2e_2.$$

It is easy to see that this is a shift. If θ_1 and θ_2 are generic (so that $\theta_1^j\theta_2^k$ is not a root of unity for any $(j,k)\neq(0,0)$), this shift is easily verified to be ergodic (as one can verify the mean ergodic theorem by hand on the generators $e_1^je_2^k$, and then argue as in the proof of Theorem 2.1 using the faithfulness of τ).

We set $a := gg^*$, where g is an element of the form $g := \sum_{k=1}^M \sum_{h\in\mathbb{Z}} c_h e_1^h e_2^k$, M is a large number (much larger than d) to be chosen later, and c_h are complex numbers to be chosen later, all but finitely many of which are zero. Clearly a is nonnegative. A computation shows that

$$a = \sum_{h,k\in\mathbb{Z}} c_{h,k} e_1^h e_2^k, \quad \text{where } c_{h,k} := M\Big(1-\frac{|k|}{M}\Big)_+ \sum_{l\in\mathbb{Z}} c_{l+h}\overline{c_l}u^{kl}. \tag{19}$$

Since

$$\alpha^n(a) = \sum_{h,k\in\mathbb{Z}} c_{h,k}\theta_1^{hn}\theta_2^{kn}e_1^he_2^k,$$

some Fourier analysis and the genericity of θ_1 and θ_2 show that the expression

$$\frac{1}{2N+1}\sum_{n=-N}^{N}\tau(a\alpha^n(a)\alpha^{2n}(a))$$

converges as $N\to\infty$ to the expression

$$\sum_{h,k} c_{h,k}c_{-2h,-2k}c_{h,k}\tau(e_1^he_2^ke_1^{-2h}e_2^{-2k}e_1^he_2^k).$$

The trace here simplifies to u^{3hk}. Inserting the expression for $c_{h,k}$ in (19), we can expand this expression as

$$M^3 \sum_{h,k,l_1,l_2,l_3\in\mathbb{Z}} \phi(k/M)c_{l_1+h}\overline{c_{l_1}}c_{l_2-2h}\overline{c_{l_2}}c_{l_3+h}\overline{c_{l_3}}u^{kl_1-2kl_2+kl_3+3hk}, \tag{20}$$

where $\phi(x) := (1-|x|)_+^2(1-|2x|)_+$. By Poisson summation, the expression

$$\sum_k \phi(k/M)u^{kl_1-2kl_2+kl_3+3hk}$$

can be computed to be $M\int_{\mathbb{R}}\phi(x)dx + O(1)$ if $l_1-2l_2+l_3+3h$ is divisible by d, and $O(1)$ otherwise, where $O(1)$ denotes a quantity that can depend on d but is bounded uniformly in M. If we then assume that the c_h vanish for h outside of $\{1,\dots,M\}$ and are bounded uniformly in M, we can thus expand (20) as

$$CM^4 \sum_{\substack{h,l_1,l_2,l_3\in\mathbb{Z}:\\ d\,|\,l_1-2l_2+l_3+3h}} c_{l_1+h}\overline{c_{l_1}}c_{l_2-2h}\overline{c_{l_2}}c_{l_3+h}\overline{c_{l_3}} + O(M^7)$$

for some absolute constant $C>0$.

If we now set $c_h := b(h)1_{[1,M]}(h)$, where $b : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}$ is a periodic function with period d and independent of M to be chosen later, we can express this as

$$C_d M^8 \sum_{\substack{h,l_1,l_2,l_3 \in \mathbb{Z}/d\mathbb{Z}: \\ l_1-2l_2+l_3+3h=0}} b(l_1+h)\overline{b(l_1)}b(l_2-2h)\overline{b(l_2)}b(l_3+h)\overline{b(l_3)} + O(M^7)$$

for some $C_d > 0$ depending on d but independent of M. Making the substitutions $l_1 = x,\ l_2 = x+k+2h$ and $l_3 = x+2k+h$, we see that we will be done as soon as we are able to find d and b for which the expression

$$X := \sum_{x,h,k \in \mathbb{Z}/d\mathbb{Z}} \overline{b(x)}b(x+h)b(x+k)\overline{b(x+k+2h)}\,\overline{b(x+2k+h)}b(x+2k+2h)$$

is negative.

To do this, we again appeal to Lemma 2.5 to find a set $F \subset \mathbb{Z}/d\mathbb{Z}$ of size at least $d^{0.99}$ (assuming d large enough), which contains no arithmetic progressions of length three, but contains at least $d^{2.99}$ hexagons $x,\ x+h,\ x+k,\ x+k+2h,\ x+2k+h,\ x+2k+2h$. We then set $b(x) := \epsilon_x 1_F(x)$, where the $\epsilon_x = \pm 1$ are independent signs; thus X is now the random variable

$$X = \sum \epsilon_x \epsilon_{x+h} \epsilon_{x+k} \epsilon_{x+2h+k} \epsilon_{x+h+2k} \epsilon_{x+2h+2k},$$

where the sum is over $\{x, h, k : x, x+h, x+k, x+k+2h, x+2k+h, x+2k+2h \in F\}$. We will show (for d large enough) that the standard deviation of X exceeds its expectation, which shows that there exists a choice of signs for which X is negative.

We first compute the expectation of X. The only summands with nonzero expectation occur when all the signs cancel, which only occurs when $h = 0$ or when $k = 0$, as can be seen by an inspection of the number of ways to collapse the hexagon in Figure 1; here we need the hypothesis that d is odd. But since F contains no nontrivial arithmetic progressions, there are no summands for which only one of the h, k are zero, so we are left only with the $h = k = 0$ terms, of which there are at most d. Thus the expectation of X is at most d.

Now we compute the variance. There are at least $d^{2.99}$ hexagons in F, and all but $O(d^2)$ of them are nondegenerate in the sense that the six vertices of the hexagon are all distinct. The summands in X corresponding to nondegenerate hexagons have variance 1, and the correlation between any two summands in X is either zero or positive (the latter occurs when two summands are permutations of each other). Thus the variance of X is $\gg d^{2.99}$, so the standard deviation is $\gg d^{1.495}$, and the claim follows. $\square$

2c. ***Negative trace for k = 5.*** Now we show negative traces can occur even in the ergodic case when $k = 5$.

Theorem 2.12. *There exists an* ergodic *von Neumann dynamical system* $(\mathcal{M}, \tau, \alpha)$ *and a nonnegative element* $a \in \mathcal{M}$, *such that* $\tau(a\alpha^n(a)\alpha^{2n}(a)\alpha^{3n}(a)\alpha^{4n}(a))$ *is negative for every nonzero* n.

This establishes the $k = 5$ case of Theorem 1.20. A similar argument holds for all larger odd values of k, which we leave to the interested reader; we restrict here to the case $k = 5$ simply for ease of notation.

To prove this theorem, our starting point is the following result of Bergelson, Host, Kra, and Ruzsa [Bergelson et al. 2005]:

Theorem 2.13. *For any* $\delta > 0$, *there is a measure-preserving system* $(X, \mathscr{X}, \mu, S)$ *and a measurable set* $A \subset X$ *with* $0 < \mu(A) < \delta$ *such that*

$$\mu(A \cap S^n(A) \cap S^{2n}(A) \cap S^{3n}(A) \cap S^{4n}(A)) \leq \mu(A)^{100}$$

(*say*) *and*

$$\mu(A \cap S^n(A)) = \mu(A)^2 \tag{21}$$

for every nonzero integer n.

Proof. This follows from [Bergelson et al. 2005, Theorem 1.3] (see also the remark immediately below that theorem). The property (21) is not explicitly stated in that theorem, but follows from the construction in [Bergelson et al. 2005, Section 2.3] (the system X is a torus $(\mathbb{R}/\mathbb{Z})^2$ with the skew shift $S : (x, y) \mapsto (x+\alpha, y+2x+\alpha)$, and the set A has the special form $A = (\mathbb{R}/\mathbb{Z}) \times B$ for some set B). □

We apply this theorem for some sufficiently small δ (to be chosen later) to obtain X, μ, S, A with the properties above. We will combine this with the group G, the automorphism T, and the elements e_0, e_1, e_2, e_3, e_4 arising from Proposition B.13 as follows.

First, we create the product space $L^\infty(X^G, d\mu^G)$, whose σ-algebra is generated up to negligible sets by the tensor products $\bigotimes_{g\in G} f_g$, where $f_g \in L^\infty(X, d\mu)$ is equal to 1 for all but finitely many g. This product has a unitary, trace-preserving action U of G, defined by

$$U(h) \bigotimes_{g\in G} f_g := \bigotimes_{g\in G} f_{h^{-1}g}.$$

We can therefore create the crossed product $\mathcal{M} := L^\infty(X^G, d\mu^G) \rtimes_U G$. Note that if we embed $L^\infty(X, \mu)$ into $L^\infty(X^G, d\mu^G)$ by using the identity component of X^G, we have

$$\bigotimes_{g\in G} f_g = \prod_{g\in G} U(g) f_g \tag{22}$$

(note that the $U(g) f_g$ necessarily commute with each other).

We define a shift α on $\mathcal{M}$ by requiring that

$$\alpha\Big(\bigotimes_{g\in G} f_g\Big) = \bigotimes_{g\in G} S(f_{T^{-1}g}) \quad \text{and} \quad \alpha(g) = Tg;$$

one can check that this is indeed a well-defined shift on $\mathcal{M}$.

We claim that α is ergodic. Indeed, if $a \in \mathcal{M}$ is of the form $a = fg$ for some $f \in L^\infty(X^G, d\mu^G)$ and $g \in G$ not equal to the identity, then since the powers of T have no nontrivial fixed points, the orbit $T^n g$ escapes to infinity, and the orbit $\alpha^n(a)$ converges weakly to zero. Meanwhile, if g is the identity, then it is classical that the Bernoulli system $G \circlearrowright L^\infty(X^G, d\mu^G)$ is ergodic, and so the ergodic theorem applies to a in this case. Putting the two facts together and arguing as for the ergodicity in Theorem 2.1 yields the ergodicity of α.

Note that 1_A lies in $L^\infty(X, d\mu)$, and can thus be identified with an element of $\mathcal{M}$ by the previous embedding. We set

$$a := \sum_{i=0}^{3} 1_A \cdot (2 - e_i - e_i^{-1}) \cdot 1_A.$$

Clearly a is nonnegative. Now let n be nonzero, and consider the expression

$$\tau(a\alpha^n(a)\alpha^{2n}(a)\alpha^{3n}(a)\alpha^{4n}(a)). \tag{23}$$

Expanding out a, we obtain a linear combination of terms of the form

$$\begin{aligned}\tau(1_A g_0 1_A 1_{S^n(A)}(T^n g_1) 1_{S^n(A)} 1_{S^{2n}(A)}(T^{2n} g_2)& \\ \cdot 1_{S^{2n}(A)} 1_{S^{3n}(A)}(T^{3n} g_3) 1_{S^{3n}(A)} 1_{S^{4n}(A)}&(T^{4n} g_4) 1_{S^{4n}(A)}),\end{aligned}$$

where $g_0, g_1, g_2, g_3, g_4 \in \{\mathrm{id}, e_0, e_1, e_2, e_3, e_4, e_0^{-1}, e_1^{-1}, e_2^{-1}, e_3^{-1}, e_4^{-1}\}$. This trace vanishes unless

$$g_0 T^n g_1 T^{2n} g_2 T^{3n} g_3 T^{4n} g_4 = \mathrm{id}\,. \tag{24}$$

By Proposition B.13, we conclude that g_0, g_1, g_2, g_3, g_4 are either all equal to the identity, or are a permutation of e_0, e_1, e_2, e_3, e_4, or are a permutation of $e_0^{-1}, e_1^{-1}, e_2^{-1}, e_3^{-1}, e_4^{-1}$. In the latter two cases, the contribution to (23) is either zero or negative (being negative the trace of the product of several nonnegative elements in a commutative von Neumann algebra). Here we are using the fact that 5 is odd. Discarding all of these contributions except the one where $g_{i,0} = e_{i,0}$ (which has a nontrivial contribution thanks to Proposition B.13), we conclude that (23) is at most

$$\begin{aligned}&10^5 \tau(1_A 1_{S^n(A)} 1_{S^{2n}(A)} 1_{S^{3n}(A)} 1_{S^{4n}(A)}) \\ &\quad - \tau(1_A e_0 1_A 1_{S^n(A)} e_1 1_{S^n(A)} 1_{S^{2n}(A)} e_2 1_{S^{2n}(A)} 1_{S^{3n}(A)} e_3 1_{S^{3n}(A)} 1_{S^{4n}(A)} e_4 1_{S^{4n}(A)}).\end{aligned}$$

By Theorem 2.13, the first expression is at most $10^5\mu(A)^{100}$. Now consider the second expression. By Proposition B.13, we see that the partial products $e_0e_1\cdots e_i$ for $i = 0, 1, 2, 3$ are distinct. Using (22), we conclude that the trace here can be computed as

$$\mu(S^{4n}(A)\cap A)\mu(A\cap S^n(A))\mu(S^n(A)\cap S^{2n}(A)) \\ \cdot\mu(S^{2n}(A)\cap S^{3n}(A))\mu(S^{3n}(A)\cap S^{4n}(A)),$$

which by (21) is equal to $\mu(A)^{10}$. Thus the expression (23) is no more than $2^{15}\mu(A)^{100} - \mu(A)^{10}$, which is negative if the upper bound δ for $\mu(A)$ is chosen to be sufficiently small.

This concludes the proof of Theorem 2.12.

Remark 2.14. Given that the counterexample in Theorem 2.13 can be extended to any $k \geq 5$, it seems reasonable to expect that Theorem 1.20 can be extended to all $k \geq 5$ (not just the odd k), though we have not pursued this issue. On the other hand, the analogue of Theorem 2.13 fails for $k = 4$, as was shown in [Bergelson et al. 2005]. Because of this, the $k = 4$ case of Theorem 1.20 remains open; the construction given here does not work, but it is possible that some other construction would suffice instead.

3. Inclusions of finite von Neumann dynamical systems

In this section we recall some fairly well-known constructions relating to von Neumann dynamical systems and their basic properties, culminating in a treatment of Popa's [2007] noncommutative version of the Furstenberg–Zimmer dichotomy. This material will be needed to establish the structure theorem, Theorem 1.14.

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. As noted in the introduction, we can embed $\mathcal{M}$ into a Hilbert space $L^2(\tau)$. In order to distinguish the algebra structure from the Hilbert space structure,[5] we shall refer in this section to the embedded copy of an element $a \in \mathcal{M}$ of the algebra in $L^2(\tau)$ as $\hat{a}$ rather than a; thus for instance $\hat{\mathcal{M}} = \{\hat{a} : a \in \mathcal{M}\}$ is a dense subspace of $L^2(\tau)$.

Clearly, $L^2(\tau)$ has the structure of an $\mathcal{M}$-bimodule, formed by extending the regular bimodule structure on $\mathcal{M}$ by density; the left-representation is, of course, the classical Gel'fand–Naimark–Segal representation associated to τ. When it is necessary to denote the copy of $\mathcal{M}$ in $B(L^2(\tau))$ consisting of the members of $\mathcal{M}$ acting by multiplication on the left (respectively, right), we will denote this algebra by $\mathcal{M}_{\text{left}}$ (respectively, $\mathcal{M}_{\text{right}}$).

[5] It is tempting to ignore these distinctions and identify $\hat{\mathcal{M}}$ with $\mathcal{M}$. While this is normally quite a harmless identification, we will take some care here because we will be studying the bimodule action of $\mathcal{M}$ on $L^2(\tau)$, and keeping track of this action can become notationally confusing if the algebra elements are identified with the vectors that they act on.

The space $L^2(\tau)$ contains a distinguished vector $\hat{1}$ — the representative of the multiplicative identity 1 in $\mathcal{M}$ — with the property that $a\hat{1} = \hat{1}a = \hat{a}$ for all $a \in \mathcal{M}$. This vector will play a prominent role in the rest of this section.

Now let $(\mathcal{N}, \tau|_{\mathcal{N}})$ be a von Neumann subalgebra of $(\mathcal{M}, \tau)$ (with the inherited trace). Then we can canonically identify $L^2(\tau|_{\mathcal{N}})$ with the closed subspace

$$\overline{\{\hat{b} : b \in \mathcal{N}\}} = \overline{\mathcal{N}\hat{1}} = \overline{\hat{1}\mathcal{N}}$$

of $L^2(\tau)$ in the obvious manner.

We will make use of certain well-known properties of these constructs, which we merely recall here. A clear account of all of them can be found in [Jones and Sunder 1997, Chapters 1 and 3].

First, it is important that there is a simple necessary and sufficient condition for a vector $\xi \in L^2(\tau)$ to lie in the dense subspace $\hat{\mathcal{M}}$: this is so if and only if the linear operator $\hat{\mathcal{M}} \to L^2(\tau)$, $\hat{x} \mapsto x\xi$ is bounded for the norm $\|\cdot\|_{L^2(\tau)}$, and so extends by continuity to a bounded operator $L^2(\tau) \to L^2(\tau)$. The necessity of this conclusion is clear, and its sufficiency requires just a little argument using the fact that for a finite von Neumann algebra $(\mathcal{M}, \tau)$ we have $\mathcal{M}_{\mathrm{right}} = \mathcal{M}''_{\mathrm{right}}$ and $\mathcal{M}_{\mathrm{left}} = \mathcal{M}''_{\mathrm{left}}$; see [Jones and Sunder 1997, Theorem 1.2.4].

A simple application of this condition now shows that the orthogonal projection $e_{\mathcal{N}} : L^2(\tau) \to \overline{\mathcal{N}\hat{1}}$ maps the dense subspace $\hat{\mathcal{M}}$ into $\hat{\mathcal{N}}$, and so defines also a linear operator $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Indeed, for $a \in \mathcal{M}$ we need only to show that the map $\hat{\mathcal{M}} \to L^2(\tau)$, $\hat{x} \mapsto xe_{\mathcal{N}}(\hat{a})$ is bounded for the norm $\|\cdot\|_{L^2(\tau)}$. Since $\mathcal{N}$ is also a von Neumann algebra and $e_{\mathcal{N}}(\hat{a}) \in \overline{\mathcal{N}\hat{1}} \cong L^2(\tau|_{\mathcal{N}})$, it actually suffices to check this for $x \in \mathcal{N}$. However, since $\overline{\mathcal{N}\hat{1}}$ is an $(\mathcal{N}, \mathcal{N})$-sub-bimodule, left multiplication by x commutes with $e_{\mathcal{N}}$, and so we have, as required,

$$\|xe_{\mathcal{N}}(\hat{a})\|_{L^2(\tau)} = \|e_{\mathcal{N}}(x\hat{a})\|_{L^2(\tau)} \leq \|\widehat{xa}\|_{L^2(\tau)} \leq \|a\|\|\hat{x}\|_{L^2(\tau)}.$$

The linear operator $E_{\mathcal{N}}$ is referred to as the *conditional expectation* of $\mathcal{M}$ onto $\mathcal{N}$ associated to τ, and it has the following readily verified properties:

Lemma 3.1 (properties of conditional expectation). *For all $a \in \mathcal{M}$, the operator $E_{\mathcal{N}}$ satisfies*

- (idempotence) $E_{\mathcal{N}}(E_{\mathcal{N}}(a)) = E_{\mathcal{N}}(a)$;
- (contractivity) $\|E_{\mathcal{N}}(a)\| \leq \|a\|$;
- (trace-preservation) $\tau|_{\mathcal{N}}(E_{\mathcal{N}}(a)) = \tau(a)$;
- (positivity) $E_{\mathcal{N}}(a^*a) \geq 0$ (*as a member of* $\mathcal{N}$); *and*
- (relation with $e_{\mathcal{N}}$) *for all* $\xi \in L^2(\tau)$,

$$e_{\mathcal{N}}(a(e_{\mathcal{N}}(\xi))) = E_{\mathcal{N}}(a)(e_{\mathcal{N}}(\xi)) = e_{\mathcal{N}}(E_{\mathcal{N}}(a)(\xi)).$$

Example 3.2. If $\mathcal{M} = L^\infty(X, \mathcal{X}, \mu)$ for some probability measure μ with the usual trace, and $(Y, \mathcal{Y}, \nu)$ is a factor space of $(X, \mathcal{X}, \mu)$ with a measurable factor map $\pi : X \to Y$ that pushes μ forward to ν, then $L^\infty(Y, \mathcal{Y}, \nu)$ can be identified with a subalgebra of $\mathcal{M}$, and the conditional expectation map becomes its classical counterpart from probability theory.

Together with $\mathcal{M}$, the orthogonal projection $e_{\mathcal{N}}$ now generates in $B(L^2(\tau))$ a larger von Neumann algebra $\langle \mathcal{M}, e_{\mathcal{N}} \rangle \supseteq \mathcal{M}$. In general $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is no longer a finite von Neumann algebra, but it does contain the dense $*$-subalgebra

$$\mathcal{A} := \mathrm{lin}(\mathcal{M} \cup \{x e_{\mathcal{N}} y : \ x, y \in \mathcal{M}\})$$

on which we define the *lifted trace* $\bar{\tau} : \mathcal{A} \to \mathbb{C}$ by specifying $\bar{\tau}(x e_{\mathcal{N}} y) = \tau(xy)$. By choosing an orthonormal basis for $L^2(\tau)$ relative to the right action of $\mathcal{N}$, and consequently realising $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ as an amplification of $\mathcal{N}$, this linear map is seen to be nonnegative and faithful, and hence defines a semifinite normal faithful $[0, +\infty]$-valued trace (which we still denote by $\bar{\tau}$) on the cone $(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)^+$ of nonnegative (and self-adjoint) elements of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$. This witnesses that the algebra $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is semifinite (that is, any positive element of it may be approximated from below by finite-$\bar{\tau}$ positive elements). We will not spell out these standard manipulations here (see, for instance, [Popa 2007, Section 1.5]), but we will invoke a notion of orthonormal basis for right-$\mathcal{N}$-submodules of $L^2(\tau)$ shortly.

Remark 3.3. In case $\mathcal{N} \subset \mathcal{M}$ is a finite-index inclusion of finite II_1 factors, then we find that $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is also a finite II_1 factor. Writing $\mathcal{M}_1$ for this factor, it follows that the construction above may be repeated with the inclusion $\mathcal{M} \hookrightarrow \mathcal{M}_1$ in place of $\mathcal{N} \hookrightarrow \mathcal{M}$, and indeed that it may be iterated to form an infinite tower of II_1 factors

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots .$$

This is Jones' *basic construction*, which underlies his famous work [1983] on the possible values of the index $[\mathcal{N} : \mathcal{M}]$, and also several more recent developments. Once again we refer the reader to [Jones and Sunder 1997] for a thorough account of its importance, and numerous further references. However, since the construction of this whole infinite tower is special to the case of II_1 factors, we will not focus on it further here.

It is easy to check that the right action of any $n \in \mathcal{N}$ commutes with any $x e_{\mathcal{N}} y$, and hence with any member of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$. In fact it can be shown that $\langle \mathcal{M}, e_{\mathcal{N}} \rangle' = \mathcal{N}_{\mathrm{right}}$ and hence that $\mathcal{N}'_{\mathrm{right}} = \langle \mathcal{M}, e_{\mathcal{N}} \rangle'' = \langle \mathcal{M}, e_{\mathcal{N}} \rangle$: first, if $A \in B(L^2(\tau))$ commutes with every $b \in \mathcal{M}_{\mathrm{left}}$, then it must be the right action of some $a \in \mathcal{M}$, and now if also $e_{\mathcal{N}}(\hat{1}a) = \hat{1}a$ then we must in fact have $a \in \mathcal{N}$; see [Jones and Sunder 1997, Proposition 3.1.2]. Let us record the following immediate but important consequence of this for our later work:

Lemma 3.4. *If $V \leq L^2(\tau)$ is a closed right-$\mathcal{N}$-submodule, then the orthogonal projection $P_V : L^2(\tau) \to V$ is a member of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$.* $\square$

Using $\bar{\tau}$ we can also define an alternative completion of $\mathcal{A} = \overline{\operatorname{lin} \mathcal{M} e_{\mathcal{N}} \mathcal{M}}$ for each $p \in [1, \infty)$ by setting $\|A\|_{p,\bar{\tau}} := \sqrt[p]{\bar{\tau}((A^*A)^{p/2})}$ for $A \in \mathcal{A}$ (where as usual the power $(A^*A)^{p/2}$ is defined using spectral theory for the self-adjoint operator A^*A, and the nonnegativity of $\bar{\tau}$ is used to show that $\bar{\tau}((A^*A)^{p/2})$ is finite even when $p/2$ is not an integer). We denote this completion by $L^p(\bar{\tau})$; it is a Hilbert space when $p = 2$. In general elements of $L^p(\bar{\tau})$ do not correspond to elements of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$, but they do give possibly unbounded but closable operators that are weakly approximable by members of this algebra, which are therefore affiliated to $\mathcal{N}_{\text{right}}$. If $A \in L^p(\bar{\tau})$ is such an operator that is self-adjoint, then it admits a spectral decomposition $A = \int_{\mathbb{R}} s\, P(\mathrm{d}s)$ for some spectral measure P on $\mathbb{R}$ taking values in the projections of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle \cap L^1(\bar{\tau})$, of possibly unbounded support in $\mathbb{R}$, but for which $\|A\|_{p,\bar{\tau}}^p = \int_{\mathbb{R}} |s|^p \,\bar{\tau} P(\mathrm{d}s) < \infty$.

If V is as in Lemma 3.4 then we may write that P_V has *finite lifted trace* if it corresponds to a member of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle \cap L^1(\bar{\tau})$.

Now let us introduce some dynamics. Suppose that α is a shift on $\mathcal{M}$ that restricts to a shift on $\mathcal{N}$. Then, as mentioned in the introduction, α induces a unitary operator acting on $L^2(\tau)$, which we shall distinguish from α by writing it as U_α; thus for instance

$$U_\alpha \hat{a} = U_\alpha(a\hat{1}) = \alpha(a)\hat{1} = \widehat{\alpha(a)} \quad \text{for all } a \in \mathcal{M}.$$

It is clear that $\overline{\mathcal{N}\hat{1}}$ is an invariant subspace for U_α, so that U_α commutes with $e_{\mathcal{N}}$. Also, conjugation by U_α agrees with the action α on $\mathcal{M}$; thus

$$U_\alpha a U_\alpha^{-1} \xi = \alpha(a)\xi \quad \text{for all } a \in \mathcal{M} \text{ and } \xi \in L^2(\tau).$$

Thus, conjugation by U_α extends the action of α to $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$.

The following special class of one-sided submodules of $L^2(\tau)$ appears here almost exactly as in the commutative setting.

Definition 3.5 (finite-rank modules). A left- (respectively, right-) $\mathcal{N}$-submodule V of $L^2(\tau)$ has *finite rank* if there are some $\xi_1, \xi_2, \ldots, \xi_r \in V$ such that $V = \overline{\sum_{i=1}^r \mathcal{N}\xi_i}$ (respectively, $V = \overline{\sum_{i=1}^r \xi_i \mathcal{N}}$), and the numerical value of its *rank* is the least $r \geq 1$ for which this is possible.

Proposition 3.6 (relativised Gram–Schmidt procedure). *If $V \leq L^2(\tau)$ is a U_α-invariant right-$\mathcal{N}$-submodule of finite rank r then there are $\xi_1, \xi_2, \ldots, \xi_r \in L^2(\tau)$ such that*

- *the subspaces $\overline{\xi_i \mathcal{N}} \leq L^2(\tau)$ are pairwise orthogonal, and*
- *$V = \sum_{i=1}^r \overline{\xi_i \mathcal{N}}$.*

Proof. This uses a relativised Gram–Schmidt argument much as in the commutative setting; see for example [Glasner 2003, Lemma 9.4]. We proceed by induction on r. If V has rank 1, then the result is immediate from the definition, so let us suppose that it has rank $r+1$ for some $r \geq 1$. Then given a representation

$$V = \overline{\sum_{i=1}^{r+1} \xi_i^\circ \mathcal{N}},$$

we know that any member of V may be approximated in $\|\cdot\|_{L^2(\tau)}$ by expressions of the form $\xi_1^\circ n_1 + \cdots + \xi_{r+1}^\circ n_{r+1}$ for $n_1, n_2, \ldots, n_{r+1} \in \mathcal{N}$. This, in turn, may be rewritten as

$$(\xi_1^\perp n_1 + \cdots + \xi_r^\perp n_r) + ((\xi_1^\circ - \xi_1^\perp) n_1 + \cdots + (\xi_r^\circ - \xi_r^\perp) n_r) + \xi_{r+1}^\circ n_{r+1}$$

where for each $i \leq r$ we have decomposed ξ_i° into its component $\xi_i^\perp$ orthogonal to $\overline{\xi_{r+1}\mathcal{N}}$ and the remainder $\xi_i^\circ - \xi_i^\perp \in \overline{\xi_{r+1}\mathcal{N}}$. Since $\overline{\xi_{r+1}\mathcal{N}}$ is a right-$\mathcal{N}$-submodule, it follows that the second and third inner sums in the decomposition above both lie in $\overline{\xi_{r+1}\mathcal{N}}$, and now since $\overline{\xi_{r+1}\mathcal{N}}^\perp$ is also a right-$\mathcal{N}$-submodule, we have in fact shown that

$$V = V_1 + \overline{\xi_{r+1}\mathcal{N}},$$

where $V_1 := \overline{\sum_{i=1}^r \xi_i^\perp \mathcal{N}}$ is a rank-r right-$\mathcal{N}$-submodule that is orthogonal to $\overline{\xi_{r+1}\mathcal{N}}$. Applying the inductive hypothesis to V_1 now completes the proof. □

The following definition is also drawn from the commutative world. This notion has previously been extended to the setting of noncommutative algebras by Popa in [2007], who discusses several other aspects and equivalent conditions in that paper. (See also [Niculescu et al. 2003; Duvenhage 2009; Beyers et al. 2010] for an analysis of the absolute analogue of weak mixing, in which the subalgebra $\mathcal{N}$ is the trivial algebra $\mathbb{C}1$.)

Definition 3.7 (relative weak mixing). If $(\mathcal{M}, \tau, \alpha)$ is a von Neumann dynamical system and $\mathcal{N} \subset \mathcal{M}$ is an α-invariant von Neumann subalgebra, then α is *weakly mixing relative to* $\mathcal{N}$ if for any $a \in \mathcal{M} \cap \mathcal{N}^\perp$ we have

$$\frac{1}{N} \sum_{n=1}^{N} \|E_{\mathcal{N}}(a^* \alpha^n(a))\|_\tau^2 \to 0 \quad \text{as } N \to \infty.$$

The basic inverse theorem that we need, extending the idea of Furstenberg and Zimmer to the noncommutative context, is contained in the following proposition, which essentially proves again part of [Popa 2007, Lemma 2.10]:

Proposition 3.8 (lack of weak mixing implies finite trace submodule). *If α is not weakly mixing relative to $\mathcal{N}$, then there is a U_α-invariant right-$\mathcal{N}$-submodule $V \leq L^2(\tau) \ominus \overline{\mathcal{N}\hat{1}}$ such that P_V has finite lifted trace.*

Proof. Suppose that $a \in \mathcal{M} \cap \mathcal{N}^\perp$ is such that

$$\frac{1}{N}\sum_{n=1}^{N} \|E_{\mathcal{N}}(a^*\alpha^n(a))\|_\tau^2 \not\to 0.$$

Define $b := ae_{\mathcal{N}}a^* \in \langle \mathcal{M}, e_{\mathcal{N}}\rangle$, and now observe (using the cyclic permutability of $\bar{\tau}$ and the identity $e_{\mathcal{N}}me_{\mathcal{N}} \equiv E_{\mathcal{N}}(m)e_{\mathcal{N}}$) that for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\bar{\tau}(b(U_\alpha^n b U_\alpha^{-n})) &= \bar{\tau}(ae_{\mathcal{N}}a^* U_\alpha^n(ae_{\mathcal{N}}a^*)U_\alpha^{-n}) = \bar{\tau}(ae_{\mathcal{N}}a^*\alpha^n(a)e_{\mathcal{N}}\alpha^n(a)^*)\\ &= \bar{\tau}\big(E_{\mathcal{N}}(a^*\alpha^n(a))e_{\mathcal{N}}\alpha^n(a)^*a\big) = \|E_{\mathcal{N}}(a^*\alpha^n(a))\|_\tau^2.\end{aligned}$$

Averaging in n, it follows that

$$\bar{\tau}\Big(b\frac{1}{N}\sum_{n=1}^{N}\alpha^n(b)\Big) \to \langle b, b_1\rangle_{\bar{\tau}} \neq 0,$$

where b_1 is the limit of the ergodic averages $N^{-1}\sum_{n=1}^{N}\alpha^n(b)$ in the Hilbertian completion $L^2(\bar{\tau})$, which is therefore invariant under the further extension of the unitary operator U_α to this Hilbert space.

This new element b_1 need not, in general, correspond to a member of $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$ (it is easily seen to be so in the commutative setting, but for special reasons); however, as a $\|\cdot\|_{2,\bar{\tau}}$-limit of members of $\langle \mathcal{M}, e_{\mathcal{N}}\rangle = \mathcal{N}'_{\mathrm{right}}$, the element can always be identified with a closed operator on $L^2(\tau)$ that is affiliated with the right action of the algebra $\mathcal{N}$, and as such it admits a spectral decomposition $b_1 = \int_0^\infty s\, P(\mathrm{d}s)$ for some resolution of the identity P on $[0, \infty)$ whose contributing spectral projections lie in $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$, and for which $\int_0^\infty s^2\bar{\tau}(P(\mathrm{d}s)) = \|b_1\|_{2,\bar{\tau}}^2 < \infty$. Hence $\bar{\tau}P(I) < \infty$ for any Borel subset $I \subseteq (0, \infty)$ bounded away from 0. Now choosing any such subset I for which $P(I) \neq 0$ gives an orthogonal projection $P(I) \in \langle M, e_{\mathcal{N}}\rangle$ of finite lifted trace that is U_α-invariant, commutes with the right-$\mathcal{N}$-action because it lies in $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$, and moreover has image orthogonal to $\widehat{1\mathcal{N}}$ because we initially chose b to lie in the orthogonal complement of this subspace. □

Remark 3.9. The implication above can in fact be reversed, and these conditions shown to be equivalent to a number of others; see [Popa 2007, Lemma 2.10] for a more complete picture.

In the next section we will push the above results a little further under the additional assumption that the subalgebra $\mathcal{N}$ is central, leading to the proof of Theorem 1.14.

4. The case of asymptotically abelian systems

We now specialise to the case of an asymptotically abelian system, with the crucial additional assumption that the subalgebra $\mathcal{N}$ is *central*.

Lemma 4.1. *Suppose that $(\mathcal{M}, \tau, \alpha)$ is a von Neumann dynamical system, $\mathcal{N} \subset \mathcal{M}$ is an α-invariant central von Neumann subalgebra and $V \leq L^2(\tau)$ is a U_α-invariant right-$\mathcal{N}$-submodule of finite lifted trace. Then for any $\varepsilon > 0$ there is a further U_α-invariant right-$\mathcal{N}$-submodule $V_1 \leq V$ such that*

- $\bar{\tau}(P_V - P_{V_1}) < \varepsilon$,
- *V_1 has finite rank, say $r \geq 1$, and*
- *there are an orthogonal right-$\mathcal{N}$-basis $\xi_1, \xi_2, \ldots, \xi_r$ and a unitary matrix of unitary operators $U = (u_{ji})_{1 \leq i,j \leq r} \in \mathcal{U}_{r \times r}(\mathcal{N})$ such that*

$$U_\alpha(\xi_i) = \sum_{j=1}^{r} \xi_j u_{ji} \quad \textit{for all } i = 1, 2, \ldots, r.$$

We refer to U as the cocycle representing the action of U_α on the basis elements ξ_i.

Proof. We will prove this invoking the picture of the representation of $\mathcal{N}$ on $L^2(\tau)$ as a direct integral coming from spectral theory. By the classical theory of direct integrals (see, for instance, [Kadison and Ringrose 1997, Chapter 14]), we can select

- a standard Borel probability space (Y, ν),
- a Borel partition $Y = \bigcup_{n \geq 1} Y_n \cup Y_\infty$,
- a collection of Hilbert spaces $\mathfrak{H}_n$ for $n \in \{1, 2, \ldots, \infty\}$ with $\dim(\mathfrak{H}_n) = n$, and
- a unitary equivalence

$$\Phi : L^2(\tau) \to \mathfrak{H} := \int_Y^{\oplus} \mathfrak{H}_y \, \nu(\mathrm{d}y),$$

  where we define $\mathfrak{H}_y$ to be $\mathfrak{H}_n$ when $y \in Y_n$,

such that $\mathcal{N}$ (acting on either the right or left, since these agree for a central subalgebra of $\mathcal{M}$) is identified with the algebra of functions $L^\infty(\nu)$ acting by pointwise multiplication. Explicitly, if we denote elements of $\mathfrak{H}$ as measurable sections $v : Y \to \coprod_{y \in Y} \mathfrak{H}_y$, then $f \in L^\infty(\nu)$ acts on $\mathfrak{H}$ by

$$M_f(v)(y) := f(y)v(y).$$

Moreover, in order to accommodate $\Phi(\overline{\mathcal{N}\hat{1}})$ we select a measurable section $v_0 \in \mathfrak{H}$ with $\|v_0(y)\|_{\mathfrak{H}_y} \equiv 1$, and now $\mathcal{N}\hat{1}$ is identified with

$$\{y \mapsto f(y)v_0(y) : f \in L^\infty(\mu)\},$$

so that the orthogonal projection $\Phi e_{\mathcal{N}} \Phi^{-1}$ acts by

$$\Phi e_{\mathcal{N}} \Phi^{-1}(v)(y) := \langle v(y), v_0(y) \rangle_{\mathfrak{H}_y} \cdot v_0(y).$$

The larger algebra $\mathcal{M}_{\mathrm{right}}$ is identified under Φ with a direct integral $\int_Y^{\oplus} \mathcal{M}_y\,\nu(\mathrm{d}y)$, so that elements of $\Phi(\mathcal{M})$ are expressed as measurable sections

$$T : Y \to \coprod_{y\in Y} B(\mathfrak{H}_y)$$

acting by $Tv(y) := T(y)(v(y))$ and such that $T(y) \in \mathcal{M}_y$ ν-almost surely, where $(\mathcal{M}_y)_{y\in Y}$ is a measurable field of finite von Neumann subalgebras of $B(\mathfrak{H}_y)$ for each of which the state

$$\mathcal{M}_y \to \mathbb{C} : T \mapsto \langle v_0(y), T(v_0(y))\rangle_{\mathfrak{H}_y}$$

is a faithful finite trace; overall we have

$$\tau(a) = \langle \hat{1}, a\hat{1}\rangle = \int_Y \langle v_0(y), \Phi(a)(y)(v_0(y))\rangle_{\mathfrak{H}_y}\,\nu(\mathrm{d}y) \quad \text{for } a \in \mathcal{M},$$

and so in particular if $n \in \mathcal{N}$ then $\Phi(n) \in L^\infty(\mu)$ and $\tau(n) = \int \Phi(n)\,\mathrm{d}\nu$.

Given these data, for $a, b \in \mathcal{M}$ we can compute that and

$$\begin{aligned}\Phi(ae_{\mathcal{N}}b)\Phi^{-1}v(y) &= \langle \Phi(b)(y)(v(y)), v_0(y)\rangle \cdot \Phi(a)(y)(v_0(y)),\\ \bar{\tau}(ae_{\mathcal{N}}b) = \tau(ab) &= \int_Y \langle v_0(y), \Phi(ab)(y)(v_0(y))\rangle_{\mathfrak{H}_y}\,\nu(\mathrm{d}y)\\ &= \int_Y \langle \Phi(a^*)(y)(v_0(y)), \Phi(b)(y)(v_0(y))\rangle_{\mathfrak{H}_y}\,\nu(\mathrm{d}y)\\ &= \int_Y \mathrm{tr}(\Phi(ae_{\mathcal{N}}b)\Phi^{-1}|_{\mathfrak{H}_y})\,\nu(\mathrm{d}y).\end{aligned}$$

In this representation an $\mathcal{N}$-submodule $V \leq L^2(\tau)$ corresponds to a subspace $\Phi(V) \leq \mathfrak{H}$ of the form $\int_Y^{\oplus} V_y\,\nu(\mathrm{d}y)$ for some measurable subfield of Hilbert spaces $V_y \leq \mathfrak{H}_y$, and the calculation above now shows that $\bar{\tau}(P_V) = \int_Y \dim(V_y)\,\nu(\mathrm{d}y)$, so P_V has finite lifted trace if and only if the function $y \mapsto \dim(V_y)$ is ν-integrable.

We can enhance this picture further by noting that since α preserves $\mathcal{N}$ it must correspond to some ν-preserving transformation $S \curvearrowright Y$, and that since it also preserves $\mathcal{M}$ and extends to a unitary operator on $L^2(\tau)$ it must also preserve each of the cells Y_n. Similarly, since V is U_α-invariant, the transformation S must preserve the function $y \mapsto \deg(V_y)$. It follows that the unitary operator $\Phi U_\alpha \Phi^{-1}$ on $L^2(\tau)$ is actually given by a measurable section of unitary operators $\Psi : Y \to \coprod_{y\in Y} \mathcal{U}(\mathfrak{H}_y)$ such that

$$\Phi U_\alpha \Phi^{-1} v(y) = \Psi(y)(v(S^{-1}y)).$$

Now, since $y \mapsto \deg(V_y)$ is ν-integrable, for sufficiently large $r \geq 1$ we know that

$$\int_{\{y\in Y:\deg(V_y)>r\}} \deg(V_y)\,\nu(\mathrm{d}y) < \varepsilon.$$

Define

$$W := \int_{\{y\in Y:\deg(V_y)\le r\}}^{\oplus} V_y \, \nu(\mathrm{d}y) \oplus \int_{\{y\in Y:\deg(V_y)>r\}}^{\oplus} \{0\} \, \nu(\mathrm{d}y)$$

and $V_1 := \Phi^{-1}(W)$. Clearly V_1 is still a right-$\mathcal{N}$-submodule that is U_α-invariant, and it clearly also has rank at most r (since it suffices to prove this for W, for which it follows by a relativised Gram–Schmidt construction of a fibrewise-orthonormal basis exactly as in the setting of commutative ergodic theory; see for instance [Glasner 2003, Lemma 9.4]). Also, we have

$$\bar{\tau}(P_V - P_{V_1}) = \int_{\{y\in Y:\deg(V_y)>r\}} \deg(V_y) \, \nu(\mathrm{d}y) < \varepsilon.$$

Finally, the selection of unitaries Ψ must preserve the field of subspaces V_y above the S-invariant set $\{y \in Y : \deg(V_y) = s\}$ for each $s \le r$. Choosing an abstract d-dimensional Euclidean space W_d for each $d \le r$ and adjusting each fibre of W by a unitary in order to identify each V_y for which $\dim(V_y) \le r$ with $W_{\dim(V_y)}$, we obtain a new representation of V_1 as a right-$\mathcal{N}$-submodule using these fibres W_d, so that the action of U_α is now described by a measurable family of unitaries $\Psi'(y) \in \mathcal{U}(W_{\dim(V_y)})$. Picking an orthonormal basis for each W_d, writing these unitary operators as unitary matrices in terms of these bases, noting that their individual entries are now identified with elements of $L^\infty(\mu) = \Phi(\mathcal{N})$, and carrying everything back to $L^2(\tau)$ using Φ^{-1} gives the desired expression for U_α. □

Remark 4.2. Frustratingly, both the fact that a U_α-invariant V of finite lifted trace may be approximated by a U_α-invariant V_1 of finite rank, and the fact that given such a module of finite rank the action of U_α on it may be described by a unitary element in $\mathcal{U}(M_{r\times r}(\mathcal{N}))$, seem to be difficult to prove without the assumption that $\mathcal{N}$ is central and the resulting representation of the action of $\mathcal{N}$ on $L^2(\mu)$ as the multiplication action of some $L^\infty(\nu)$ on a measurable field of Hilbert spaces. It would be interesting to settle this issue more generally:

Question 4.3. Do these conclusions hold for a finite-lifted-trace invariant submodule corresponding to an arbitrary inclusion of finite von Neumann algebras with a trace-preserving automorphism?

Before moving on let us quickly note an important difference from the setting of abelian von Neumann algebras.

Example 4.4. If $\mathcal{M}$ is abelian, then it is well known from commutative ergodic theory that all the intermediate U_α-invariant submodules $V \le L^2(\tau)$ that have finite-rank over $\mathcal{N}$ together generate an intermediate sub*algebra* between $\mathcal{N}$ and $\mathcal{M}$, and that this then corresponds to an intermediate measure-preserving system. We will see shortly that an analogous conclusion can sometimes be recovered in the asymptotically abelian setting, but it is certainly not true for general finite-rank submodules, even when the smaller algebra $\mathcal{N}$ is abelian.

Consider, for example, the inclusion $i : L\mathbb{Z} \cong L^\infty(m_{\mathbb{T}}) \hookrightarrow L\mathbb{F}_2$ corresponding to the embedding of $\mathbb{Z}$ as the cyclic subgroup $a^{\mathbb{Z}}$ of the free group $\mathbb{F}_2 = \langle a, b\rangle$. Here LG is the group von Neumann algebra of G, defined in Section 2a. In this case we can identify $L^2(\tau)$ as $\ell^2(\mathbb{F}_2)$ and $L^2(\tau|_{\mathcal{N}})$ as the subspace spanned by $\{\xi_{a^n}\}_{n\in\mathbb{Z}}$. Now define $\alpha \in \mathrm{Aut}\, L\mathbb{F}_2$ simply by lifting the group automorphism of $\mathbb{F}_2$ that fixes a and maps $b \mapsto ba$. Now the subspace $V := \overline{\mathrm{lin}}\{\xi_{ba^n} : n \in \mathbb{Z}\} \leq \ell^2(\mathbb{F}_2)$ is a U_α-invariant right $\mathcal{N}$-module of rank one which is orthogonal to $L^2(\tau|_{\mathcal{N}})$. On the other hand, although $\xi_b \in \hat{\mathcal{M}} \cap V$, we have $\alpha^m(\xi_b^2) = \alpha^m(\xi_{b^2}) = \xi_{ba^mba^m}$ for $m \in \mathbb{Z}$, and it is easy to see that these elements of $\mathcal{M}$ do not remain within any finite-rank right-$\mathcal{N}$-submodule.

It is true that if $L^2(\tau) \ominus L^2(\tau|_{\mathcal{N}})$ contains a finite-rank right-$\mathcal{N}$-submodule V, then it also contains a finite-rank left-$\mathcal{N}$-module in the form of $J(V)$, where J is the *modular automorphism* on V, defined by extending the conjugation map $a \mapsto a^*$ on $\mathcal{M} \equiv \hat{\mathcal{M}}$ by density. The point is that it can happen that $J(V) \perp V$, and that all elements of $J(V)$ are weakly mixed by U_α: it is the right-module V, and no other, that serves as the obstruction to overall relative weak mixing coming from Theorem 1.13.

Definition 4.5. A vector $\xi \in L^2(\tau)$ is *central* if $m\xi = \xi m$ for all $m \in \mathcal{M}$.

Lemma 4.6 (no nonobvious central vectors). *The closure $\overline{\mathfrak{Z}(\mathcal{M})\hat{1}} = \overline{\hat{1}\mathfrak{Z}(\mathcal{M})}$ is equal to the set of all central vectors in $L^2(\tau)$.*

Proof. Suppose that $\xi \in L^2(\tau)$ is central. Define $a_\xi : \mathcal{M}\hat{1} \to L^2(\tau)$ by $a_\xi(m\hat{1}) := \xi m$. This is a densely defined linear operator on $L^2(\tau)$, and it is closable because if $m_n\hat{1} = \hat{1}m_n \to 0$ in $\|\cdot\|_{L^2(\tau)}$ for some sequence $(m_n)_{n\geq 1}$ in $\mathcal{M}$ and also $\xi m_n \to \xi'$ in $\|\cdot\|_{L^2(\tau)}$, then we have

$$\langle m'\hat{1}, \xi'\rangle = \lim_{n\to\infty} \langle m'\hat{1}, \xi m_n\rangle = \lim_{n\to\infty} \langle \hat{1}m_n^*, (m')^*\xi\rangle = 0 \quad \text{for every } m' \in \mathcal{M},$$

and so in fact we must have $\xi' = 0$. Also, we clearly have

$$a_\xi(m\hat{1}) = a_\xi(\hat{1}m) = \xi m = m\xi = (a_\xi(\hat{1}))m = m(a_\xi(\hat{1})) \quad \text{for every } m \in \mathcal{M},$$

so a_ξ is affiliated with both the right- and left-actions of $\mathcal{M}$ on $L^2(\tau)$. The same therefore holds for $a_\xi + a_\xi^*$ and $\mathrm{i}(a_\xi - a_\xi^*)$, and now these are self-adjoint and so each of them may be expressed as an unbounded spectral integral all of whose contributing spectral projections must lie in $\mathcal{M}'_{\mathrm{left}} \cap \mathcal{M}'_{\mathrm{right}} = \mathfrak{Z}(\mathcal{M})$. Therefore, approximating $a_\xi = \frac{1}{2}(a_\xi + a_\xi^*) + \frac{1}{2}(a_\xi - a_\xi^*)$ by a sum of two large but bounded integrals with respect to the respective resolutions of the identity, we get a sequence of elements $a_n \in \mathfrak{Z}(\mathcal{M})$ such that $a_n \to a_\xi$ pointwise on $\mathrm{dom}(\mathrm{clos}(a_\xi)) \supseteq \mathcal{M}\hat{1}$, and hence such that $a_n\hat{1} \to \xi$ in $\|\cdot\|_{L^2(\tau)}$. Hence $\xi \in \overline{\mathfrak{Z}(\mathcal{M})\hat{1}}$, as required. $\square$

Proposition 4.7. *If $(\mathcal{M}, \tau, \alpha)$ is an asymptotically abelian von Neumann dynamical system, $\mathcal{N}$ is a shift-invariant central von Neumann subalgebra, and $V \leq L^2(\tau)$ is an α-invariant right-$\mathcal{N}$-submodule of $\mathcal{M}$ having finite lifted trace, then all elements of V are central vectors.*

Proof. Clearly it will suffice to prove this for all finite-rank approximants V_1 to V as given by Lemma 4.1. Thus we may assume that V actually has finite rank. Let $\xi_1, \xi_2, \dots, \xi_r$ and $U = (u_{ji})_{1\leq i,j\leq r} \in \mathcal{M}_{r\times r}(\mathcal{N})$ be as given by the third part of that lemma.

Since α is asymptotically abelian, we have for any $a\hat{1} \in \mathcal{M}\hat{1}$ and $b \in \mathcal{M}$ that

$$\frac{1}{N}\sum_{n=1}^{N}\|bU_\alpha^n(a\hat{1}) - U_\alpha^n(a\hat{1})b\|_{L^2(\tau)} = \frac{1}{N}\sum_{n=1}^{N}\|b\alpha^n(a) - \alpha^n(a)b\|_{L^2(\tau)} \to 0.$$

Approximating an arbitrary $\xi \in L^2(\tau)$ by elements of $\mathcal{M}\hat{1}$, it follows that for each fixed $b \in \mathcal{M}$ and $\xi \in L^2(\tau)$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\|bU_\alpha^n(\xi) - U_\alpha^n(\xi)b\|_{L^2(\tau)} = 0.$$

On the other hand, we know that

$$U_\alpha(\xi_i) = \sum_{j=1}^{r}\xi_j u_{ji} \quad \text{for all } i = 1, 2, \dots, r,$$

and so, writing $U^n = (u_{ji}^{(n)})_{1\leq i,j\leq r}$, we have

$$U_\alpha^{-n}(\xi_i) = \sum_{j=1}^{r}\xi_j u_{ji}^{(-n)} \quad \text{implies} \quad \xi_i = \sum_{j=1}^{r} U_\alpha^n(\xi_j)\alpha^n(u_{ji}^{(-n)}) \quad \text{for all } i = 1, 2, \dots, r.$$

Clearly each $u_{ji}^{(-n)}$ is still a unitary, and so from this, averaging in n and the centrality of $\mathcal{N}$, we obtain

$$\begin{aligned}
\|b\xi_i - \xi_i b\|_{L^2(\tau)} &= \Big\|\frac{1}{N}\sum_{n=1}^{N}\Big(\sum_{j=1}^{r} bU_\alpha^n(\xi_j)\alpha^n(u_{ji}^{(-n)}) - \sum_{j=1}^{r} U_\alpha^n(\xi_j)\alpha^n(u_{ji}^{(-n)})b\Big)\Big\|_{L^2(\tau)} \\
&= \Big\|\frac{1}{N}\sum_{n=1}^{N}\sum_{j=1}^{r}(bU_\alpha^n(\xi_j) - U_\alpha^n(\xi_j)b)\alpha^n(u_{ji}^{(-n)})\Big\|_{L^2(\tau)} \\
&\leq \sum_{j=1}^{r}\frac{1}{N}\sum_{n=1}^{N}\|bU_\alpha^n(\xi_j) - U_\alpha^n(\xi_j)b\|_{L^2(\tau)},
\end{aligned}$$

and now since each of the summands in j tends to 0 as $N \to \infty$, it follows that we must in fact have $b\xi_i = \xi_i b$ for every $i \le r$, and hence (taking $\mathcal{N}$-linear combinations, which have central coefficients, and then a completion) that all vectors in V are central, as required. $\square$

Corollary 4.8. *If* $(\mathcal{M}, \tau, \alpha)$ *is an asymptotically abelian von Neumann dynamical system, then the subalgebra* $\mathcal{M}^\alpha := \{a \in \mathcal{M} : \alpha(a) = a\}$ *of individually* α*-invariant elements is central.*

Proof. Of course, if $\alpha(a) = a$, then $\mathrm{lin}\{\hat{1}a\}$ is a rank one α-invariant submodule of $L^2(\tau)$ for the trivial central subalgebra $\mathcal{N} := \mathbb{C}\hat{1}$, and the claim follows from Proposition 4.7. This claim can also be easily verified directly from the definition of asymptotic abelianness. $\square$

Proof of Theorem 1.14. Suppose, for the sake of contradiction, that α were not weakly mixing relative to $\mathfrak{Z}(\mathcal{M}) \subset \mathcal{M}$. Then Proposition 3.8 gives a nontrivial right-$\mathfrak{Z}(\mathcal{M})$-submodule $V \le L^2(\tau) \ominus \overline{\mathfrak{Z}(\mathcal{M})\hat{1}}$ of finite lifted trace, and now Proposition 4.7 tells us that V must consist of central vectors. However, Lemma 4.6 now gives $V \le \overline{\mathfrak{Z}(\mathcal{M})\hat{1}}$, implying a contradiction with our assumption that $V \perp \overline{\mathfrak{Z}(\mathcal{M})\hat{1}}$. $\square$

For the results in this section it suffices to assume that for every $a \in \mathcal{M}$ there exists a sequence $\{n_j\}$ such that $\lim_{j\to\infty} \|[\alpha^{n_j}(a), b]\|_{L^2(\tau)} = 0$ for every $b \in \mathcal{M}$. We do not know whether this condition is strictly weaker than asymptotically abelianness.

Remark 4.9. A variant of Theorem 1.14 can also be deduced from the results in [Niculescu et al. 2003] (and more specifically, Theorem 4.2 and Proposition 5.5 of that paper); we thank the anonymous referee for pointing out this fact. More specifically, the result is that if α is an automorphism of a finite von Neumann algebra $\mathcal{M}$ that leaves invariant a faithful normal trace τ, and E_τ is the conditional expectation to the factor

$$\mathcal{M}_r := \overline{\mathrm{lin}}^{\mathrm{wot}}\{a \in \mathcal{M} : \alpha(a) = \lambda a \text{ for some } \lambda \in \mathbb{T}\},$$

then for any $a, b \in \mathcal{M}$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle E_\tau(a^*\alpha^n(a)) - E_\tau(a)^*\alpha^n(E_\tau(a)), b\rangle_{L^2(\tau)}| = 0;$$

in particular, for N going to infinity along a density one set of integers, the expression $E_\tau(a^*\alpha^n(a)) - E_\tau(a)^*\alpha^n(E_\tau(a))$ converges to zero in the weak operator topology. This property is weaker than the relative weak mixing property with respect to this factor (which one does not expect to hold in general, even in the abelian case), but on the other hand does not require any hypothesis of asymptotic abelianness.

5. Triple averages for nonasymptotically abelian systems

The use to which we put relative weak mixing in the preceding section is very special to asymptotically abelian systems: in general there seems to be no way to track the error term resulting from the rearrangement at the heart of the proof of Theorem 1.13 without this assumption. However, in the special case of triple averages this problem does simplify somewhat, provided we assume instead that our system $(\mathcal{M}, \tau, \alpha)$ is *ergodic*, so that $\mathcal{M}^\alpha = \mathbb{C}1$. In this case we will be able to obtain convergence weakly and in norm, as well as recurrence on a dense set (Theorem 1.17).

This assumption is not so innocuous as might be expected from its analogue in the world of commutative ergodic theory. In that setting it is possible quite generally to decompose a system (that is, more precisely, to decompose its invariant measure) into ergodic components, and then many assertions about the whole system, including multiple recurrence and the convergence of multiple averages, follow if they can be proved for each ergodic component separately. However, in the setting of a general von Neumann dynamical system, this decomposition is available only if $\mathcal{M}^\alpha$ is central in $\mathcal{M}$; otherwise the automorphism α can exhibit genuinely new phenomena precisely by virtue of having the nontrivial fixed subalgebra $\mathcal{M}^\alpha$ "move around". This was already seen in the failure of recurrence on a dense set when the ergodicity hypothesis is dropped (Theorem 1.19).

The key for convergence of triple averages is the following decomposition that is similar to the commutative case, first established (in a slightly more general setting) in [Niculescu et al. 2003] (and more specifically, from Theorem 4.2 and Proposition 5.5 in that paper); for the convenience of the reader we give a short proof of that decomposition here. The result does not require ergodicity of the system. A closely related decomposition was also used in [Fidaleo 2009].

Proposition 5.1 (decomposition of von Neumann dynamical systems). *Suppose $(\mathcal{M}, \tau, \alpha)$ is a von Neumann dynamical system. Then one has the orthogonal decomposition $\mathcal{M} = \mathcal{M}_r \oplus \mathcal{M}_s$, where*

$$\begin{aligned}&\mathcal{M}_r := \overline{\operatorname{lin}}^{\mathrm{wot}}\{a \in \mathcal{M} : \alpha(a) = \lambda a \text{ for some } \lambda \in \mathbb{T}\} \quad \text{and}\\ &\mathcal{M}_s := \Big\{a \in \mathcal{M} : \lim_{N\to\infty} N^{-1} \textstyle\sum_{n=1}^N |\tau(b\alpha^n(a))| = 0 \text{ for every } b \in \mathcal{M}\Big\},\end{aligned}$$

that is, $\mathcal{M}_r$ is the von Neumann subalgebra spanned by the eigenvectors of α and $\mathcal{M}_s$ is the subspace of the elements of $\mathcal{M}$ that are weakly mixed by α. The corresponding projection onto $\mathcal{M}_r$ is the conditional expectation of $\mathcal{M}$ onto $\mathcal{M}_r$ and in particular preserves positivity.

Proof. Since the continuation U_α of α to $L^2(\tau)$ is a unitary operator, the Jacobs–Glicksberg–de Leeuw decomposition holds for U_α (see for example [Krengel 1985, Section 2.4]), that is, $L^2(\tau) = L^2_r(\tau) \oplus L^2_s(\tau)$, where the *reversible part* $L^2_r(\tau)$ is defined as

$$L^2_r(\tau) = \overline{\text{lin}}\{x : U_\alpha(x) = \lambda x \text{ for some } \lambda \in \mathbb{T}\}$$

and the *stable part* $L^2_s(\tau)$ is defined as the space of all $x \in L^2(\tau)$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U^n_\alpha(x), y\rangle| = 0 \quad \text{for every } y \in L^2(\tau).$$

Moreover, this decomposition is orthogonal since U_α is unitary. We do not need here the Jacobs–Glicksberg–de Leeuw decomposition in full generality but only its version for unitary operators, which can be also proved via the spectral theorem.

By a result of Størmer [1974], the eigenvectors of U_α belong to $\mathcal{M}$. We thus have $\mathcal{M}_r = \mathcal{M} \cap L^2_r(\tau)$ and $\mathcal{M}_s = \mathcal{M} \cap L^2_s(\tau)$. The fact that the weak operator closure and the closure in the $L^2(\tau)$-topology coincide for self-adjoint subalgebras implies the second formula for $\mathcal{M}_r$ and thus $\mathcal{M}_r$ is a von Neumann subalgebra of $\mathcal{M}$. The conditional expectation now maps $\mathcal{M}$ onto $\mathcal{M}_r$ assuring the orthogonal decomposition $\mathcal{M} = \mathcal{M}_r \oplus \mathcal{M}_s$. □

In the remainder of this section we assume our system is ergodic.

Proposition 5.2 (convergence of triple averages). *Let $(\mathcal{M}, \tau, \alpha)$ be an ergodic von Neumann dynamical system. Then the averages*

$$\frac{1}{N} \sum_{n=1}^{N} \alpha^n(a)\alpha^{2n}(b) \tag{25}$$

converge in $\|\cdot\|_{L^2(\tau)}$ as $N \to \infty$ for every $a, b \in \mathcal{M}$.

Proof. By the proposition above, it suffices to assume that a and b each belong to $\mathcal{M}_r$ or $\mathcal{M}_s$. Suppose first that $a \in \mathcal{M}_r$, and fix b. The operators S_N given by

$$S_N x = \frac{1}{N} \sum_{n=1}^{N} \alpha^n(x)\alpha^{2n}(b)$$

are linear and bounded on $\mathcal{M}$ for the norm $\|\cdot\|_{L^2(\tau)}$, so we may assume that $\alpha(a) = \lambda a$ for some $\lambda \in \mathbb{T}$. Then $S_N a = (N+1)^{-1} \sum_{n=0}^{N} a(\lambda\alpha^2)^n(b)$, which converges in $L^2(\tau)$ by the mean ergodic theorem.

Suppose now that $a \in \mathcal{M}_s$. We show that the desired limit is zero. Consider $u_n := \alpha^n(a)\alpha^{2n}(b)\hat{1}$ and observe that

$$\begin{aligned}\langle u_n, u_{n+j}\rangle &= \tau(\alpha^{2n}(b^*)\alpha^n(a^*)\alpha^{n+j}(a)\alpha^{2n+2j}(b)) \\ &= \tau(\alpha^n(b^*)a^*\alpha^j(a)\alpha^{n+2j}(b)) = \tau(a^*\alpha^j(a)\alpha^n(\alpha^{2j}(b)b^*)).\end{aligned}$$

The ergodicity of the system implies

$$\gamma_j := \lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle u_n, u_{n+j}\rangle\right| = \left|\tau(a^*\alpha^j(a)\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\alpha^n(\alpha^{2j}(b)b^*))\right|$$
$$= |\tau(a^*\alpha^j(a))|\cdot|\tau(\alpha^{2j}(b)b^*)|.$$

Since $a\in\mathcal{M}_s$ and $\tau(\alpha^{2j}(b)b^*)$ are bounded in j, $\lim_{N\to\infty}N^{-1}\sum_{j=1}^{N}\gamma_j=0$, and therefore by the classical van der Corput lemma for Hilbert spaces (see for example [Furstenberg 1977] or [Bergelson 1987]), we have $\lim_{N\to\infty}N^{-1}\sum_{n=1}^{N}u_n=0$. □

Remarks 5.3. (1) For compact nonergodic systems the averages (25) converge as well, since $\mathcal{M}=\mathcal{M}_r$ in this case; this was observed in [Beyers et al. 2010].

(2) As in the commutative case we see that the Kronecker subalgebra $\mathcal{M}_r$ is characteristic for (25), that is, the limit of the averages in (25) does not change when replacing a by $E_{\mathcal{M}_r}a$ and b by $E_{\mathcal{M}_r}b$.

As was shown in Corollary 2.7, one cannot expect that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tau(a\alpha^n(a)\alpha^{2n}(a))>0 \quad \text{for every positive } a.$$

However, a modification extending [Beyers et al. 2010, Theorem 5.13] is still true.

Proposition 5.4. *For an ergodic von Neumann system* $(\mathcal{M},\tau,\alpha)$, *one has*

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\operatorname{Re}\tau(a\alpha^n(a)\alpha^{2n}(a)))_+>0 \quad \textit{for every } 0<a\in\mathcal{M}.$$

In particular, one has recurrence on a dense set.

Proof. Decompose $a=b+c$ with $b\in\mathcal{M}_r$ and $c\in\mathcal{M}_s$ as in Proposition 5.1, with $b>0$ by Lemma 3.1. We first show that there exists a compact abelian group G, an open set $U\subset G$, and $g\in G$ such that for the 1-step Bohr set $K_U:=\{n\in\mathbb{N}:g^n\in U\}$ one has

$$\operatorname{Re}\tau(b\alpha^n(b)\alpha^{2n}(b))>\tfrac{1}{2}\tau(b^3)>0 \quad \text{for every } n\in K_U. \tag{26}$$

Take $\varepsilon:=\tau(b^3)/(18\|b\|^2)$. Since $b\in\mathcal{M}_r$, we find $k\in\mathbb{N}$, $\lambda_1,\dots,\lambda_k\in\mathbb{T}$ and $b_1,\dots,b_k\in\mathcal{M}\setminus\{0\}$ such that $\alpha(b_j)=\lambda_jb_j$ for every $j=1,\dots,k$ and such that $\|b-(b_1+\cdots+b_k)\|_{L^2(\tau)}<\varepsilon$. Set now $G:=\mathbb{T}^k$, $g:=(\lambda_1,\dots,\lambda_k)$ and $U:=U_{\varepsilon/(k\max\|b_j\|)}(1)\subset\mathbb{T}^k$. Observe that for every n such that $g^n\in U$, we have

$|\lambda_j^n - 1| < \varepsilon/(k \max\|b_j\|)$ for every $j = 1, \ldots, k$ and therefore

$$\begin{aligned}\|\alpha^n(b) - b\|_{L^2(\tau)} &\le \|a^n(b_1 + \cdots + b_k) - (b_1 + \cdots + b_k)\|_{L^2(\tau)} \\ &\qquad + 2\|b_1 + \cdots + b_k - b\|_{L^2(\tau)} \\ &\le \max\|b_j\|_{L^2(\tau)}(|\lambda_1^n - 1| + \cdots + |\lambda_k^n - 1|) + 2\varepsilon \\ &< \max\|b_j\| \frac{k\varepsilon}{k \max\|b_j\|} + 2\varepsilon = 3\varepsilon.\end{aligned}$$

So we can prove (26) by the Cauchy–Schwarz inequality:

$$\begin{aligned}|\tau(b\alpha^n(b)\alpha^{2n}(b)) - \tau(b^3)| &\le |\tau(b\alpha^n(b)(\alpha^{2n}(b) - b))| + |\tau(b(\alpha^n(b) - b)b)| \\ &\le \|b\|^2(\|\alpha^{2n}(b) - b\|_{L^2(\tau)} + \|\alpha^n(b) - b\|_{L^2(\tau)}) \\ &\le 3\|b\|^2\|\alpha^n(b) - b\|_{L^2(\tau)} < 9\|b\|^2\varepsilon = \tfrac{1}{2}\tau(b^3).\end{aligned}$$

Take now $V := U_{\varepsilon/(2k \max\|b_j\|)}(1) \subset U$ and a continuous function $f : G \to [0, 1]$ satisfying $\mathbf{1}_V \le f \le \mathbf{1}_U$. Then by (26), $\operatorname{Re}\tau(b\alpha^n(b)\alpha^{2n}(b))$ is positive whenever $f(g^n) \ne 0$ and therefore

$$\begin{aligned}&\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(g^n)\operatorname{Re}\tau(b\alpha^n(b)\alpha^{2n}(b)) \\ &\qquad\qquad \ge \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_V(g^n)\operatorname{Re}\tau(b\alpha^n(b)\alpha^{2n}(b)).\end{aligned}$$

Since the set $K_V := \{n \in \mathbb{N} : g^n \in V\} \subset K_U$ is syndetic (that is, has bounded gaps) in $\mathbb{N}$, this implies by (26)

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(g^n)\operatorname{Re}\tau(b\alpha^n(b)\alpha^{2n}(b)) > 0. \tag{27}$$

Next, we show that

$$\|\cdot\|_{L^2(\tau)} - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(g^n)\alpha^n(b)\alpha^{2n}(c) = 0. \tag{28}$$

We first consider a character $\gamma \in \hat{G}$ and define $u_n := \gamma(g^n)\alpha^n(b)\alpha^{2n}(c)\hat{1}$. We have

$$\begin{aligned}\langle u_n, u_{n+j}\rangle &= \overline{\gamma(g^n)}\gamma(g^{n+j})\gamma(\alpha^{2n}(c^*)\alpha^n(b^*)\alpha^{n+j}(b)\alpha^{2n+2j}(c)) \\ &= \gamma(g^j)\tau(\alpha^n(c^*)b^*\alpha^j(b)\alpha^{n+2j}(c)) = \gamma(g^j)\tau(b^*\alpha^j(b)\alpha^n(\alpha^{2j}(c)c^*)).\end{aligned}$$

By ergodicity of α,

$$\begin{aligned}\gamma_j &:= \lim_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\langle u_n, u_{n+j}\rangle\Big| = \Big|\gamma(g^j)\tau(b^*\alpha^j(b) \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\alpha^n(\alpha^{2j}(c)c^*))\Big| \\ &= |\tau(b^*\alpha^j(b))| \cdot |\tau(\alpha^{2j}(c)c^*)|,\end{aligned}$$

and the assumption $c \in \mathcal{M}_s$ implies $\lim_{N\to\infty} N^{-1}\sum_{j=1}^N \gamma_j = 0$. By the van der Corput estimate, we thus have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N u_n = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \gamma(g^n)\alpha^n(b)\alpha^{2n}(c)\hat{1} = 0.$$

We have now proved (28), since the characters form a total set in $C(G)$ and the operators $S_N f := N^{-1}\sum_{n=1}^N f(g^n)\alpha^n(b)\alpha^{2n}(c)$ are uniformly bounded on $C(G)$. Analogously one also has

$$\begin{aligned}\|\cdot\|_{L^2(\tau)} - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(g^n)\alpha^n(c)\alpha^{2n}(b) \\ = \|\cdot\|_{L^2(\tau)} - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(g^n)\alpha^n(c)\alpha^{2n}(c) = 0.\end{aligned}$$

The Cauchy–Schwarz inequality implies now that

$$\begin{aligned}\limsup_{N\to\infty}&\Big|\frac{1}{N}\sum_{n=1}^N f(g^n)\tau(c\alpha^n(b)\alpha^{2n}(c))\Big| \\ &= \limsup_{N\to\infty}\Big|\tau\Big(c\frac{1}{N}\sum_{n=1}^N f(g^n)\alpha^n(b)\alpha^{2n}(c)\Big)\Big| \\ &\le \|c\|_{L^2(\tau)} \limsup_{N\to\infty}\Big\|\frac{1}{N}\sum_{n=1}^N f(g^n)\alpha^n(b)\alpha^{2n}(c)\Big\|_{L^2(\tau)} = 0.\end{aligned}$$

Analogously, the Cesàro sums of $f(g^n)\tau(c\alpha^n(c)\alpha^{2n}(b))$, $f(g^n)\tau(c\alpha^n(c)\alpha^{2n}(c))$ and $f(g^n)\tau(b\alpha^n(c)\alpha^{2n}(c))$ vanish, while

$$\tau(c\alpha^n(b)\alpha^{2n}(b)) = \tau(b\alpha^n(b)\alpha^{2n}(c)) = \tau(b\alpha^n(c)\alpha^{2n}(b)) = 0$$

follows from the orthogonality of $\mathcal{M}_r$ and $\mathcal{M}_s$ and the fact that $\mathcal{M}_r$ is an α-invariant self-adjoint subalgebra of $\mathcal{M}$.

Combining this with (27), we obtain by the linearity of τ

$$\begin{aligned}\liminf_{N\to\infty}&\frac{1}{N}\sum_{n=1}^N (\operatorname{Re}\tau(a\alpha^n(a)\alpha^{2n}(a)))_+ \\ &\ge \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^N f(g^n)(\operatorname{Re}\tau(a\alpha^n(a)\alpha^{2n}(a)))_+ \\ &= \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^N f(g^n)(\operatorname{Re}\tau(b\alpha^n(b)\alpha^{2n}(b)))_+ > 0. \qquad \square\end{aligned}$$

6. Closing remarks

We present some remarks concerning Question 1.22. By Theorem 1.17, we have a positive answer to this question when the invariant algebra $\mathcal{M}^\alpha$ is trivial. One can also extend these arguments to cover the case when the invariant algebra $\mathcal{M}^\alpha$ is central by representing $\mathcal{M}$ as a direct integral over $\mathcal{M}^\alpha$, see Kadison, Ringrose [Kadison and Ringrose 1997, Chapter 14].

It is clear that if the answer to Question 1.23 is always positive, then the same is true for Question 1.22. What is less obvious is that the converse is true; if the answer to 1.22 is always true, then the answer to 1.23 is always true. To see this, let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra with two commuting shifts α_1 and α_2. We then form the infinite tensor product $\mathcal{M}^{\mathbb{Z}} := \bigotimes_{n\in\mathbb{Z}} \mathcal{M}$, which is another finite von Neumann algebra, which contains an embedded copy of $\mathcal{M}$ by using the 0 coordinate of $\mathbb{Z}$. Next, let G be the free abelian group on two generators e and f, and let U be the action of G on $\mathcal{M}^{\mathbb{Z}}$ defined by

$$U(e) \bigotimes_{n\in\mathbb{Z}} a_n := \bigotimes_{n\in\mathbb{Z}} \alpha_1^2 \alpha_2^{-1}(a_n) \quad \text{and} \quad U(f) \bigotimes_{n\in\mathbb{Z}} a_n := \bigotimes_{n\in\mathbb{Z}} a_{n-1}$$

for all $a_n \in \mathcal{M}$ with all but finitely many a_n equal to 1. If we define a shift α' to $\mathcal{M}^{\mathbb{Z}}$ by the formula

$$\alpha' \bigotimes_{n\in\mathbb{Z}} a_n := \bigotimes_{n\in\mathbb{Z}} \alpha_1^{2(n+1)} \alpha_2^{-n}(a_n),$$

we then observe the identities

$$\alpha' U(e) (\alpha')^{-1} = U(e) \quad \text{and} \quad \alpha' U(f) (\alpha')^{-1} = U(fe)$$

(here we use the hypothesis that α_1 and α_2 commute). Because of this, we can define a shift α on the crossed product $\mathcal{M}^{\mathbb{Z}} \rtimes_U G$ by declaring α to equal α' on $\mathcal{M}^{\mathbb{Z}}$, and

$$\alpha(e) := e \quad \text{and} \quad \alpha(f) := fe.$$

If a_1 and a_2 lie in $\mathcal{M}^{\mathbb{Z}}$, we observe that

$$\alpha^n(a_1 f^2) \alpha^{2n}(f^{-2} a_2 f) = (\alpha')^n(a_1)((\alpha')^{2n} U(e)^{-2n}(a_2)) f.$$

If we assume that a_1 and a_2 in fact lie in $\mathcal{M}$, we can simplify this as

$$\alpha_1^{2n}(a_1) \alpha_2^{2n}(a_2) f.$$

Thus, if we assume 1.22 has an affirmative answer for the system $\mathcal{M}^{\mathbb{Z}} \rtimes_U G$, we see that the averages of $\alpha_1^{2n}(a_1)\alpha_2^{2n}(a_2) f$ (and hence of $\alpha_1^{2n}(a_1)\alpha_2^{2n}(a_2)$) converge for arbitrary $a_1, a_2 \in \mathcal{M}$; from this one easily deduces (after dividing n into even and odd classes) that 1.23 has an affirmative answer for the system $\mathcal{M}$.

In particular, we see that the task of establishing Question 1.22 in the affirmative for arbitrary von Neumann dynamical systems is at least as hard as that of achieving convergence for two commuting shifts in the abelian case, a result first obtained in [Conze and Lesigne 1984].

One can also cover some other (nonergodic, nonabelian) cases of Question 1.22 by ad hoc methods. Suppose that $\mathcal{M}$ is a group von Neumann algebra LG, with shift α given by automorphisms $\alpha_1, \alpha_2 : G \to G$ of the group. Then one can affirmatively answer 1.22 as follows. First, by density and linearity we may assume that a_1 and a_2 are themselves group elements: $a_1 = g_1 \in G$ and $a_2 = g_2 \in G$. We then see that the means of $\alpha^n(g_1)\alpha^{2n}(g_2)$ will converge to zero unless there exists a group element g_0 for which

$$\alpha^n(g_1)\alpha^{2n}(g_2) = g_0 \tag{29}$$

for all n in a set of positive upper density. But such sets contain nontrivial parallelograms n, $n+h$, $n+k$, $n+h+k$ for $h, k > 0$. Applying (29) for n and $n+h$ and rearranging, one obtains

$$\alpha^n(g_2\alpha^{2h}(g_2^{-1})) = g_1^{-1}\alpha^h(g_1).$$

Similarly, applying (29) for $n+k$ and $n+h+k$, one has

$$\alpha^{n+k}(g_2\alpha^{2h}(g_2^{-1})) = g_1^{-1}\alpha^h(g_1).$$

Writing $u := g_1^{-1}\alpha^h(g_1)$, one thus has

$$\alpha^h(g_1) = g_1 u \quad \text{and} \quad \alpha^k(u) = u.$$

If we then write

$$v := g_1^{-1}\alpha^{hk}(g_1) = u\alpha^h(u)\cdots\alpha^{(k-1)h}(u),$$

we see that $\alpha^{hkn}(g_1) = g_1 v^n$ for all n, and $\alpha(v) = v$. Thus we have

$$\alpha^{hkn+j}(g_1)\alpha^{2hkn+2j}(g_2) = \alpha^j(g_1(\alpha^{2hk}(v))^n\alpha^j(g_2)) \quad \text{for any } n, j.$$

The means of this in n converge in $L^2(\tau)$ by the mean ergodic theorem. Summing over all $0 \le j < hk$, we obtain weak convergence, thus answering Question 1.22 affirmatively in this case. The same type of argument also lets one deal with crossed products of abelian systems by groups, in which the shift acts as an automorphism on the group; we omit the details.

Finally, we remark that the results on asymptotically abelian systems, while stated for $\mathbb{Z}^k$-systems, should in fact be valid for any commuting action of a general locally compact second countable (lcsc) abelian group.

Appendix A. An application of the van der Corput lemma

The purpose of this appendix is to establish Theorem 1.13. Our arguments follow [Niculescu et al. 2003, Proposition 7.4 and Theorem 7.5] closely (for another adaptation of the same argument, see also [Beyers et al. 2010, Proposition 4.4]). We may normalise α_0 to be the identity.

We induct on $k \geq 2$. When $k = 2$ we know from the usual mean ergodic theorem for von Neumann algebras (see for example [Krengel 1985, Section 9.1]) that

$$\frac{1}{N}\sum_{n=1}^{N}\alpha^n(a) \to E_{\mathcal{M}^\alpha}(a) \quad \text{in } \|\cdot\|_{L^2(\tau)},$$

and since $\mathcal{M}^\alpha \subseteq \mathcal{N}$ by the relative weak mixing assumption, we also have

$$\frac{1}{N}\sum_{n=1}^{N}\alpha^n(E_{\mathcal{N}}(a)) \to E_{\mathcal{M}^\alpha}(E_{\mathcal{N}}(a)) = E_{\mathcal{M}^\alpha}(a) \quad \text{in } \|\cdot\|_{L^2(\tau)},$$

so combining these conclusions gives the result.

Now suppose that $k \geq 3$ and that we know the desired conclusion for any similar family of $\ell < k$ automorphisms. By decomposing each a_i as $(a_i - E_{\mathcal{N}}(a_i)) + E_{\mathcal{N}}(a_i)$ and expanding out the expression $\prod_{i=1}^{k-1}\alpha_i^n(a_i)$, we find that it suffices to show that for any $i \leq k-1$,

$$a_i \perp \mathcal{N} \quad \text{implies} \quad \frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k-1}\alpha_i^n(a_i) \to 0 \quad \text{in } \|\cdot\|_{L^2(\tau)};$$

let us argue the case $i = 1$, the others following at once by symmetry.

By the Hilbert-space-valued version of the classical van der Corput estimate (see, for instance, [Furstenberg 1977] or [Bergelson 1987]) this will follow if we show that

$$\begin{aligned}
&\frac{1}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=1}^{N}\Big\langle\prod_{i=1}^{k-1}\alpha_i^{n+h}(a_i), \prod_{i=1}^{k-1}\alpha_i^n(a_i)\Big\rangle_\tau\Big| \\
&= \frac{1}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=1}^{N}\tau\big(\alpha_{k-1}^n(\alpha_{k-1}^h(a_{k-1}^*))\cdots\alpha_1^n(\alpha_1^h(a_1^*))\cdot\alpha_1^n(a_1)\cdots\alpha_{k-1}^n(a_{k-1})\big)\Big| \to 0
\end{aligned}$$

as $N \to \infty$ and then $H \to \infty$.

Let us now set $b_i := \alpha_i^n(\alpha_i^h(a_i^*))$ and $c_i := \alpha_i^n(\alpha_i^h(a_i))$ to lighten notation. Having done so, we now set ourselves up for applying the asymptotic abelianness property by observing that

$$\begin{aligned}
&b_{k-1}b_{k-2}b_{k-3}\cdots c_1c_2\cdots\\
&\qquad = (b_{k-2}b_{k-1}b_{k-3}\cdots c_1c_2\cdots) + ([b_{k-1},b_{k-2}]b_{k-3}\cdots c_1c_2\cdots)\\
&\qquad = (b_{k-2}b_{k-3}b_{k-1}b_{k-4}\cdots c_1c_2\cdots) + (b_{k-2}[b_{k-1},b_{k-3}]b_{k-4}\cdots c_1c_2\cdots)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + ([b_{k-1},b_{k-2}]b_{k-3}b_{k-4}\cdots c_1c_2\cdots)\\
&\qquad \vdots\\
&\qquad = b_{k-2}b_{k-3}b_{k-4}\cdots b_1c_1c_2\cdots c_{k-2}(b_{k-1}c_{k-1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{j=1}^{k-2} x_j[b_{k-1},b_j]y_j + \sum_{j=1}^{k-2} u_j[b_{k-1},c_j]v_j,
\end{aligned}$$

where each x_j, y_j, u_j and v_j for $1 \le j \le k-2$ is some product of a subset of the elements $\{b_i, c_i : i \le k-2\}$.

Importantly, there is some $M > 0$ such that $\|x_j\|, \|y_j\|, \|u_j\|, \|v_j\| \le M$ for all $j \le k-2$, and not depending on n or h, while on the other hand for any $j \le k-2$ we have

$$[b_{k-1}, b_j] = [\alpha_{k-1}^n(\alpha_{k-1}^h(a_{k-1}^*)), \alpha_j^n(\alpha_j^h(a_j^*))],$$

and hence overall we have

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}\Big\|\sum_{j=1}^{k-2} x_j[b_{k-1},b_j]y_j\Big\|_{L^2(\tau)}\\
&\qquad\qquad \le M^2\sum_{j=1}^{k-2}\frac{1}{N}\sum_{n=1}^{N}\|[b_{k-1},b_j]\|_{L^2(\tau)}\\
&\qquad\qquad = M^2\sum_{j=1}^{k-2}\frac{1}{N}\sum_{n=1}^{N}\|[\alpha_{k-1}^h(a_{k-1}^*), (\alpha_{k-1}^{-1}\alpha_j)^n(\alpha_j^h(a_j^*))]\|_{L^2(\tau)} \to 0
\end{aligned}$$

as $N \to \infty$, by the asymptotic abelianness of $\alpha_{k-1}^{-1}\alpha_j$. The same reasoning applies to the term $\sum_{j=1}^{k-2} u_j[b_{k-1}, c_j]v_j$, and now applies again to show that in the scalar average of interest to us we may also commute b_{k-2} from the left end of our product over to be immediately on the left of c_{k-2}, and then move b_{k-3} to c_{k-3}, and so on. Overall, this shows that

$$\begin{aligned}
&\frac{1}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=1}^{N}\tau\big(\alpha_{k-1}^n(\alpha_{k-1}^h(a_{k-1}^*))\cdots\alpha_1^n(\alpha_1^h(a_1^*))\cdot\alpha_1^n(a_1)\cdots\alpha_{k-1}^n(a_{k-1})\big)\Big|\\
&\quad \sim \frac{1}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=1}^{N}\tau\big(\alpha_1^n(\alpha_1^h(a_1^*)a_1)\cdots\alpha_{k-1}^n(\alpha_{k-1}^h(a_{k-1}^*)a_{k-1})\big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=1}^{N}\tau\big(\alpha_1^h(a_1^*)a_1\cdot(\alpha_2\alpha_2^{-1})^n(\alpha_2^h(a_2^*)a_2)\\
&\qquad\qquad\cdots(\alpha_{k-1}\alpha_1^{-1})^n(\alpha_{k-1}^h(a_{k-1}^*)a_{k-1})\big)\Big|\\
&= \frac{1}{H}\sum_{h=1}^{H}\Big|\tau\Big(\alpha_1^h(a_1^*)a_1\cdot\Big(\frac{1}{N}\sum_{n=1}^{N}(\alpha_2\alpha_1^{-1})^n(\alpha_2^h(a_2^*)a_2)\\
&\qquad\qquad\cdots(\alpha_{k-1}\alpha_1^{-1})^n(\alpha_{k-1}^h(a_{k-1}^*)a_{k-1})\Big)\Big)\Big|
\end{aligned}
$$

as $N\to\infty$ and then $H\to\infty$. However, now we notice that the inner average of operators with respect to N here is precisely of the form hypothesized by the theorem, but involving only the $k-1$ automorphisms $\alpha_j\alpha_1^{-1}$ for $j=1,2,\ldots,k-1$, which still satisfy the necessary hypotheses of relative weak mixing and asymptotic abelianness. Hence this operator average asymptotically agrees with

$$
\begin{aligned}
&\frac{1}{H}\sum_{h=1}^{H}\Big|\tau\Big(\alpha_1^h(a_1^*)a_1\cdot\Big(\frac{1}{N}\sum_{n=1}^{N}(\alpha_2\alpha_1^{-1})^n(E_{\mathcal{N}}(\alpha_2^h(a_2^*)a_2))\\
&\qquad\qquad\cdots(\alpha_{k-1}\alpha_1^{-1})^n(E_{\mathcal{N}}(\alpha_{k-1}^h(a_{k-1}^*)a_{k-1}))\Big)\Big|\\
&\quad= \frac{1}{H}\sum_{h=1}^{H}\Big|\tau\Big(E_{\mathcal{N}}(\alpha_1^h(a_1^*)a_1)\cdot\Big(\frac{1}{N}\sum_{n=1}^{N}(\alpha_2\alpha_1^{-1})^n(E_{\mathcal{N}}(\alpha_2^h(a_2^*)a_2))\\
&\qquad\qquad\cdots(\alpha_{k-1}\alpha_1^{-1})^n(E_{\mathcal{N}}(\alpha_{k-1}^h(a_{k-1}^*)a_{k-1}))\Big)\Big)\Big|,
\end{aligned}
$$

where the second equality holds because the operator average in the inner brackets now lies in $\mathcal{N}$, and so we apply the usual identity for conditional expectations $\tau(aE_{\mathcal{N}}(b))=\tau(E_{\mathcal{N}}(aE_{\mathcal{N}}(b)))=\tau(E_{\mathcal{N}}(a)E_{\mathcal{N}}(b))$.

Writing

$$
s_N := \frac{1}{N}\sum_{n=1}^{N}(\alpha_2\alpha_1^{-1})^n(E_{\mathcal{N}}(\alpha_2^h(a_2^*)a_2))\cdots(\alpha_{k-1}\alpha_1^{-1})^n(E_{\mathcal{N}}(\alpha_{k-1}^h(a_{k-1}^*)a_{k-1})),
$$

we see that $\|s_N\|\le C$ for some fixed C and all $N\in\mathbb{N}$, and now combining this bound with the Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
\frac{1}{H}\sum_{h=1}^{H}|\tau(E_{\mathcal{N}}(\alpha_1^h(a_1^*)a_1)\cdot s_n)| &= \frac{1}{H}\sum_{h=1}^{H}|\langle s_n^*\hat{1},(E_{\mathcal{N}}(\alpha_1^h(a_1^*)a_1)\hat{1}\rangle_{L^2(\tau)}|\\
&\le \frac{1}{H}\sum_{h=1}^{H}C\cdot\|E_{\mathcal{N}}(\alpha_1^h(a_1^*)a_1\|_{L^2(\tau)}.
\end{aligned}
$$

Finally, it follows that this tends to 0 as $H\to\infty$ by the our assumption that $a_1\perp\mathcal{N}$ and the relative weak mixing hypothesis. This completes the proof of Theorem 1.13.

Appendix B. A group theory construction

The purpose of this appendix is to explicitly describe a certain type of group, which we shall term a *square group*, generated by relations involving quadruples of generators. In particular, we will be able to solve the equality problem for such groups. Our arguments here are motivated by an observation of Grothendieck that groups can be identified with the sheaf of their flat connections on simplicial complexes, and experts will be able to detect the ideas of sheaf theory lurking beneath the surface of the material here, although we will not use that theory explicitly.

Definition B.1 (square groups). A *square base* $\square = (H \cup V, \square)$ consists of the following data:

- A set $H \cup V$ of generators, partitioned into a subset H of *horizontal generators* and a subset V of *vertical generators*.
- A set $\square \subset (H \times V \times H \times V) \cup (V \times H \times V \times H)$ of quadruples (e_0, e_1, e_2, e_3) of alternating orientation (thus if e_0 is horizontal then e_1 must be vertical, and so forth).

Furthermore, we require the two axioms on the set $\square$:

- (Cyclic symmetry.) If $(e_0, e_1, e_2, e_3) \in \square$, then $(e_1, e_2, e_3, e_0) \in \square$.
- (Unique continuation.) If $e_0, e_1 \in H \cup V$, then there is at most one quadruple $(e_0, e_1, e_2, e_3) \in \square$ with the first two components e_0 and e_1.

If $\square$ is a square base, we define the *square group* $G_\square$ associated to that base to be the group generated by the generators $H \cup V$, subject to the relations $e_0 e_1 e_2 e_3 = \mathrm{id}$ for all $(e_0, e_1, e_2, e_3) \in \square$. We define the *alphabet* of the square base (or square group) to be the set $H \cup V \cup H^{-1} \cup V^{-1}$ consisting of the horizontal and vertical generators and their formal inverses.

To describe square groups explicitly, we shall need some notation of a combinatorial and geometric nature. Let $\mathbb{N} := \{0, 1, 2, \dots\}$ denote the natural numbers.

Definition B.2 (monotone paths and regions). A *monotone path* is a finite path in the discrete quadrant $\mathbb{N}^2$ from $(0, 0)$ to some endpoint (n, m) that consists only of rightward edges $(i, j) \to (i + 1, j)$ and upward edges $(i, j) \to (i, j + 1)$ (in particular, the path will have length $n + m$). Given a monotone path γ from $(0, 0)$ to (n, m), the *shadow* of γ is defined to be all the pairs $(i, j) \in \mathbb{N}^2$ such that $(i, j') \in \gamma$ for some $j' \geq j$. We say that one monotone path γ' *lies above* another monotone path γ with the same endpoint (n, m) if the shadow of γ' contains the shadow of γ. In such cases, we refer to the set-theoretic difference between the two shadows as a *monotone region* from $(0, 0)$ to (n, m), with γ' and γ referred to as the *upper boundary* and *lower boundary* of the region, respectively.

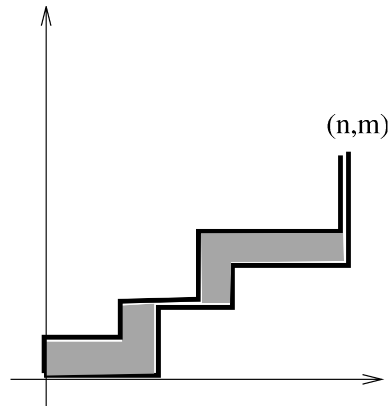

Figure 2. A monotone region, bounded above and below by two monotone paths. Note the horizontal and vertical convexity of the monotone region.

We will also consider a monotone path as a degenerate example of a monotone region. Monotone regions are horizontally and vertically convex: if two endpoints of a horizontal or vertical line segment in $\mathbb{N}^2$ lie in a monotone region, then the interior of that segment does also.

Definition B.3 (flat connections). Fix a square base $\square$, and let $\Omega \subset \mathbb{N}^2$ be a set. A *connection* Γ on Ω is an assignment $\Gamma((i, j) \to (i+1, j)) \in H \cup H^{-1}$ of a horizontal element of the alphabet to every horizontal edge $(i, j), (i+1, j) \in \Omega$, and an assignment $\Gamma((i, j) \to (i, j+1)) \in V \cup V^{-1}$ of a vertical element of the alphabet to every vertical edge $(i, j) \mapsto (i, j+1) \in \Omega$. We adopt the convention that

$$\Gamma((i+1, j) \to (i, j)) := \Gamma((i, j) \to (i+1, j))^{-1},$$
$$\Gamma((i, j+1) \to (i, j)) := \Gamma((i, j) \to (i, j+1))^{-1},$$

where $(e^{-1})^{-1} := e$ for $e \in H \cup V$ of course.

We say that the connection Γ is *flat* if for every square $(i, j), (i+1, j), (i, j+1), (i+1, j+1)$ in Ω, there exists an oriented loop f_0, f_1, f_2, f_3 of horizontal and vertical edges around the square (in either orientation) such that

$$(\Gamma(f_0), \Gamma(f_1), \Gamma(f_2), \Gamma(f_3)) \in \square.$$

We call a flat connection on a monotone region from $(0, 0)$ to (n, m) *maximal* if it cannot be extended to any strictly larger monotone region with the same endpoints. It is *reduced* if there does not exist a triple $(i, j), (i+1, j), (i+2, j)$ or $(i, j), (i, j+1), (i, j+2)$ in Ω such that $\Gamma((i, j) \to (i+1, j))\Gamma((i+1, j) \to (i+2, j)) = \mathrm{id}$ or $\Gamma((i, j+1) \to (i, j))\Gamma((i, j+1) \to (i, j+2)) = \mathrm{id}$.

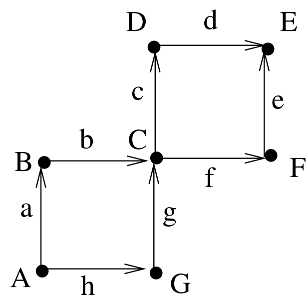

Figure 3. A monotone region $\{A, B, C, D, E, F, G\}$ (with $A = (0, 0)$, $B = (0, 1)$, and so on) with a connection Γ defined by the group elements $a, b, c, d, e, f, g, h \in G_\Box$; thus for instance $\Gamma(B \to C) = b$ and $\Gamma(C \to B) = b^{-1}$. If say (a, b, g^{-1}, h^{-1}) and (f, e, d^{-1}, c^{-1}) are in $\Box$, then this connection is flat.

In the degenerate case when Ω is just a monotone path, every connection is automatically flat, as there are no squares.

Let Γ be a flat connection on a monotone region Ω. Then one can *integrate* this connection to produce a map $\Phi_\Gamma : \Omega \to G_\Box$ by setting $\Phi_\Gamma(0, 0) := \mathrm{id}$ and $\Phi_\Gamma(v) = \Phi_\Gamma(u)\Gamma(u \to v)$ for all horizontal and vertical edges $(u \to v)$ in Ω. From the flatness of Γ and the "connected" nature of Ω it is easy to see that Φ_Γ exists and is unique. In particular, we can define the *definite integral* $|\Gamma|$ of Γ to be the group element $|\Gamma| := \Phi_\Gamma(n, m)$, where (n, m) is the endpoint of Ω.

Example B.4. The definite integral of the flat connection in Figure 3 is equal to $abcd = abfe = hgcd = hgfe$.

Every group element g in $G_\Box$ can arise as a definite integral of some flat connection, simply by expressing g as a word in the alphabet $H \cup V \cup H^{-1} \cup V^{-1}$, and creating an associated monotone path and connection for that word. Later on we shall see that the definite integral will provide a one-to-one correspondence between group elements and maximal reduced flat connections (Corollary B.10).

Lemma B.5. *Let* $\Box$ *be a square base, and let* $(n, m) \in \mathbb{N}^2$.

- (Unique continuation.) *If* Ω *is a monotone region from* $(0, 0)$ *to* (n, m), *and* γ *is a path from* $(0, 0)$ *to* (n, m) *in* Ω, *then any flat connection on* Ω *is uniquely determined by its restriction to* γ. *In other words, if* Γ *and* Γ' *are two flat connections on* Ω *that agree on* γ, *then they agree on all of* Ω.
- (Maximality.) *If* Ω_0 *is a monotone region from* $(0, 0)$ *to* (n, m), *and* Γ *is a flat connection on* Ω_0, *then there exists a unique extension of* Γ *to a maximal flat connection on a monotone region* Ω *from* $(0, 0)$ *to* (n, m) *containing* Ω_0.

Proof. We first establish unique continuation. This is best explained visually. The key observation is that if two flat connections on a square agree on two adjacent sides of a square, then they must agree on the whole square. This is ultimately a consequence of the unique continuation property of the square base $\Box$, and can be verified by a routine case check. Thus, if Γ and Γ' are two connections on Ω that agree on γ, they also agree on any perturbation of γ in Ω formed by taking an adjacent pair of horizontal and vertical edges in γ and "popping" them by replacing them by the other two edges of the square that they form; note that this retains the property of being a monotone path. One can check that after a sufficient number of upward and downward "popping" operations one can cover the upper and lower boundaries of Γ, and everything in between, and the claim follows.

Example B.6. We continue working with Figure 3. Suppose two flat connections Γ and Γ' on the indicated region agree on the upper boundary $ABCDE$, with the indicated connection values a, b, c, d. By unique continuation of $\Box$, the only possible values available for Γ and Γ' on the remaining two edges CF, FE of the square $CDEF$ are f and e. Thus we may "pop" the upper square and obtain that Γ and Γ' also agree on the monotone path $ABCFE$. After popping the lower square also we obtain that Γ and Γ' agree on the entire monotone region.

To prove the second claim, we simply observe that if Γ can be extended to two monotone regions Ω and Ω' containing Ω_0, then by unique continuation they agree on the intersection $\Omega \cap \Omega'$ (which is also a monotone region), and can thus be glued to form a flat connection on the union $\Omega \cup \Omega'$ (which is also a monotone region[6]). Since there are only finitely many monotone regions from $(0, 0)$ to (n, m), the claim then follows from the greedy algorithm. $\Box$

Definition B.7 (concatenation). Let Γ be a maximal reduced flat connection on some monotone region Ω from $(0, 0)$ to (n, m), and let $x \in H \cup V \cup H^{-1} \cup V^{-1}$ be a symbol in the alphabet. We define the *concatenation* $\Gamma \cdot x$ of Γ with x to be the maximal flat connection $\Gamma' = \Gamma \cdot x$ on a monotone region Ω' from $(0, 0)$ to (n', m') generated by the following rule.

- (Collapse.) If x is horizontal (that is, $x \in H \cup H^{-1}$), if $(n-1, m)$ lies in Ω, and if $\Gamma((n-1, m) \to (n, m)) = x^{-1}$, then one sets $(n', m') := (n-1, m)$, sets Ω' to be the restriction of Ω to the region $\{(i, j) \in \mathbb{N}^2 : i \leq n-1\}$ (that is, one deletes the rightmost column of Ω), and sets Γ' to be the restriction of Γ to Ω'.
- (Extension.) If x is horizontal, and either $(n-1, m)$ lies outside of Ω or $\Gamma((n-1, m) \to (n, m)) \neq x^{-1}$, then one sets $(n', m') := (n+1, m)$, and

[6] One way to see this is to rotate the plane by 45 degrees, so that monotone paths become graphs of discrete Lipschitz functions with Lipschitz constant 1, and monotone regions become the regions between two such functions.

extends Γ to $\Omega \cup \{(n+1, m)\}$ by setting $\Gamma((n, m) \to (n+1, m)) := x$; note that this is still flat because it does not create any squares. One then extends Γ further by the second part of Lemma B.5 to create the maximal flat connection Γ' on Ω' that extends Γ.

- If x is vertical instead of horizontal, one follows the analogue of the above rules but with the roles of n and m reversed.

Example B.8. Imagine one concatenated a horizontal edge x to the flat connection in Figure 3, which we shall assume to be maximal reduced. If x is not equal to d^{-1}, then the concatenated connection would thus extend one unit to the right of E to the endpoint $(3, 2)$, and may possibly extend also to the square to the right of EF if there is an appropriate tuple in $\Box$ to achieve this extension. If instead x was equal to d^{-1}, then the connection would collapse to the region $\{A, B, C, D, G\}$, so that the endpoint is now $D = (1, 2)$.

This definition gives a representation of $G_\Box$:

Lemma B.9. *Let $\Box$ be a square base and Γ a maximal reduced flat connection.*

- (Preservation of reducibility.) $\Gamma \cdot x$ *is reduced for any* $x \in H \cup V \cup H^{-1} \cup V^{-1}$.
- (Invertibility.) *We have* $(\Gamma \cdot x) \cdot x^{-1} = \Gamma$ *for any* $x \in H \cup V \cup H^{-1} \cup V^{-1}$.
- (Square relations.) *We have* $(((\Gamma \cdot e_0) \cdot e_1) \cdot e_2) \cdot e_3 = \Gamma$ *for any* $(e_0, e_1, e_2, e_3) \in \Box$.

In particular, the group $G_\Box$ acts on the space $\mathbb{O}$ of maximal reduced flat connections in a unique manner, sending Γ to $\Gamma \cdot g$ for any $\Gamma \in \mathbb{O}$ and $g \in G_\Box$.

Proof. We begin with the preservation of reducibility claim. If $\Gamma \cdot x$ is formed by collapsing Γ, the claim is clear, so suppose instead that $\Gamma \cdot x$ is formed by extension. By symmetry we may assume that x is horizontal. Let (n, m) denote the endpoint of Γ, and let Ω' be the domain of $\Gamma \cdot x$ (which then has endpoint $(n+1, m)$).

Assume for contradiction that $\Gamma \cdot x$ is not reduced. Since Γ was reduced, there are only two possibilities: either one has a vertical degeneracy

$$\Gamma((n+1, j) \to (n+1, j+1))\Gamma((n+1, j+1) \to (n+1, j+2)) = \mathrm{id} \tag{30}$$

for some $(n+1, j), (n+1, j+1), (n+1, j+2) \in \Omega'$, or else one has a horizontal degeneracy

$$\Gamma((n-1, j) \to (n, j))\Gamma((n, j) \to (n+1, j)) = \mathrm{id} \tag{31}$$

for some $(n-1, j), (n, j), (n+1, j) \in \Omega'$.

Suppose first that one has a vertical degeneracy (30). Consider the restrictions Γ_1 and Γ_2 of the connection Γ on the adjacent squares

$$((n, j), (n, j+1), (n+1, j), (n+1, j+1)) \quad \text{and}$$
$$((n, j+1), (n, j+2), (n+1, j+1), (n+1, j+2)).$$

By construction Γ_1 and Γ_2 agree on their common edge $((n, j+1) \to (n+1, j+1))$, and $\Gamma_1((n+1, j+1) \to (n+1, j))$ is equal to $\Gamma_2((n+1, j+1) \to (n+1, j+2))$. By the unique continuation property of $\Box$, this implies that Γ_1 and Γ_2 are reflections of each other; in particular $\Gamma_1((n, j+1) \to (n, j))$ equals $\Gamma_2((n, j+1) \to (n, j+2))$. But this implies that Γ is not reduced, a contradiction.

Suppose instead that one has a horizontal degeneracy (31). From Definition B.7 we know that j cannot equal m, otherwise we would have collapsed rather than extended Γ. Let $0 \leq j < m$ be the largest j for which (31) holds. By repeating the argument in the previous paragraph, we see that the restrictions of Γ to the adjacent squares

$$\begin{gathered}((n-1, j), (n, j), (n-1, j+1), (n, j+1)) \quad \text{and} \\ ((n, j), (n+1, j), (n, j+1), (n+1, j+1))\end{gathered}$$

are reflections of each other, which implies that (31) also holds for $j+1$, contradicting the maximality of j. This establishes the preservation of reducibility.

Now we establish the invertibility. Again, by symmetry we may assume that x is horizontal.

If $\Gamma \cdot x$ is a (horizontal) extension of Γ, then it is easy to see from Definition B.7 that $(\Gamma \cdot x) \cdot x^{-1}$ will be the (horizontal) collapse of $\Gamma \cdot x$, which is Γ. Conversely, if $\Gamma \cdot x$ is the (horizontal) collapse of Γ, then $(\Gamma \cdot x) \cdot x^{-1}$ will be the (horizontal) extension (because Γ was reduced), which will equal Γ again (by uniqueness of maximal extension).

Finally, we establish the square relations. From cyclic symmetry and invertibility we may assume that e_0 and e_2 are horizontal and e_1 and e_3 are vertical. From invertibility again, it suffices to show that

$$(\Gamma \cdot e_0) \cdot e_1 = (\Gamma \cdot e_3^{-1}) \cdot e_2^{-1}$$

for any maximal reduced flat connection Γ. We denote the endpoint of Γ by (n, m).

We divide into four cases. Suppose first that $\Gamma \cdot e_0$ is an extension of Γ, and that $(\Gamma \cdot e_0) \cdot e_1$ is an extension of $\Gamma \cdot e_0$. Then we claim that $\Gamma \cdot e_3^{-1}$ is an extension of Γ. If this were not the case, then $\Gamma((n, m-1) \to (n, m))$ must equal e_3, but then since $(\Gamma \cdot e_0)((n, m) \to (n+1, m))$ equals e_0 by construction, the domain of $\Gamma \cdot e_0$ must include the square $(n, m-1)$, (n, m), $(n+1, m-1)$, $(n+1, m)$ with

$$(\Gamma \cdot e_0)((n+1, m-1) \to (n+1, m)) = e_1^{-1},$$

causing $(\Gamma \cdot e_0) \cdot e_1$ to be a collapse rather than an extension, a contradiction. Thus $\Gamma \cdot e_3^{-1}$ extends Γ. A similar argument shows that $(\Gamma \cdot e_3^{-1}) \cdot e_2^{-1}$ extends $\Gamma \cdot e_3^{-1}$ (otherwise $\Gamma((n-1, m) \to (n, m))$ would equal e_0^{-1}, causing $\Gamma \cdot e_0$ to be a collapse rather than an extension). It is then easy to verify that $(\Gamma \cdot e_3^{-1}) \cdot e_2^{-1}$ and $(\Gamma \cdot e_0) \cdot e_1$

are the same since they glue together to form a flat connection on Γ and on the square (n, m), $(n+1, m)$, $(n, m+1)$, $(n+1, m+1)$.

Now suppose that $\Gamma \cdot e_0$ is an extension of Γ, but that $(\Gamma \cdot e_0) \cdot e_1$ is a collapse of $\Gamma \cdot e_0$. Arguing as before, we conclude that $\Gamma((n, m-1) \to (n, m))$ equals e_3, and so $\Gamma \cdot e_3^{-1}$ is a collapse of Γ; similarly, $(\Gamma \cdot e_3^{-1}) \cdot e_2^{-1}$ cannot be a collapse of $\Gamma \cdot e_3^{-1}$ (this would force $\Gamma \cdot e_0$ to be a collapse also) and so is an extension. It is again easy to verify that $(\Gamma \cdot e_3^{-1}) \cdot e_2^{-1}$ and $(\Gamma \cdot e_0) \cdot e_1$ are the same.

The remaining two cases (when $\Gamma \cdot e_0$ is a collapse of Γ, and $(\Gamma \cdot e_0) \cdot e_1$ is either an extension or collapse of $\Gamma \cdot e_0$) are similar to the preceding two, and are left to the reader. $\square$

This gives us a satisfactory explicit description of a square group:

Corollary B.10. *Let $\square$ be a square group. Then the definite integral map $\Gamma \mapsto |\Gamma|$ is a bijection from $\mathbb{O}$ to $G_\square$; thus every group element has a unique representation as the definite integral of a maximal reduced flat connection.*

Proof. The surjectivity of this map was already established in the discussion after Definition B.3, so it suffices to establish the injectivity. We will establish this via the identity $\Gamma = \varnothing \cdot |\Gamma|$ for all for all $\Gamma \in \mathbb{O}$, where $\varnothing$ is the trivial flat connection over the monotone region $\{(0, 0)\}$ from $(0, 0)$ to $(0, 0)$. This identity shows that Γ can be reconstructed from $|\Gamma|$, demonstrating injectivity.

Let Ω be the domain of Γ, which by definition is a monotone region from $(0, 0)$ to some point (n, m). Let γ be some monotone path in Ω from $(0, 0)$ to (n, m) (for example, one could take γ to be the upper or lower boundary of Ω). We label the vertices of γ in order as $(0, 0) = (i_0, j_0), (i_1, j_1), \ldots, (i_{n+m}, j_{n+m}) = (n, m)$. From definition of $|\Gamma|$, we see that

$$|\Gamma| = \Gamma((i_0, j_0) \to (i_1, j_1))\Gamma((i_1, j_1) \to (i_2, j_2)) \cdots \Gamma((i_{n+m-1}, j_{n+m-1}) \\ \to (i_{n+m}, j_{n+m})).$$

For each $0 \leq k \leq n+m$, defined Ω_k to be the portion of Ω that is in the region $\{(i, j) : i \leq i_k, j \leq j_k\}$; thus Ω_k is a monotone region from $(0, 0)$ to (i_k, j_k) that is increasing in k. Let Γ_k be the restriction of Γ to Ω_k. Since Γ was maximal and reduced, each of the Γ_k is also. Since $\Gamma_{n+m} = \Gamma$, it will suffice to establish that

$$\Gamma_k = \varnothing \cdot \Gamma((i_0, j_0) \to (i_1, j_1))\Gamma((i_1, j_1) \to (i_2, j_2)) \cdots \Gamma((i_{k-1}, j_{k-1}) \to (i_k, j_k))$$

for all $0 \leq k \leq n+m$. But this is easily established by induction (the reduced nature of the Γ_k is necessary to avoid the collapse case in Definition B.7). $\square$

As a consequence of this corollary, we can distinguish any two elements in $G_\square$ from each other as long as we can express them as the definite integrals of distinct maximal reduced flat connections.

Applications. We now specialise the abstract group-theoretic machinery above to the application at hand. We begin with a proposition that will be used to show nonconvergence of quadruple recurrence (Theorem 2.1).

Proposition B.11 (independence of AP4 relations). *Let $A \subset \mathbb{Z}$ be a (possibly infinite) set of integers. Then there exist a group G with elements e_0, e_1, e_2, e_3, together with an automorphism $T : G \to G$, such that for $r \in \mathbb{N}$, the relation*

$$e_0(T^r e_1)(T^{2r} e_2)(T^{3r} e_3) = \mathrm{id} \tag{32}$$

holds if and only if $r \in A$. Furthermore, no power T^k of T with $k \neq 0$ has any fixed points other than the identity element id.

Remark B.12. Informally, this proposition asserts that the algebraic relations (32) for various $r \in \mathbb{Z}$ are independent of each other. In contrast, with progressions of length three (that is, in the case $k = 3$) the analogous relations are highly degenerate. Indeed, suppose that

$$e_0(T^r e_1)(T^{2r} e_2) = \mathrm{id} \tag{33}$$

for all $r \in A$. Then if $r, r+h$ lie in A, we have

$$e_0(T^r e_1)(T^{2r} e_2) = e_0(T^r T^h e_1)(T^{2r} T^{2h} e_2),$$

which we can rearrange as $(T^h e_1^{-1})e_1 = T^r((T^{2h} e_2)e_2^{-1})$. If $r, r+h, r', r'+h$ lie in A, we thus have

$$T^r((T^{2h} e_2)e_2^{-1}) = T^{r'}((T^{2h} e_2)e_2^{-1}).$$

Assuming that $T^{r'-r}$ has no fixed points, we conclude that $(T^{2h} e_2)e_2^{-1}$ is the identity; assuming that T^{2h} has no fixed points either, we conclude that e_2 is the identity. Similar arguments can be used to show that e_0 and then e_1 are also the identity. Thus the relations (33) and the no-fixed-points hypothesis lead to a total collapse of the group generated by e_0, e_1, e_2 as soon as A contains even a single nontrivial parallelogram $r, r+h, r', r'+h$. (A variant of this argument also shows that if (33) is obeyed for r and $r+h$, then it is also obeyed for $r+2h$ even without the fixed point hypothesis.) This algebraic distinction between triple recurrence and quadruple recurrence can be viewed as the primary reason why recurrence and convergence results continue to hold for triple products, but not for quadruple products even under the assumption of ergodicity (which is reflected here in the no-fixed-points assumption).

Proof. We let G be the group generated by the generators $e_{i,n}$ for $i = 0, 1, 2, 3$ and $n \in \mathbb{Z}$, subject to the relations

$$e_{0,n}e_{1,n+r}e_{2,n+2r}e_{3,n+3r} = \mathrm{id} \quad \text{for all } n \in \mathbb{Z} \text{ and } r \in A.$$

Since the set of such relations is invariant under the shift $e_{i,n} \mapsto e_{i,n+1}$, we see that we can define an automorphism $T : G \to G$ by setting $Te_{i,n} := e_{i,n+1}$. If we then set $e_i := e_{i,0}$, it is clear that (32) holds for all $r \in A$.

To see that (32) fails for $r \notin A$, we note that G can be viewed as a square group, with horizontal generators $\{e_{i,n} : i = 0, 2; n \in \mathbb{Z}\}$ and vertical generators $\{e_{i,n} : i = 1, 3; n \in \mathbb{Z}\}$ and square relations $\square$ made of $(e_{0,n}, e_{1,n+r}, e_{2,n+2r}, e_{3,n+3r})$ and its cyclic permutations for all $n \in \mathbb{Z}$ and $r \in A$; note that the crucial unique continuation property follows from the basic observation that an arithmetic progression is determined by any two of its elements ("two points determine a line"). If $n \in \mathbb{Z}$ and $r \notin A$, one sees that the connection on the path of length four from $(0, 0)$ to $(2, 2)$ associated to the word $e_{0,n}e_{1,n+r}e_{2,n+2r}e_{3,n+3r}$ is already a maximal reduced flat connection (as none of the three squares that share two edges with the path can be completed to a square from $\square$) and so by Corollary B.10, its definite integral $e_{0,n}e_{1,n+r}e_{2,n+2r}e_{3,n+3r}$ is not equal to the identity, as required.

Finally, to show that T^k has no nontrivial fixed points, one simply observes that T^k will shift any nontrivial maximal reduced flat connection to a different maximal reduced flat connection, and then invokes Corollary B.10 again. $\square$

Next, we establish a variant that is useful for showing negative averages for quintuple recurrence (Theorem 2.12).

Proposition B.13 (independence of AP5 relations). *There exists a group G with distinct elements e_0, e_1, e_2, e_3, e_4, together with an automorphism $T : G \to G$, such that the relation*

$$e_0(T^r e_1)(T^{2r} e_2)(T^{3r} e_3)(T^{4r} e_4) = \mathrm{id} \tag{34}$$

holds for all $r \in \mathbb{Z}$. Furthermore, no power T^k of T with $k \neq 0$ has any fixed points other than the identity element id. *Finally, if $r \in \mathbb{Z}$ is nonzero, and*

$$g_0, g_1, g_2, g_3, g_4 \in \{\mathrm{id}, e_0, e_1, e_2, e_3, e_4, e_0^{-1}, e_1^{-1}, e_2^{-1}, e_3^{-1}, e_4^{-1}\}$$

are such that

$$g_0(T^r g_1)(T^{2r} g_2)(T^{3r} g_3)(T^{4r} g_4) = \mathrm{id}, \tag{35}$$

then g_0, g_1, g_2, g_3, g_4 are either equal to the identity, or are a permutation of $\{e_0, e_1, e_2, e_3, e_4\}$ or of $\{e_0^{-1}, e_1^{-1}, e_2^{-1}, e_3^{-1}, e_4^{-1}\}$.

Proof. For each $i = 0, 1, 2, 3, 4$, we define $G^{(i)}$ to be the group generated by the generators $e^{(i)}_{j,n}$ for $j \in \{0, 1, 2, 3, 4\} \setminus \{i\}$ and $n \in \mathbb{Z}$ subject to the relations

$$e^{(i)}_{0,n}e^{(i)}_{1,n+r}e^{(i)}_{2,n+2r}e^{(i)}_{3,n+3r}e^{(i)}_{4,n+4r} = \mathrm{id} \quad \text{for all } n, r \in \mathbb{Z}, \tag{36}$$

with the convention that $e^{(i)}_{i,n} = \mathrm{id}$ for all n. This group has an automorphism $T^{(i)} : G^{(i)} \to G^{(i)}$ that maps $e^{(i)}_{j,n}$ to $e^{(i)}_{j,n+1}$ for all n.

We now set G to be the product group $G := G^{(0)} \times G^{(1)} \times \cdots \times G^{(4)}$, and set

$$e_j := (e^{(0)}_{j,0}, e^{(1)}_{j,0}, \ldots, e^{(4)}_{j,0}) \quad \text{for } j = 0, 1, 2, 3, 4.$$

We also set

$$T(g^{(0)}, g^{(1)}, \ldots, g^{(4)}) := (T^{(0)} g^{(0)}, T^{(1)} g^{(1)}, \ldots, T^{(4)} g^{(4)});$$

thus T is an automorphism on G. By construction it is clear that (34) holds. Also, by the arguments in Proposition B.11, no nonzero power of $T^{(i)}$ has any nontrivial fixed points, and so the same is also true of T.

Now we establish the final claim of the proposition. Suppose $g_0, \ldots, g_4$ obey the stated properties. Let $i = 0, 1, 2, 3, 4$, and let $g^{(i)}_j$ be the $G^{(i)}$ component of g_j for $j = 0, 1, 2, 3, 4$; thus

$$g^{(i)}_0 ((T^{(i)})^r g^{(i)}_1)((T^{(i)})^{2r} g^{(i)}_2)((T^{(i)})^{3r} g^{(i)}_3)((T^{(i)})^{4r} g^{(i)}_4) = \mathrm{id}\,. \tag{37}$$

From the construction of $G^{(i)}$, we see that for any distinct $j, k \in \{0, 1, 2, 3, 4\} \setminus \{i\}$, there is a homomorphism $\phi^{(i)}_{j,k} : G^{(i)} \to \mathbb{Z}$ to the additive group $\mathbb{Z}$ mapping $e^{(i)}_{j,n}$ to $+1$, $e^{(i)}_{k,n}$ to -1, and all other $e^{(i)}_{l,n}$ to zero for $n \in \mathbb{Z}$ and $l \in \{0, 1, 2, 3, 4\} \setminus \{i, j, k\}$ (note that these requirements are compatible with the defining relations (36)). This homomorphism is $T^{(i)}$ invariant. Applying this homomorphism to (37), we obtain $\sum_{l=0}^{4} \phi^{(i)}_{j,k}(g^{(i)}_l) = 0$.

In other words, the number of times g_l for $l = 0, 1, 2, 3, 4$ equals e_j, minus the number of times it equals e_j^{-1}, is equal to the number of times g_l equals e_k, minus the number of times it equals e_k^{-1}. Letting j, k, i vary, we thus see that this number is independent of j. It is easy to see that this number cannot exceed 1 in magnitude, and if it is equal to $+1$ or -1, then g_0, g_1, g_2, g_3, g_4 is a permutation of $\{e_0, e_1, e_2, e_3, e_4\}$ or of $\{e_0^{-1}, e_1^{-1}, e_2^{-1}, e_3^{-1}, e_4^{-1}\}$, respectively. (Note that this argument also ensures that e_0, e_1, e_2, e_3 and e_4 are distinct.) The remaining possibility to eliminate is when this number is zero, thus each e_i occurs in g_0, g_1, g_2, g_3, g_4 as often as e_i^{-1}. Suppose for instance that g_0, g_1, g_2, g_3, g_4 contains one occurrence each of $e_0, e_0^{-1}, e_1, e_1^{-1}$. Applying (37) with $i = 4$ (say), and then applying the homomorphism that maps $e^{(4)}_{0,n}$ to zero, $e^{(4)}_{1,n}$ to n, $e^{(4)}_{2,n}$ to $-2n$, and $e^{(4)}_{3,n}$ to n (here we use the identity $(n+r)-2(n+2r)+(n+3r)=0$ to ensure consistency with (36)), we obtain a contradiction. We argue similarly if g_0, g_1, g_2, g_3, g_4 contains any other combination of one or two distinct pairs e_j, e_j^{-1}. The remaining case to eliminate is if g_0, g_1, g_2, g_3, g_4 contains e_j and e_j^{-1} twice each for some j, say $j = 0$. Applying (37) with $i = 4$ again, we can use Corollary B.10 to contradict (37), since the right side is a definite integral of a maximal flat connection on a horizontal path of length four. We argue similarly for other values of j, and the claim follows. □

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TIM AUSTIN
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
151 THAYER ST, BOX 1917
PROVIDENCE, RI 02912
UNITED STATES
timaustin@math.brown.edu
http://www.math.brown.edu/∼timaustin

TANJA EISNER
KORTEWEG-DE VRIES INSTITUTE FOR MATHEMATICS
UNIVERSITY OF AMSTERDAM
P.O. BOX 94248
1090 GE AMSTERDAM
THE NETHERLANDS
t.eisner@uva.nl
http://staff.science.uva.nl/∼eisner

TERENCE TAO
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA AT LOS ANGELES
405 HILGARD AVENUE
LOS ANGELES, CA 90095-1555
UNITED STATES
tao@math.ucla.edu
http://www.math.ucla.edu/∼tao/

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PRINCIPAL CURVATURES OF FIBERS AND HEEGAARD SURFACES

WILLIAM BRESLIN

We study principal curvatures of fibers and Heegaard surfaces smoothly embedded in hyperbolic 3-manifolds. It is well known that a fiber or a Heegaard surface in a hyperbolic 3-manifold cannot have principal curvatures everywhere less than one in absolute value. We show that given an upper bound on the genus of a minimally embedded fiber or Heegaard surface and a lower bound on the injectivity radius of the hyperbolic 3-manifold, there exists a $\delta > 0$ such that the fiber or Heegaard surface must contain a point at which one of the principal curvatures exceeds $1 + \delta$ in absolute value.

1. Introduction

The principal curvatures of a surface or lamination smoothly embedded in a hyperbolic 3-manifold are related to the topology of the surface and the 3-manifold. For example in [Breslin 2010] we show that incompressible surfaces and strongly irreducible Heegaard surfaces embedded in hyperbolic 3-manifolds can always be isotoped to a surface with principal curvatures bounded in absolute value by a fixed constant that does not depend on the surface or the 3-manifold. In [Breslin 2009] we show that laminations in hyperbolic 3-manifolds with principal curvatures everywhere close to zero have boundary leaves with noncyclic fundamental group and that laminations in hyperbolic 3-manifolds with principal curvatures everywhere in the interval $(-1, 1)$ have boundary leaves with nontrivial fundamental group.

This note was motivated by a question about surfaces with principal curvatures near the interval $(-1, 1)$. It is well known that a closed orientable surface smoothly embedded in a finite-volume complete hyperbolic 3-manifold with principal curvatures everywhere in the interval $(-1, 1)$ is incompressible and lifts to a quasiplane in $\mathbb{H}^3$ (see [Thurston 1979] or [Leininger 2006] for a proof). Thus Heegaard surfaces and fibers in hyperbolic 3-manifolds cannot have principal curvatures everywhere in the interval $(-1, 1)$. We are interested in finding obstructions to isotoping Heegaard surfaces and fibers in hyperbolic 3-manifolds to have principal

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curvatures close to the interval $(-1, 1)$. See [Rubinstein 2005] or [Krasnov and Schlenker 2007] for more on surfaces in hyperbolic 3-manifolds with principal curvatures in the interval $(-1, 1)$.

It follows from work of Freedman, Hass, and Scott [Freedman et al. 1983] that an incompressible surface in a closed Riemannian 3-manifold can be isotoped to a minimal surface. It follows from work of Pitts-Rubinstein that a strongly irreducible Heegaard surface in a closed Riemannian 3-manifold can be isotoped to either a minimal surface or the boundary of a regular neighborhood of a minimal surface (see [Rubinstein 2005] for a sketch of the proof). We show that given an upper bound on the genus of a minimally embedded fiber or Heegaard surface and a lower bound on the injectivity radius of the hyperbolic 3-manifold, there exists a $\delta > 0$ such that the fiber or Heegaard surface must contain a point at which one of the principal curvatures is greater than $1 + \delta$ in absolute value.

Theorem 1. *For each $g \geq 2$, $\epsilon > 0$, there exists $\delta := \delta(g, \epsilon)$ such that if S is a genus g minimally embedded fiber in a closed hyperbolic mapping torus M with $\mathrm{inj}(M) > \epsilon$, then S contains a point at which one of the principal curvatures is at least $1 + \delta$ in absolute value.*

Theorem 2. *For each $g \geq 2$, $\epsilon > 0$, there exists $\delta := \delta(g, \epsilon)$ such that if S is a genus g minimally embedded Heegaard surface in a closed hyperbolic 3-manifold M with $\mathrm{inj}(M) > \epsilon$, then S contains a point at which one of the principal curvatures is at least $1 + \delta$ in absolute value.*

The proofs of Theorem 1 and Theorem 2 both use geometric limit arguments. Assuming that no such $\delta > 0$ exists, we consider a sequence of hyperbolic 3-manifolds as in the statement with minimally embedded fibers or Heegaard surfaces whose principal curvatures are closer and closer to the interval $[-1, 1]$. After possibly passing to a subsequence, the sequence of manifolds converges geometrically to a hyperbolic 3-manifold M and the surfaces converge to an incompressible surface S in M with principal curvatures everywhere in the interval $[-1, 1]$. This implies that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$. In either case, we show that the cover of M corresponding to the image of $\pi_1(S)$ in $\pi_1(M)$ has a doubly degenerate hyperbolic structure contradicting that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.

2. Preliminaries

Let M be a hyperbolic 3-manifold with no cusps and finitely generated fundamental group. By a result of Scott, M has a *compact core* which is a compact submanifold C of M whose inclusion into M is a homotopy equivalence. The connected components of $M \setminus C$ are called the *ends* of M. It follows from the positive solution of the tameness conjecture by Agol [2004] and by Calegari and Gabai [2006] that an end of M is homeomorphic to $\Sigma \times [0, \infty)$ where Σ is a closed orientable surface. The convex core, $CC(M)$, of M is the smallest convex submanifold of M whose inclusion is a homotopy equivalence. An end E of M is *convex-cocompact* if $E \cap CC(M)$ is compact and E is *degenerate* otherwise. Given a closed orientable surface Σ of genus greater than one, a hyperbolic structure on $\Sigma \times \mathbb{R}$ such that both ends are degenerate is called *doubly degenerate*.

A sequence of pointed hyperbolic n-manifolds (M_i, p_i) *converges geometrically* to the pointed hyperbolic n-manifold (M, p) if for every sufficiently large R and each $\epsilon > 0$, there exists i_0 such that for every $i \geq i_0$, there is a $(1+\epsilon)$-bilipschitz pointed diffeomorphism $\kappa_i : (B(p, R), p) \to M_i$, where $B(p, R) \subset M$ is the ball of radius R centered at p and $B(p_i, R) \subset M_i$ is the ball of radius R centered at p_i. We call the maps κ_i *almost isometries*.

We will use the fact that minimal surfaces have bounded diameter in the presence of a lower bound on injectivity radius. See [Rubinstein 2005] or [Souto 2007] for more on minimal surfaces in hyperbolic 3-manifolds.

Lemma 1. *Let S be a connected minimal surface in a complete hyperbolic* 3-*manifold M with* $\operatorname{inj}(M) \geq \epsilon$. *Then the diameter of S is at most* $4|\chi(F)|/\epsilon + 2\epsilon$.

We will also use the following Lemma in the proofs of Theorems 1 and 2.

Lemma 2. *If S is a closed orientable surface smoothly immersed with principal curvatures everywhere in the interval* $[-1, 1]$ *in a complete hyperbolic* 3-*manifold M with no cusps, then the limit set of a lift of S to* $\mathbb{H}^3$ *is a proper subset of* $\partial\mathbb{H}^3$.

Proof. Let $\tilde{S}$ be a lift of S to $\mathbb{H}^3$. Assume that $\tilde{S}$ is not a horosphere, as otherwise we are done. Thus the principal curvatures of S cannot be everywhere equal to 1 or everywhere equal to -1. If the principal curvatures at every point of S are -1 and 1, then there is a pair of line fields defined on the entire surface, implying that S is a torus. Since closed surfaces in M with all principal curvatures in $[-1, 1]$ are incompressible and M has no cusps, S cannot be a torus. Thus there is a point p in $\tilde{S}$ at which one of the principal curvatures is in $(-1, 1)$. Assume that the other principal curvature at p is in $[-1, 1)$. Let H be a horosphere tangent to $\tilde{S}$ at p. Use an upper half space model of $\mathbb{H}^3$ in which H is a horizontal plane and $\tilde{S}$ is below H. Let l be a simple loop in $\tilde{S}$ which contains p such that the principal curvatures at each point on l are in $[-1, 1)$ with at least principal curvature in $(-1, 1)$. At each point x in l, let H_x be the horosphere above $\tilde{S}$ tangent to $\tilde{S}$ at x. For each x in l, let $c_x \in \partial\mathbb{H}^3$ be the center of the horosphere H_x. The set of points $C = \{c_x | x \in l\}$ forms a closed curve in $\partial\mathbb{H}^3$. Since the principal curvatures of $\tilde{S}$ are everywhere in the interval $[-1, 1]$, $\tilde{S}$ cannot transversely intersect any of the horospheres H_x. Thus, the limit set of $\tilde{S}$ cannot cross the closed curve C, so that the limit set of $\tilde{S}$ is a proper subset of $\partial\mathbb{H}^3$. □

It is well-known that the limit set of a lift to $\mathbb{H}^3$ of a fiber Σ in a doubly degenerate hyperbolic $\Sigma \times \mathbb{R}$ is the entire boundary $\partial\mathbb{H}^3$. By Lemma 2, such a fiber Σ cannot be smoothly embedded with principal curvatures everywhere in the interval $[-1, 1]$.

3. Principal curvatures of fibers

In the proof of Theorem 1, we will use the following fact about geometric limits of hyperbolic mapping tori.

Theorem. *Let (M_i, p_i) be a sequence of pairwise distinct pointed hyperbolic mapping tori with genus g fibers and* $\operatorname{inj}(M_i) > \epsilon$ *for all i. Then a subsequence of (M_i, p_i) converges geometrically to a pointed hyperbolic 3-manifold (M, p) homeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a closed genus g surface and M has a doubly degenerate hyperbolic structure.*

Proof of Theorem 1. Suppose, for contradiction, that Theorem 1 does not hold. Then there exists a sequence of hyperbolic mapping tori (M_i) with $\operatorname{inj}(M_i) > \epsilon$ such that M_i has a genus g minimal surface fiber with principal curvatures less than $1 + 1/i$ in absolute value. For each i, let p_i be a point in S_i. By Theorem A the sequence (M_i, p_i) has a subsequence, say the entire sequence, which converges to a doubly degenerate pointed hyperbolic 3-manifold (M, p) homeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a genus g closed surface. By Lemma 1, the diameters of the surfaces S_i are uniformly bounded. Thus we can find a compact subset K of M homeomorphic to $\Sigma \times [-1, 1]$ such that for i large enough, say for all i, S_i is contained in $\kappa_i(K)$. The surface $S := \Sigma \times \{0\}$ in M is isotopic to $\kappa_i^{-1}(S_i)$ for each i. Since the surfaces $\kappa_i^{-1}(S_i)$ have bounded area and curvature, a subsequence converges to a smoothly immersed surface with principal curvatures in $[-1, 1]$ which is homotopic to S. Lemma 2 implies that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$, contradicting the fact that M is doubly degenerate. □

4. Principal curvatures of Heegaard surfaces

In the proof of Theorem 2, we will use the following fact about geometric limits.

Theorem. *Every sequence (M_i, p_i) of pointed hyperbolic 3-manifolds such that* $\operatorname{inj}(M_i, p_i)$ *is bounded away from 0 has a geometrically convergent subsequence.*

Lemma 3 [Souto 2006, Lemma 2.1]. *Let (M_i) be a sequence of hyperbolic 3-manifolds converging to a hyperbolic manifold M. Assume that there is a compact subset $K \subset M$ such that for all sufficiently large i the homomorphism $\pi_1(K) \to \pi_1(M_i)$ provided by geometric convergence is surjective. Then, if the cover of M corresponding to the image of $\pi_1(K)$ into $\pi_1(M)$ has a convex-cocompact end, so does M_i for all but finitely many i.*

Proof of Theorem 2. Suppose for contradiction that Theorem 2 does not hold. Then there exists a sequence (M_i) of closed hyperbolic 3-manifolds with $\mathrm{inj}(M_i) > \epsilon$ such that M_i has a genus g minimal Heegaard surface S_i with principal curvatures less than $1 + 1/i$ in absolute value. For each i let p_i be a point in S_i. By Theorem B the sequence (M_i, p_i) has a convergent subsequence, say the entire sequence, which converges geometrically to a pointed hyperbolic 3-manifold (M, p). By Lemma 1, the diameters of the surfaces S_i are uniformly bounded. Thus each M_i contains a compact subset K_i homeomorphic to $S_i \times [-1, 1]$ with uniformly bounded diameter. For i large enough the pull-back $\kappa_i^{-1}(K_i)$ of K_i through the almost isometries provided by geometric convergence are embedded compact subsets homeomorphic to $\Sigma \times [-1, 1]$ where Σ is a closed surface of genus g. For i large enough the surfaces $\kappa_i^{-1}(S_i)$ are all isotopic to a fixed embedded genus g surface S in M. Since the surfaces $\kappa_i^{-1}(S_i)$ have bounded area and curvature, a subsequence converges to a smoothly immersed surface with principal curvatures in $[-1, 1]$ which is homotopic to S. Thus the surface S is incompressible in M and by Lemma 2 the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.

To arrive at a contradiction we will show that the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ is doubly degenerate, implying that the limit set of a lift of S to $\mathbb{H}^3$ is all of $\partial\mathbb{H}^3$. For i large enough $\kappa_i(S)$ is isotopic to the Heegaard surface S_i in M_i, so that the homomorphism $(\kappa_i)_* : \pi_1(S) \to \pi_1(M_i)$ provided by geometric convergence is surjective. By Lemma 3, if the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ has a convex-cocompact end, so does M_i for all but finitely many i. Since each M_i is closed we have that the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ cannot have a convex-cocompact end. Thus the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ is doubly degenerate contradicting the fact that S is isotopic to a surface with principal curvatures everywhere in $[-1, 1]$. □

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WILLIAM BRESLIN
breslin@umich.edu
UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
530 CHURCH STREET
ANN ARBOR, MI 48109-1043
UNITED STATES
http://www-personal.umich.edu/~breslin/index.html

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SELF-IMPROVING PROPERTIES OF INEQUALITIES OF POINCARÉ TYPE ON s-JOHN DOMAINS

SENG-KEE CHUA AND RICHARD L. WHEEDEN

We derive weak- and strong-type global Poincaré estimates over s-John domains in spaces of homogeneous type. The results show that Poincaré inequalities over quasimetric balls with given exponents and weights are self-improving in the sense that they imply global inequalities of a similar kind, but with improved exponents and larger classes of weights. The main theorems are applications of a geometric construction for s-John domains together with self-improving results in more general settings, both derived in our companion paper *J. Funct. Anal.* 255 (2008), 2977–3007. We have reduced our assumption on the principal measure μ to be just reverse doubling on the domain instead of the usual assumption of doubling. While the primary case considered in the literature is $p \le q$, we will also study the case $1 \le q < p$.

0. Introduction

This is a companion paper to [Chua and Wheeden 2008], where we established the self-improving nature of Poincaré inequalities over domains in general measure spaces. The self-improving nature of Poincaré estimates was observed initially by Saloff-Coste [1992] in the setting of Riemannian manifolds and has been extensively studied in other general settings; see examples in [Chua and Wheeden 2008]. The main goal of this paper is to apply our previous results to derive global Poincaré estimates on s-John domains (see Definition 1.2) in spaces of homogeneous type for reverse doubling measures (see Definition 1.4) instead of the usual doubling measures; see [Franchi et al. 1998; 2003].

The notion of an s-John domain was introduced by Smith and Stegenga [1990], while the terminology John domain introduced by Martio and Sarvas [1979]. John domains are the same as s-John domains in case $s = 1$. In spaces of homogeneous type with the segment (geodesic) property, John domains are the same as Boman domains; see [Buckley et al. 1996]. It is easy to see that bounded Lipschitz domains

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(including all bounded domains with smooth boundaries) and bounded domains that satisfy the interior cone condition are John domains. When $s > 1$, the notion of an s-John domain is a generalization of that of a John domain, a weakening of requirements relative to the case $s = 1$ in order to accommodate domains with rougher boundaries. Some examples of s-John domains in case $s > 1$ are given in [Hajłasz and Koskela 1998]. There have been many studies concerning (bounded) John domains; see for example [Buckley and Koskela 1995; Chua 2001; Acosta et al. 2006] and references therein. Results for John domains have also been generalized in [Hurri-Syrjänen 2004; Väisälä 1994; Chua 2009] to "unbounded John domains" or "generalized John domains". On the other hand, for bounded convex domains, sharp estimates have been obtained in [Chua and Wheeden 2006; Chua and Duan 2009; Chua and Wheeden 2010]

One of our main goals in this paper is to extend the following Poincaré estimate for s-John domains stated in [Kilpeläinen and Malý 2000, Theorem 2.3].

Theorem A. *Suppose that $\Omega \subset \mathbb{R}^n$ is an s-John domain. Let a, b, p, q be real numbers that satisfy*

$$a \ge 0, \quad b \ge 1-n, \quad 1 \le p < q < \infty, \quad 1/q \ge 1/p - 1/n$$

and

$$\frac{1}{q} \ge \frac{s(n+b-1)-p+1}{(n+a)p}. \tag{0-1}$$

Then there is a constant $C = C(n, a, b, p, q, \Omega) > 0$ such that

$$\|f - f_{\Omega,\rho^a dx}\|_{L^q_{\rho^a dx}(\Omega)} \le C\|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} \quad \text{for all } f \in C^1(\Omega), \tag{0-2}$$

where $\rho(x) = \operatorname{dist}(x, \Omega^c)$ and $f_{\Omega,\rho^a dx} = \int_\Omega f(x)\rho(x)^a dx \,\big/ \int_\Omega \rho(x)^a dx$.

The assumption that $f \in C^1(\Omega)$ in Theorem A does not automatically imply that the norm on the right side of (0-2) or the average $f_{\Omega,\rho^a dx}$ on the left side is finite. However, as we shall see in Theorem 1.12, (0-2) holds under the weaker hypothesis that $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, that is, it holds for all f that are locally Lipschitz continuous on Ω, provided the average on the left side is replaced by the average $|B'|^{-1}\int_{B'} f(x)dx$ over a "central" ball $B' \subset \Omega$, which is always finite for such f. If $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and the right side of (0-2) is finite, it follows that $f \in L^q_{\rho^a dx}(\Omega)$, and then $f_{\Omega,\rho^a dx}$ is finite and it is possible to replace the average over the central ball by this average in (0-2). The inequality (0-2) was also proved in [Hajłasz and Koskela 1998] except that when $p > 1$, they required strict inequality in (0-1). The necessity of the conditions $1/q \ge 1/p - 1/n$ and $1 + (n+a)/q - (n+b)/p \ge 0$ is easy to see as usual by considering Lipschitz functions that vanish outside balls in Ω; see [Chua and Duan 2009, Final Remark]. Condition (0-1) is sharp too as can be seen by considering mushroom-like domains; see [Hajłasz and Koskela 1998] for details. On the other hand, for special s-John domains such as s-cusp domains, condition (0-1) can be relaxed; see Theorem 1.14 for an estimate of this kind.

In this paper, we will apply results from [Chua and Wheeden 2008], where we use a different approach from those in [Hajłasz and Koskela 1998] and [Kilpeläinen and Malý 2000] to obtain self-improving properties of Poincaré-type inequalities in measure spaces. The approach modifies one used in [Franchi et al. 2003]. We now apply the outcome to derive global Poincaré inequalities on s-John domains Ω (including 1-John domains) in spaces of homogeneous type and for measures that are doubling, δ-doubling or just reverse doubling on Ω; see Definition 1.4. The notions of δ-doubling and doubling on Ω are equivalent on 1-John domains. We note that power-type weights of the form $\mathrm{dist}(x, \Omega_0)^a$, with $a \ge 0$ and $\Omega_0 \subset \Omega^c$, are examples of δ-doubling measures. We are also able to prove Theorem A without the assumption $b \ge 1-n$. Moreover, we will consider the case $1 \le q \le p$.

1. Definitions and main results

Definition 1.1. A pair $\langle H, d\rangle$ is a *quasimetric space* if d is a *quasimetric* on the set H, that is, if there exists a constant κ such that for all $x, y, z \in H$,

(1) $d(y, x) = d(x, y)$ is positive if $x \ne y$ and vanishes if $x = y$, and

(2) $d(x, y) \le \kappa(d(x, z) + d(y, z))$.

For a quasimetric space $\langle H, d\rangle$, any $x \in H$ and $r > 0$, we write

$$B(x, r) = \{y \in H : d(x, y) < r\}$$

and call $B(x, r)$ the ball with center x and radius r. If $B = B(x, r)$ is a ball and c is a positive constant, we use cB to denote $B(x, cr)$. If B is a ball, we use $r(B)$ and x_B to denote the radius and center of B.

Definition 1.2. Let $\langle H, d\rangle$ be a quasimetric space. Fix $\Omega \subset H$, and for $x \in H$, set

$$d(x) = \mathrm{dist}(x, \Omega^c) = \inf_{y\in\Omega^c} d(x, y).$$

Let ϕ be a strictly increasing function on $[0, \infty)$ such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. We say that Ω is a ϕ-John domain with *central point* (or *center*) $x' \in \Omega$ if for all $x \in \Omega$ with $x \ne x'$, there is a curve $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x$, $\gamma(l) = x'$,

$$d(\gamma(b), \gamma(a)) \le b - a \qquad \text{for all } [a, b] \subset [0, l], \text{ and} \tag{1-1}$$

$$d(\gamma(t)) > \phi(t) \qquad \text{for all } t \in [0, l]. \tag{1-2}$$

If Ω is a ϕ-John domain for the function $\phi = \phi_s$ defined by $\phi_s(t) = c_s t^s$ for $t \le 1$ and $\phi_s(t) = c_s t$ for $t > 1$, with $s \ge 1$, we say Ω is an s-John domain. We may assume that $0 < c_s < 1$. This definition is essentially the same as those by Smith and Stegenga [1990] and Hajłasz and Koskela [1998], who instead assume that $\phi_s(t) = c_0 t^s$ for some $c_0 > 0$ and all $t \geq 0$. For any $M > 1$, we will write $\mathcal{J}_M(t) = t/M$. As M varies, the class of $\mathcal{J}_M$-John domains is the same as the class of 1-John domains. If Ω is a $\mathcal{J}_M$-John domain for some M, then we will refer to M as the 1-John constant of Ω.

Note that (1-2) implies that $d(x) > 0$ for all $x \in \Omega$.

Definition 1.3. Let $\langle H, d\rangle$ be a quasimetric space. Given $\Omega \subset H$ and $\delta > 0$, we say that a ball $B(x, r)$ is a δ-ball if $x \in \Omega$ and $0 < r \leq \delta d(x)$. Balls of the form $B(x, r)$ with $x \in \Omega$ and $r = \delta d(x)$ will be called δ-Whitney balls.

Some useful properties of δ-balls are listed in Observation 2.1 in the next section. See also [Sawyer and Wheeden 2006], where such balls play a role in proving regularity of solutions of subelliptic equations.

For technical reasons (see, for example, the proof of Observation 2.1), whenever we consider δ-balls, we will always assume that $0 < \delta < 1/(2\kappa^2)$, where κ is the quasimetric constant in Definition 1.1. We note now that the weaker restriction $0 < \delta < 1/\kappa$ guarantees that every δ-ball is contained in Ω. In fact, let $x \in \Omega$ and $B(x, r)$ be a δ-ball with $\kappa\delta < 1$. If $y \in B(x, r)$, then

$$d(x) \leq \kappa(d(x, y) + d(y)) < \kappa(r + d(y)) \leq \kappa(\delta d(x) + d(y)).$$

Hence, $d(y) > [(1/\kappa) - \delta]d(x)$. In particular, $d(y) > 0$ and therefore $y \in \Omega$.

We next define what we mean by δ-doubling, doubling and reverse doubling.

Definition 1.4. Let $\langle H, d\rangle$ be a quasimetric space. A nonnegative finite functional σ defined on balls in H, that is, $\sigma : \{B : B \text{ is a ball in } H\} \to [0, \infty)$, will be called a *ball set* function (or a set function on balls). In practice, given $\Omega \subset H$, we will only consider balls B with $x_B \in \Omega$ and $r(B) \leq \operatorname{diam}(\Omega)$, where $\operatorname{diam}(\Omega)$ is defined using the quasimetric d. Given $\Omega \subset H$, $0 < \delta < 1/(2\kappa^2)$, and a ball set function σ, we say that σ is *δ-doubling* on Ω if there are positive constants A_σ and D_σ such that for all δ-balls $B(x, r)$ in Ω,

$$\frac{\sigma(B(x, \tilde{r}))}{\sigma(B(x, r))} \leq A_\sigma \left(\frac{\tilde{r}}{r}\right)^{D_\sigma} \quad \text{for all } 0 < r < \tilde{r} \leq \operatorname{diam}(\Omega).$$

If this inequality holds for all balls with center in Ω and $\tilde{r} \leq \operatorname{diam}(\Omega)$, we say that σ is *doubling* on Ω. If σ is also a measure[1] on Ω, we say that σ is a *δ-doubling* measure or *doubling* measure on Ω. Note that this definition is equivalent to the one in [Chua and Wheeden 2008, Definition 1.7]. In case σ is a ball set function or measure and there is a constant C such that $\sigma(2B) \leq C\sigma(B)$ for all balls $B \subset H$,

[1]Except in Theorem B below, we will assume that all measures are defined on a fixed σ-algebra that contains all balls.

we say simply that σ is doubling instead of doubling on H. Moreover, we say that σ is *reverse doubling* on Ω if there exist $A, D > 0$ such that

$$\frac{\sigma(B(x,r))}{\sigma(B(x,\tilde{r}))} \le A\Big(\frac{r}{\tilde{r}}\Big)^D \quad \text{for all } x \in \Omega, \text{ with } 0 < r < \tilde{r} \le \operatorname{diam}(\Omega). \tag{1-3}$$

Björn and Shanmugalingam [2007] gave a similar definition of doubling on Ω. Some properties of δ-doubling ball set functions are given in Proposition 2.2.

If $B(x,r) \setminus B(x,r') \neq \varnothing$ for all $0 < r' < r$, $x \in H$, we say the quasimetric space satisfies the *nonempty annuli property* in H. Similarly, we say that a set $\Omega \subset H$ has the nonempty annuli property if $(\Omega \cap B(x,r)) \setminus B(x,r') \neq \varnothing$ for all $0 < r' < r$ and $x \in \Omega$ for which Ω is not a subset of $B(x,r')$. A doubling measure on Ω satisfies a reverse condition of the same type provided Ω has the nonempty annuli property; this is similar to a fact from [Wheeden 1993, page 269].

We say that a family of balls (or cubes in the usual Euclidean case) has *bounded intercepts* if there exists a constant N such that each ball in the family intersects at most N other balls in the family. Such a family also has bounded overlaps in the pointwise sense since no point belongs to more than $N+1$ balls in the family.

Given an s-John domain with central point x' and a number $M > 1$, we distinguish two types of points x, depending on whether or not x can be connected to x' by a curve satisfying the $\mathcal{J}_M$-John condition:

Definition 1.5. Let $M > 1$ and Ω be an s-John domain with central point x'. Let Ω_g^M be the set of points x in Ω such that there is $\gamma_x : [0, l_x] \to \Omega$ such that $\gamma_x(0) = x$ and $\gamma_x(l_x) = x'$, and

$$\begin{aligned} d(\gamma_x(t_1), \gamma_x(t_2)) &\le |t_1 - t_2| \quad \text{for } t_1, t_2 \in [0, l_x], \\ d(\gamma_x(t)) &> \mathcal{J}_M(t) \qquad \text{for all } t \in [0, l_x]. \end{aligned}$$

We will say points in Ω_g^M are *M-good* points of Ω, and points in $\Omega \setminus \Omega_g^M = \Omega_\flat^M$ are *M-bad* points of Ω. Note that if $\Omega_g^M = \Omega$, then Ω is a 1-John domain.

A nonempty subset Ω_0 of Ω^c will be said to *confine the M-bad points of* Ω if there exists a constant $\overline{M} > 0$ such that

$$\sup_{x \in \Omega_\flat^M} \sup_{t \in [0, l_x]} d(\gamma_x(t), \Omega_0)/d(\gamma_x(t)) \le \overline{M}. \tag{1-4}$$

Note that (1-4) is the same as $d(B, \Omega_0) \le C(\kappa, \delta)\overline{M} r(B)$ for all $x \in \Omega_\flat^M$ and all δ-Whitney balls B with center along the s-John curve that connects x to x'.

Similar definitions can be given for ϕ-John domains.

In case Ω is a 1-John domain, there exists $M > 1$ such that $\Omega_g^M = \Omega$, and hence any nonempty set $\Omega_0 \subset \Omega^c$ confines the M-bad points of Ω. For any s-John domain, the choice $\Omega_0 = \Omega^c$ confines all the M-bad points of Ω, and

$d(B, \Omega^c) \le d(x_B) = \delta^{-1} r(B)$ for any δ-Whitney ball B. Moreover, in case $\Omega \subset \mathbb{R}^n$ with the usual Euclidean metric, we will show in the proof of Theorem 1.12 that if there exists $\varepsilon > 0$ such that $\Omega_0 \supset \partial\Omega \cap (\bigcup_{x \in \Omega_\flat^M} B(x, \varepsilon))$, then Ω_0 confines the M'-bad points of Ω for some $M' \ge M$.

If Ω is an s-John domain and c is a positive constant, then any point $x \in \Omega$ with $d(x) \ge c$ is an M-good point for suitably large M depending only on c, κ, s and c_s; the simple proof is given at the beginning of the proof of Theorem 1.12.

Before we state our first main theorem, we need to describe some chains of balls. We say a measure μ satisfies the *ratio condition* (R) on Ω if there are constants $0 < \theta_1 < \theta_2 < 1$ and $\alpha \ge 2$ such that for each $x \in \Omega$, there exists a strictly decreasing sequence $\{r_j^x\}_{j \in N}$ of positive real numbers such that

$$r_j^x \to 0, \quad r_1^x = \operatorname{diam}(\Omega), \quad r_j^x/\alpha < r_{j+1}^x < r_j^x$$

and

$$\theta_1 \le \frac{\mu(B(x, r_{j+1}^x))}{\mu(B(x, r_j^x))} \le \theta_2 \quad \text{for all } j. \tag{1-5}$$

It follows from (1-5) that $\mu(B(x, r_j^x)) \to 0$, and then the fact that $r_j^x \to 0$ is automatic since r_j^x decreases and we always assume that balls have positive μ-measure. See parts (1) and (2) of Remark 1.7 for further comments about (1-5).

Next, given any $\delta < 1/(2\kappa^2)$ and $1 \le \tau < 1/(2\delta\kappa^2)$, Proposition 2.3(c) implies that for any ϕ-John domain Ω, there is a sequence of δ-balls $\{Q_i^x\}_{i=1}^\infty$ with centers along the curve γ from x to x' guaranteed by the ϕ-John condition, such that $Q_1^x = B(x', \delta d(x'))$ and $\{Q_i^x\}$ has the intersection property

$$Q_i^x \cap Q_{i+1}^x \quad \text{contains a } \delta\text{-ball } Q_i' \text{ with } Q_i^x \cup Q_{i+1}^x \subset N Q_i'$$

for some positive constant N independent of x and i. Moreover, Q_i^x is centered at x for large i; in fact, for balls $B_j^x = B(x, r_j^x)$ as in (1-5), there exist $K_x, K_x' \in \mathbb{N}$ such that $\tau Q_{i+K_x}^x = B_{i+K_x'}^x$ for $i \ge 0$, B_j^x is a $\tau\delta$-ball if $j \ge K_x$, and Q_i^x is not centered at x if $i \le K_x$. We associate with each ball $B_j^x = B(x, r_j^x)$ for $j \ge 1$ the following special subcollection of $\{Q_i^x\}$:

$$\mathscr{C}(B_j^x) = \mathscr{C}_\phi(B_j^x) = \{Q_i^x : \tau Q_i^x \subset B_j^x \text{ and } \tau Q_i^x \not\subset B_{j+1}^x\}. \tag{1-6}$$

In case $j \ge K_x$, the set $\mathscr{C}(B_j^x)$ consists of just the single ball $\tau^{-1} B_j^x = Q_j^x$.

Our first self-improving result for s-John domains will be a consequence of a general weak-type theorem [Chua and Wheeden 2008, Theorem 1.2]; this theorem, which we now recall, is measure-theoretic and does not require the underlying structure of an s-John domain or even of a quasimetric space. In it, the sets Q_i^x and B_j^x are merely measurable sets, generally unrelated to the balls of (1-5) and (1-6).

Theorem B. *Let σ and μ be measures on a σ-algebra Σ of subsets of X. Let Ω be a measurable subset of X and f be a fixed measurable function such that the following assumptions hold for some constants satisfying*

$$0 < p_0, q < \infty, \quad 0 < \theta_1 < \theta_2 < 1,$$
$$0 < A_1, A_2 < \infty, \quad 0 < \theta < 1, \qquad C_\sigma \geq 1, \quad \wp \geq 1.$$

(1) *For each $x \in \Omega$, there is a sequence of measurable sets $\{Q_i^x\}_{i=1}^\infty$, depending on x, and a fixed set $B' \subset X$ such that $Q_1^x = B'$,*

$$0 < \sigma(Q_i^x \cup Q_{i+1}^x) \leq C_\sigma \sigma(Q_i^x \cap Q_{i+1}^x) < \infty \quad \text{for } i = 1, 2, \ldots, \tag{1-7}$$

and

$$\left(\frac{1}{\sigma(Q_i^x)} \int_{Q_i^x} |f - f_{Q_i^x}|^{p_0} d\sigma \right)^{1/p_0} \leq a(Q_i^x), \tag{1-8}$$

where $\{f_{Q_i^x}\}$ is a sequence of constants that converges to $f(x)$ and $\{a(Q_i^x)\}$ is a sequence of nonnegative numbers.

(2) *For each $x \in \Omega$, there is a sequence $\{B_j^x\}_{j=1}^\infty$ of measurable sets and a sequence $\{\mu^*(B_j^x)\}$ of positive numbers such that*

$$\mu(\Omega) \leq \wp \mu^*(B_1^x) \quad \text{and} \quad A_1 \theta_1^k \leq \frac{\mu^*(B_{j+k}^x)}{\mu^*(B_j^x)} \leq A_2 \theta_2^k \quad \text{for } j, k \in \mathbb{N}. \tag{1-9}$$

(3) *Let $\mathfrak{F} = \{B_j^x\}_{x \in \Omega, j \in \mathbb{N}}$. Assume for any $B_j^x \in \mathfrak{F}$, there is $\mathscr{C}(B_j^x) \subset \{Q_l^x\}_{l \in \mathbb{N}}$ such that $\bigcup_{j \in \mathbb{N}} \mathscr{C}(B_j^x) = \{Q_l^x\}_{l \in \mathbb{N}}$ and $\mathscr{C}(B_i^x) \cap \mathscr{C}(B_j^x) = \varnothing$ for each $x \in \Omega$ when $i \neq j$. Further, for any countable subcollection I of pairwise disjoint sets $\{B_\alpha\}$ in $\mathfrak{F}$, let*

$$A(B_\alpha) = \sum_{Q \in \mathscr{C}(B_\alpha)} a(Q)$$

and assume that

$$\sum_{B_\alpha \in I} \big(A(B_\alpha)^q \mu^*(B_\alpha) \big)^\theta \leq (C_0^q \mu(\Omega))^\theta. \tag{1-10}$$

(4) *Let the collection $\mathfrak{F}$ be a cover of Vitali type of subsets of Ω with respect to (μ, μ^*), that is, given any measurable set $E \subset \Omega$ and a collection $\mathscr{B}_E = \{B_{i(x)}^x : x \in E\}$, there is a countable, pairwise disjoint collection $\mathscr{B}'_E \subset \mathscr{B}_E$ such that*

$$\mu(E) \leq V_\mu \sum_{B_\alpha \in \mathscr{B}'_E} \mu^*(B_\alpha) \quad \text{and} \quad V_\mu \geq 1.$$

Then

$$\sup_{t>0} t\mu\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le CC_0\big(\wp V_\mu \mu(\Omega)\big)^{1/q}, \tag{1-11}$$

where C depends on C_σ, p_0, q, A_1, A_2, θ, θ_1 and θ_2.

Note that [Chua and Wheeden 2008, Theorem 1.8] can be generalized by assuming that μ^* satisfies condition (R) instead of (1.14) there. Of course, one must change the B_j in (1.15) there accordingly.

We now revert to the context of an s-John domain and to the choice of balls made in (1-5) and (1-6). Our first self-improving result is as follows.

Theorem 1.6. *Let Ω be an s-John domain with central point x' in a quasimetric space $\langle H, d\rangle$. Let $0 < \delta < 1/(2\kappa^2)$, $1 \le \tau < 1/(2\delta\kappa^2)$ and $M > 1$. Suppose σ, μ and w are measures, σ is δ-doubling on Ω, and $a_*(B)$ is a nonnegative functional defined for all δ-balls B. Let $0 < p_0 < \infty$ and $1 \le p < \infty$, and let f and g be fixed measurable functions such that*

$$\frac{1}{\sigma(B)^{1/p_0}}\|f - f_B\|_{L^{p_0}_\sigma(B)} \le a_*(B)\|g\|_{L^p_w(\tau B)} \tag{1-12}$$

for all δ-balls B in Ω with $f_{B(x,r)} \to f(x)$ as $r \to 0$ for μ-almost all $x \in \Omega$, that is, such that (1-8) *holds with $a(B) = a_*(B)\|g\|_{L^p_w(\tau B)}$. Let Ω_0 be a nonempty subset of Ω^c that confines* (*with constant $\overline{M}$*) *the M-bad points of Ω. Set $\rho(x) = d(x, \Omega_0)$,*[2] *and for real numbers a and b, define measures μ_a and w_b by $d\mu_a = \rho^a d\mu$ and $dw_b = \rho^b dw$. Let*

$$\rho(\Omega) = \sup\{\rho(x) : x \in \Omega\}.$$

Suppose μ satisfies condition (*R*) *on Ω and there are constants η, η', β and β' with $\beta' \ge 0$ such that for all pairs of balls (B, Q) with $B = B^x_j = B(x, r^x_j)$ as in* (1-5) *and $Q \in \mathscr{C}(B)$,*

$$\mu(B)^{1/q} a_*(Q) \le C_1 r(B)^{\beta'} \tag{1-13}$$

if either $x \in \Omega^M_g$ or B is any $\tau\delta$-ball (*then $B = \tau Q$*), *and*

$$\mu(B)^{1/q} a_*(Q) \le C_1 r(B)^{\eta/q} r(Q)^{\beta-\eta'/p} \tag{1-14}$$

if $x \in \Omega^M_\flat$ and $r(B) \ge \tau\delta d(x)$. Let $a \ge 0$, $\eta + a \ge 0$, and

$$\varepsilon' = \beta' + \frac{a}{q} - \frac{b}{p} \ge 0, \tag{1-15}$$

$$\varepsilon = \frac{\eta+a}{q} + \min\{\chi s, \chi\} \ge 0 \quad \textit{where } \chi = \frac{s(\beta p - b - \eta') - (s-1)(p-1)}{sp}, \tag{1-16}$$

[2] See Remark 3.2 concerning the choice of $\rho(x)$ in our theorems.

and $\chi > 0$ if $\eta + a = 0$. Assume further that μ_a satisfies the following Vitali-type condition (*compare with condition* (4) *of Theorem B*): *given any measurable set $E \subset \Omega$ and a collection $\mathcal{B}_E = \{B^x_{j(x)} : x \in E\}$, there is a countable pairwise disjoint collection $\mathcal{B}'_E \subset \mathcal{B}_E$ such that*

$$\mu_a(E) \le V_a \sum_{B_\alpha \in \mathcal{B}'_E} \mu_a(B_\alpha) \quad \textit{and} \quad V_a \ge 1.$$

(i) If $p < q < \infty$, then

$$\sup_{t>0} t\mu_a\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le CC_1\left(\frac{\mu_a(\Omega)}{\mu_a(B')}\right)^{1/q} \|g\|_{L^p_{w_b}(\Omega)} \times \begin{cases} \max\{\rho(\Omega)^{\varepsilon'}, \operatorname{diam}(\Omega)^{\varepsilon}\} & \text{if } \chi \ne 0, \\ \max\{\rho(\Omega)^{\varepsilon'}, \operatorname{diam}(\Omega)^{\varepsilon}(1 + |\log \operatorname{diam}(\Omega)|)^{(p-1)/p}\} & \text{if } \chi = 0, \end{cases} \tag{1-17}$$

where $B' = B(x', \delta d(x'))$ and C depends on all parameters in the conditions but is independent of $\rho(\Omega)$ and $\operatorname{diam}(\Omega)$. If $s = 1$, neither (1-14) nor (1-16) is needed (see Remark 1.7(3)), and the weak-type constant can be chosen to have the form

$$CC_1\left(\frac{\mu_a(\Omega)}{\mu_a(B')}\right)^{1/q} \rho(\Omega)^{\varepsilon'}. \tag{1-18}$$

Here C is also independent of M, $\overline{M}$, η, η' and β.

(ii) Suppose $1 \le q \le p$ and there exist $M_1, M_2, \tilde{\eta}, \tilde{\eta}' > 0$ such that for $\lambda = \kappa + 2\kappa^2$ and all $k \in \mathbb{Z}$, the number of disjoint balls $B(x, r)$ with center $x \in \Omega^M_\flat$ and $r \ge \max\{\tau\delta d(x), \lambda^k\}$ is at most $M_1\lambda^{-\tilde{\eta}k}$, and the number of disjoint $\tau\delta$-balls B with $r(B) \ge \lambda^k$ is at most $M_2\lambda^{-\tilde{\eta}'k}$. If

$$(p-q)\tilde{\eta}/(pq) < \varepsilon \quad \text{and} \quad (p-q)\tilde{\eta}'/(pq) < \min\{\varepsilon', \beta'\}, \tag{1-19}$$

then

$$\sup_{t>0} t\mu_a\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le CC_1\left(\frac{\mu_a(\Omega)}{\mu_a(B')}\right)^{1/q} \|g\|_{L^p_{w_b}(\Omega)}, \tag{1-20}$$

where C depends on all parameters in the conditions and on $\operatorname{diam}(\Omega)$ and $\rho(\Omega)$.

Remark 1.7. (1) When Ω satisfies the nonempty annuli property, condition (1-5) will hold for $r^x_i = 2^{-i+1} \operatorname{diam}(\Omega)$ if we assume that μ is doubling on Ω since the first inequality of (1-5) will then hold because of doubling, and the second will hold since doubling implies reverse doubling; see [Chua and Wheeden 2008, Proposition 2.3].

(2) Condition (R) is implied by weaker assumptions than doubling and nonempty annuli. In fact, suppose μ is reverse doubling on Ω, and there exists $0 < \theta' < 1$ such

that for each fixed $x \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$, there exists r' with $r < r' < \operatorname{diam}(\Omega)$ and

$$\theta' \mu(B(x, r')) \le \mu(B(x, r)). \tag{1-21}$$

Then (R) holds for μ. Note that (1-21) is true for any $\theta' < 1$ if $\mu(B(x, r))$ is a right-continuous function of $r \le \operatorname{diam}(\Omega)$ for each fixed x in Ω; for Euclidean balls, this is the case whenever μ is absolutely continuous with respect to Lebesgue measure. To show that (R) holds, first choose $\alpha > 2$ such that $\theta_2 = A\alpha^{-D} < 1$, where A and D are constants in (1-3). Note that $\mu(B(x, r/\alpha))/\mu(B(x, r)) \le \theta_2$ for any $x \in \Omega$ and $0 < r \le \operatorname{diam}(\Omega)$ by (1-3). Fix any $0 < \theta_1 < \theta'\theta_2$ and define

$$r_- = \sup\{t \in [r/\alpha, r] : \mu(B(x, t)) \le \theta_2 \mu(B(x, r))\},$$
$$r_+ = \inf\{t \in [r/\alpha, r] : \mu(B(x, t)) \ge (\theta_1/\theta')\mu(B(x, r))\}.$$

Note that $r/\alpha \le r_- < r$ by left-continuity, and also that $r/\alpha \le r_+ \le r$. If $r_+ < r_-$, then for any r' with $r_+ < r' < r_-$, we have

$$\theta_1 < \theta_1/\theta' \le \mu(B(x, r'))/\mu(B(x, r)) \le \theta_2.$$

It is impossible that $r_+ > r_-$ since otherwise there exists t with $r_- < t < r_+$, and consequently $\mu(B(x, t)) > \theta_2 \mu(B(x, r))$ and $\mu(B(x, t)) < (\theta_1/\theta')\mu(B(x, r))$, yielding the contradiction $\theta_1/\theta' > \theta_2$. We now only need to handle the case $r_+ = r_-$. But, by monotonicity of measure, in case $r_- > r/\alpha$ we have

$$\mu(B(x, r_-)) = \lim_{t \to (r_-)^-} \mu(B(x, t)) \le \theta_2 \mu(B(x, r)),$$

while in case $r_- = r/\alpha$ we have $\mu(B(x, r_-)) \le \theta_2 \mu(B(x, r))$ by (1-3) as above. On the other hand, by (1-21), there exists $r' > r_- = r_+$ (and $r' < r$ as $r_- < r$) such that

$$\theta' \mu(B(x, r')) \le \mu(B(x, r_-)).$$

But $\mu(B(x, r')) \ge (\theta_1/\theta')\mu(B(x, r))$ as $r' > r_+$. Then r_- itself has the desired properties $r/\alpha \le r_- < r$ and $\theta_1 \le \mu(B(x, r_-))/\mu(B(x, r)) \le \theta_2$. In any case we can find $r/\alpha \le r' < r$ such that

$$\theta_1 \le \frac{\mu(B(x, r'))}{\mu(B(x, r))} \le \theta_2.$$

(3) Conditions (1-14) and (1-16) are not required for 1-John domains since then $\Omega_\flat^M$ is empty if M is large. Thus we only need condition (1-13) if $s = 1$. For any $s \ge 1$, we have $r(Q) \sim r(B)$ in condition (1-13), with constants depending only on τ and M, no matter whether $x \in \Omega_g^M$ and $Q \in \mathscr{C}(B)$, or whether $x \in \Omega$ and $B = \tau Q$. Hence, if μ is δ-doubling, (1-13) is equivalent to the simpler condition

$$\mu(B)^{1/q} a_*(B) \le C_1 r(B)^{\beta'} \tag{1-22}$$

for all δ-balls B.

(4) Condition (1-13) can often be replaced by the simpler (1-22) even when μ has no doubling properties. For example, suppose that $a_*(B)$ has the special monotonicity property that given $M' > 1$, there exists $c \geq 1$ such that

$$a_*(B_1) \leq c a_*(B_2) \quad \text{if } B_1 \subset B_2 \subset M' B_1. \tag{1-23}$$

Then, whether or not μ is doubling, (1-13) follows easily if (1-22) holds with $B = M'Q$ for all δ-balls Q and an appropriate constant M' depending on M, τ, s.

As an application, we obtain results about 1-John domains of the type studied in [Drelichman and Durán 2008] and [Hurri-Syrjänen 2004]. We illustrate this now in the form of a weak-type statement; however, the analogous strong-type statement is also true by using ideas related to Theorem 1.12. Consider for simplicity the case of Euclidean balls $B \subset \mathbb{R}^n$, and let

$$p_0 = 1, \quad \beta = 1, \quad d\sigma = dx, \quad 1 < p < \infty, \quad p' = p/(p-1).$$

For nonnegative locally integrable weights w_1, w_2 such that $w_2^{-1/(p-1)}$ is locally integrable, let $d\mu = w_1 dx$ and

$$\tilde{a}_*(B) = C\frac{r(B)}{|B|}\Big(\int_B w_2^{-1/(p-1)} dx\Big)^{1/p'}.$$

It is easy to see that $\tilde{a}_*(B)$ has the special monotonicity property (1-23) since $\int_B w_2^{-1/(p-1)} dx$ is truly monotone increasing in B. On the other hand, Hölder's inequality applied to the L^1, L^1 Poincaré estimate for Euclidean balls yields the following version of (1-12) involving $\tilde{a}_*(B)$, with $\beta = p_0 = 1$ and $d\sigma = dx$:

$$\begin{aligned}\frac{1}{|B|}\int_B |f - f_B| dx &\leq C r(B)\frac{1}{|B|}\int_B |\nabla f| dx \\ &\leq C\frac{r(B)}{|B|}\Big(\int_B |\nabla f|^p w_2 dx\Big)^{1/p}\Big(\int_B w_2^{-1/(p-1)} dx\Big)^{1/p'} \\ &= \tilde{a}_*(B)\Big(\int_B |\nabla f|^p w_2 dx\Big)^{1/p}.\end{aligned}$$

Condition (1-22) takes the form

$$\Big(\int_B w_1 dx\Big)^{1/q} r(B)^{1-n}\Big(\int_B w_2^{-1/(p-1)} dx\Big)^{1/p'} \leq C r(B)^{\beta'} \tag{1-24}$$

for $B = M'Q$ and all δ-balls Q. If $s = 1$, (1-14) is not needed as a hypothesis in Theorem 1.6 (since $\Omega_\flat^M$ is empty for 1-John domains), and if we assume the remaining hypothesis (R) for the measure $d\mu = w_1 dx$, for example if we assume (see Remark 1.7(2)) the reverse doubling condition (1-3) and note that (1-21) is automatically true since μ is absolutely continuous with respect to the Lebesgue measure, then we obtain as a corollary of Theorem 1.6(i) that for a 1-John domain Ω and $1 < p < q < \infty$,

$$\sup_{t>0} t(w_1)_a\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le C\|\nabla f\|_{L^p_{(w_2)_b}(\Omega)}$$

for the same range of a and b as in Theorem 1.6 with $\beta = 1$ and $\eta' = \eta = n$. In fact, our hypotheses are weaker than those in [Drelichman and Durán 2008], where (1-24) with $\beta' = 0$ is assumed for *all* balls B, and where both absolute continuity and reverse doubling of μ are assumed, whereas we require (1-24) for a more restricted class of balls and can assume (R) for μ rather than absolute continuity and reverse doubling.

(5) If μ_a is doubling on Ω or if Ω has the Besicovitch covering property (for example, Euclidean space has the Besicovitch property), then μ_a will satisfy the Vitali covering condition in Theorem 1.6. See also [Sawyer and Wheeden 1992; Di Fazio et al. 2008].

(6) The exponents ε and ε' in (1-17) are nonnegative by (1-15) and (1-16). We also note for future reference that (1-16) implies $(\eta + a)/q - (\eta' + b)/p + \beta \ge 0$; in fact, this is the same as $(\eta + a)/q \ge (\eta' + b - \beta p)/p$, which follows from (1-16) when $\eta' + b - \beta p > 0$ (since $s, p \ge 1$) and is obvious when $\eta' + b - \beta p \le 0$. Moreover, if we assume (1-14) for *all* B, Q with $Q \in \mathscr{C}(B)$, then (1-13) follows in case $\beta' \le \beta + \eta/q - \eta'/p$.

(7) In (1-17) of Theorem 1.6, it is often true that $\rho(\Omega) \le C(\kappa, \overline{M}) \operatorname{diam}(\Omega)$. This clearly occurs when $\partial\Omega \cap \Omega_0 \ne \varnothing$. It is also the case when $\Omega^M_\flat \ne \varnothing$ since if there is $x_1 \in \Omega^M_\flat$, then $d(x_1, \Omega_0) \le \overline{M} d(x_1)$ by (1-4), and hence

$$d(x, \Omega_0) \le \kappa(\operatorname{diam}(\Omega) + d(x_1, \Omega_0)) \le C(\kappa, \overline{M}) \operatorname{diam}(\Omega) \quad \text{for all } x \in \Omega.$$

Recall that $\Omega^M_\flat$ is nonempty unless Ω is a 1-John domain. If also $\operatorname{diam}(\Omega) \le 1$ and $\beta' \ge \beta + \eta/q - \eta'/p$, then $\varepsilon' = \beta' + a/q - b/p \ge \beta + (\eta + a)/q - (\eta' + b)/p$ and so both maximums in (1-17) are the corresponding last terms that involve $\operatorname{diam}(\Omega)$ because

$$\beta + \frac{\eta + a}{q} - \frac{\eta' + b}{p} = \left(\frac{\eta + a}{q} + \chi\right) + \frac{s - 1}{sp'} \ge \varepsilon.$$

(8) By definition, $\rho(x) = \operatorname{dist}(x, \Omega_0)$ for *any* subset Ω_0 of Ω^c that confines the M-bad points of Ω. Hajłasz and Koskela [1998] and Kilpeläinen and Malý [2000] assume Ω_0 to be all of Ω^c.

(9) For particular choices of η and η', condition (1-14) is a corollary of (1-13) if μ satisfies the doubling condition $\mu(\tilde{B}) \le C(r(\tilde{B})/r(B))^{D_1}\mu(B)$ for some D_1 and all pairs B, $\tilde{B}$ of balls with $B \subset \tilde{B}$, $\tilde{B}$ centered in Ω and $r(\tilde{B}) \le \operatorname{diam}(\Omega)$. In fact,

fix a ball B centered in Ω and let Q be a δ-ball in B. Then

$$\begin{aligned}\mu(B)^{1/q}a_*(Q) &\le C\left(\frac{r(B)}{r(Q)}\right)^{D_1/q}\mu(Q)^{1/q}a_*(Q)\\ &\le C\left(\frac{r(B)}{r(Q)}\right)^{D_1/q}r(Q)^{\beta'} \quad \text{by (1-13)},\end{aligned}$$

which gives (1-14) with $\eta = D_1$ and $\eta' = p\big(\beta - \beta' + D_1/q\big)$. In particular, with this version of (1-14), condition (1-16) implies

$$\frac{D_1+a}{q} \ge \frac{s(pD_1/q + b - \beta' p) + (s-1)(p-1)}{p}.$$

While these estimates are often not sharp and the η and η' obtained in this way are often undesirable, nevertheless, in the usual Euclidean case, where $\beta = 1$, $\mu = w = 1$, $D_1 = n$ and (1-13) holds with $\beta' = 1 + n/q - n/p$, they yield the same conditions as in Theorem A. In fact, the version of (1-16) given above reduces to

$$\frac{n+a}{q} \ge \frac{s(n+b-1) - p + 1}{p},$$

and the restriction $\beta' \ge 0$ is the same as $1/q \ge 1/p - 1/n$. Finally, (1-15) becomes $(n+a)/q - (n+b)/p + 1 \ge 0$, which follows from (1-16) as explained in part (6) of this remark.

(10) By using standard interpolation techniques, we find the weak L^q estimate (1-17) implies a strong-type inequality in which the left side of (1-17) is replaced by $\|f - f_{B'}\|_{L^{q_0}_{\mu_a}(\Omega)}$ for any q_0 with $0 < q_0 < q$; see [Chua and Wheeden 2008, Remark 1.13].

(11) In Theorem 1.6, the condition $a \ge 0$ can be replaced by assuming that (1-5), (1-13) and (1-14) hold for μ_a (instead of μ), as will be clear from the proof. When Ω is a 1-John domain in $\mathbb{R}^n$ with Euclidean distance, there exists $\varepsilon > 0$ such that if $-\varepsilon < a < 0$, the weighted Lebesgue measure $\rho(x)^a dx = \operatorname{dist}(x, \Omega^c)^a dx$ is δ-doubling (and hence doubling) on Ω; see [Hajłasz and Koskela 1998, Theorem 6 and Lemma 6]. Thus, for such a (set $a = -\varepsilon_0$ for convenience), Theorem 1.8(i) below with $d\sigma = \rho^{-\varepsilon_0}dx$, $dw = dx$ and $a = 0$ can be used to deduce [Hajłasz and Koskela 1998, Theorem 8] as it is easy to see (1-12) holds with $d\sigma = \rho^{-\varepsilon_0}dx$, $dw = dx$ and $\beta = p = p_0 = 1$ and (1-13) holds with $\mu = \sigma$ and $\beta' = (n - \varepsilon_0)/q - n + 1$. Note that when $\Omega_0 = \Omega^c$, we do not need to assume $\beta' \ge 0$ since then $r(B) \sim \rho(B)$ for all δ-Whitney balls. Moreover, the argument works for $1 < p \le q$ by choosing $\beta' = (n - \varepsilon_0)/q - n/p + 1$.

(12) The measure μ can be replaced in (1-5), (1-13), (1-14) and the Vitali-type condition of Theorem 1.6 by $\mu|_\Omega$ since the conclusions (for example, (1-17)) are relative to Ω.

(13) For any ball $B = B(x,r)$ with $x \in \Omega$ and $r \geq 1$, the set $\mathscr{C}(B)$ will contain a δ-ball of comparable size. Since σ is δ-doubling on Ω, there can be at most a bounded number (with bound depending on A_σ, D_σ and $\operatorname{diam}(\Omega)$) of pairwise disjoint such balls $B(x,r)$.

(14) When d is a metric, the first ball B_1^x in the ratio condition (1-5) satisfies $\Omega \subset B_1^x$ and hence the factor $(\mu_a(\Omega)/\mu_a(B'))^{1/q}$ in (1-17), (1-18) and (1-20) can be replaced by 1; see the proof of Theorem 1.6 concerning the estimate of $\wp$ in (1-9).

Next, we discuss some strong-type inequalities in the special cases when $\mu=\sigma$ and $p=q=1$ or $s=1$. Other estimates of strong type are given in later theorems.

Theorem 1.8. *Let Ω be an s-John domain with central point x' in a quasimetric space $\langle H,d\rangle$, and let δ, τ, M, $a_*(B)$, p_0, p, β' and B' be as in Theorem 1.6. Suppose σ and w are measures and σ is δ-doubling on Ω. Also, let f and g be as before, that is,* (1-12) *holds for all δ-balls B in Ω, but we do not assume $f_{B(x,r)} \to f(x)$ σ-almost everywhere.*

(i) *Suppose $s=1$, $q=p_0 \geq p$, $\beta' \geq 0$ and*

$$\sigma(B)^{1/q} a_*(B) \leq C_1 r(B)^{\beta'} \tag{1-25}$$

for all balls B for which there is a concentric δ-Whitney ball $\tilde{B}$ with $\lambda^{-2}\tilde{B} \subset B \subset \tilde{B}$, where $\lambda = \kappa + 2\kappa^2$. If $a \geq 0$ and $\varepsilon' = \beta' + a/q - b/p \geq 0$, then the strong-type estimate

$$\|f - f_{B'}\|_{L^q_{\sigma_a}(\Omega)} \leq C C_1\, \rho(\Omega)^{\varepsilon'} \|g\|_{L^p_{w_b}(\Omega)} \tag{1-26}$$

holds with C depending on all relevant parameters but not on $\rho(\Omega)$ or $\operatorname{diam}(\Omega)$. *The condition $\beta' \geq 0$ is not needed when $\Omega_0 = \Omega^c$ or when $r(B) \leq c\rho(B)$ for all balls B as above.*

(ii) *Suppose $s \geq 1$, $q = p_0 = p = 1$, $\beta \in \mathbb{R}$ and* (1-14) *holds with μ replaced by σ for any pair (B,Q) of balls such that $Q \subset B$, Q satisfies $\lambda^{-2}\tilde{Q} \subset Q \subset \tilde{Q}$, where $\tilde{Q}$ is the δ-Whitney ball concentric with Q, and B is a ball centered in Ω with $r(B) \leq$* $\operatorname{diam}(\Omega)$. *If $a \geq 0$, $\beta+\eta-\eta' \geq 0$ and* (1-16) *holds, then*

$$\begin{aligned} \|f - f_{B'}\|_{L^1_{\sigma_a}(\Omega)} &\\ \leq C C_1 \max\{\rho(\Omega)^{(\eta+a-s(\eta'+b-\beta))/s},\ &\rho(\Omega)^{\eta+a-\eta'-b+\beta}\}\|g\|_{L^1_{w_b}(\Omega)}, \end{aligned} \tag{1-27}$$

where C depends on all relevant parameters but not on $\rho(\Omega)$ or $\operatorname{diam}(\Omega)$. *Also,* (1-27) *holds even if Ω_0 does not confine the M-bad points provided*

$$\beta + \eta/s - \eta' \geq 0. \tag{1-28}$$

Again, one can replace the conditions for σ by the corresponding ones for $\sigma|_\Omega$.

To derive a strong-type version of (1-17) better than the one in Remark 1.7(10), we recall from [Chua and Wheeden 2008] a strong-type analogue of Theorem B.

Given $\omega > 0$ and a nonnegative function g, the truncation $\tau_\omega g$ is defined by

$$\tau_\omega g(x) = \min\{g(x), 2\omega\} - \min\{g(x), \omega\} = \begin{cases} \omega & \text{if } g(x) \geq 2\omega, \\ g(x) - \omega & \text{if } \omega \leq g(x) < 2\omega, \\ 0 & \text{if } g(x) < \omega. \end{cases}$$

Let f be a fixed measurable function on Ω and B' be a fixed measurable set in Ω. Let $f_{B',\sigma} = \int_{B'} f\, d\sigma/\sigma(B')$. For each function $\tau_\omega|f - f_{B',\sigma}|$, $\omega > 0$, and each $x \in \Omega$, we assume the existence of sequences $\{B_i^x\}$, $\{Q_i^x\}$ and $\{a(Q_i^x)\}$ with properties as in Theorem B, but as there, these sequences as well as $\mathfrak{F}$ and the sets $\mathscr{C}(B)$ may depend on $\tau_\omega|f - f_{B',\sigma}|$. For easy reference, we will denote $f^\omega = \tau_\omega|f - f_{B',\sigma}|$ and write $b(Q_i^x, f^\omega)$ instead of $a(Q_i^x)$, and $\mathfrak{F}(f^\omega)$ instead of $\mathfrak{F}$, but we will not adopt new notation to indicate that $\{B_i^x\}$ and $\{Q_i^x\}$ may vary with ω. A typical example of $b(Q, g)$ is

$$b(Q, g) = b_Y(Q, g) = r(Q)^\beta \left(\frac{1}{w(Q)} \int_Q |Yf|^p dw \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

where Y is a differential operator with $Y1 = 0$, that is, with no zero order term.

Given f and $f^\omega = \tau_\omega|f - f_{B',\sigma}|$, the analogue of (1-8) that we will assume in our strong-type analogue of Theorem B is

$$\begin{aligned} \frac{1}{\sigma(Q_i^x)^{1/p_0}} \| f^\omega - (f^\omega)_{Q_i^x,\sigma} \|_{L_\sigma^{p_0}(Q_i^x)} &\leq b(Q_i^x, f^\omega), \\ (f^\omega)_{Q_i^x,\sigma} &= \frac{1}{\sigma(Q_i^x)} \int_{Q_i^x} f^\omega d\sigma \end{aligned} \tag{1-29}$$

for all $\omega > 0$. We will also assume an analogue of (1-10): For some constants $q > 0$ and $0 < \theta < 1$,

$$\begin{aligned} \sum_{B_\alpha \in I} \big(A(B_\alpha, f^\omega)^q \mu^*(B_\alpha) \big)^\theta &= \sum_{B_\alpha \in I} \Big(\Big[\sum_{Q \in \mathscr{C}(B_\alpha)} b(Q, f^\omega) \Big]^q \mu^*(B_\alpha) \Big)^\theta \\ &\leq \big(h(\Omega, f^\omega)^q \mu(\Omega) \big)^\theta \end{aligned} \tag{1-30}$$

for every disjoint subcollection I of $\mathfrak{F}(f^\omega)$ and all $\omega > 0$. Here $h(\Omega, \cdot)$ is a constant that is assumed to satisfy

$$h^*(\Omega, f)^q := \sup_{\omega>0} \sum_{k=1}^{\infty} h(\Omega, f^{2^k\omega})^q < \infty. \tag{1-31}$$

Conditions (1-30) and (1-31) are stability properties of the functional b_Y under truncation similar to ones that were introduced in [Long and Nie 1991; Maz'ja 1985] and exploited in many papers such as [Franchi et al. 1995; Franchi et al. 1998; 2003].

The following strong-type analogue of Theorem B extends both [Franchi et al. 2003, Corollary 3] and [Franchi et al. 1998, Theorem 3.1].

Theorem 1.9 [Chua and Wheeden 2008, Theorem 1.10]. *Let σ and μ be measures on a σ-algebra of subsets of X, let Ω be a measurable set, and let f be a fixed measurable function. Suppose that for each $f^\omega = \tau_\omega|f - f_{B',\sigma}|$ with $\omega > 0$, there are sets $\{Q_i^x\}$ and $\{B_i^x\}$ (possibly depending on ω and f in addition to x, but with $Q_1^x = B'$ for all x) satisfying the conditions of Theorem B, but now assuming* (1-29) *instead of* (1-8), *and* (1-30) *for all $\omega > 0$ instead of* (1-10). *If* (1-31) *is true, then the strong-type Poincaré inequality*

$$\frac{1}{\mu(\Omega)}\|f - f_{B',\sigma}\|^q_{L^q_\mu(\Omega)} \le C\wp V_\mu h^*(\Omega, f)^q + \left(\frac{8}{\sigma(B')}\|f - f_{B',\sigma}\|_{L^1_\sigma(B')}\right)^q \tag{1-32}$$

holds with C as in Theorem B.

We will derive the following result as a corollary of Theorem 1.9 and use it to prove the strong-type estimate given below in Theorem 1.12.

Theorem 1.10. *Suppose that the conditions of Theorem 1.6(i) hold except that* (1-12) *is replaced by*

$$\frac{1}{\sigma(B)^{1/p_0}}\|f^\omega - f^\omega_{B,\sigma}\|_{L^{p_0}_\sigma(B)} \le a_*(B)\|Yf^\omega\|_{L^p_w(\tau B)} \tag{1-33}$$

for all $f^\omega = \tau_\omega|f - f_{B',\sigma}|$ with $\omega > 0$ (where f is a fixed function), and all δ-balls B, where Yf^ω is some function. Then when $q > p$, instead of (1-17), *the strong-type inequality*

$$\begin{aligned}\frac{1}{\mu_a(\Omega)}&\|f - f_{B',\sigma}\|^q_{L^q_{\mu_a}(\Omega)} \\ &\le \frac{\tilde{C}}{\mu_a(B')}\sup_{\omega>0}\sum_{k=1}^{\infty}\|Yf^{2^k\omega}\|^q_{L^p_{w_b}(\Omega)} + \frac{C}{\sigma(B')^q}\|f - f_{B',\sigma}\|^q_{L^1_\sigma(B')}\end{aligned} \tag{1-34}$$

holds, where $\tilde{C}$ is an absolute constant times those in (1-17) *and* (1-18).

Remark 1.11. In many applications, the right side of (1-34) can be reduced to a multiple of $\|Yf\|^q_{L^p_{w_b}(\Omega)}$. For example, since $q > p$, it is true for a differential operator Y on Euclidean space that

$$\sum_k \|Yf^{2^k\omega}\|^q_{L^p_{w_b}(\Omega)} \le \Big(\sum_k \|Yf^{2^k\omega}\|^p_{L^p_{w_b}(\Omega)}\Big)^{q/p} \le C\|Yf\|^q_{L^p_{w_b}(\Omega)} \quad \text{for } \omega > 0. \tag{1-35}$$

Moreover, the second term on the right side of (1-34) is often bounded by a multiple of $\|Yf\|_{L^p_{w_b}(\Omega)}$. For instance, if we assume (1-33) holds for f on the ball B' and $p_0 \ge 1$, then

$$\begin{aligned}\frac{1}{\sigma(B')}\|f - f_{B',\sigma}\|_{L^1_\sigma(B')} &\le \frac{1}{\sigma(B')^{1/p_0}}\|f - f_{B',\sigma}\|_{L^{p_0}_\sigma(B')} \\ &\le a_*(B')\|Yf\|_{L^p_w(\tau B')} \le a_*(B')\|Yf\|_{L^p_{w_b}(\Omega)}.\end{aligned}$$

Our next result, a corollary of Theorem 1.10, contains Theorem A in the special case that $\Omega_0 = \Omega^c$. We do not require that $b \ge 1-n$ and we consider more general types of distance weights than those in Theorem A. Also, we include the case $p \ge q \ge 1$.

Theorem 1.12. *Suppose that $s \ge 1$ and $\Omega \subset \mathbb{R}^n$ is an s-John domain with respect to ordinary Euclidean distance d_E. Let $0 < \delta < 1/2$ and $B' = B(x', \delta d_E(x'))$ be the δ-Whitney ball centered at the central point x' of Ω. Suppose $\varepsilon > 0$, $M > 1$ and that Ω_0 satisfies*

$$\partial\Omega \cap \Big(\bigcup_{x\in\Omega^M_\flat} B(x,\varepsilon)\Big) \subset \Omega_0 \subset \Omega^c, \tag{$*$}$$

and set $\rho(x) = d_E(x, \Omega_0)$. Let $a \ge 0$, $b \in \mathbb{R}$, and p, q satisfy $1 \le p, q < \infty$ and $1/q \ge 1/p - 1/n$. If either $q > p$ and

$$\frac{s(n+b-1)-p+1}{(n+a)p} \le \frac{1}{q}, \tag{1-37}$$

or if $p \ge q$ and both

$$\frac{s(n+b-1)-p-n+1}{p} < \frac{a}{q} \quad \text{and} \quad 1 + \frac{a}{q} - \frac{b}{p} > 0, \tag{1-38}$$

then there is a constant C, depending on all relevant parameters, diam(Ω) *and $\rho(\Omega)$, such that*

$$\|f - C(\Omega, f)\|_{L^q_{\rho^a dx}(\Omega)} \le C\|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} \quad \text{for } f \in \operatorname{Lip}_{\mathrm{loc}}(\Omega). \tag{1-39}$$

Here $C(\Omega, f)$ can be chosen to be either

$$\frac{1}{|B'|}\int_{B'} f\,dx \quad \text{or} \quad f_{\mathfrak{D},\rho^a dx} = \frac{1}{|\mathfrak{D}|_{\rho^a dx}}\int_{\mathfrak{D}} f\rho^a\,dx$$

for any $\mathcal{D} \subset \Omega$ with $|\mathcal{D}| > 0$. In case $C(\Omega, f) = f_{\mathcal{D},\rho^a dx}$, the constant C also depends on the ratio $|\Omega|_{\rho^a dx}/|\mathcal{D}|_{\rho^a dx}$. Furthermore, for $s = 1$, (1-39) *remains valid even if $p = q$ when* (1-37) *holds. In case $p = q = 1$,* (1-39) *holds if*

$$n + a + s(1 - b - n) \geq 0,$$

as opposed to the strict inequality required in (1-38) *when $p = q$. Moreover, it remains true for any nonempty set $\Omega_0 \subset \Omega^c$ if $1 \leq s \leq n/(n-1)$, that is, for such s, the restriction that $\partial\Omega \cap (\bigcup_{x \in \Omega_\flat^M} B(x, \varepsilon)) \subset \Omega_0$ is not needed.*

Remark 1.13. (1) The average $f_{\mathcal{D},\rho^a dx}$ is well defined if

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} < \infty;$$

this follows as usual by first applying (1-39) with $C(\Omega, f)$ chosen to be $|B'|^{-1} \int_{B'} f\,dx$.

(2) The case $p = q = 1$ is also considered in [Hajłasz and Koskela 1998], except that $b \geq 1 - n$ is assumed there.

(3) If $s = 1$, then $\Omega_g^M = \Omega$ for some $M > 1$, and Theorems 1.12 and 1.10 are generalizations of results in [Chua 2006] and [Chua 2001], where the weights are assumed to be doubling on all of $\mathbb{R}^n$.

(4) The range of q is sharp; see [Hajłasz and Koskela 1998] for the case $q > p$.

As mentioned earlier, the q range in Theorem 1.12 can be enlarged for special s-John domains. Some results of this type are given in Section 3. In particular, for $s > 1$, we will consider the following typical s-cusp domain, which is an s-John domain:

$$D = \{(z, z') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < z < 4, |z'| < z^s\}.$$

The next result extends [Kilpeläinen and Malý 2000, Example 2.4], where the case $\mathcal{D}_0 = \mathcal{D}^c$ (equivalently, $\mathcal{D}_0 = \partial\mathcal{D}$) is mentioned.

Theorem 1.14. *Let $\mathcal{D}$ be the s-cusp domain above, let $\mathcal{D}_0$ be a subset of $\mathcal{D}^c$, and let $\rho(x) = d_E(x, \mathcal{D}_0)$. Suppose $a \geq 0$, $b \in \mathbb{R}$, $1 \leq p < q$, and*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n}. \tag{1-40}$$

(1) *If there exists $\varepsilon > 0$ such that $B((0, 0), \varepsilon) \cap \partial\mathcal{D} \subset \mathcal{D}_0$ and*

$$\frac{1}{q} \geq \frac{s(n + b - p) + (s - 1)(p - 1)}{p(s(n - 1) + 1 + a)}, \tag{1-41}$$

then

$$\|f - f_{\mathcal{D},\rho^a dx}\|_{L^q_{\rho^a dx}(\mathcal{D})} \leq C\|\nabla f\|_{L^p_{\rho^b dx}(\mathcal{D})} \quad \textit{for all } f \in Lip_{\mathrm{loc}}(\mathcal{D}). \tag{1-42}$$

(2) *If* $\mathcal{D}_0 = \partial\mathcal{D}$ *and*

$$\frac{1}{q} \geq \frac{s(n+b-p)+(s-1)(p-1)}{p(s(n-1+a)+1)}, \tag{1-43}$$

then (1-42) *holds for all* $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{D})$.

The q range in (1-43) is larger than in (1-41), and the range in (1-41) is larger than in Theorem 1.12. Results for $p \geq q$ can also be obtained by similar methods.

2. Preliminaries

In general, we will not attempt to give very precise values for constants that arise in the proofs, although we will keep track of important parameters on which constants depend. We will consistently use the notation

$$\lambda = \kappa + 2\kappa^2.$$

The constant λ arises naturally in Observation 2.1 and Proposition 2.2 and for simplicity we often use it in estimates in which better constants could be chosen.

We now recall several useful geometric facts, which require only that d be a quasimetric.

Observation 2.1 [Chua and Wheeden 2008, Observation 2.1]. (1) *If* $z \in B(x, r)$, *then*

$$B(z, r) \subset 2\kappa B(x, r) \subset \lambda B(z, r).$$

(2) *Let* B_1 *and* B_2 *be balls with* $B_1 \cap B_2 \neq \varnothing$. *Then*

(a) $B_2 \subset \lambda \max\{r(B_2)/r(B_1), 1\} B_1$.

(b) *If in addition both* B_1 *and* B_2 *are* δ*-balls with* $\delta < 1/(2\kappa^2)$, *then*

$$\lambda^{-1} d(x_{B_2}) \leq d(x_{B_1}) \leq \lambda d(x_{B_2}).$$

Thus if B_1 *and* B_2 *are intersecting* δ*-Whitney balls, then*

$$\lambda^{-1} \leq r(B_2)/r(B_1) \leq \lambda \quad \textit{and} \quad \lambda^{-2} B_1 \subset B_2 \subset \lambda^2 B_1.$$

(c) *If* $\delta < 1/(2\kappa^2)$ *and* z *is in a* δ*-ball* $B(x, r)$, *then*

$$\frac{1}{2\kappa} \leq \frac{d(x)}{d(z)} \leq 2\kappa.$$

Next, we list some facts about δ-doubling set functions on balls.

Proposition 2.2. (1) *If* $0 < \delta_1, \delta_2 < 1/\kappa$ *and* σ *is* δ_1*-doubling on* Ω, *then* σ *is also* δ_2*-doubling on* Ω.

(2) *Let* σ *be a measure on* Ω. *If* σ *is* δ*-doubling on balls in* Ω *and* $\sigma|_\Omega$ *is defined by* $\sigma|_\Omega(B) = \sigma(B \cap \Omega)$ *for balls* $B \subset H$, *then* $\sigma|_\Omega$ *is also* δ*-doubling since* $\sigma|_\Omega$ *and* σ *are the same on* δ*-balls.*

(3) *If* Ω *is a* ϕ*-John domain and* $M > 1$, *then for any* x *and* r *that satisfy* $x \in \Omega_g^M$ *and* $\delta d(x) \le r \le \operatorname{diam}(\Omega)$, *there is a* δ*-ball* Q *such that* $Q \subset B(x,r)$ *and* $r \le c_2 r(Q)$ *with* c_2 *depending only on* κ, δ, $\operatorname{diam}(\Omega)/d(x')$ *and* M.

(4) *If* Ω *is a* 1*-John domain, then the notions of* δ*-doubling on* Ω *and doubling on* Ω *are equivalent.*

Proof. Parts (1) and (2) are easy to show, and we will only prove (3) and (4).

Proof of (3): Let x, r be as in part (3) and $B' = B(x', \delta d(x'))$. If $B' \subset B(x,r)$, then since

$$r \le \operatorname{diam}(\Omega) = \frac{\operatorname{diam}(\Omega)}{d(x')} d(x') = \frac{1}{\delta}\frac{\operatorname{diam}(\Omega)}{d(x')} r(B'),$$

we may choose $Q = B'$ and $c_2 \ge \operatorname{diam}(\Omega)/(\delta d(x'))$. If $B' \not\subset B(x,r)$, we let $\gamma : [0,l] \to \Omega$ be a 1-John curve that connects x to x' and define

$$t_0 = \sup\{t \in [0,l] : B(\gamma(t), \delta d(\gamma(t))) \subset B(x,r)\}.$$

Clearly $0 \le t_0 \le l$.

Claim: There exist t_1 and t_2 with $0 \le t_1 \le t_0 \le t_2 \le l$ such that the balls

$$Q_1 = B(\gamma(t_1), \delta d(\gamma(t_1))) \quad \text{and} \quad Q_2 = B(\gamma(t_2), \delta d(\gamma(t_2)))$$

satisfy

$$Q_1 \subset B(x,r), \quad Q_2 \not\subset B(x,r), \quad x_{Q_2} = \gamma(t_2) \in Q_1.$$

We will prove the claim by considering 2 cases.

Case (i): $B(\gamma(t_0), \delta d(\gamma(t_0))) \subset B(x,r)$. In this case, $t_0 < l$ since we have assumed $B' \not\subset B(x,r)$. We then choose $t_1 = t_0$ and $t_2 = t_0 + \varepsilon < l$ for sufficiently small $\varepsilon > 0$ such that $\gamma(t_2) \in B(\gamma(t_0), \delta d(\gamma(t_0)))$, using the fact that $d(\gamma(t_2), \gamma(t_0)) \le |t_2 - t_0|$.

Case (ii): $B(\gamma(t_0), \delta d(\gamma(t_0))) \not\subset B(x,r)$. In this case, $t_0 > 0$ since $\gamma(0) = x$ and $\delta d(x) \le r$. We then let $t_2 = t_0$ and pick $t_1 < t_0$ such that $Q_1 \subset B(x,r)$ and $|t_1 - t_0| < \delta d(\gamma(t_0))/\lambda$. Clearly $\gamma(t_1) \in B(\gamma(t_0), \delta d(\gamma(t_0)))$, and hence by Observation 2.1(2b),

$$d(\gamma(t_1), \gamma(t_0)) \le |t_1 - t_0| < \delta d(\gamma(t_0))/\lambda \le \delta d(\gamma(t_1)).$$

Therefore $\gamma(t_2) = \gamma(t_0) \in Q_1$. This completes the proof of the claim.

With t_1 and t_2 as in the claim, set $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$. Let us show that there exists $c_1 > 0$ depending on κ, δ and M such that $d(x_1) > c_1 r$. To this end, pick $z \in Q_2$ with $z \notin B(x,r)$. Then

$$r \le d(z,x) \le \kappa(d(x_1,x) + \kappa(d(x_2,x_1) + d(x_2,z))) \le \kappa(t_1 + \kappa(\delta d(x_1) + \delta\lambda d(x_1)))$$

by Observation 2.1(2b). Also $d(x_1) = d(\gamma(t_1)) > t_1/M$, and it is now clear that $d(x_1) \geq C(M, \kappa, \delta)r$. Thus, after choosing $Q = Q_1$, we have $Q \subset B(x, r)$, $r(Q) = \delta d(x_1)$ and $r \leq c_2 r(Q)$.

Proof of (4): It is clear that if σ is doubling on Ω, then it is also δ-doubling on Ω. Next, suppose σ is δ-doubling on Ω with $\sigma(2^k B) \leq c^k \sigma(B)$ for all δ-balls B in Ω and all positive integers k. Let us show that $d(x') \sim \operatorname{diam}(\Omega)$ with constants of equivalence depending only on κ and M. Indeed, choose $x_0 \in \Omega$ such that $d(x_0, x') > C(\kappa) \operatorname{diam}(\Omega)$ and let $\gamma : [0, l] \to \Omega$ be a 1-John curve that connects x_0 to x', that is,

$$d(\gamma(s_1), \gamma(s_2)) \leq |s_1 - s_2| \quad \text{and} \quad d(\gamma(t)) > t/M \quad \text{if } t, s_1, s_2 \in [0, l].$$

Then $l \geq d(x_0, x')$ and $d(x') = d(\gamma(l)) > l/M \geq C(\kappa, M) \operatorname{diam}(\Omega)$, while the opposite inequality $d(x') \leq \operatorname{diam}(\Omega)$ is obvious.

Let $B(x, r)$ be a ball with $\delta d(x) < r \leq \operatorname{diam}(\Omega)$ and $x \in \Omega$. By part (3), we can find a δ-ball Q such that $Q \subset B(x, r)$ and $r \leq c_2 r(Q)$, with c_2 depending only on κ, δ and the 1-John constant M of Ω. Hence by Observation 2.1(2a), $B(x, r) \subset CQ$ with C depending on κ, δ and M, and by Observation 2.1(1), $2^k B(x, r) \subset C 2^k Q$. Consequently,

$$\sigma(2^k B(x, r)) \leq \sigma(C 2^k Q) \leq C(\kappa, \delta, M) c^k \sigma(Q) \leq C(\kappa, \delta, M) c^k \sigma(B(x, r)),$$

where the second inequality follows from the fact that Q is a δ-ball. This completes the proof of part (4). □

The next proposition guarantees the existence of a covering of a ϕ-John domain by balls with Whitney-like properties, as well as with extra properties that are useful for deriving weighted Poincaré estimates.

Proposition 2.3 [Chua and Wheeden 2008, Proposition 2.6]. *Let $\langle H, d\rangle$ be a quasi-metric space and $0 < \delta < 1/(2\kappa^2)$. Suppose $\Omega \subset H$, there is a δ-doubling measure μ on Ω with doubling constant D_μ, and $d(x) = d(x, \Omega^c) > 0$ for all $x \in \Omega$. Then there exists a covering $W = \{B_i\}$ of Ω by δ-balls B_i with the following properties*:

(a) *$r(B_i) \leq \delta d(x_{B_i}) \leq \lambda^2 r(B_i)$, where x_{B_i} is the center of B_i.*

(b) *For every $\tau \geq 1$ that satisfies $\tau\delta < 1/(2\kappa^2)$, there is a constant K depending only on τ, κ and D_μ such that the balls $\{\tau B_i : B_i \in W\}$ have bounded intercepts with bound K; in particular, the balls $\{\tau B_i : B_i \in W\}$ also have pointwise bounded overlaps with overlap constant K.*

(c) *Let $x' \in \Omega$ and ϕ be a strictly increasing function on $[0, \infty)$ that satisfies $\phi(0) = 0$ and $\phi(t) < t$ for all t. Then for each $x \in \Omega$ for which there is a curve $\gamma : [0, l] \to \Omega$ satisfying $\gamma(0) = x$ and $\gamma(l) = x'$ and the ϕ-John properties* (1-1) *and* (1-2), *there exists a finite chain of δ-balls $\{B_i\}_{i=0}^L \subset W$, depending* *on x and with $L = L_x$, such that $x \in B_0$, $x' \in B_L$, B_L is independent of x and satisfies $\lambda^{-2}B(x', \delta d(x')) \subset B_L \subset B(x', \delta d(x'))$, $B_i \cap B_{i+1}$ contains a δ-ball B_i' with $B_i \cup B_{i+1} \subset \lambda^4 B_i'$ for all i, and*

$$B_0 \subset \frac{\lambda^2 \phi^{-1}(2\kappa\lambda^2 r(B_i)/\delta)}{r(B_i)} B_i \quad \textit{for all } i. \tag{2-1}$$

Also, there is a finite chain of δ-Whitney balls $\{\mathcal{Q}_i\}_{i=0}^{L}$ depending on x with bounded intercepts and centers on γ such that

$$\mathcal{Q}_0 = B(x, \delta d(x)), \quad \mathcal{Q}_L = B(x', \delta d(x')), \quad \lambda^{-2}\mathcal{Q}_i \subset B_i \subset \mathcal{Q}_i,$$

and $\mathcal{Q}_i \cap \mathcal{Q}_{i+1}$ contains a δ-ball $\mathcal{Q}_i'$ with $\mathcal{Q}_i \cup \mathcal{Q}_{i+1} \subset \lambda^6 \mathcal{Q}_i'$. Note that the last ball $\mathcal{Q}_L$ in the chain does not depend on x.

(d) *Let x', ϕ, x and $\{\mathcal{Q}_i\}$ be as in (c). If $\mathcal{Q}_i \not\subset B(x, r)$, then $r(\mathcal{Q}_i) \geq \delta\phi(r/(2\kappa))$.*

(e) *Let x, γ and $\{\mathcal{Q}_i\}$ be as in* (c). *For all $\varepsilon > 0$, the number of* disjoint *$\mathcal{Q}_i$ having radius between ε and 2ε is at most $2\phi^{-1}(2\varepsilon/\delta)/\varepsilon$. In particular, if $\phi = \mathcal{J}_M$, the number of disjoint $\mathcal{Q}_i$ with radius between $\delta\varepsilon/(4\kappa^2 M)$ and $4\kappa\varepsilon$ is at most a constant depending only on δ, κ and M.*

Finally, the next result gives a simple extension of [Chua 1993, Theorem 1.5]:

Proposition 2.4 [Chua and Wheeden 2008, Theorem 2.9]. *Let Ω be a domain in a quasimetric space with quasimetric constant κ, and let $0 < \delta < 1/(2\kappa^2)$. Suppose Ω is covered by a countable collection W of δ-balls such that for some $N \geq 1$,*

(i) $\sum_{B \in W} \chi_B \leq N\chi_\Omega$, *and*

(ii) *there is a central ball $B_0 \in W$ that can be connected with every ball $B \in W$ by a finite chain of balls $B_0, B_1, \ldots, B_{k(B)} = B$ from W such that $B \subset NB_j$ for all j and each $B_j \cap B_{j+1}$ contains a ball B_j' with $B_j \cup B_{j+1} \subset NB_j'$.*

(*Domains satisfying* (i) *and* (ii) *are often called Boman chain domains.*)

Let f be a function on Ω and f_B be an associated constant for every $B \in W$. If w is a δ-doubling measure on Ω and $1 \leq q < \infty$, then

$$\|f - f_{B_0}\|^q_{L^q_w(\Omega)} \leq C \sum_{B \in W} \|f - f_B\|^q_{L^q_w(B)}, \tag{2-2}$$

where C depends only on κ, q, N and the doubling constant of w.

Remark 2.5. It is easy to see from parts (a)–(c) of Proposition 2.3 with $\phi = \mathcal{J}_M$ that 1-John domains satisfy the Boman chain condition. The converse is also true if the domain is assumed to satisfy a segmental geodesic condition; for this fact on metric spaces, see [Buckley et al. 1996].

The proof mentioned in [Chua and Wheeden 2008] only works if w is a doubling measure on Ω. It is true that δ-doubling measures are doubling on 1-John domains.

However, a Boman chain domain may not be a 1-John domain. Thus, in order to prove Proposition 2.4, one must modify [Chua and Wheeden 2008, Lemma 2.8] by assuming that w is δ-doubling and that the family of balls consists of δ-balls. The modified lemma can be proved by considering the Hardy–Littlewood maximal function with respect to δ-balls instead of all balls.

3. Proofs of the main theorems

Proof of Theorem 1.6. Let Ω be an s-John domain, let $M > 1$ and let Ω_0 be a nonempty subset of Ω^c that confines the M-bad points of Ω. Set $\rho(x) = d(x, \Omega_0)$ and $d\mu_a = \rho^a\, d\mu$. For any ball B, let

$$\rho(B) = \sup\{\rho(x) : x \in B\}, \quad \rho^*(B) = \rho(B) + r(B), \quad \mu_a^*(B) = \rho^*(B)^a \mu(B).$$

Note that $\mu_a(B) \le \mu_a^*(B)$ when $a \ge 0$.

Let us show that ρ is essentially constant on any δ-ball B for $\delta < 1/(2\kappa)^2$. In fact, if $x, y \in B$, then

$$\rho(y) = d(y, \Omega_0) \le \kappa(d(y, x) + d(x, \Omega_0)) \le \kappa(2\kappa r(B) + \rho(x)).$$

But

$$r(B) \le \delta d(x_B) \sim d(x) \le d(x, \Omega_0) = \rho(x),$$

and we get $\rho(y) \le C\rho(x)$ by combining inequalities. It's also true that $r(B) \le \rho(B)$ for any δ-ball B. Otherwise we would have $d(x_B, \Omega_0) < r(B)$, so there would exist $z \in \Omega_0$ with $d(x_B, z) < r(B)$, and then $z \in B \cap \Omega_0$, while B must lie in Ω since it is a δ-ball. Hence, if B is a δ-ball and $\delta < 1/(2\kappa)^2$, there is a positive constant $C(\kappa) \le 1$ such that

$$C(\kappa)\rho(B) \le \rho(x) \le \rho(B) \text{ for all } x \in B, \quad \text{and} \quad r(B) \le \rho(B). \tag{3-1}$$

Let us show that

$$C(\kappa)\frac{r(B)}{r(\tilde{B})} \le \frac{\rho^*(B)}{\rho^*(\tilde{B})} \le 1 \quad \text{for all concentric balls } B \subset \tilde{B}. \tag{3-2}$$

The second inequality holds since $\rho(B) \le \rho(\tilde{B})$ and $r(B) \le r(\tilde{B})$ for such B and $\tilde{B}$. Also then $\rho(\tilde{B}) \le \kappa(\rho(B) + 2\kappa r(\tilde{B}))$ and hence there is a constant c_1 depending on κ such that

$$\rho^*(\tilde{B}) \le c_1(\rho(B) + r(\tilde{B})).$$

We now consider two cases:

Case $\rho(B) \ge r(\tilde{B})$: Then

$$\frac{\rho^*(B)}{\rho^*(\tilde{B})} \ge \frac{\rho(B)}{c_1(\rho(B) + r(\tilde{B}))} \ge \frac{1}{2c_1} \ge \frac{r(B)}{2c_1 r(\tilde{B})}.$$

Case $\rho(B) < r(\tilde{B})$: Then

$$\frac{\rho^*(B)}{\rho^*(\tilde{B})} \geq \frac{r(B)}{c_1(\rho(B)+r(\tilde{B}))} \geq \frac{r(B)}{2c_1 r(\tilde{B})}.$$

It follows that (3-2) holds.

Let $a \geq 0$. By hypothesis, μ satisfies (1-5). We will now show that under the hypothesis of Theorem 1.6, conditions (1)–(4) in Theorem B hold with μ, μ^* there replaced by μ_a, μ_a^*, and with $B' = B(x', \delta d(x'))$. Recall that $B_j^x = B(x, r_j^x)$ for $x \in \Omega$ are balls as in (1-5) and satisfy $r_1^x = \operatorname{diam}(\Omega)$, $r_j^x/\alpha < r_{j+1}^x < r_j^x$ for some $\alpha \geq 2$, and $r_j^x \to 0$. We next define $\{Q_i^x\}_{i=1}^\infty$ for $x \in \Omega$ by letting $\{\mathcal{Q}_i\}_{i=0}^L$ be as in Proposition 2.3 and defining $\{Q_i^x\}_{i=1}^{L+1}$ by

$$Q_1^x = \mathcal{Q}_L = B', \quad Q_2^x = \mathcal{Q}_{L-1}, \quad \ldots, \quad Q_{L+1}^x = \mathcal{Q}_0 = B(x, \delta d(x)).$$

Note that there exists l such that $B_{l+1}^x \subset \tau\mathcal{Q}_0 \subset B_l^x$ and $B_{l+1}^x \neq \tau\mathcal{Q}_0$. We then define $Q_{L+i+1}^x = \tau^{-1} B_{l+i}^x$ for $i \geq 1$. Then since σ is δ-doubling by hypothesis, (1-7) follows from $r_j^x \sim r_{j+1}^x$ and from Proposition 2.3(c) since the balls $\mathcal{Q}_i'$ there are δ-balls. Also (1-8) holds with $a(Q) = a_*(Q)\|g\|_{L^p_w(\tau Q)}$ by (1-12), so condition (1) of Theorem B holds for $\{Q_i^x\}$ with this choice of $a(Q)$.

By (1-5) for μ and (3-2), μ_a^* satisfies the ratio estimate in (1-9) for $\{B_j^x\}$ with θ_1 be replaced by θ_1/α^a and θ_2 remaining the same. Moreover, since $B' = Q_1^x \subset \cup_i Q_i^x \subset B_1^x$, we have $\mu_a(B') \leq \mu_a(B_1^x) \leq \mu_a^*(B_1^x)$ and

$$\mu_a(\Omega) = \frac{\mu_a(\Omega)}{\mu_a^*(B_1^x)}\mu_a^*(B_1^x) \leq \frac{\mu_a(\Omega)}{\mu_a(B')}\mu_a^*(B_1^x).$$

Hence the first estimate in (1-9) holds for the pair μ_a, μ_a^* with $\wp = \mu_a(\Omega)/\mu_a(B')$, and we have verified condition (2) of Theorem B.

We will now verify condition (3). The partitioning properties follow easily and we only need to check (1-10) for (μ_a, μ_a^*). Let us show that (1-10) holds with $\theta = p/q$ for p and q as in Theorem 1.6. Let I be a collection of disjoint balls $\{B_j\}$ in $\{B_l^x : x \in \Omega, l \in \mathbb{N}\}$. Consider first those B_j that are $\tau\delta$-balls, so that $A(B_j) = a(Q_j)$ where $B_j = \tau Q_j$. Since $\rho(B_j) \geq r(B_j)$ by (3-1), we have

$$\begin{aligned} A(B_j)\mu_a^*(B_j)^{1/q} &= a(Q_j)\mu_a^*(B_j)^{1/q} = a(Q_j)\mu(B_j)^{1/q}\rho^*(B_j)^{a/q} \\ &\leq a_*(Q_j)\|g\|_{L^p_w(\tau Q_j)}\mu(B_j)^{1/q}(2\rho(B_j))^{a/q} \\ &\leq C a_*(Q_j)\|g\|_{L^p_{w_b}(\tau Q_j)}\mu(B_j)^{1/q}\rho(B_j)^{a/q-b/p} \end{aligned}$$

since $\rho(z) \sim \rho(B_j)$ for all $z \in B_j$ by (3-1). Here C and the constants of equivalence depend at most on p, q, a, b, κ and τ. For the rest of the proof, C and various constants of equivalence are positive and may depend on these parameters and many others, but not on the constant C_1 in (1-13) and (1-14). Continuing the estimate above, we obtain

$$\begin{aligned}
&\le CC_1 \|g\|_{L^p_{w_b}(\tau Q_j)} \rho(B_j)^{a/q-b/p} r(B_j)^{\beta'} && \text{by (1-13)}\\
&\le CC_1 \|g\|_{L^p_{w_b}(\tau Q_j)} \rho(B_j)^{a/q-b/p+\beta'} && \text{since } \beta' \ge 0\\
&\le CC_1\, \rho(\Omega)^{a/q-b/p+\beta'} \|g\|_{L^p_{w_b}(\tau Q_j)} && \text{since } \beta' + \frac{a}{q} - \frac{b}{p} \ge 0 \text{ by (1-15)}\\
&= CC_1 \rho(\Omega)^{a/q-b/p+\beta'} \|g\|_{L^p_{w_b}(B_j\cap\Omega)}
\end{aligned}$$

since $B_j = \tau Q_j \subset \Omega$ in the present case. Here $\rho(\Omega) = \sup_{z\in\Omega} \rho(z)$ as usual.

Next consider a typical B_j that is not a $\tau\delta$-ball: $B_j = B(x_j, r_j)$, $x_j = x_{B_j} \in \Omega$ and $r_j = r(B_j) > \tau\delta d(x_j)$. We will now use the notion of M-good and M-bad points to extend the notion of an s-John domain by allowing the function ϕ to vary with the starting point x of the curve γ, using ϕ_s and $\mathcal{J}_M$ in Definition 1.2 for M-bad points and M-good points respectively:

Convention 3.1. We adopt a convention for choosing curves that connect points x of the s-John domain Ω to the central point x': If $x \in \Omega_g^M$, we choose the curve γ from x to x' that corresponds to picking $\phi = \mathcal{J}_M$, while if $x \in \Omega_\flat^M$, we choose the curve corresponding to $\phi = \phi_s$.

We abuse earlier terminology by referring to these curves as curves from x to x' *guaranteed by the $\phi_{s,M}$-John condition*. Furthermore, given x and t, we denote by $\phi_{s,M}(t)$ either $\phi_s(t)$ or $\mathcal{J}_M(t)$, depending on whether $x \in \Omega_\flat^M$ or $x \in \Omega_g^M$.

Let γ be the $\phi_{s,M}$-John curve connecting x_j to x', with $\phi_{s,M}$ equal to either $\mathcal{J}_M$ or ϕ_s depending on whether x_j is an M-good or M-bad point. The balls Q_i in $\mathscr{C}(B_j)$ are δ-Whitney balls centered on γ, and they lie in B_j and (by (1-6) and Proposition 2.3(d)) satisfy $\tau r(Q_i) \ge \delta\phi_{s,M}(r_j/(2\alpha\kappa))$. Furthermore, the enlarged balls τQ_i lie in B_j by definition (see (1-6)), and they have bounded intercepts as we now show. In fact, Observation 2.1(2b) applied with δ replaced by $\tau\delta$ to the $\tau\delta$-balls τQ_i shows that if two such $\tau Q_i, \tau Q_k$ intersect, then $d(x_{Q_i}) \sim d(x_{Q_k})$, and therefore $r(Q_i) \sim r(Q_k)$. Then τQ_i have bounded intercepts (for example, this follows from [Chua and Wheeden 2008, Lemma 2.5] applied with $\mathscr{F} = \{Q_i\}$ and $N = \tau$).

Let $I_j = \{Q_i\}$ be a family of *disjoint* balls in $\mathscr{C}_{\phi_{s,M}}(B_j)$. Since balls in $\mathscr{C}_{\phi_{s,M}}(B_j)$ have bounded intercepts uniformly in j, it is a union of a bounded number of families of disjoint balls; see the proof of [Chua and Wheeden 2008, Lemma 2.5]. Because of disjointness, we know by Proposition 2.3(e) that the number of Q_i in I_j with radius between ε and 2ε is at most $(2/\varepsilon)\phi_{s,M}^{-1}(2\varepsilon/\delta)$. Our strategy for verifying (1-10) will be first to estimate the portion of $A(B_j)\mu_a^*(B_j)^{1/q}$ that corresponds to summing over I_j, and then to sum over different I_j and different B_j.

First suppose that x_j is an M-bad point of Ω and $r_j \le 1$. In case $s = 1$, we choose M (depending on c_s) so large that there are no M-bad points, and so we may now assume $s > 1$. Since x_j is M-bad, we have $r(Q_i) \ge Cr_j^s$ for all $Q_i \in I_j$. For the part of $A(B_j)\mu_a^*(B_j)^{1/q}$ that corresponds to summing over I_j, we have in case $p > 1$,

$$\begin{aligned}\sum_{Q_i\in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &= \sum_{Q_i\in I_j} a_*(Q_i)\|g\|_{L^p_w(\tau Q_i)}\mu_a^*(B_j)^{1/q}\\ &\le C\sum_{Q_i\in I_j} a_*(Q_i)\rho(Q_i)^{-b/p}\|g\|_{L^p_{w_b}(\tau Q_i)}\,\mu_a^*(B_j)^{1/q}\\ &\qquad\qquad \text{since } \rho(z_1)\sim\rho(z_2) \text{ for any two } z_1, z_2\in\tau Q_i \text{ by (3-1)}\\ &\le C\Big(\sum_{Q_i\in I_j}\|g\|^p_{L^p_{w_b}(\tau Q_i)}\Big)^{1/p}\Big(\sum_{Q_i\in I_j}\Big(\frac{a_*(Q_i)}{\rho(Q_i)^{b/p}}\Big)^{p'}\Big)^{1/p'}\mu_a^*(B_j)^{1/q}.\end{aligned}$$

We now show that $\mu_a^*(B_j)\le Cr_j^a\mu(B_j)$ since x_j is an M-bad point and Ω_0 confines the M-bad points. By definition of ρ^*, it suffices to show that $\rho(B_j)\le Cr_j$. We first estimate $\rho(B_j')$, where B_j' is the δ-Whitney ball concentric with B_j:

$$\begin{aligned}\rho(B_j') &= \sup_{B_j'}\rho \le C(\kappa)\inf_{B_j'}\rho && \text{by (3-1)}\\ &= C(\kappa)d(B_j',\Omega_0)\le C(\kappa)\overline{M}r(B_j') && \text{by (1-4)}\\ &= C(\kappa)\overline{M}\delta d(x_j)\le Cr_j\end{aligned}$$

by the assumptions about B_j presently in force. Thus

$$\rho(B_j)\le\kappa(\rho(B_j')+r_j)\le C\,r_j$$

as desired. It follows that the earlier expression is at most

$$\le C\|g\|_{L^p_{w_b}(\bigcup\tau Q_i)}\bigg(\sum_{l=0}^{L_j}\sum_{2^lr_0\le r(Q_i)<2^{l+1}r_0}\Big(\frac{a_*(Q_i)}{\rho(Q_i)^{b/p}}\mu(B_j)^{1/q}\Big)^{p'}\bigg)^{1/p'}r_j^{a/q},$$

where $r_0=\min\{r(Q_i):Q_i\in I_j\}\ge Cr_j^s$ and $2^{L_j}r_0\sim\max\{r(Q_i):Q_i\in I_j\}\le 2\kappa r_j$, and where we used the fact that the τQ_i have bounded overlaps.

Note that $\rho(Q_i)\sim r(Q_i)$ by applying (3-1) and the argument just used showing that $\rho(B_j')$ is less than a multiple of $r(B_j')$. Also, the number of terms in the inner sum above is at most $C(2^lr_0)^{-1+(1/s)}$. Therefore, by (1-14) and since $\bigcup\tau Q_i\subset B_j$, the last expression is bounded by

$$CC_1\|g\|_{L^p_{w_b}(B_j\cap\Omega)}\bigg(\sum_{l=0}^{L_j}(2^lr_0)^{(\beta-b/p-\eta'/p)p'-(s-1)/s}\bigg)^{1/p'}r_j^{(\eta+a)/q}.$$

Recall that

$$\chi = \frac{s(\beta p - b - \eta') - (s-1)(p-1)}{sp} = \beta - \frac{b}{p} - \frac{\eta'}{p} - \frac{s-1}{sp'}. \tag{3-3}$$

Therefore,

$$\sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} \le CC_1 \|g\|_{L^p_{w_b}(B_j\cap\Omega)} \left(\sum_{l=0}^{L_j} (2^l r_0)^{p'\chi}\right)^{1/p'} r_j^{(\eta+a)/q}.$$

In case $\chi \ge 0$, this is at most

$$\begin{aligned} CC_1\|g\|_{L^p_{w_b}(B_j\cap\Omega)}(1+L_j)^{1/p'} r_j^{\chi+(\eta+a)/q} \\ \le CC_1 r_j^{\chi+(\eta+a)/q}(1+|\log r_j|)^{1/p'}\|g\|_{L^p_{w_b}(B_j\cap\Omega)}, \end{aligned}$$

where the $(1+L_j)^{1/p'}$ term is present only if $\chi = 0$ and where we used that $2^{L_j} \le 2\kappa(r_j/r_0) \le Cr_j^{1-s}$; recall that $r_j \le 1$ in the present case. For any positive real numbers u, v, α with $u < v \le 1$, we have $u^\alpha(1+|\log u|) \le c_\alpha v^\alpha(1+|\log v|)$, and therefore, since by assumption $\chi \ge 0$ and $\eta + a \ge 0$ with $\chi > 0$ if $\eta + a = 0$, we obtain the estimate

$$\le CC_1 \operatorname{diam}(\Omega)^{\chi+(\eta+a)/q}(1+|\log \operatorname{diam}(\Omega)|)^{1/p'}\|g\|_{L^p_{w_b}(B_j\cap\Omega)}.$$

Again, the factor $(1+\log\operatorname{diam}(\Omega))^{1/p'}$ is needed only when $\chi = 0$.

In case $\chi < 0$,

$$\begin{aligned} \sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &\le CC_1\|g\|_{L^p_{w_b}(B_j\cap\Omega)} r_0^\chi r_j^{(\eta+a)/q} \\ &\le CC_1 r_j^{s\chi+(\eta+a)/q}\|g\|_{L^p_{w_b}(B_j\cap\Omega)} \\ &\le CC_1 \operatorname{diam}(\Omega)^{s\chi+(\eta+a)/q}\|g\|_{L^p_{w_b}(B_j\cap\Omega)} \end{aligned}$$

since $s\chi + (\eta + a)/q \ge 0$ by (1-16).

Similarly, when $p = 1$ (still assuming x_j is an M-bad point and $r_j \le 1$, and recalling that $r(Q_i) \ge Cr_j^s$ if $B_i \in I_j$),

$$\begin{aligned} \sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &\le C\sum \|g\|_{L^1_{w_b}(\tau Q_i)} \left(\sup_{Q_i\in I_j} \frac{a_*(Q_i)}{\rho(Q_i)^b}\right)\mu_a^*(B_j)^{1/q} \\ &\le C\|g\|_{L^1_{w_b}(B_j\cap\Omega)}\left(\sup_{Q_i\in I_j}\frac{a_*(Q_i)\mu(B_j)^{1/q}}{\rho(Q_i)^b}\right) r_j^{a/q} \\ &\le CC_1\|g\|_{L^1_{w_b}(B_j\cap\Omega)}\left(\sup_{Q_i\in I_j} r(Q_i)^{\beta-b-\eta'}\right) r_j^{(\eta+a)/q}. \end{aligned}$$

Since $p=1$, we have $\chi=\beta-b-\eta'$, and $\sup_{Q_i\in I_j} r(Q_i)^{\beta-b-\eta'}$ is at most a multiple of r_j^{χ} if $\chi\ge 0$ and a multiple of $r_j^{s\chi}$ if $\chi<0$. Thus the last expression is at most

$$CC_1\|g\|_{L^1_{w_b}(B_j\cap\Omega)}\begin{cases}\operatorname{diam}(\Omega)^{\chi+(\eta+a)/q} & \text{if } \chi\ge 0,\\ \operatorname{diam}(\Omega)^{s\chi+(\eta+a)/q} & \text{if } \chi<0,\end{cases}$$

which is equal to

$$CC_1\operatorname{diam}(\Omega)^{\varepsilon}\|g\|_{L^1_{w_b}(B_j\cap\Omega)}.$$

Our estimation of the portion of $A(B_j)\mu_a^*(B_j)^{1/q}$ that corresponds to summing over I_j is now complete in case x_j is an M-bad point and $r_j\le 1$.

Next we will estimate the same portion of $A(B_j)\mu_a^*(B_j)^{1/q}$ in the remaining cases that $x_j\in\Omega_g^M$, or both $x_j\in\Omega_\flat^M$ and $r_j>1$. The case $s=1$ is included in the first of these by choosing M to be the value in the definition of a 1-John domain. In either case, $\Phi_{s,M}(r_j/(2\alpha\kappa))\sim r_j$ (for the second case, recall that $\phi_s(t)=c_st$ when $t\ge 1$). Thus $r(Q_i)\sim r_j$ if $Q_i\in I_j$, and consequently while the argument will be similar to the one above, it will be simpler.

Let us show that $\rho^*(B_j)\sim\rho(Q_i)$ for such Q_i. Since $\rho^*(B_j)=\rho(B_j)+r_j$ and $\rho(B_j)\ge\rho(Q_i)\ge r(Q_i)\sim r_j$, then $\rho^*(B_j)\sim\rho(B_j)$, and it suffices to show that $\rho(B_j)\le C\rho(Q_i)$. But the quasitriangle inequality gives the desired

$$\rho(B_j)\le C(\kappa)(\rho(Q_i)+r_j)\sim\rho(Q_i).$$

Thus, in either of the remaining cases,

$$\begin{aligned}\sum_{Q_i\in I_j}a(Q_i)\mu_a^*(B_j)^{1/q}&=\sum_{Q_i\in I_j}\|g\|_{L^p_w(\tau Q_i)}a_*(Q_i)\mu_a^*(B_j)^{1/q}\\&\le C\sum_{Q_i\in I_j}\|g\|_{L^p_{w_b}(\tau Q_i)}a_*(Q_i)\mu(B_j)^{1/q}\rho(Q_i)^{a/q-b/p}\\&\le CC_1\sum_{Q_i\in I_j}r_j^{\beta'}\|g\|_{L^p_{w_b}(\tau Q_i)}\rho(Q_i)^{a/q-b/p}\quad\text{by (1-13)}\\&\le CC_1\sum_{Q_i\in I_j}\rho(Q_i)^{\beta'+a/q-b/p}\|g\|_{L^p_{w_b}(\tau Q_i)}\quad\text{since }\beta'\ge 0\\&\le CC_1\rho(\Omega)^{\beta'+a/q-b/p}\|g\|_{L^p_{w_b}(B_j\cap\Omega)}\quad\text{by (1-15).}\end{aligned}\tag{3-4}$$

We have now estimated $\sum_{Q_i\in I_j}a(Q_i)\mu_a^*(B_j)^{1/q}$ in all cases. The corresponding estimates of the full sum

$$A(B_j)\mu_a^*(B_j)^{1/q}=\sum_{Q_i\in\mathscr{C}(B_j)}a(Q_i)\mu_a^*(B_j)^{1/q},\quad\text{with }\mathscr{C}(B_j)=\mathscr{C}_{\phi_{s,M}}(B_j),$$

are comparable. To verify (1-10), it remains to raise these estimates to the power p and add them over those B_j in a disjoint collection $I=\{B_j\}$.

Thus, if $s = 1$,

$$\sum_I A(B_j)^p \mu_a^*(B_j)^{p/q} \le C(C_1)^p \rho(\Omega)^{p(\beta'+a/q-b/p)} \|g\|^p_{L^p_{w_b}(\Omega)} = C(C_1)^p \rho(\Omega)^{p\epsilon'} \|g\|^p_{L^p_{w_b}(\Omega)}$$

by (1-15) since the B_j are disjoint. Note that C is independent of M, $\overline{M}$, η, $\tilde{\eta}$, β. In any of the other cases,

$$\sum_I A(B_j)^p \mu_a^*(B_j)^{p/q} \le$$

$$C(C_1)^p \|g\|^p_{L^p_{w_b}(\Omega)} \begin{cases} \max\{\rho(\Omega)^{p\varepsilon'}, \operatorname{diam}(\Omega)^{p\varepsilon}\} & \text{if } \chi \ne 0, \\ \max\{\rho(\Omega)^{p\varepsilon'}, \operatorname{diam}(\Omega)^{p\varepsilon}(1+|\log \operatorname{diam}(\Omega)|)^{p-1}\} & \text{if } \chi = 0. \end{cases}$$

It now follows that (1-10) holds with $C_0^p \mu(\Omega)^{p/q}$ there taken to be the right sides of the estimates above. This verifies condition (3) of Theorem B.

Then (1-11) of Theorem B implies that (1-17) and (1-18) hold provided

$$\{B(x, r_j^x) : x \in \Omega,\ j \in \mathbb{N}\}$$

is a Vitali-type cover with respect to (μ_a, μ_a^*). However, this follows from the analogous assumption in Theorem 1.6 for μ_a and the fact that $\mu_a \le \mu_a^*$. The proof of part (i) of Theorem 1.6 is now complete.

We next prove part (ii), that is, the case $1 \le q \le p$. Recall that we have obtained the following estimates in the proof of part (i) for the balls

$$\{B_\alpha\} = \{B_j^x : x \in \Omega,\ j \in \mathbb{N}\}$$

as in (1-5):

First, if $B_\alpha = B(x, r)$ is a $\tau\delta$-ball, or if $x \in \Omega_g^M$, or if both $x \in \Omega_\flat^M$ and $r \ge 1$, then — see the reasoning before (3-4), note that $r(Q_i) \sim r(B_\alpha)$ and $\rho(Q_i) \sim \rho(B_\alpha)$ for any $Q_i \in \mathscr{C}(B_\alpha)$ now, and recall that $r(Q_i) \le \rho(Q_i)$ by (3-1) —

$$\begin{aligned} A(B_\alpha)\mu_a^*(B_\alpha)^{1/q} &\le CC_1 \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)} r(B_\alpha)^{\beta'} \rho(B_\alpha)^{a/q-b/p} \\ &\le CC_1 \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)} r(B_\alpha)^{\min\{\varepsilon',\beta'\}} \rho(\Omega)^{\max\{0,a/q-b/p\}}, \end{aligned}$$

where $\varepsilon' = \beta' + a/q - b/p$.

Second, suppose $B_\alpha = B(x, r)$, $x \in \Omega_\flat^M$, and $1 > r \ge \tau\delta d(x)$. If $\chi \ne 0$,

$$A(B_\alpha)\mu_a^*(B_\alpha)^{1/q} \le CC_1\, r(B_\alpha)^\varepsilon \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)},$$

but if $\chi = 0$,

$$A(B_\alpha)\mu_a^*(B_\alpha)^{1/q} \le CC_1 r(B_\alpha)^\varepsilon (1 + |\log r(B_\alpha)|)^{1/p'} \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)}.$$

Assuming (1-19), there exists $0 < \theta < 1$ such that $(p-q\theta)\tilde{\eta}/(pq\theta) < \varepsilon$ and $(p-q\theta)\tilde{\eta}'/(pq\theta) < \min\{\varepsilon', \beta'\}$. Part (ii) will then also follow from Theorem B. Indeed, for example, when $\chi \neq 0$, if I is a collection of pairwise disjoint balls with center in $\Omega^M_\flat$ and $\tau\delta d(x) \le r < 1$, then it follows from Hölder's inequality that

$$\sum_{B_\alpha\in I}(A(B_\alpha)\mu_a^*(B_\alpha)^{1/q})^{q\theta} \le CC_1^{q\theta}\Big(\sum_{B_\alpha\in I}\|g\|^p_{L^p_{w_b}(B_\alpha\cap\Omega)}\Big)^{q\theta/p}\Big(\sum_{B_\alpha\in I}(r(B_\alpha)^{\varepsilon q\theta})^{p/(p-q\theta)}\Big)^{(p-q\theta)/p}.$$

However, by (1-19) and the hypothesis of Theorem 1.6(ii),

$$\begin{aligned}\sum(r(B_\alpha)^{\varepsilon q\theta})^{p/(p-q\theta)} &= \sum_{k=-\infty}^{0}\ \sum_{\lambda^k<r(B_\alpha)\le\lambda^{k+1}} r(B_\alpha)^{\varepsilon q\theta p/(p-q\theta)}\\ &\le \sum_{k=-\infty}^{0} M_1\lambda^{-\tilde{\eta}k}\lambda^{(k+1)\varepsilon q\theta p/(p-q\theta)} \le C.\end{aligned}$$

When $s=1$, the constant C is independent of M, M_1, $\overline{M}$, β, η, $\tilde{\eta}$, η'. □

Remark 3.2. Checking through the proof of Theorem 1.6, we note that instead of requiring $\rho(x) = d(x, \Omega_0)$, it suffices to assume that ρ is any nonnegative function satisfying the following properties (with $\rho(B)$ defined to be $\sup_{x\in B}\rho(x)$):

(i) $\rho(x) \sim \rho(B)$ if $x \in B$ for any δ-ball B in Ω;

(ii) $r(B) \le C\rho(B)$ for any δ-ball B in Ω;

(iii) $\rho(\tilde{B}) \le C(\rho(B) + r(\tilde{B}))$ for all balls $B \subset \tilde{B}$ with both centers in Ω;

(iv) $\rho(Q) \sim r(Q)$ for all δ-Whitney balls Q along s-John curves from M-bad points.

In case Ω is a 1-John domain, (iv) is redundant as there are then no M-bad points. If $\rho(x) = d(x, \Omega_0)$ with $\Omega_0 \subset \Omega^c$, the first three properties of course hold, and (iv) will hold if Ω_0 confines all the M-bad points of Ω.

The same remark applies also to Theorem 1.10. Furthermore, in Theorem 1.8(i), only the first three properties are needed since $s = 1$, while in Theorem 1.8 part (ii), one can substitute (i)–(iii) above for the condition that $\rho(x) = d(x, \Omega_0)$ with $\Omega_0 \subset \Omega^c$, and substitute (iv) for the condition that Ω_0 confines all M-bad points. Finally, Theorem 1.12 remains valid for any nonnegative function ρ that satisfies all four properties (on Euclidean balls instead of quasimetric balls) instead of choosing $\rho(x) = d_E(x, \Omega_0)$ and assuming condition $(*)$ there. In fact, condition $(*)$ is used in Theorem 1.12 to ensure that Ω_0 confines all M-bad points.

Proof of Theorem 1.8. For part (i), let Ω be a 1-John domain, and fix τ and δ with $\tau \geq 1$ and $0 < \tau\delta < 1/(2\kappa^2)$. As noted in the remark following Proposition 2.4, Proposition 2.3 provides a collection $W = \{B\}$ of δ-balls for which the Boman chain conditions listed in the hypothesis of Proposition 2.4 hold, with B_0 in the proposition chosen to be the ball $\mathcal{Q}_L = B(x', \delta d(x'))$ of Proposition 2.3(c), which we denote by B'. Moreover, by Proposition 2.3(a), each ball $B \in W$ contains a concentric δ/λ^2-Whitney ball and lies inside a concentric δ-Whitney ball. By part (b) of the same proposition, the enlarged balls $\{\tau B\}_{B\in W}$ have bounded overlaps. Thus, assuming the hypothesis of Theorem 1.8(i) and applying Proposition 2.4, we have, with C depending on c_s, q, κ, A_σ, D_σ, a and δ,

$$\begin{aligned}\|f - f_{B'}\|^q_{L^q_{\sigma_a}(\Omega)} &\leq C \sum_{B\in W} \|f - f_B\|^q_{L^q_{\sigma_a}(B)} \\ &\leq C \sum_{B\in W} \rho(B)^a \|f - f_B\|^q_{L^q_\sigma(B)} && \text{by (3-1) and } a \geq 0 \\ &\leq C \sum_{B\in W} \rho(B)^a \sigma(B)\big(a_*(B)\|g\|_{L^p_w(\tau B)}\big)^q && \text{by (1-12) since now } p_0 = q.\end{aligned}$$

Recall that if $\lambda^{-2}\tilde{B} \subset B \subset \tilde{B}$, where $\tilde{B}$ is the δ-Whitney ball concentric with B, then by (1-25),

$$\sigma(B)a_*(B)^q \leq C_1^q r(B)^{\beta' q}.$$

Combining estimates, we obtain

$$\begin{aligned}\|f - f_{B'}\|^q_{L^q_{\sigma_a}(\Omega)} &\leq C(C_1)^q \sum_{B\in W} \rho(B)^a r(B)^{\beta' q} \|g\|^q_{L^p_w(\tau B)} \\ &\leq C(C_1)^q \sum_{B\in W} \rho(B)^{a+\beta' q} \rho(B)^{-bq/p} \|g\|^q_{L^p_{w_b}(\tau B)},\end{aligned}$$

with C depending also on b and τ, since $r(B) \leq \rho(B)$ by (3-1), $\beta' \geq 0$ and ρ is essentially constant on τB by (3-1) applied to $\tau\delta$-balls. Note that the condition $\beta' \geq 0$ need not hold if $\rho(B) \leq c r(B)$ for all δ/λ^2-Whitney balls, and then the constant C also depends on c. Finally, since $\varepsilon' \geq 0$, we obtain the bound

$$C(C_1)^q \rho(\Omega)^{a+\beta' q - bq/p} \sum_{B\in W} \|g\|^q_{L^p_{w_b}(\tau B)} \leq C(C_1)^q \rho(\Omega)^{a+\beta' q - bq/p} \|g\|^q_{L^p_{w_b}(\Omega)}$$

using the bounded overlap property of $\{\tau B\}_{B\in W}$ and the fact that $q \geq p$. Now Theorem 1.8(i) follows.

Next, let us prove part (ii). Thus suppose $s \geq 1$, $p_0 = p = q = 1$ and the hypotheses of Theorem 1.8(ii) hold. Let W be a covering of Ω that satisfies the properties in Proposition 2.3. Fix M and for each $x \in \Omega$, let

$$\mathscr{C}_x = \{R_0, R_1, \ldots, R_L\}, \quad \text{where } L = L_x,$$

be a chain of δ-balls as in the first part of property (c) in Proposition 2.3 with $\phi = \phi_{s,M}$; this chain is denoted there by $\{B_i\}_{i=0}^{L}$. The point x itself lies in the first ball R_0 in $\mathscr{C}_x$, and the last ball R_L satisfies $\lambda^{-2}B' \subset R_L \subset B'$ where $B' = B(x', \delta d(x'))$ is the "central" ball. Moreover, R_L is the same for all $x \in \Omega$, and we denote $R_L = B''$. As in the proof of [Chua and Wheeden 2008, Lemma 3.1],

$$\begin{aligned}
\|f_{R_0} - f_{B''}\|_{L^1_{\sigma_a}(R_0)} &\le \sum_{j=1}^{L} \|f_{R_j} - f_{R_{j-1}}\|_{L^1_{\sigma_a}(R_0)} \\
&= \sum_{j=1}^{L} \frac{\sigma_a(R_0)}{\sigma_a(R_j \cap R_{j-1})} \|f_{R_j} - f_{R_{j-1}}\|_{L^1_{\sigma_a}(R_j \cap R_{j-1})} \\
&\le \sum_{j=1}^{L} \frac{\sigma_a(R_0)}{\sigma_a(R_j \cap R_{j-1})} \big(\|f - f_{R_{j-1}}\|_{L^1_{\sigma_a}(R_j \cap R_{j-1})} + \|f - f_{R_j}\|_{L^1_{\sigma_a}(R_j \cap R_{j-1})} \big) \\
&\le C(a, A_\sigma, D_\sigma, \kappa) \sum_{j=0}^{L} \frac{\sigma_a(R_0)}{\sigma_a(R_j)} \|f - f_{R_j}\|_{L^1_{\sigma_a}(R_j)} \qquad \text{(3-5)}
\end{aligned}$$

since σ_a is δ-doubling.

Let $W_\flat = \{R : R \in \mathscr{C}_x, x \in \Omega_\flat^M\}$ and $W_g = \{R : R \in \mathscr{C}_x, x \in \Omega_g^M\}$. Also, let $W_{\flat 0}$ and W_{g0} be the subsets of $W_\flat$ and W_g consisting of those R_0 that are the first entry in $\mathscr{C}_x$ as x ranges over $\Omega_\flat^M$ or over Ω_g^M respectively. We will not distinguish between $W_{\flat 0}$ and the subset of Ω that is covered by the balls in $W_{\flat 0}$, and similarly for W_{g0}. Then $\Omega_\flat^M \subset W_{\flat 0}$, $\Omega_g^M \subset W_{g0}$, and $\Omega = W_{\flat 0} \cup W_{g0}$. Hence

$$\|f - f_{B''}\|_{L^1_{\sigma_a}(\Omega)} \le \|f - f_{B''}\|_{L^1_{\sigma_a}(W_{\flat 0})} + \|f - f_{B''}\|_{L^1_{\sigma_a}(W_{g0})}. \qquad \text{(3-6)}$$

For the first term on the right of (3-6), we have

$$\begin{aligned}
\|f - f_{B''}\|_{L^1_{\sigma_a}(W_{\flat 0})} &\le \sum_{R_0 \in W_{\flat 0}} \|f - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \\
&\le \sum_{R_0 \in W_{\flat 0}} \|f - f_{R_0}\|_{L^1_{\sigma_a}(R_0)} + \sum_{R_0 \in W_{\flat 0}} \|f_{R_0} - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

To estimate II, note by (3-5) that if $R_0 \in W_{\flat 0}$, then

$$\|f_{R_0} - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \le C(a, A_\sigma, D_\sigma, \kappa) \sum_{R \in W_\flat;\, R_0 \subset R^*} \frac{\sigma_a(R_0)}{\sigma_a(R)} \|f - f_R\|_{L^1_{\sigma_a}(R)},$$

where $R^* \subset C[\phi_s^{-1}(Cr(R))/r(R)]R \cap B(x_R, \operatorname{diam}(\Omega))$. In fact, by (2-1), R^* is chosen (depending at most on δ, κ, ϕ_s and M) so that for each ball R_j in (3-5), we have $R_0 \subset R_j^*$ assuming that $R_0 \in W_{\flat 0}$. Adding over R_0 gives

$$\begin{aligned} \mathrm{II} &\le C \sum_{R \in W_\flat} \Big(\sum_{R_0 \in W_{\flat 0};\, R_0 \subset R^*} \sigma_a(R_0) \Big) \frac{1}{\sigma_a(R)} \|f - f_R\|_{L^1_{\sigma_a}(R)} \\ &\le C \sum_{R \in W_\flat} \frac{\sigma_a(R^*)}{\sigma_a(R)} \|f - f_R\|_{L^1_{\sigma_a}(R)} \end{aligned}$$

since the balls in W have bounded overlaps. Clearly term I has the same bound, and consequently the first term on the right of (3-6) satisfies

$$\begin{aligned} \|f - f_{B''}\|_{L^1_{\sigma_a}(W_{\flat 0})} &\le C \sum_{R \in W_\flat} \frac{\sigma_a(R^*)}{\sigma_a(R)} \|f - f_R\|_{L^1_{\sigma_a}(R)} \\ &\le C \sum_{R \in W_\flat} \frac{\sigma_a(R^*) a_*(R)}{\rho(R)^b} \|g\|_{L^1_{w_b}(\tau R)} \qquad \text{by (1-12) and (3-1)} \\ &\le C \Big(\sup_{R \in W_\flat} \frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b} \Big) \sum_{R \in W} \|g\|_{L^1_{w_b}(\tau R)} \\ &\qquad\qquad\qquad\qquad \text{since } \sigma_a(R^*) \le \rho(R^*)^a \sigma(R^*) \\ &\le C \Big(\sup_{R \in W_\flat} \frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b} \Big) \|g\|_{L^1_{w_b}(\Omega)} \end{aligned} \tag{3-7}$$

since $\{\tau R : R \in W\}$ has bounded overlaps.

The same argument with R^* replaced by CR can be used to estimate the second term on the right of (3-6) since (2-1) guarantees that in (3-5) we have $R_0 \subset CR_j$ when $R_0 \in W_{g0}$. This gives

$$\|f - f_{B''}\|_{L^1_{\sigma_a}(W_{g0})} \le C \Big(\sup_{R \in W_g} \frac{\sigma(CR)\rho(CR)^a a_*(R)}{\rho(R)^b} \Big) \|g\|_{L^1_{w_b}(\Omega)}. \tag{3-8}$$

To estimate the supremum in (3-8), note that every $R \in W$ is a δ-ball and so satisfies $r(R) \le \rho(R)$ by (3-1). Also, by Proposition 2.3, $\lambda^{-2}Q \subset R \subset Q$ for the δ-Whitney ball Q concentric with R. Recall that we now assume a version of (1-14) for such balls with μ replaced by σ and $p = q = 1$. Also $\rho(CR) \le C\rho(R)$ from the definition of $\rho(R)$, and $\sigma(CR) \le C\sigma(R)$ since σ is δ-doubling. Thus

$$\frac{\sigma(CR)\rho(CR)^a a_*(R)}{\rho(R)^b} \le CC_1\, \rho(R)^{a-b} r(R)^{\beta+\eta-\eta'} \le CC_1\, \rho(R)^{a-b+\beta+\eta-\eta'}$$

since $\beta + \eta - \eta' \ge 0$ by hypothesis. Using $a - b + \beta + \eta - \eta' \ge 0$ due to (1-16) with $p = q = 1$ (see Remark 1.7(6)), we obtain

$$\sup_{R \in W_g} \frac{\sigma(CR)\rho(CR)^a a_*(R)}{\rho(R)^b} \le CC_1\, \rho(\Omega)^{a-b+\beta+\eta-\eta'}. \tag{3-9}$$

The same estimate holds for the part of the supremum in (3-7) that is extended over those $R \in W_\flat$ with $r(R) \geq 1$, since $R^* \subset CR \subset B(x_R, \operatorname{diam}(\Omega))$ for such R. To estimate the remaining part, namely the part corresponding to $r(R) \leq 1$, we first apply our version of (1-14) to the pair (R^*, R) and note that $r(R^*) = Cr(R)^{1/s}$ when $r(R) \leq 1$, obtaining

$$\sup_{R \in W_\flat;\, r(R) \leq 1} \frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b} \leq C \sup_{R \in W_\flat;\, r(R) \leq 1} \frac{\rho(R^*)^a}{\rho(R)^b}\, r(R)^{\beta - \eta' + \eta/s}. \tag{3-10}$$

To further estimate (3-10), let us show that $r(R) \sim \rho(R)$ for any $R \in W_\flat$. In fact, $\rho(R) \geq r(R)$ by (3-1). Also

$$\rho(R) = \sup_{z \in R} \rho(z) \leq C \inf_{z \in R} \rho(z) = Cd(R, \Omega_0) \quad \text{by (3-1)},$$

and it is enough to show that $d(R, \Omega_0) \leq Cr(R)$ if $R \in W_\flat$. This follows directly from (1-4) if $R \in W_\flat$ is centered on an s-John curve leading from an M-bad point, and it then follows for general $R \in W_\flat$ by using Proposition 2.3(c) to find a subball of R of comparable radius that is centered on such a curve. Then if $R \in W_\flat$ and $r(R) \leq 1$,

$$\rho(R^*) \leq \kappa\,(\rho(R) + r(R^*)) \leq C(\rho(R) + r(R)^{1/s}) \leq C\rho(R)^{1/s},$$

and consequently, by (3-10),

$$\begin{aligned} \sup_{R \in W_\flat; r(R) \leq 1} \frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b} &\leq C \sup_{R \in W} \rho(R)^{a/s - b + \beta - \eta' + \eta/s} \\ &\leq C\rho(\Omega)^{a/s - b + \beta - \eta' + \eta/s} \end{aligned} \tag{3-11}$$

since

$$a/s - b + \beta - \eta' + \eta/s \geq 0$$

by (1-16) (with $p = q = 1$). The estimate (3-11) holds even if Ω_0 does not confine the M-bad points provided we assume in addition that $\beta - \eta' + \eta/s \geq 0$; this follows by simply majorizing the factor $r(R)^{\beta - \eta' + \eta/s}$ in (3-10) by $\rho(R)^{\beta - \eta' + \eta/s}$ and using the inequality $r(R) \leq \rho(R)$ when estimating $\rho(R^*)$ above.

Combining (3-6)–(3-11) gives

$$\|f - f_{B''}\|_{L^1_{\sigma_a}(\Omega)} \leq CC_1 \max\big\{\rho(\Omega)^{a/s - b + \beta - \eta' + \eta/s}, \rho(\Omega)^{a - b + \beta + \eta - \eta'}\big\} \|g\|_{L^1_{w_b}(\Omega)}.$$

Finally, using a similar approach as in [Chua and Wheeden 2008, Lemma 3.1], we have (recall that $\lambda^{-2}B' \subset B'' \subset B'$)

$$\begin{aligned}
\|f_{B'} - f_{B''}\|_{L^1_{\sigma_a}(\Omega)} &= \sigma_a(\Omega)|f_{B'} - f_{B''}| \\
&\le \frac{\sigma_a(\Omega)}{\sigma(B'')}\Big(\int_{B''}(|f - f_{B'}| + |f - f_{B''}|)d\sigma\Big) \\
&\le \frac{\sigma_a(\Omega)}{\sigma(B'')}\big(\|f - f_{B'}\|_{L^1_\sigma(B')} + \|f - f_{B''}\|_{L^1_\sigma(B'')}\big) \\
&\le C\sigma_a(\Omega)(a_*(B') + a_*(B''))\|g\|_{L^1_w(\tau B')} \\
&\qquad\qquad \text{by (1-12) and the fact that } \sigma \text{ is } \delta\text{-doubling} \\
&\le C\sigma(\Omega)\rho(\Omega)^a \frac{1}{\rho(B')^b}\big(a_*(B') + a_*(B'')\big)\|g\|_{L^1_{w_b}(\tau B')} \\
&\le CC_1\rho(\Omega)^{a-b+\beta+\eta-\eta'}\|g\|_{L^1_{w_b}(\Omega)}
\end{aligned}$$

using (1-14) for B' and B'' (with μ replaced by σ and $p = q = 1$). This completes the proof of (1-27) by the triangle inequality. □

Proof of Theorem 1.10. For each $x \in \Omega$, choose $\{Q_i^x\}_{i=1}^\infty$ and $\{B_j^x\}_{j=1}^\infty$ as in the proof of Theorem 1.6. For any $\omega > 0$, set

$$b(Q, f^\omega) = a_*(Q)\|Yf^\omega\|_{L^p_w(\tau Q)}.$$

Note that (1-29) holds with this $b(\cdot\,,\cdot)$ by the hypothesis of Theorem 1.10. Also, by the proof of Theorem 1.6(i) (with g there replaced by $|Yf^\omega|$),

$$\sum_{B\in I} A(B, f^\omega)^p \mu_a^*(B)^{p/q} \le (C^*)^p \|Yf^\omega\|^p_{L^p_{w_b}(\Omega)} \tag{3-12}$$

for any collection I of disjoint balls B_j^x. Here C^* is the constant in either (1-17) or (1-18), respectively. This shows that (1-30) holds with (μ_a^*, μ_a) in place of (μ^*, μ), with $\theta = p/q$, and with $h(\Omega, f^\omega)$ defined by

$$h(\Omega, f^\omega) = C^*\|Yf^\omega\|_{L^p_{w_b}(\Omega)}\mu_a(\Omega)^{-1/q}.$$

Then (1-31) requires that

$$h^*(\Omega, f)^q = \sup_{\omega>0}\sum_{k=1}^\infty \|Yf^{2^k\omega}\|^q_{L^p_{w_b}(\Omega)} < \infty.$$

Theorem 1.9 now gives (noting that $\wp = \mu_a(\Omega)/\mu_a(B')$ as in the proof of Theorem 1.6)

$$\frac{1}{\mu_a(\Omega)}\|f - f_{B',\sigma}\|^q_{L^q_{\mu_a}(\Omega)} \le C\frac{\mu_a(\Omega)}{\mu_a(B')}\,C^{*q}h^*(\Omega, f)\frac{1}{\mu_a(\Omega)} + \Big(\frac{8}{\sigma(B')}\|f - f_{B',\sigma}\|_{L^1_\sigma(B')}\Big)^q,$$

which proves Theorem 1.10. □

Proof of Theorem 1.12. Suppose $\varepsilon > 0$, $M > 1$ and Ω_0 is a subset of Ω^c with $\partial\Omega \cap \big(\bigcup_{x\in\Omega_\flat^M} B(x, \varepsilon)\big) \subset \Omega_0$. We will show that Ω_0 confines the M'-bad points of Ω for suitable M'. Let us first show that if $d(x) \ge \varepsilon/3$ then x is an M'-good point for some $M' > 1$ depending on ε. Indeed, since Ω is an s-John domain, there is a curve $\gamma : [0, l] \to \Omega$ with $\gamma(0) = x$ and $\gamma(l) = x'$ such that

$$|\gamma(t_1) - \gamma(t_2)| \le |t_1 - t_2| \quad \text{and} \quad d(\gamma(t)) \ge c_s \min\{t^s, t\}.$$

If $t \le \varepsilon/(6\kappa)$, then

$$\tfrac{1}{3}\varepsilon \le d(x) \le \kappa(d(\gamma(0), \gamma(t)) + d(\gamma(t))) \le \kappa(t + d(\gamma(t))) \le \kappa(\varepsilon/(6\kappa) + d(\gamma(t))),$$

and consequently, $d(\gamma(t)) \ge \varepsilon/(6\kappa) \ge t$. On the other hand, if $\kappa t \ge \varepsilon/6$, then

$$d(\gamma(t)) > c_s t \min\{1, t^{s-1}\} \ge c_s \min\{1, (\varepsilon/(6\kappa))^{s-1}\}t.$$

Combining estimates shows that $x \in \Omega_g^{M'}$ for suitably large M' depending only on ε, κ, s and c_s. We may assume $M' \ge M$, so that $\Omega_\flat^{M'} \subset \Omega_\flat^M$.

We now show that there is a constant $C > 0$ (independent of x) such that if $x \in \Omega_\flat^{M'}$ and $\gamma : [0, l] \to \Omega$ is the s-John curve connecting x to x', then $d(\gamma(t)) \ge C d(\gamma(t), \Omega_0)$. We will use the fact that $\Omega_0 \supset \partial\Omega \cap B(x, \varepsilon)$. First, recall that we must have $d(x) < \varepsilon/3$ since $x \in \Omega_\flat^{M'}$. Let us consider two cases.

Case (i): $t < \varepsilon/3$. Then $|\gamma(t) - x| \le t < \varepsilon/3$ and hence

$$d(\gamma(t)) \le |\gamma(t) - x| + d(x) < 2\varepsilon/3.$$

Pick $z \in \partial\Omega$ such that $d(\gamma(t)) = |\gamma(t) - z|$. Then $z \in B(x, \varepsilon)$ since

$$|z - x| \le |z - \gamma(t)| + |\gamma(t) - x| < \varepsilon.$$

Thus $z \in \Omega_0$ and $d(\gamma(t)) \ge d(\gamma(t), \Omega_0)$.

Case (ii): $t \ge \varepsilon/3$. We combine the facts that $d(\gamma(t)) > c_s \min\{t^s, t\} \ge c_{s,\varepsilon} t$ and

$$d(\gamma(t), \Omega_0) \le |\gamma(t) - x| + d(x, \Omega_0) \le t + \varepsilon \le 4t.$$

It follows that Ω_0 confines the M'-bad points of Ω, as desired.

For all $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and all Euclidean balls $B \subset \Omega$, the L^1, L^1 version of Poincaré's inequality together with Hölder's inequality yield the L^1, L^p version

$$\frac{1}{|B|}\|f - f_B\|_{L^1(B)} \le C\frac{r(B)}{|B|^{1/p}}\|\nabla f\|_{L^p(B)}.$$

We will apply various earlier results with $\sigma = \mu = 1$, $Y = \nabla$, $\beta = 1$, $s \ge 1$, $a \ge 0$, $b \in \mathbb{R}$, $1/q \ge 1/p - 1/n$, $\tilde{\eta} = \tilde{\eta}' = \eta = \eta' = n$, $\beta' = 1 - n/p + n/q$, and $a_*(B) = Cr(B)^{1-n/p}$. Let us first consider the case $C(\Omega, f) = |B'|^{-1}\int_{B'} f\,dx$. In case $q > p$, we apply Theorem 1.10 to obtain (1-39); note that (1-37) now agrees with (1-16).

For the case $p = q > 1$, note that (1-38) implies (1-37) with strict inequality, that is,

$$\frac{s(n+b-1) - p + 1}{(n+a)p} < \frac{1}{p}.$$

It follows that there exists $q_0 > p$ such that

$$\frac{s(n+b-1) - p + 1}{(n+a)p} \le \frac{1}{q_0},$$

and we can then apply the result from the first part and then use Hölder's inequality to conclude this case. For the case $p = q \ge 1$ and $s = 1$, where we assume (1-37), that is, $b - a \le p$, just apply Theorem 1.8(i), noting that $\beta' + a/q - b/p \ge 0$ follows from (1-37).

For the case $p > q \ge 1$, we apply Theorem 1.6(ii). Conditions (1-38) now agree with (1-19) by arguments like those in Remark 1.7(6). Of course, we will only get a weak-type estimate instead of a strong-type one in this way. However, as the conditions (1-38) are strict inequality, the weak-type estimate will be valid for some $q_0 > q$. Then (1-39) follows from interpolation; see Remark 1.7(10). Finally, in case $q = p = 1$ and $s \ge 1$, recall that we assume $n + a \ge s(n+b-1)$. In fact, (1-39) with $q = p = 1$ and $C(\Omega, f) = |B'|^{-1}\int_{B'} f\,dx$ is true by Theorem 1.8(ii); now $\beta' = 1$ and $a_*(B) = Cr(B)^{1-n}$. Note that (1-16) follows from $n + a \ge s(n+b-1)$ since then $n + a \ge n + b - 1$ if $n + b - 1 \ge 0$, while $n + a \ge n + b - 1$ holds trivially if $n + b - 1 < 0$.

Now (1-39) is clear with $C(\Omega, f) = |B'|^{-1}\int_{B'} f$ in all cases. By the same argument used in [Chua and Wheeden 2008, Remark 1.3], we see that (1-39) also holds with $C(\Omega, f) = (|\mathcal{D}|_{\rho^a dx})^{-1}\int_{\mathcal{D}} f\rho^a dx$ for any $\mathcal{D} \subset \Omega$ such that $|\mathcal{D}| > 0$, provided the constant C in (1-39) also depends on $|\Omega|_{\rho^a dx}/|\mathcal{D}|_{\rho^a dx}$.

Finally, the last sentence in the statement of Theorem 1.12 follows directly from the result in the last sentence of Theorem 1.8(ii) applied with the standard Euclidean structure, that is, with $\beta = 1$, $\eta = \eta' = n$ and $w = \sigma = 1$, since then the requirement in that sentence that $\beta + \eta/s - \eta' \ge 0$ is guaranteed by assuming that $s \le n/(n-1)$. $\square$

Proof of Theorem 1.14.. We will prove the result by applying Theorem 1.10 with $a_*(B) = Cr(B)^{1-n/p}$. For any $(z, z') \in \mathfrak{D}$, we first connect (z, z') to the axis $z' = 0$ (say to $(z_1, 0)$) along the line through (z, z') that is orthogonal to the boundary, and then connect $(z_1, 0)$ to $(2, 0)$ by a segment of the axis. Clearly, there exists $1 \geq \tilde{c} = \tilde{c}(s) > 0$ small enough such that $z_1 < 2$ whenever $z < \tilde{c}$, and then the path from (z, z') to $(2, 0)$ lies in $\mathfrak{D}$ and is an s-John curve. Moreover, if $z \geq \tilde{c}$ then (z, z') can be connected to $(2, 0)$ by a 1-John curve with constant M depending on $\tilde{c}$. Hence the set of M-bad points $\mathfrak{D}_\flat^M \subset \{(z, z') \in \mathfrak{D} : z < \tilde{c}\}$, and it is clear that $\mathfrak{D}_0$ confines the M-bad points when $B((0, 0), \varepsilon) \cap \partial\mathfrak{D} \subset \mathfrak{D}_0$ for some $\varepsilon > 0$. Moreover, recall that $\partial\mathfrak{D}$ always confines the M-bad points.

Let $\delta = 1/4$. First note that the measure $\mu(E) = |E \cap \mathfrak{D}|_{\rho^a dx}$ is δ-doubling for any $a \geq 0$. Let us show that it is also doubling on $\mathfrak{D}$ when either $a = 0$ or when $\mathfrak{D}_0 = \partial\mathfrak{D}$ and $a \geq 0$. By Proposition 2.2(3) and the fact that μ is δ-doubling, we only need to show that there exists $c_1 \geq 1$ such that $\mu(B(x, 2r)) \leq c_1\mu(B(x, r))$ for all $d(x)/4 \leq r \leq 2$ and $x = (z, z')$ with $z < \tilde{c}$ as $\{(z, z') \in \mathfrak{D} : z \geq \tilde{c}\} \subset \mathfrak{D}_g^M$.

Consider first the case $x = (z_0, 0)$. Using the fact that $y = t^s$ is convex and $z_0 < 1$, we see by elementary calculus that $d(x)$ is at least the distance between $x = (z_0, 0)$ and the straight line passing through the point (z_0, z_0^s) with slope s; note it suffices to consider only $n = 2$ here. Hence $d(x) \geq z_0^s/\sqrt{1 + s^2} \geq z_0^s/(2s)$. Again by calculus, if $z_0^s/(2s) \leq r \leq 2$ then

$$r^2 \geq \left(\frac{z_0/(2s)^{1/s} + r/2}{2}\right)^{2s} + (r/2)^2$$

since the analogous inequality with $r^{1/s}$ in place of $z_0/(2s)^{1/s}$ is true if $0 \leq r \leq 2$. Hence when $z_0 < 1$ and $x = (z_0, 0)$, the cylinder

$$\left\{(z, z') \in \mathbb{R} \times \mathbb{R}^{n-1} : z_0 + r/4 \leq z \leq z_0 + r/2, |z'| < \left(\frac{z_0/(2s)^{1/s} + r/2}{2}\right)^s\right\}$$

lies inside $B(x, r) \cap \mathfrak{D}$ when $d(x) \leq r \leq 2$. It follows that if $x = (z_0, 0)$ and $d(x) \leq r \leq 2$, then

- when $a = 0$, $\mu(B(x, r)) \geq Cr(z_0 + r)^{(n-1)s}$;
- when $a \geq 0$ and $\mathfrak{D}_0 = \partial\mathfrak{D}$ (so $\rho((z, z')) = d_E((z, z'))$),

$$\mu(B(x, r)) \geq Cr(z_0 + r)^{(n-1+a)s}$$

  since $d_E((z, z')) \geq C(z_0 + r)^s$ on a proportional part of the cylinder.

It is easy to see that both of these remain true (for a larger constant C) even if $d(x)/4 \leq r \leq 4$ when $x = (z_0, 0)$, $z_0 < 1$.

Next note that if $a \geq 0$, $\mathfrak{D}_0 = \mathfrak{D}^c$ and $0 < R < 4$, then

$$\mu(B(x, R)) \leq R(z_0 + R)^{(n-1+a)s}. \tag{3-13}$$

Moreover, if $a=0$ but $\mathcal{D}_0$ may not be $\mathcal{D}^c$, then for $0<R<4$,

$$\mu(B(x,R)) \le CR(z_0+R)^{(n-1)s}.$$

Clearly (3-13) and the last estimate remain valid for $x=(z_0,0)\in\mathcal{D}$ without the restriction $z_0<1$. It is now easy to see that if either $\mathcal{D}_0=\mathcal{D}^c$ or $a=0$, then $\mu(B(x,2r))\le c_1\mu(B(x,r))$ for any $x=(z_0,0)$ with $z_0<\tilde{c}$ and $d(x)/4\le r\le 2$.

Finally, it remains to consider the case $x=(z_0,z')$, $z'\neq 0$ and $z_0<\tilde{c}$. Recall that there is $z_1>z_0$ such that the line connecting $x=(z_0,z')$ to $x_1=(z_1,0)$ is orthogonal to the boundary. If $r<2|(z_0,z')-(z_1,0)|$, it is easy to see that $B(x,r)$ contains a δ-ball Q with $r(Q)\ge c_2 r$, and hence $\mu(B(x,2r))\le c_1\mu(B(x,r))$ since μ is δ-doubling for $\delta=1/4$. On the other hand, when $r\ge 2|(z_0,z')-(z_1,0)|$, we have $B(x_1,r/2)\subset B(x,r)\subset B(x_1,2r)$, and consequently $\mu(B(x,2r))\le c_1\mu(B(x,r))$ with a larger c_1 if necessary. We conclude that μ is doubling on $\mathcal{D}$ if either $a=0$ or $\mathcal{D}_0=\mathcal{D}^c$ and $a\ge 0$. In particular, by Remark 1.7(1), (1-5) holds for μ in these cases.

We are now ready to show part (2) of Theorem 1.14. Let a,b,p,q satisfy (1-40) and (1-43). It will be convenient to rename a and b by $\tilde{a}$ and $\tilde{b}$ respectively. We will apply Theorem 1.10 with $a=b=0$ there to the measures

$$\mu(E)=|E\cap\mathcal{D}|_{\rho^{\tilde{a}}dx} \quad\text{and}\quad w(E)=|E\cap\mathcal{D}|_{\rho^{\tilde{b}}dx},$$

where $\rho(x)=d(x,\Omega_0)$. First, let B be any δ-ball in $\mathcal{D}$ and f be any locally Lipschitz function on $\mathcal{D}$. By the unweighted L^1,L^1 Poincaré inequality and the fact that $\rho(x)\sim\rho(B)$ on the δ-ball B, we have for $p\ge 1$ that

$$\frac{1}{\mu(B)}\|f-f_B\|_{L^1_\mu(B)} \le C\frac{r(B)}{w(B)^{1/p}}\|\nabla f\|_{L^p_w(B)}. \tag{3-14}$$

Thus, (1-12) holds with $f_B=|B|^{-1}\int_B f\,dx$, $\tau=1$, $\sigma=\mu$ and $p_0=1$.

To verify (1-14), suppose as in (1-14) that $B=B(x,r)$ and Q satisfy $x\in\mathcal{D}_\flat^M$, $d(x)/4\le r<2$ and $Q\in\mathscr{C}(B)$. We first consider the case $x=(z_0,0)$, with $z_0<\tilde{c}<1$. Observe that

$$\mu(B)^{1/q}\le Cr(B)^{1/q}r(Q)^{(n-1+\tilde{a})/q}$$

by (3-13) and the fact that $r(Q)\ge C(z_0+r(B))^s$ (instead of the usual $cr(B)^s$) since Q has center on the axis between $(z_0,0)$ and $(2,0)$. Now, since $r(Q)\sim d(Q,\partial\mathcal{D})$, we have $w(Q)\sim r(Q)^{n+\tilde{b}}$ and hence

$$\mu(B)^{1/q}\le Cr(B)^{1/q}w(Q)^{1/p}r(Q)^{(n-1+\tilde{a})/q-(n+\tilde{b})/p}. \tag{3-15}$$

Since μ is δ-doubling, we can as before extend (3-15) to include the case when $x = (z_0, z') \in \mathcal{D}_b^M$, with $z' \neq 0$. Thus

$$(\mu(B)/r(B))^{1/q} \le C(w(Q)/r(Q)^{n+\tilde{b}-p(n-1+\tilde{a})/q})^{1/p},$$

which verifies (1-14) with $\eta = 1$ and $\eta' = n + \tilde{b} - p(n-1+\tilde{a})/q$.

Now suppose B is a δ-ball. Then $\mu(B) \sim r(B)^n \rho(B)^{\tilde{a}}$ and $w(B) \sim r(B)^n \rho(B)^{\tilde{b}}$. Hence,

$$\mu(B)^{1/q} w(B)^{-1/p} \le C\rho(B)^{\tilde{a}/q-\tilde{b}/p} r(B)^{n/q-n/p} \le C_\Omega r(B)^{\beta'-1},$$

where

$$\beta' = 1 + \frac{n}{q} - \frac{n}{p} + \min\left\{0, \frac{\tilde{a}}{q} - \frac{\tilde{b}}{p}\right\} \quad \text{and} \quad C_\Omega = \begin{cases} \rho(\Omega)^{\tilde{a}/q-\tilde{b}/p} & \text{if } \tilde{a}/q - \tilde{b}/p > 0, \\ 1 & \text{if } \tilde{a}/q - \tilde{b}/p \le 0 \end{cases}$$

and in case $\tilde{a}/q - \tilde{b}/p > 0$ we have used $r(B) \le \rho(B) \le \rho(\Omega)$. Since

$$1 + n(1/q - 1/p) \ge 0 \quad \text{and} \quad 1 + \frac{n+\tilde{a}}{q} - \frac{n+\tilde{b}}{p} \ge 0$$

by (1-43), we have $\beta' \ge 0$.

We now check that by (1-43), with η and η' as above,

$$\frac{\eta + 0}{q} \ge \frac{s(\eta' + 0 - p) + (s-1)(p-1)}{p}.$$

Hence (1-16) holds with a, b, β there chosen as $a = b = 0$ and $\beta = 1$. Part (2) of Theorem 1.14 then follows from Theorem 1.10; see Remark 1.11.

To prove part (1), we will use $\mu(E) = w(E) = |E \cap \mathcal{D}|$. It is clear that (3-14) remains true for all δ-balls B. Next note that instead of (3-15), we have, for all balls B and Q such that $Q \in \mathcal{C}(B)$,

$$(\mu(B)/r(B))^{1/q} \le C(w(Q)/r(Q)^{n-p(n-1)/q})^{1/p}.$$

Part (1) now follows from Theorem 1.10 (see Remark 1.11) with $a = a$, $b = b$, $\beta = 1$, $\eta = 1$ and $\eta' = n - p(n-1)/q$. □

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SENG-KEE CHUA
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
10, LOWER KENT RIDGE ROAD
SINGAPORE 119076
SINGAPORE
matcsk@nus.edu.sg

RICHARD L. WHEEDEN
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
PISCATAWAY, NJ 08854
UNITED STATES
wheeden@math.rutgers.edu

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THE ORBIT STRUCTURE OF THE GELFAND–ZEITLIN GROUP ON $n \times n$ MATRICES

MARK COLARUSSO

Dedicated to Bertram Kostant on the occasion of his 80th birthday.

Recently, Kostant and Wallach constructed an action of a simply connected Lie group $A \cong \mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$ using a completely integrable system derived from the Poisson analogue of the Gelfand–Zeitlin subalgebra of the enveloping algebra. They show that A-orbits of dimension $n(n-1)/2$ form Lagrangian submanifolds of regular adjoint orbits in $\mathfrak{gl}(n)$ and describe the orbits of A on a certain Zariski open subset of regular semisimple elements. In this paper, we describe all A-orbits of dimension $n(n-1)/2$ and thus all polarizations of regular adjoint orbits obtained using Gelfand–Zeitlin theory.

1. Introduction

For $n \in \mathbb{N}$, let $\Delta^n_{i,j}$ be the set of ordered pairs of indices (i, j) such that $1 \le j \le i \le n$.

In [2006a; 2006b], Bertram Kostant and Nolan Wallach constructed an action of a complex, commutative, simply connected Lie group $A \cong \mathbb{C}^{n(n-1)/2}$ on the Lie algebra of $n \times n$ complex matrices $\mathfrak{gl}(n)$. The dimension of this group is exactly half the dimension of a regular adjoint orbit in $\mathfrak{gl}(n)$, and orbits of A of dimension $n(n-1)/2$ are Lagrangian submanifolds of regular adjoint orbits. We refer to the group A introduced by Kostant and Wallach as the Gelfand–Zeitlin group because of its connection with the Gelfand–Zeitlin algebra, as we will explain in Section 2.

The group A and its action are constructed as follows. Given $i < n$, we can think of $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ as a subalgebra by embedding an $i \times i$ matrix into the upper left corner of an $n \times n$ matrix. For $(i, j) \in \Delta^n_{i,j}$, let $f_{i,j}(x)$ be the polynomial on $\mathfrak{gl}(n)$ defined by $f_{i,j}(x) = \mathrm{tr}(x_i^j)$, where x_i denotes the $i \times i$ submatrix in the upper left corner of x. In [2006a], Konstant and Wallach showed that the functions $\{f_{i,j} \mid (i, j) \in \Delta^n_{i,j}\}$ are algebraically independent and Poisson commute with respect to the Lie–Poisson structure on $\mathfrak{gl}(n) \cong \mathfrak{gl}(n)^*$. The corresponding

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Hamiltonian vector fields $\xi_{f_{i,j}}$ generate a commutative Lie algebra $\mathfrak{a}$ of dimension $n(n-1)/2$. The group A is defined to be the simply connected, complex Lie group that corresponds to the Lie algebra $\mathfrak{a}$. The vector fields $\xi_{f_{i,j}}$ are complete [Kostant and Wallach 2006a, Theorem 3.5], and therefore $\mathfrak{a}$ integrates to a global action of $\mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$, thus defining the action of the group A on $\mathfrak{gl}(n)$.

Our goal in this paper is to describe all A-orbits of dimension $n(n-1)/2$. An element $x \in \mathfrak{gl}(n)$ is called strongly regular if and only if its A-orbit is of dimension $n(n-1)/2$. We denote the set of strongly regular elements by $\mathfrak{gl}(n)^{\text{sreg}}$. One way of studying such orbits is to study the action of A on fibers of the map $\Phi : \mathfrak{gl}(n) \to \mathbb{C}^{n(n+1)/2}$

$$\Phi(x) = (p_{1,1}(x_1), p_{2,1}(x_2), \dots, p_{n,n}(x)), \tag{1-1}$$

where $p_{i,j}(x_i)$ is the coefficient of t^{j-1} in the characteristic polynomial of x_i.

In [2006a, Theorem 2.3], Kostant and Wallach show that this map is surjective and that every fiber of this map $\Phi^{-1}(c) = \mathfrak{gl}(n)_c$ contains strongly regular elements. Following them, we denote the strongly regular elements in the fiber $\mathfrak{gl}(n)_c$ by $\mathfrak{gl}(n)_c^{\text{sreg}}$. By [2006a, Theorem 3.12], the A-orbits in $\mathfrak{gl}(n)^{\text{sreg}}$ are precisely the irreducible components of the fibers $\mathfrak{gl}(n)_c^{\text{sreg}}$. Thus, our study of the action of A on $\mathfrak{gl}(n)^{\text{sreg}}$ is reduced to studying the A-orbit structure of the fibers $\mathfrak{gl}(n)_c^{\text{sreg}}$. In [2006a], the authors also describe the A-orbit structure on a special class of fibers that consist of certain regular semisimple elements. In this paper, we describe the A-orbit structure of $\mathfrak{gl}(n)_c^{\text{sreg}}$ for any $c \in \mathbb{C}^{n(n+1)/2}$.

In Section 2, we describe the construction in [Kostant and Wallach 2006a] of the group A in more detail, and in Section 3, we describe their results about the A-orbits. We summarize these results briefly here. For any $x \in \mathfrak{gl}(i)$, let $\sigma(x)$ denote the spectrum of x. Kostant and Wallach describe the action of the group A on a Zariski open subset of regular semisimple elements defined by

$$\mathfrak{gl}(n)_\Omega = \{x \in \mathfrak{gl}(n) \mid x_i \text{ is regular semisimple, } \sigma(x_{i-1}) \cap \sigma(x_i) = \varnothing,\ 2 \le i \le n\}.$$

Let $c_i \in \mathbb{C}^i$ and consider $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2}$. Regard $c_i = (z_1, \dots, z_i)$ as the coefficients of the degree i monic polynomial

$$p_{c_i}(t) = z_1 + z_2 t + \cdots + z_i t^{i-1} + t^i. \tag{1-2}$$

Let Ω_n denote the Zariski open subset of $\mathbb{C}^{n(n+1)/2}$ given by the tuples c such that $p_{c_i}(t)$ has distinct roots and $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common. Clearly, $\mathfrak{gl}(n)_\Omega = \bigcup_{c \in \Omega_n} \mathfrak{gl}(n)_c$. The action of A on $\mathfrak{gl}(n)_\Omega$ is described as follows.

Theorem 1.1 [Kostant and Wallach 2006a, Theorems 3.23 and 3.28]. *The elements of $\mathfrak{gl}(n)_\Omega$ are strongly regular. If $c \in \Omega_n$, then $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{\text{sreg}}$ is precisely one orbit under the action of the group A. Moreover, $\mathfrak{gl}(n)_c$ is a homogeneous space for a free, algebraic action of the torus $(\mathbb{C}^\times)^{n(n-1)/2}$.*

In Section 4, we give a construction that describes an A-orbit in an arbitrary fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$ as the image of a certain morphism of a commutative, connected algebraic group into $\mathfrak{gl}(n)_c^{\text{sreg}}$. The construction gives a bijection between A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$ and orbits of a product of connected, commutative algebraic groups acting freely on a concrete, well-understood variety, but it does not enumerate the A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$. In Section 5, we use the construction and combinatorial data of the fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$ to give explicit descriptions of the A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$. The main result is Theorem 5.11, which contrasts substantially with the generic case described in Theorem 1.1.

Theorem 1.2. *Let $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2}$ be such that there are $0 \le j_i \le i$ roots in common between the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Then the number of A-orbits in $\mathfrak{gl}(n)_c^{\text{sreg}}$ is exactly 2^j, where $j = \sum_{i=1}^{n-1} j_i$. For $x \in \mathfrak{gl}(n)_c^{\text{sreg}}$, let Z_i denote the centralizer of the Jordan form of x_i in $\mathfrak{gl}(i)$. The orbits of A on $\mathfrak{gl}(n)_c^{\text{sreg}}$ are the orbits of a free algebraic action of the complex, commutative, connected algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{\text{sreg}}$.*

Remark 1.3. After the results of this paper were established, a very interesting paper by Roger Bielawski and Victor Pidstrygach [2008] appeared proving similar results. They show there are 2^j orbits, where $j = \sum_{i=1}^{n-1} j_i$, using symplectic reduction and rational maps of fixed degree from the Riemann sphere into the flag manifold for $\mathrm{GL}(n+1)$. Their arguments are independent and completely different from ours. Our work is more precise in that we provide explicit representatives for the A-orbits, and show that the A-orbits have a simply transitive algebraic action of $Z_1 \times \cdots \times Z_{n-1}$. These ideas were useful in the writing of [Colarusso and Evens 2010].

The nilfiber $\mathfrak{gl}(n)_0 = \Phi^{-1}(0)$ contains some of the most interesting structure in regard to the action of A. The fiber $\mathfrak{gl}(n)_0$ has been studied extensively by Lie theorists and numerical linear algebraists. Parlett and Strang [2008] studied matrices in $\mathfrak{gl}(n)_0$ and obtained interesting results. Ovsienko [2003] also studied $\mathfrak{gl}(n)_0$ and showed that it is a complete intersection. It turns out that the A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ correspond to 2^{n-1} Borel subalgebras of $\mathfrak{gl}(n)$. The main results are contained in Theorems 5.2 and 5.5. We combine them into a single statement here.

Theorem 1.4. *The nilfiber $\mathfrak{gl}(n)_0^{\text{sreg}}$ contains 2^{n-1} A-orbits. For $x \in \mathfrak{gl}(n)_0^{\text{sreg}}$, let $\overline{A \cdot x}$ denote the Zariski closure of $A \cdot x$ (which is the same as its Hausdorff closure). Then $\overline{A \cdot x}$ is a nilradical of a Borel subalgebra in $\mathfrak{gl}(n)$ that contains the standard Cartan subalgebra of diagonal matrices.*

The nilradicals obtained as closures of A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ are described explicitly in Theorem 5.5. In Theorem 5.7, we also describe the permutations that conjugate the strictly lower triangular matrices into each of these 2^{n-1} nilradicals.

Theorem 1.2 lets us identify exactly where the action of the group A is transitive on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. (See Corollary 5.13 and Remark 5.14.)

Corollary 1.5. *The action of A is transitive on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ if and only if $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ are relatively prime for each i in $1 \leq i \leq n-1$. Moreover, for such $c \in \mathbb{C}^{n(n+1)/2}$, we have $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{\mathrm{sreg}}$.*

This corollary allows us to identify the maximal subset of $\mathfrak{gl}(n)$ on which the action of A is transitive on the fibers of the map Φ in (1-1) over this set. The set $\mathfrak{gl}(n)_\Omega$ is a proper open subset of this maximal set. This is discussed in detail in Section 5c.

2. The group A

We briefly discuss the construction of an analytic action of a group $A \cong \mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$ that appears in [Kostant and Wallach 2006a]; see also [Colarusso 2009].

We regard $\mathfrak{gl}(n)^*$ as a Poisson manifold with the Lie–Poisson structure; see [Vaisman 1994; Chriss and Ginzburg 1997]. The Lie–Poisson structure is the unique Poisson structure on the symmetric algebra $S(\mathfrak{gl}(n)) = \mathbb{C}[\mathfrak{gl}(n)^*]$ such that for $x, y \in S^1(\mathfrak{gl}(n))$, the Poisson bracket $\{x, y\}$ is the Lie bracket $[x, y]$. We use the trace form to transfer the Poisson structure from $\mathfrak{gl}(n)^*$ to $\mathfrak{gl}(n)$. For $i \leq n$, we can view $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ as a subalgebra simply by embedding an $i \times i$ matrix in the upper left corner of an $n \times n$ matrix, that is, via

$$Y \hookrightarrow \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}. \tag{2-1}$$

We have a corresponding embedding of the adjoint groups $\mathrm{GL}(i) \hookrightarrow \mathrm{GL}(n)$ via

$$g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & \mathrm{Id}_{n-i} \end{bmatrix}.$$

In this paper, we always think of $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ and $\mathrm{GL}(i) \hookrightarrow \mathrm{GL}(n)$ via these embeddings, unless otherwise stated.

We can use the embedding (2-1) to realize $\mathfrak{gl}(i)$ as a summand of $\mathfrak{gl}(n)$. Indeed, we have

$$\mathfrak{gl}(n) = \mathfrak{gl}(i) \oplus \mathfrak{gl}(i)^{\perp}, \tag{2-2}$$

where $\mathfrak{gl}(i)^{\perp}$ denotes the orthogonal complement of $\mathfrak{gl}(i)$ in $\mathfrak{gl}(n)$ with respect to the trace form. It is convenient for us to have a coordinate description of this decomposition.

Definition 2.1. For $x \in \mathfrak{gl}(n)$, we let $x_i \in \mathfrak{gl}(i)$ be the upper left corner of x, that is, $(x_i)_{k,l} = x_{k,l}$ for $1 \leq k, l \leq i$. We refer to x_i as the $i \times i$ cutoff of x.

The decomposition of $y \in \mathfrak{gl}(n)$ in (2-2) is written $y = y_i \oplus y_i^\perp$, where $y_i^\perp$ denotes the entries $y_{k,l}$ where k and l are not both in the set $\{1, \dots, i\}$. Using this decomposition, we can think of the polynomials on $\mathfrak{gl}(i)$, which we denote by $P(\mathfrak{gl}(i))$, as a Poisson subalgebra of $P(\mathfrak{gl}(n))$, the polynomials on $\mathfrak{gl}(n)$. Explicitly, if $f \in P(\mathfrak{gl}(i))$, then (2-2) gives $f(x) = f(x_i)$ for $x \in \mathfrak{gl}(n)$. The Poisson structure on $P(\mathfrak{gl}(i))$ inherited from $P(\mathfrak{gl}(n))$ agrees with the Lie–Poisson structure on $P(\mathfrak{gl}(i))$; see [Kostant and Wallach 2006a, page 330].

Since $\mathfrak{gl}(n)$ is a Poisson manifold, we have the notion of a Hamiltonian vector field ξ_f for any holomorphic function $f \in \mathbb{O}(\mathfrak{gl}(n))$. If $g \in \mathbb{O}(\mathfrak{gl}(n))$, then $\xi_f(g) = \{f, g\}$. The group A is defined as the simply connected, complex Lie group that corresponds to a certain Lie algebra of Hamiltonian vector fields on $\mathfrak{gl}(n)$. To define this Lie algebra of vector fields, we consider the subalgebra of $P(\mathfrak{gl}(n))$ generated by the adjoint invariant polynomials for each of the n subalgebras $\mathfrak{gl}(i)$. We denote this subalgebra by $J(\mathfrak{gl}(n))$. We will soon see that

$$J(\mathfrak{gl}(n)) \cong P(\mathfrak{gl}(1))^{\mathrm{GL}(1)} \otimes \cdots \otimes P(\mathfrak{gl}(n))^{\mathrm{GL}(n)}. \tag{2-3}$$

This algebra may be viewed as a classical analogue of the Gelfand–Zeitlin subalgebra of the universal enveloping algebra $U(\mathfrak{gl}(n))$; see [Drozd et al. 1994]. Since $P(\mathfrak{gl}(i))^{\mathrm{GL}(i)}$ is the Poisson center of $P(\mathfrak{gl}(i))$, it is easy to see that $J(\mathfrak{gl}(n))$ is Poisson commutative; see [Kostant and Wallach 2006a, Proposition 2.1]. Let $f_{i,1}, \dots, f_{i,i}$ generate the ring $P(\mathfrak{gl}(i))^{\mathrm{GL}(i)}$. Then the set $f_{i,i} \mid 1 \le i \le n\}$ generates $J(\mathfrak{gl}(n))$. Note that the sum $\sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1) = n(n-1)/2$ is half the dimension of a regular adjoint orbit in $\mathfrak{gl}(n)$. We will see shortly that the functions $\{f_{i,1}, \dots, f_{i,i} \mid 1 \le i \le n-1\}$ form a completely integrable system on a regular adjoint orbit.

The surprising fact about this integrable system proved by Kostant and Wallach is that the corresponding Hamiltonian vector fields $\xi_{f_{i,j}}$ for $(i, j) \in \Delta_{i,j}^{n-1}$ are complete; [Kostant and Wallach 2006a, Theorem 3.5]. Let $f_{i,j} = \mathrm{tr}(x_i^j)$ and $\mathfrak{a} = \mathrm{span}\{\xi_{f_{i,j}} \mid (i, j) \in \Delta_{i,j}^{n-1}\}$. We define A as the simply connected, complex Lie group corresponding to the Lie algebra $\mathfrak{a}$. Since the vector fields $\xi_{f_{i,j}}$ commute for all i and j, the corresponding (global) flows define a global action of $\mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n)$. The group $A \cong \mathbb{C}^{n(n-1)/2}$, and it acts on $\mathfrak{gl}(n)$ by composing these flows in any order. The action of A also preserves adjoint orbits [Kostant and Wallach 2006a, Theorems 3.3 and 3.4].

The action of $A \cong \mathbb{C}^{n(n-1)/2}$ may seem at first glance to be noncanonical. However, one can show that the orbit structure of $\mathbb{C}^{n(n-1)/2}$ given by integrating the complete vector fields $\xi_{f_{i,j}}$ is independent of the choice of generators $f_{i,j}$ for $P(\mathfrak{gl}(i))^{\mathrm{GL}(i)}$. See [Kostant and Wallach 2006a, Theorem 3.5]. Since we are interested in studying the geometry of these orbits, we lose no information by fixing a choice of generators.

Remark 2.2. Using the Gelfand–Zeitlin algebra for complex orthogonal Lie algebras $\mathfrak{so}(n)$, we can define an analogous group $\mathbb{C}^d$, where d is half the dimension of a regular adjoint orbit in $\mathfrak{so}(n)$. The construction of the group and the study of its orbit structure on certain regular semisimple elements of $\mathfrak{so}(n)$ are discussed in detail in [Colarusso 2009].

For our choice of generators, we can write down the Hamiltonian vector fields $\xi_{f_{i,j}}$ in coordinates and their corresponding global flows. We use the following notation. Given $x, z \in \mathfrak{gl}(n)$, we denote the directional derivative in the direction of z evaluated at x by ∂_x^z. Its action on function on a holomorphic function f is

$$\partial_x^z f = \frac{d}{dt}\Big|_{t=0} f(x+tz). \tag{2-4}$$

By [Kostant and Wallach 2006a, Theorem 2.12],

$$(\xi_{f_{i,j}})_x = \partial_x^{[-jx_i^{j-1},x]}. \tag{2-5}$$

We see that $\xi_{f_{i,j}}$ integrates to an action of $\mathbb{C}$ on $\mathfrak{gl}(n)$ given by

$$\mathrm{Ad}(\exp(tjx_i^{j-1}))\cdot x \quad \text{for } t \in \mathbb{C}, \tag{2-6}$$

where $x_i^0 = \mathrm{Id}_i \in \mathfrak{gl}(i)$.

Remark 2.3. The orbits of A are the composition of the (commuting) flows in (2-6) for $(i, j) \in \Delta_{i,j}^{n-1}$, in any order acting on $x \in \mathfrak{gl}(n)$. Clearly, the action of A stabilizes adjoint orbits.

Equation (2-5) gives us a convenient description of the tangent space to the action of A on $\mathfrak{gl}(n)$. We first need some notation. For $x \in \mathfrak{gl}(n)$, let Z_{x_i} be the associative subalgebra of $\mathfrak{gl}(i)$ generated by the elements $\mathrm{Id}_i, x_i, x_i^2, \dots, x_i^{i-1}$. We then let $Z_x = \sum_{i=1}^n Z_{x_i}$. Let $x \in \mathfrak{gl}(n)$ and let $A\cdot x$ denote its A-orbit. Then (2-5) gives us

$$T_x(A\cdot x) = \mathrm{span}\{(\xi_{f_{i,j}})_x \mid (i, j) \in \Delta_{i,j}^{n-1}\} = \mathrm{span}\{\partial_x^{[z,x]} \mid z \in Z_x\}.$$

Following the notation in [Kostant and Wallach 2006a], we let

$$V_x := \mathrm{span}\{\partial_x^{[z,x]} \mid z \in Z_x\} = T_x(A\cdot x) \subset T_x(\mathfrak{gl}(n)). \tag{2-7}$$

Our work focuses on orbits of A of maximal dimension $n(n-1)/2$, since such orbits form Lagrangian submanifolds of regular adjoint orbits. (If such orbits exist, they are the leaves of maximal dimension of the Gelfand–Zeitlin integrable system.) Accordingly, we make the following theorem/definition. See [Kostant and Wallach 2006a, Theorem 2.7 and Remark 2.8].

Theorem/definition 2.4. *An element $x \in \mathfrak{gl}(n)$ is called strongly regular if and only if the differentials $\{(df_{i,j})_x \mid (i,j) \in \Delta^n_{i,j}\}$ are linearly independent at x. Equivalently, x is strongly regular if the A-orbit $A \cdot x$ of x has $\dim(A \cdot x) = n(n-1)/2$. We denote the set of strongly regular elements of $\mathfrak{gl}(n)$ by $\mathfrak{gl}(n)^{\mathrm{sreg}}$.*

Our goal is to determine the A-orbit structure of $\mathfrak{gl}(n)^{\mathrm{sreg}}$. In [Kostant and Wallach 2006a], Kostant and Wallach produce strongly regular elements using the map

$$\Phi : \mathfrak{gl}(n) \to \mathbb{C}^{n(n+1)/2}, \quad x \mapsto (p_{1,1}(x_1), p_{2,1}(x_2), \ldots, p_{n,n}(x)), \tag{2-8}$$

where $p_{i,j}(x_i)$ is the coefficient of t^{j-1} in the characteristic polynomial of x_i.

Theorem 2.5 [Kostant and Wallach 2006a, Theorem 2.3]. *Let $\mathfrak{b} \subset \mathfrak{gl}(n)$ denote the standard Borel subalgebra of upper triangular matrices in $\mathfrak{gl}(n)$. Let f be the sum of the negative simple root vectors. Then the restriction of Φ to the affine variety $f + \mathfrak{b}$ is an algebraic isomorphism.*

We will refer to the elements of $f + \mathfrak{b}$ as Hessenberg matrices. They are matrices of the form

$$f + \mathfrak{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ 0 & 1 & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{nn} \end{bmatrix}.$$

Theorem 2.5 implies that if $x \in f + \mathfrak{b}$, the differentials $(dp_{i,j})_x$ for $(i,j) \in \Delta^n_{i,j}$ are linearly independent. For the same range of i and j, the sets of functions $\{f_{i,j}\}$ and $\{p_{i,j}\}$ both generate the classical analogue of the Gelfand–Zeitlin algebra $J(\mathfrak{gl}(n))$. It follows that

$$\operatorname{span}\{(df_{i,j})_x \mid (i,j) \in \Delta^n_{i,j}\} = \operatorname{span}\{(dp_{i,j})_x \mid (i,j) \in \Delta^n_{i,j}\} \quad \text{for any } x \in \mathfrak{gl}(n)$$

by the Leibniz rule. Theorem 2.5 then implies $f + \mathfrak{b} \subset \mathfrak{gl}(n)^{\mathrm{sreg}}$ and therefore $\mathfrak{gl}(n)^{\mathrm{sreg}}$ is a nonempty Zariski open subset of $\mathfrak{gl}(n)$. Thus the functions $\{f_{i,j} \mid (i,j) \in \Delta^n_{i,j}\}$ are algebraically independent, and we obtain (2-3).

For $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C} \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2}$, we denote the fiber $\Phi^{-1}(c)$ by $\mathfrak{gl}(n)_c$, with Φ as in (2-8). For $c_i \in \mathbb{C}^i$, we define a monic polynomial $p_{c_i}(t)$ with coefficients given by c_i as in (1-2). Then $x \in \mathfrak{gl}(n)_c$ if and only if x_i has characteristic polynomial $p_{c_i}(t)$ for all i. Then for any $c \in \mathbb{C}^{n(n+1)/2}$, Theorem 2.5 says that $\mathfrak{gl}(n)_c$ is nonempty and contains a unique Hessenberg matrix. We denote the strongly regular elements of the fiber $\mathfrak{gl}(n)_c$ by $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$, that is,

$$\mathfrak{gl}(n)^{\mathrm{sreg}}_c = \mathfrak{gl}(n)_c \cap \mathfrak{gl}(n)^{\mathrm{sreg}}.$$

Since Hessenberg matrices are strongly regular, $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ is a nonempty Zariski open subset of $\mathfrak{gl}(n)_c$ for any $c \in \mathbb{C}^{n(n+1)/2}$.

Theorem 2.5 implies that every regular adjoint orbit contains strongly regular elements. This follows from the fact that a regular adjoint orbit contains a companion matrix, which is Hessenberg. We can then use A-orbits of dimension $n(n-1)/2$ to construct polarizations of dense, open submanifolds of regular adjoint orbits. Hence, the Gelfand–Zeitlin system is completely integrable on each regular adjoint orbit [Kostant and Wallach 2006a, Theorem 3.36].

Our goal is to give a complete description of the A-orbit structure of $\mathfrak{gl}(n)^{\text{sreg}}$. It follows from the Poisson commutativity of the algebra $J(\mathfrak{gl}(n))$ in (2-3) that the fibers $\mathfrak{gl}(n)_c$ are A-stable. Whence, the fibers $\mathfrak{gl}(n)_c^{\text{sreg}}$ are A-stable. Moreover, we have by [Kostant and Wallach 2006a, Theorem 3.12] that the A-orbits in $\mathfrak{gl}(n)^{\text{sreg}}$ are the irreducible components of the fibers $\mathfrak{gl}(n)_c^{\text{sreg}}$. From this it follows that

$$\text{there are only finitely many } A\text{-orbits in the fiber } \mathfrak{gl}(n)_c^{\text{sreg}}\,.$$

In this paper, we describe the A-orbit structure of an arbitrary fiber $\mathfrak{gl}(n)_c^{\text{sreg}}$ and count the exact number of A-orbits in the fiber. This gives a complete description of the A-orbit structure of $\mathfrak{gl}(n)^{\text{sreg}}$.

Remark 2.6. The set of fibers of the map Φ is the same as the set of fibers of the moment map for the A-action $x \to (f_{1,1}(x_1), f_{2,1}(x_2), \ldots, f_{n,n}(x))$. Thus, studying the action of A on the fibers of Φ is essentially studying the action of A on the fibers of the corresponding moment map. We use the map Φ instead of the moment map, since it is easier to describe the fibers of Φ.

For our purposes, it is convenient to have a more concrete characterization of strongly regular elements.

Proposition 2.7 [Kostant and Wallach 2006a, Theorem 2.14]. *Let $x \in \mathfrak{gl}(n)$ and let $\mathfrak{z}_{\mathfrak{gl}(i)}(x_i)$ denote the centralizer in $\mathfrak{gl}(i)$ of x_i. Then x is strongly regular if and only if the following two conditions hold.*

(a) *$x_i \in \mathfrak{gl}(i)$ is regular for all $1 \le i \le n$.*

(b) *$\mathfrak{z}_{\mathfrak{gl}(i-1)}(x_{i-1}) \cap \mathfrak{z}_{\mathfrak{gl}(i)}(x_i) = 0$ for all $2 \le i \le n$.*

3. The action of A on generic matrices

For $x \in \mathfrak{gl}(i)$, let $\sigma(x)$ denote the spectrum of x, where x is viewed as an $i \times i$ matrix. We consider the following Zariski open subset of regular semisimple elements of $\mathfrak{gl}(n)$

$$\begin{aligned} &\mathfrak{gl}(n)_\Omega \\ &\quad = \{x \in \mathfrak{gl}(n) \mid x_i \text{ is regular semisimple}, \sigma(x_{i-1}) \cap \sigma(x_i) = \varnothing,\ 2 \le i \le n\}. \end{aligned} \tag{3-1}$$

Kostant and Wallach give a complete description of the action of A on $\mathfrak{gl}(n)_\Omega$. We give an example of a matrix in $\mathfrak{gl}(3)_\Omega$.

Example 3.1. Consider the matrix in $\mathfrak{gl}(3)$

$$X = \begin{bmatrix} 1 & 2 & 16 \\ 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}.$$

Because X has eigenvalues $\sigma(X) = \{-3, 3, -2\}$, it is regular semisimple and $\sigma(X_2) = \{2, -1\}$. Clearly, $\sigma(X_1) = \{1\}$. Thus $X \in \mathfrak{gl}(3)_\Omega$.

We recall the notation introduced in (1-2). (If $c_i = (z_1, z_2, \dots, z_i) \in \mathbb{C}^i$, then $p_{c_i}(t) = z_1 + z_2 t + \cdots + z_i t^{i-1} + t^i$.) Let $\Omega_n \subset \mathbb{C}^{n(n+1)/2}$ be the Zariski open subset consisting of $c \in \mathbb{C}^{n(n+1)/2}$ with $c = (c_1, \dots, c_i, \dots, c_n)$ such that $p_{c_i}(t)$ has distinct roots and $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common [Kostant and Wallach 2006a, Remark 2.16]. It is easy to see that $\mathfrak{gl}(n)_\Omega = \bigcup_{c \in \Omega_n} \mathfrak{gl}(n)_c$.

Kostant and Wallach described the A-orbit structure on $\mathfrak{gl}(n)_\Omega$, as summarized in Theorem 1.1. We sketch the ideas behind a possible proof in the case of $\mathfrak{gl}(3)$. See [Kostant and Wallach 2006a] or [Colarusso 2009] for complete proofs and a more thorough explanation.

The A-orbit of $x \in \mathfrak{gl}(3)$ is

$$\operatorname{Ad}\left(\begin{bmatrix} z_1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} z_2 & & \\ & z_2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \exp(t x_2) & \\ & 1 \end{bmatrix} \right) \cdot x, \tag{3-2}$$

where $z_1, z_2 \in \mathbb{C}^\times$ and $t \in \mathbb{C}$; see Equation (2-6).

If we let $Z_i \subset \mathrm{GL}(i)$ be the centralizer of x_i in $\mathrm{GL}(i)$, we notice from (3-2) that the action of A appears to push down to an action of $Z_1 \times Z_2$. For $x \in \mathfrak{gl}(3)_\Omega$, we should then expect to see an action of $(\mathbb{C}^\times)^3$ as realizing the action of A.

Working directly from the definition of the action of A in (3-2) is cumbersome. The action of Z_2 on x_2 would be much easier to write down if x_2 were diagonal. However, x_2 is not diagonal for $x \in \mathfrak{gl}(3)_\Omega$, but it is diagonalizable. So, we first diagonalize x_2 and then conjugate by the centralizer $Z_2 = (\mathbb{C}^\times)^2$. If $\gamma(x) \in \mathrm{GL}(2)$ is such $(\operatorname{Ad}(\gamma(x)) \cdot x)_2$ is diagonal, then we can define an action of $(\mathbb{C}^\times)^3$ on $\mathfrak{gl}(3)_c$ for $c \in \Omega_3$ by

$$(z_1', z_2', z_3') \cdot x = \operatorname{Ad}\left(\begin{bmatrix} z_1' & & \\ & 1 & \\ & & 1 \end{bmatrix} \gamma(x)^{-1} \begin{bmatrix} z_2' & & \\ & z_3' & \\ & & 1 \end{bmatrix} \gamma(x) \right) \cdot x, \tag{3-3}$$

with $z_i' \in \mathbb{C}^\times$.

We can show (3-3) is a simply transitive algebraic group action on $\mathfrak{gl}(3)_c$ by explicit computation. Comparing (3-3) and (3-2), it is not hard to believe that the action of $(\mathbb{C}^\times)^3$ in (3-3) has the same orbits as the action of A on $\mathfrak{gl}(3)_c$. To prove this precisely, one needs to see that $\mathfrak{gl}(3)_c^{\text{sreg}} = \mathfrak{gl}(3)_c$. This can be proved by computing the tangent space to the action of $(\mathbb{C}^\times)^3$ in (3-3) and showing that it is same as the subspace V_x in (2-7), or by appealing to [Kostant and Wallach 2006a, Theorem 2.17]. The fact that $\mathfrak{gl}(3)_c$ is one A-orbit then follows by applying [Kostant and Wallach 2006a, Theorem 3.12].

This line of argument is not the one used in [Kostant and Wallach 2006a] to prove Theorem 1.1. The ideas here go back to a preliminary approach by Kostant and Wallach. However, it is this method that generalizes to describe all orbits of A in $\mathfrak{gl}(n)^{\mathrm{sreg}}$. We describe the general construction in the next section.

4. Constructing nongeneric A-orbits

4a. ***Overview.*** In the next three sections, we classify A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}$ by determining the A-orbit structure of an arbitrary fiber $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. Let $c_i \in \mathbb{C}^i$ and $p_{c_i}(t) = (t-\lambda_1)^{n_1}\cdots(t-\lambda_r)^{n_r}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$; see (1-2). To study the action of A on $\mathfrak{gl}(n)_c$ with

$$c = (c_1, \dots, c_i, c_{i+1}, \dots, c_n) \in \mathbb{C}^1 \times \cdots \times \mathbb{C}^i \times \mathbb{C}^{i+1} \times \cdots \times \mathbb{C}^n = \mathbb{C}^{n(n+1)/2},$$

we consider elements of $\mathfrak{gl}(i+1)$ of the form

$$\left[\begin{array}{ccccc} \begin{bmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_1 \end{bmatrix} & & 0 & & \begin{matrix} y_{1,1} \\ \vdots \\ \vdots \\ y_{1,n_1} \end{matrix} \\ & \ddots & & & \vdots \\ 0 & & \begin{bmatrix} \lambda_r & 1 & \cdots & 0 \\ 0 & \lambda_r & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_r \end{bmatrix} & & \begin{matrix} y_{r,1} \\ \vdots \\ \vdots \\ y_{r,n_r} \end{matrix} \\ z_{1,1} \;\cdots\;\cdots\; z_{1,n_1} & \cdots & z_{r,1} \;\cdots\;\cdots\; z_{r,n_r} & & w \end{array}\right] \tag{4-1}$$

with characteristic polynomial $p_{c_{i+1}}(t)$.

To avoid ambiguity, it is necessary to order the Jordan blocks of the $i \times i$ cutoff of the matrix in (4-1). To do this, we introduce a lexicographical ordering on $\mathbb{C}$ defined as follows. Let $z_1, z_2 \in \mathbb{C}$. We say that $z_1 > z_2$ if and only if $\operatorname{Re} z_1 > \operatorname{Re} z_2$, or $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 > \operatorname{Im} z_2$.

Definition 4.1. Let $c_i \in \mathbb{C}^i$ be such that $p_{c_i}(t) = (t-\lambda_1)^{n_1}\cdots(t-\lambda_r)^{n_r}$ with $\lambda_j \neq \lambda_k$, as in (1-2), and let $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ in the lexicographical ordering on $\mathbb{C}$. For $c_{i+1} \in \mathbb{C}^{i+1}$, we define $\Xi^i_{c_i,c_{i+1}}$ as the set of elements $x \in \mathfrak{gl}(i+1)$ of the form (4-1) whose characteristic polynomial is $p_{c_{i+1}}(t)$. We refer to $\Xi^i_{c_i,c_{i+1}}$ as the solution variety at level i.

We know from Theorem 2.5 that $\Xi^i_{c_i,c_{i+1}}$ is nonempty for any $c_i \in \mathbb{C}^i$ and any $c_{i+1} \in \mathbb{C}^{i+1}$. Let us denote the regular Jordan form that is the $i \times i$ cutoff of the matrix in (4-1) by J. Let Z_i denote the centralizer of J in $\mathrm{GL}(i)$. Since J is regular, Z_i is a connected, abelian algebraic group [Kostant 1963, Proposition 14]. The group Z_i acts algebraically on the solution variety $\Xi^i_{c_i,c_{i+1}}$ by conjugation. In the remainder of Section 4, we give a bijection between A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ and free $Z_1 \times \cdots \times Z_{n-1}$ orbits on $\Xi^1_{c_1,c_2} \times \cdots \times \Xi^{n-1}_{c_{n-1},c_n}$. In Section 5, we will classify the Z_i-orbits on $\Xi^i_{c_i,c_{i+1}}$ using combinatorial data of the tuple $c \in \mathbb{C}^{n(n+1)/2}$. We will then have a complete picture of the A-action on $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$.

We now briefly outline the construction, which gives the bijection between A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ and $Z_1 \times \cdots \times Z_{n-1}$ orbits in $\Xi^1_{c_1,c_2} \times \cdots \times \Xi^{n-1}_{c_{n-1},c_n}$. This construction not only describes A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$, but all A-orbits in the larger set $\mathfrak{gl}(n)_c \cap S$, where S is the Zariski open subset of $\mathfrak{gl}(n)$ consisting of elements x whose cutoffs x_i for $1 \le i \le n-1$ are regular. We know by Proposition 2.7(a) that $\mathfrak{gl}(n)^{\mathrm{sreg}}_c \subset \mathfrak{gl}(n)_c \cap S$, and it is in general a proper subset; see Example 5.4.

For $1 \le i \le n-2$, choose a Z_i-orbit $\mathbb{O}^i_{a_i} \in \Xi^i_{c_i,c_{i+1}}$ consisting of regular elements of $\mathfrak{gl}(i+1)$. For $i = n-1$, choose any orbit $\mathbb{O}^{n-1}_{a_{n-1}}$ of Z_{n-1} in $\Xi^{n-1}_{c_{n-1},c_n}$. Then define a morphism

$$
\begin{aligned}
\text{(4-2)}\quad \Gamma_n^{a_1,a_2,\dots,a_{n-1}} &: \mathbb{O}^1_{a_1} \times \cdots \times \mathbb{O}^{n-1}_{a_{n-1}} \to \mathfrak{gl}(n)_c \cap S,\\
&(x_1,\dots,x_{n-1}) \mapsto \mathrm{Ad}(g_{1,2}(x_1)^{-1} g_{2,3}(x_2)^{-1} \cdots g_{n-2,n-1}(x_{n-2})^{-1}) x_{n-1}.
\end{aligned}
$$

where $g_{i,i+1}(x_i)$ conjugates x_i into Jordan canonical form (with eigenvalues in decreasing lexicographical order). For brevity, we define

$$\mathbf{\Gamma}_n := \Gamma_n^{a_1,a_2,\dots,a_{n-1}}.$$

We denote the image of this morphism by $\mathrm{im}\,\mathbf{\Gamma}_n$.

Theorem 4.2. *Every A-orbit in $\mathfrak{gl}(n)_c \cap S$ is of the form $\mathrm{im}\,\mathbf{\Gamma}_n$ for some choice of orbits $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$, with $\mathbb{O}^i_{a_i}$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \le i \le n-2$.*

In Section 4c, we prove Theorem 4.2 for A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$ (see Theorem 4.9). In Section 4d, we establish the results needed to prove Theorem 4.2 for $\mathfrak{gl}(n)_c \cap S$.

4b. ***Definition and properties of the $\mathbf{\Gamma}_n$ maps.*** We first define the map $\mathbf{\Gamma}_n$ only for Z_i-orbits $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ on which Z_i acts freely. To define $\mathbf{\Gamma}_n$, we must define a morphism $\mathbb{O}^i_{a_i} \to \mathrm{GL}(i+1)$ that sends $y \mapsto g_{i,i+1}(y)$, where $g_{i,i+1}(y)$ conjugates y into Jordan form with eigenvalues in decreasing lexicographical order. Since Z_i acts freely on $\mathbb{O}^i_{a_i}$, we can identify $\mathbb{O}^i_{a_i} \cong Z_i$ as algebraic varieties. Let x_{a_i} be an arbitrary choice of base point for the orbit $\mathbb{O}^i_{a_i}$, that is, $\mathbb{O}^i_{a_i} = \mathrm{Ad}(Z_i) \cdot x_{a_i}$. We choose an element $g_{i,i+1}(x_{a_i}) \in \mathrm{GL}(i+1)$ that conjugates the base point x_{a_i} into Jordan form (with eigenvalues in decreasing lexicographical order). For $y = \mathrm{Ad}(k_i) \cdot x_{a_i}$, with $k_i \in Z_i$, we define

$$g_{i,i+1}(y) = g_{i,i+1}(x_{a_i}) k_i^{-1}. \tag{4-3}$$

For each choice of orbit $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ for $1 \le i \le n-1$, we define a morphism $\mathbf{\Gamma}_n : Z_1 \times \cdots \times Z_{n-1} \to \mathfrak{gl}(n)$ by

$$\begin{aligned} \mathbf{\Gamma}_n(k_1, \dots, k_{n-1}) \\ = \mathrm{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} k_2 g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}. \end{aligned} \tag{4-4}$$

We want to give a more intrinsic characterization of $\operatorname{im} \mathbf{\Gamma}_n$.

Proposition 4.3. *We have*

$$\operatorname{im} \mathbf{\Gamma}_n \subset \mathfrak{gl}(n)_c \cap S,$$
$$\operatorname{im} \mathbf{\Gamma}_n = \{x \in \mathfrak{gl}(n) \mid x_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i} \textit{ for all } 1 \le i \le n-1\}. \tag{4-5}$$

Thus, $\operatorname{im} \mathbf{\Gamma}_n$ *is a quasiaffine subvariety of* $\mathfrak{gl}(n)$.

The following simple observation is useful in proving Proposition 4.3.

Remark 4.4. Let $x \in \mathfrak{gl}(n)_c \cap S$, and suppose that $g \in \mathrm{GL}(i)$ is such that $[\mathrm{Ad}(g) \cdot x]_i = \mathrm{Ad}(g) \cdot x_i$ is in Jordan canonical form with eigenvalues in decreasing lexicographical order for $1 \le i \le n-1$. Then $[\mathrm{Ad}(g) \cdot x]_{i+1} = \mathrm{Ad}(g) \cdot x_{i+1} \in \Xi^i_{c_i,c_{i+1}}$.

Proof of Proposition 4.3. Denote the set on the right side of (4-5) by T. We note $T \subset \mathfrak{gl}(n)_c \cap S$. Indeed, let $Y \in T$. Then $Y_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i}$ for $1 \le i \le n-1$. Since $x_{a_i} \in \Xi^i_{c_i,c_{i+1}}$, the characteristic polynomial of Y_{i+1} is $p_{c_{i+1}}(t)$. For $1 \le i \le n-2$, note that x_{a_i} is regular and hence so is Y_{i+1}. Lastly, using the fact that $k_1 \in \mathrm{GL}(1) = Z_1$ centralizes the $(1, 1)$ entry of $x_{a_1} \in \Xi^1_{c_1,c_2}$, it follows that the $(1, 1)$ entry of Y is given by $c_1 \in \mathbb{C}$.

The inclusion $\operatorname{im} \mathbf{\Gamma}_n \subset T$ is clear from the definition of $\mathbf{\Gamma}_n$ in (4-4). To see the opposite inclusion we use induction. Let $y \in T$. Then y_2 is in $\mathrm{Ad}(\mathrm{GL}(1)) \cdot x_{a_1} = \mathbb{O}^1_{a_1}$ since $Z_1 = \mathrm{GL}(1)$. Thus, there exists a $k_1 \in Z_1$ such that $y_2 = \mathrm{Ad}(k_1) \cdot x_{a_1}$. It follows that

$$z_2 = [\mathrm{Ad}(g_{1,2}(x_{a_1}))\, \mathrm{Ad}(k_1^{-1}) \cdot y]_3 = [\mathrm{Ad}(g_{1,2}(x_{a_1}))\, \mathrm{Ad}(k_1^{-1}) \cdot y_3] \in \Xi^2_{c_2,c_3}.$$

But $y_3 \in \mathrm{Ad}(\mathrm{GL}(2)) \cdot x_{a_2}$, so that $z_2 \in \Xi^2_{c_2,c_3} \cap \mathrm{Ad}(\mathrm{GL}(2)) \cdot x_{a_2}$, from which it follows easily that $z_2 \in \mathbb{O}^2_{a_2}$. Thus, there exists a $k_2 \in Z_2$ such that

$$[\mathrm{Ad}(g_{2,3}(x_{a_2}))\, \mathrm{Ad}(k_2^{-1})\, \mathrm{Ad}(g_{1,2}(x_{a_1}))\, \mathrm{Ad}(k_1^{-1}) \cdot y]_4 \in \Xi^3_{c_3,c_4}.$$

This completes the first two steps of the induction. We now assume that there exist $k_1, \ldots, k_{j-1} \in Z_1, \ldots, Z_{j-1}$ respectively such that

$$\begin{aligned} z_j &= [\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\,\mathrm{Ad}(k_{j-1}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1}) \cdot y]_{j+1} \\ &\in \Xi^{j}_{c_j, c_{j+1}}. \end{aligned} \tag{4-6}$$

Since $y_{j+1} \in \mathrm{Ad}(\mathrm{GL}(j)) \cdot x_{a_j}$, it follows that $z_j \in \Xi^j_{c_j,c_{j+1}} \cap \mathrm{Ad}(\mathrm{GL}(j)) \cdot x_{a_j}$. As above, it follows that $z_j \in \mathbb{O}^j_{a_j}$, so that there exists an element $k_j \in K_j$ such that

$$\begin{aligned} [\mathrm{Ad}(g_{j,j+1}(x_{a_j}))\,\mathrm{Ad}(k_j^{-1})\,\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\,\mathrm{Ad}(k_{j-1}^{-1}) \\ \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1}) \cdot y]_{j+2} \in \Xi^{j+1}_{c_{j+1},c_{j+2}}. \end{aligned}$$

We have made use of Remark 4.4 throughout. By induction, we conclude that there exist $k_1, \ldots, k_{n-1} \in Z_1, \ldots, Z_{n-1}$ respectively such that

$$x_{a_{n-1}} = \mathrm{Ad}(k_{n-1}^{-1})\,\mathrm{Ad}(g_{n-2,n-1}(x_{a_{n-1}}))\,\mathrm{Ad}(k_{n-2}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\,\mathrm{Ad}(k_1^{-1}) \cdot y,$$

from which it follows that $y = \mathbf{\Gamma}_n(k_1, \ldots, k_{n-1})$.

To see the final statement, we observe T is a Zariski locally closed subset of $\mathfrak{gl}(n)$. Indeed, the set $U_i = \{x \mid x_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i}\}$ is locally closed, since it is the preimage of the orbit $\mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i} \subset \mathfrak{gl}(i+1)$ under the projection morphism $\pi_{i+1}(x) = x_{i+1}$. The set $T = U_1 \cap \cdots \cap U_{n-1}$ is locally closed. □

Remark 4.5. From Proposition 4.3 it follows that the set $\operatorname{im} \mathbf{\Gamma}_n$ depends only on the orbits $\mathbb{O}^i_{a_i}$ for $1 \leq i \leq n-1$, and is thus independent of the choices involved in defining the map $\mathbf{\Gamma}_n$ in (4-4).

4c. $\mathbf{\Gamma}_n$ *and A-orbits in* $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. In this section, we show that the image of the morphism $\mathbf{\Gamma}_n$ is an A-orbit in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. The first step is to see $\operatorname{im} \mathbf{\Gamma}_n$ is smooth variety.

Theorem 4.6. *The morphism*

$$\mathbf{\Gamma}_n : Z_1 \times \cdots \times Z_{n-1} \to \mathfrak{gl}(n)_c \cap S$$

is an isomorphism onto its image. Hence, $\operatorname{im} \mathbf{\Gamma}_n$ *is a smooth*, *irreducible subvariety of* $\mathfrak{gl}(n)$ *of dimension* $n(n-1)/2$.

Proof. We explicitly construct an inverse $\Psi : \operatorname{im} \mathbf{\Gamma}_n \to Z_1 \times \cdots \times Z_{n-1}$ of $\mathbf{\Gamma}_n$ and show that it is a morphism. Specifically, we show that there exist morphisms $\psi_i : \operatorname{im} \mathbf{\Gamma}_n \to Z_i$ for $1 \leq i \leq n-1$ such that the morphism

$$\Psi = (\psi_1, \ldots, \psi_{n-1}) : \operatorname{im} \mathbf{\Gamma}_n \to Z_1 \times \cdots \times Z_{n-1} \tag{4-7}$$

is an inverse of $\mathbf{\Gamma}_n$. The morphisms ψ_i are constructed inductively.

Given $y \in \operatorname{im} \boldsymbol{\Gamma}_n$, we have $y_2 \in \mathbb{O}^1_{a_1} \subset \Xi^1_{c_1,c_2}$ by Proposition 4.3. Thus, $y_2 = \operatorname{Ad}(k_1) \cdot x_{a_1}$ for a unique k_1 in Z_1. The map $\mathbb{O}^1_{a_1} \to Z_1$, $\operatorname{Ad}(k_1) \cdot x_{a_1} \mapsto k_1$ is an isomorphism of smooth affine varieties. Hence, the map $\psi_1(y) = k_1$ is a morphism.

Arguing as in the proof of Proposition 4.3, suppose we have defined morphisms $\psi_1, \dots, \psi_{j-1}$, with $\psi_i : \operatorname{im} \boldsymbol{\Gamma}_n \to Z_i$ for $1 \le i \le j-1$. Then the function $\operatorname{im} \boldsymbol{\Gamma}_n \to \mathbb{O}^j_{a_j}$ given by (4-6), that is,

$$y \mapsto [\operatorname{Ad}(g_{j-1,j}(x_{a_{j-1}})) \operatorname{Ad}(\psi_{j-1}(y)^{-1}) \cdots \operatorname{Ad}(g_{1,2}(x_{a_1})) \operatorname{Ad}(\psi_1(y)^{-1}) \cdot y]_{j+1},$$

is a morphism. We can then define a morphism $\psi_j : \operatorname{im} \boldsymbol{\Gamma}_n \to Z_j$ by $y \mapsto k_j$, where k_j is the unique element of Z_j such that

$$\begin{aligned} \text{(4-8)} \quad \operatorname{Ad}(k_j) \cdot x_{a_j} = [&\operatorname{Ad}(g_{j-1,j}(x_{a_{j-1}})) \operatorname{Ad}(\psi_{j-1}(y)^{-1}) \\ &\cdots \operatorname{Ad}(g_{1,2}(x_{a_1})) \operatorname{Ad}(\psi_1(y)^{-1}) \cdot y]_{j+1}. \end{aligned}$$

This completes the induction.

We now show that Ψ is an inverse of $\boldsymbol{\Gamma}_n$. That $\boldsymbol{\Gamma}_n(\psi_1(y), \dots, \psi_{n-1}(y)) = y$ follows exactly as in the proof of the inclusion $T \subset \operatorname{im} \boldsymbol{\Gamma}_n$ in Proposition 4.3.

Finally, we show that $\Psi(\boldsymbol{\Gamma}_n(k_1, \dots, k_{n-1})) = (k_1, \dots, k_{n-1})$. Consider the element

$$\operatorname{Ad}(k_j g_{j,j+1}(x_{a_j})^{-1} \cdots g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) \cdot x_{a_{n-1}}.$$

The $(j+1) \times (j+1)$ cutoff of this element is equal to $k_j \cdot x_{a_j}$. This fact with $j = 1$ gives $\psi_1(y) = k_1$. Assume that we have $\psi_2(y) = k_2, \dots, \psi_l(y) = k_l$ for $2 \le l \le j-1$. Using the definition of ψ_j in (4-8), we obtain

$$\begin{aligned} \operatorname{Ad}(\psi_j(y)) \cdot x_{a_j} &= [\operatorname{Ad}(k_j) \operatorname{Ad}(g_{j,j+1}(x_{a_j})^{-1} \cdots g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}]_{j+1} \\ &= \operatorname{Ad}(k_j) \cdot x_{a_j}. \end{aligned}$$

Thus by induction, $\Psi \circ \boldsymbol{\Gamma}_n$ is the identity. Hence, Ψ is a regular inverse of the map $\boldsymbol{\Gamma}_n$ and Ψ is an isomorphism of varieties. □

The image of $\boldsymbol{\Gamma}_n$ is a smooth irreducible quasiaffine subvariety of $\mathfrak{gl}(n)$. Thus $\operatorname{im} \boldsymbol{\Gamma}_n$ has the structure of a connected analytic submanifold of $\mathfrak{gl}(n)$, and $\boldsymbol{\Gamma}_n$ is an analytic isomorphism.

Proposition 4.7. *The action of the analytic group A on $\mathfrak{gl}(n)$ preserves the submanifolds* $\operatorname{im} \boldsymbol{\Gamma}_n$.

Proof. The action of A on $\mathfrak{gl}(n)$ is given by the composition of the flows in (2-6) in any order; see Remark 2.3. Thus, to see that the action of A preserves $\operatorname{im} \boldsymbol{\Gamma}_n$, it suffices to see that the action of $\mathbb{C}$ in (2-6) preserves $\operatorname{im} \boldsymbol{\Gamma}_n$ for any $(i, j) \in \Delta^{n-1}_{i,j}$. Suppose that $x \in \operatorname{im} \boldsymbol{\Gamma}_n$. Then by Proposition 4.3, $x_{k+1} \in \operatorname{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$ for any $1 \le k \le n-1$. Define an element $h = \exp(t j x_i^{j-1}) \in \mathrm{GL}(i)$ with $t \in \mathbb{C}$ fixed and consider $\operatorname{Ad}(h) \cdot x$ as in (2-6). We claim that $(\operatorname{Ad}(h) \cdot x)_{k+1} \in \operatorname{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$ for

$1 \le k \le n-1$. We consider two cases. Suppose $k \ge i$ and consider $(\mathrm{Ad}(h) \cdot x)_{k+1}$. We have $(\mathrm{Ad}(h) \cdot x)_{k+1} = \mathrm{Ad}(h) \cdot x_{k+1}$. But $x_{k+1} \in \mathrm{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$, so that $\mathrm{Ad}(h) \cdot x_{k+1} \in \mathrm{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}$, since $\mathrm{GL}(i) \subset \mathrm{GL}(k)$. Next, we suppose that $k < i$, so that $k+1 \le i$. Since $h \in \mathrm{GL}(i)$ centralizes x_i,

$$(\mathrm{Ad}(h)x)_{k+1} = (\mathrm{Ad}(h)(x_i))_{k+1} = (x_i)_{k+1} = x_{k+1} \in \mathrm{Ad}(\mathrm{GL}(k)) \cdot x_{a_k}.$$

By Proposition 4.3, $\mathrm{Ad}(h) \cdot x \in \mathrm{im}\,\mathbf{\Gamma}_n$. □

The main theorem of this section depends on a technical result about the action of Z_i on the solution varieties $\Xi^i_{c_i,c_{i+1}}$; this result will be proved independently in Section 4d.

Lemma 4.8. *For $x \in \Xi^i_{c_i,c_{i+1}}$, the isotropy group* $\mathrm{Stab}(x)$ *of x under the action of Z_i is a connected algebraic group.*

Thus, given an orbit $\mathbb{O} \subset \Xi^i_{c_i,c_{i+1}}$ of Z_i,

$$\dim(\mathbb{O}) = i \quad \text{if and only if } Z_i \text{ acts freely on } \mathbb{O}. \tag{4-9}$$

Theorem 4.9. *The submanifold* $\mathrm{im}\,\mathbf{\Gamma}_n \subset \mathfrak{gl}(n)_c \cap S$ *is a single A-orbit in* $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. *Every A-orbit in* $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ *is of the form* $\mathrm{im}\,\mathbf{\Gamma}_n$ *with* $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$, *where* $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ *are free Z_i-orbits consisting of regular elements of* $\mathfrak{gl}(i+1)$ *for* $1 \le i \le n-1$.

Proof. First, we show that $\mathrm{im}\,\mathbf{\Gamma}_n$ is an A-orbit. For this, we need to describe the tangent space $T_y(\mathrm{im}\,\mathbf{\Gamma}_n) = (d\mathbf{\Gamma}_n)_{\underline{k}}$, where $\underline{k} = (k_1, \dots, k_{n-1}) \in Z_1 \times \cdots \times Z_{n-1}$ and $y = \mathbf{\Gamma}_n(\underline{k})$. Let $\{\alpha_{i1}, \dots, \alpha_{ii}\}$ be a basis for $\mathrm{Lie}(Z_i) = \mathfrak{z}_i$. Working analytically, we compute

$$(d\mathbf{\Gamma}_n)_{\underline{k}}(0, \dots, \alpha_{ij}, \dots, 0) = \tfrac{d}{dt}\big|_{t=0} \mathbf{\Gamma}_n(k_1, \dots, k_i \exp(t\alpha_{ij}), \dots, k_{n-1})$$

for $1 \le j \le i$. Using the definition of the morphism $\mathbf{\Gamma}_n$, the right side of this becomes

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} \cdots k_i \exp(t\alpha_{ij}) g_{i,i+1}(x_{a_i})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}},$$

which, after defining

$$l_i = k_1 g_{1,2}(x_{a_1})^{-1} \cdots k_i \quad \text{and} \quad h_i = g_{i,i+1}(x_{a_i})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1},$$

becomes

$$\tfrac{d}{dt}\big|_{t=0} \mathrm{Ad}(l_i \exp(t\alpha_{ij}) h_i) \cdot x_{a_{n-1}},$$

which in turn has differential

$$\mathrm{ad}(\mathrm{Ad}(l_i) \cdot \alpha_{ij}) \cdot (\mathrm{Ad}(l_i h_i) \cdot x_{a_{n-1}}). \tag{4-10}$$

By definition of the element $l_i \in \mathrm{GL}(i)$, the $i \times i$ cutoff of $\mathrm{Ad}(l_i^{-1}) \cdot y = \mathrm{Ad}(l_i^{-1}) \cdot y_i$ is in Jordan form (with eigenvalues in decreasing lexicographical order). Hence elements of the form $\mathrm{Ad}(l_i) \cdot \alpha_{ij} = \gamma_{ij}$ for $1 \leq j \leq i$ form a basis for $\mathfrak{z}_{\mathfrak{gl}(i)}(y_i)$. Since $\mathrm{Ad}(l_i h_i) \cdot x_{a_{n-1}} = y$, (4-10) implies that the image of $(d\mathbf{\Gamma}_n)_{\underline{k}}$ is

$$\text{(4-11)} \qquad \mathrm{im}((d\mathbf{\Gamma}_n)_{\underline{k}}) = \mathrm{span}\{\partial_y^{[\gamma_{i,j},y]} \mid (i,j) \in \Delta_{i,j}^{n-1}\} = T_y(\mathrm{im}\,\mathbf{\Gamma}_n).$$

Equation (2-7), with $y \in \mathrm{im}\,\mathbf{\Gamma}_n$ instead of x, reads

$$T_y(A \cdot y) = \mathrm{span}\{\partial_y^{[z,y]} \mid z \in Z_y\} := V_y.$$

Now, y has the property that y_i is regular for all $i \leq n-1$, so that $\mathfrak{z}_{\mathfrak{gl}(i)}(y_i)$ has basis $\{\mathrm{Id}_i, y_i, \dots, y_i^{i-1}\}$; see [Kostant 1963, page 382]. Thus,

$$T_y(\mathrm{im}\,\mathbf{\Gamma}_n) = \mathrm{span}\{\partial_y^{[z,y]} \mid z \in Z_y\} = V_y.$$

This gives

$$\text{(4-12)} \qquad \dim V_y = \dim(A \cdot y) = n(n-1)/2,$$

which implies $\mathrm{im}\,\mathbf{\Gamma}_n \subset \mathfrak{gl}(n)_c^{\mathrm{sreg}}$. By Proposition 4.7, A acts on $\mathrm{im}\,\mathbf{\Gamma}_n$. We claim that the action of A is transitive on $\mathrm{im}\,\mathbf{\Gamma}_n$. Indeed, an A-orbit $A \cdot y$ with $y \in \mathrm{im}\,\mathbf{\Gamma}_n$ is a submanifold of $\mathrm{im}\,\mathbf{\Gamma}_n$ of the same dimension as $\mathrm{im}\,\mathbf{\Gamma}_n$ by (4-12), and thus must be open. The action of A is then clearly transitive on $\mathrm{im}\,\mathbf{\Gamma}_n$ since $\mathrm{im}\,\mathbf{\Gamma}_n$ is connected.

We now show that every A-orbit in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is obtained in this manner. For $x \in \mathfrak{gl}(n)_c^{\mathrm{sreg}}$, by Proposition 2.7(a) and Remark 4.4 there exists a matrix $g_i \in \mathrm{GL}(i)$ such that $z_i = \mathrm{Ad}(g_i) \cdot x_{i+1} \in \Xi_{c_i,c_{i+1}}^i$ and z_i is regular for each $1 \leq i \leq n-1$. Thus $z_i \in \mathbb{O}_{a_i}^i$, with $\mathbb{O}_{a_i}^i$ an orbit of Z_i in $\Xi_{c_i,c_{i+1}}^i$ consisting of regular elements of $\mathfrak{gl}(i+1)$. We claim that Z_i must act freely on $\mathbb{O}_{a_i}^i$. Suppose to the contrary that $\mathrm{Stab}(x_{a_i})$ is nontrivial. Lemma 4.8 gives that $\dim(\mathrm{Stab}(x_{a_i})) \geq 1$. But, this implies $\dim(Z_{\mathrm{GL}(i)}(x_i) \cap Z_{\mathrm{GL}(i+1)}(x_{i+1})) \geq 1$, contradicting Proposition 2.7(b). By Proposition 4.3, x is in $\mathrm{im}\,\mathbf{\Gamma}_n$, with $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$ for some choice of free Z_i-orbits $\mathbb{O}_{a_i}^i \subset \Xi_{c_i,c_{i+1}}^i$. □

Remark 4.10. Let $\mathbf{\Gamma}_n$ be defined using Z_i-orbits $\mathbb{O}_{a_i}^i$, and let $\tilde{\mathbf{\Gamma}}_n := \Gamma_{n-1}^{\tilde{a}_1,\tilde{a}_2,\dots,\tilde{a}_{n-1}}$ be defined using Z_i-orbits $\mathbb{O}_{\tilde{a}_i}^i = \mathrm{Ad}(Z_i) \cdot x_{\tilde{a}_i}$, where $\mathbb{O}_{a_i}^i \cap \mathbb{O}_{\tilde{a}_i}^i = \varnothing$ for some i in $1 \leq i \leq n-1$. Then the A-orbits $\mathrm{im}\,\mathbf{\Gamma}_n$ and $\mathrm{im}\,\tilde{\mathbf{\Gamma}}_n$ are distinct: Suppose to the contrary that $y \in \mathrm{im}\,\mathbf{\Gamma}_n \cap \mathrm{im}\,\tilde{\mathbf{\Gamma}}_n$. By Proposition 4.3, $y_{i+1} \in \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{a_i} \cap \mathrm{Ad}(\mathrm{GL}(i)) \cdot x_{\tilde{a}_i}$. This implies that there exists $h \in \mathrm{GL}(i)$ such that $\mathrm{Ad}(h) \cdot x_{a_i} = x_{\tilde{a}_i}$. Since $x_{a_i}, x_{\tilde{a}_i} \in \Xi_{c_i,c_{i+1}}^i$, the previous equation forces $h \in Z_i$, which implies $\mathbb{O}_{a_i}^i = \mathbb{O}_{\tilde{a}_i}^i$, a contradiction. We have thus established a bijection between free $Z_1 \times \cdots \times Z_{n-1}$ orbits on the product of solution varieties $\Xi_{c_1,c_2}^1 \times \cdots\cdots \times \Xi_{c_{n-1},c_n}^{n-1}$ and A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$.

On the subvariety $\operatorname{im} \boldsymbol{\Gamma}_n$, we have a free and transitive algebraic action of the algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$. This action is defined as follows:

$$\text{(4-13)} \qquad \begin{aligned} &\text{if} && {\boldsymbol{\Gamma}_n}^{-1}(y) = (k_1, \ldots, k_{n-1}), \\ &\text{then} && (k_1', \ldots, k_{n-1}') \cdot y = \boldsymbol{\Gamma}_n(k_1'k_1, \ldots, k_{n-1}'k_{n-1}). \end{aligned}$$

Remark 4.11. The action in (4-13) generalizes the action of $(\mathbb{C}^\times)^3$ in (3-3) to the nongeneric case.

Thus, the A-orbit $\operatorname{im} \boldsymbol{\Gamma}_n$ is the orbit of an algebraic group acting on a quasiaffine variety. We now show that $Z = Z_1 \times \cdots \times Z_{n-1}$ acts algebraically on the fiber $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. By [Kostant and Wallach 2006a, Theorem 3.12], the A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ are the irreducible components of $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. Since they are disjoint, these components are both open and closed in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ (in the Zariski topology on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$). Following [Kostant and Wallach 2006a], we index these components by $\mathfrak{gl}_{c,i}^{\mathrm{sreg}}(n) = A \cdot x(i)$, with $x(i) \in \mathfrak{gl}(n)_c^{\mathrm{sreg}}$. We have morphisms $\phi_i : Z \times \mathfrak{gl}_{c,i}^{\mathrm{sreg}}(n) \to \mathfrak{gl}_c^{\mathrm{sreg}}(n)$ given by the action of Z on $\operatorname{im} \boldsymbol{\Gamma}_n$. The sets $Z \times \mathfrak{gl}_{c,i}^{\mathrm{sreg}}(n)$ are (Zariski) open in the product $Z \times \mathfrak{gl}(n)_c^{\mathrm{sreg}}$ and are disjoint. Thus, the morphisms ϕ_i glue to a unique morphism

$$\Phi : Z \times \mathfrak{gl}(n)_c^{\mathrm{sreg}} \to \mathfrak{gl}(n)_c^{\mathrm{sreg}} \quad \text{such that } \Phi\big|_{Z \times \mathfrak{gl}_{c,i}^{\mathrm{sreg}}(n)} = \phi_i.$$

The morphism Φ defines an algebraic action of the group Z on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ whose orbits are the orbits of A in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. We have thus proved the following theorem.

Theorem 4.12. *Let $x \in \mathfrak{gl}(n)_c^{\mathrm{sreg}}$ be arbitrary and let Z_i be the centralizer in* $\mathrm{GL}(i)$ *of the Jordan form of x_i (with eigenvalues in decreasing lexicographical order). On $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ the orbits of the group A are orbits of a free algebraic action of the connected abelian algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$.*

We end this section with a result that will be of great use in Section 5 where we count the number of A-orbits in the fiber $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$.

It turns out that the condition in Theorem 4.9 that $\mathbb{O}_{a_i}^i \subset \mathfrak{gl}(i+1)^{\mathrm{reg}}$ is superfluous.

Theorem 4.13. *If $\mathbb{O}_{a_i}^i \subset \Xi_{c_i, c_{i+1}}^i$ is a free Z_i-orbit, then $\mathbb{O}_{a_i}^i \subset \mathfrak{gl}(i+1)^{\mathrm{reg}}$.*

Proof. Let $c = (c_1, c_2, \ldots, c_j, c_{j+1}, \ldots, c_n) \in \mathbb{C}^{n(n+1)/2}$, with $c_j \in \mathbb{C}^j$, be given. By Theorem 2.5, there is a unique upper Hessenberg matrix $h \in \mathfrak{gl}(n)_c^{\mathrm{sreg}}$. This implies by Remark 4.4 that for any j in $1 \le j \le n-1$, there exists a $g_j \in \mathrm{GL}(j)$ such that $(\mathrm{Ad}(g_j) \cdot h)_{j+1} \in \Xi_{c_j, c_{j+1}}^j$. Thus, $\mathrm{Ad}(g_j) \cdot h_{j+1} \in Z_j \cdot x_{a_j} = \mathbb{O}_{a_j}^j$ for some $x_{a_j} \in \Xi_{c_j, c_{j+1}}^j$. But $h \in \mathfrak{gl}(n)^{\mathrm{sreg}}$ and therefore h_{j+1} is regular by Proposition 2.7(a), which implies that $\mathbb{O}_{a_j}^j \subset \mathfrak{gl}(j+1)^{\mathrm{reg}}$. Also, by Proposition 2.7(b), Z_j acts freely on $\mathbb{O}_{a_j}^j$, as in the proof of the last statement of Theorem 4.9. Thus, for any j in $1 \le j \le n-1$, there exists a free Z_j-orbit in $\Xi_{c_j, c_{j+1}}^j$ consisting of regular elements of $\mathfrak{gl}(j+1)$.

Now, let $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ be any free Z_i-orbit. Now, we use the free Z_j-orbit $\mathbb{O}^j_{a_j} \subset \mathfrak{gl}(j+1)^{\mathrm{reg}}$ as above for $1 \leq j \leq i-1$ and we use $\mathbb{O}^i_{a_i}$ to construct a morphism

$$\mathbf{\Gamma}_{i+1} := \Gamma^{a_1,a_2,\ldots,a_i}_{i+1} : Z_1 \times \cdots \times Z_i \to \mathfrak{gl}(n)_c \cap S.$$

By Theorem 4.9, $\operatorname{im} \mathbf{\Gamma}_{i+1} \subset \mathfrak{gl}(i+1)^{\mathrm{sreg}}$. Proposition 2.7(a) then implies

$$\operatorname{im} \mathbf{\Gamma}_{i+1} \subset \mathfrak{gl}(i+1)^{\mathrm{reg}}.$$

Then $\mathbb{O}^i_{a_i} \subset \mathfrak{gl}(i+1)^{\mathrm{reg}}$ since elements of $\mathbb{O}^i_{a_i}$ are conjugate to those of $\operatorname{im} \mathbf{\Gamma}_{i+1}$. □

4d. ***A-orbits in $\mathfrak{gl}(n)_c \cap S$.*** We now discuss how the construction in Sections 4b and 4c can be generalized to describe A-orbits of dimension strictly less than $n(n-1)/2$ in the Zariski open subset of the fiber $\mathfrak{gl}(n)_c \cap S$. In this case, it is more difficult to define the morphism $\mathbf{\Gamma}_n$ of (4-2). The problem is that it is not clear how to define a morphism $\mathbb{O}^i_{a_i} \to \mathrm{GL}(i+1)$ that sends $x \to g_{i,i+1}(x)$, where $\mathrm{Ad}(g_{i,i+1}(x)) \cdot x$ is in Jordan form (with eigenvalues in decreasing lexicographical order). This is not difficult in the strongly regular case, since we are dealing with free Z_i-orbits $\mathbb{O}^i_{a_i} \cong Z_i$ so that $g_{i,i+1}(x)$ can be defined as in (4-3). The fortunate fact is that even for an orbit $\mathbb{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$ of dimension strictly less than i, there exists a connected, Zariski closed subgroup $K_i \subset Z_i$ with K_i acting freely on $\mathbb{O}^i_{a_i} \cong K_i$. Therefore, we can mimic what we did in (4-3).

To prove this, we need to understand better the action of Z_i on $\Xi^i_{c_i,c_{i+1}}$. As in Section 4a, let $J = J_1 \oplus \cdots \oplus J_r$ be the $i \times i$ cutoff of the matrix in (4-1), where $J_j \in \mathfrak{gl}(n_j)$ is the Jordan block corresponding to eigenvalue λ_j. We note since J is regular, Z_i is an abelian connected algebraic group, which is the product $\prod_{j=1}^r Z_{J_j}$ of groups, where Z_{J_j} denotes the centralizer of J_j. It is then easy to see that the action of Z_i is the diagonal action of the product $\prod_{j=1}^r Z_{J_j}$ on the last column of $x \in \Xi^i_{c_i,c_{i+1}}$ and the dual action on the last row of x; see (4-1). In other words, Z_{J_j} acts only on the columns and rows of x that contain the Jordan block J_j; see (4-1). This leads us to define an action of Z_{J_j} on $\mathbb{C}^{2n_j}$ by

(4-14) $$z \cdot ([t_1, \ldots, t_{n_j}], [s_1, \ldots, s_{n_j}]^T) = ([t_1, \ldots, t_{n_j}] \cdot z^{-1}, z \cdot [s_1, \cdots, s_{n_j}]^T).$$

Let $\mathbb{O}$ be the Z_i-orbit of some $x \in \Xi^i_{c_i,c_{i+1}}$, and let $\mathbb{O}_j \subset \mathbb{C}^{2n_j}$ be the Z_{J_j}-orbit of

$$x[j] = ([z_{j,1}, \ldots, z_{j,n_j}], [y_{j,1}, \ldots, y_{j,n_j}]),$$

where the coordinates for x are as in (4-1). It follows directly from our remarks above that

(4-15) $$\mathbb{O} \cong \mathbb{O}_1 \times \cdots \times \mathbb{O}_r,$$

where the isomorphism is Z_i-equivariant. It is easy to describe the structure of the isotropy groups for the Z_i-action using this description of a Z_i-orbit $\mathbb{O} \subset \Xi^i_{c_i,c_{i+1}}$.

Lemma 4.14. *Let* $x \in \Xi^i_{c_i, c_{i+1}}$ *and let* $\mathrm{Stab}(x) \subset Z_i$ *be its isotropy group under the action of* Z_i *on* $\Xi^i_{c_i, c_{i+1}}$. *Then, up to reordering,*

$$\mathrm{Stab}(x) = \prod_{j=1}^{q} Z_{J_j} \times \prod_{j=q+1}^{r} U_j, \tag{4-16}$$

where $U_j \subset Z_{J_j}$ *is a unipotent Zariski closed subgroup (possibly trivial) for some* q *in* $0 \le q \le r$.

Proof. Suppose that $x \in \Xi^i_{c_i, c_{i+1}}$ is given by (4-1). By (4-15), to compute the stabilizer of x we need only compute the stabilizers for each of the Z_{J_k} orbits $\mathbb{O}_k = Z_{J_k} \cdot x[k]$, where $1 \le k \le r$. To compute the stabilizer of $x[k]$, suppose that there exists an i with $1 \le i \le n_k$ such that $y_{k,i} \ne 0$ and $y_{k,l} = 0$ for $i < l \le n_k$. We consider the matrix equation

$$A_k \cdot \underline{y}_k = \underline{y}_k, \tag{4-17}$$

where $A_k \in Z_{J_k}$ is an invertible upper triangular Toeplitz matrix and $\underline{y}_k \in \mathbb{C}^{n_k}$ is the column vector $\underline{y}_k = (y_{k,1}, \ldots, y_{k,i}, 0, \ldots, 0)^T$. Since A_k is an upper triangular Toeplitz matrix, we see by considering the i-th row in (4-17) that A_k is forced to be unipotent. If on the other hand, all $y_{k,j} = 0$ for $1 \le j \le n_k$, we can argue similarly using the $z_{k,j}$ and the dual action.

If $y_{k,l} = 0$ for all l and $z_{k,l} = 0$ for all l, then clearly the stabilizer of $x[k]$ is Z_{J_k} itself. Repeating this analysis for each k in $1 \le k \le r$ and after possibly reordering the Jordan blocks of x_i, we get the desired result. □

Proof of Lemma 4.8. Upon reordering the eigenvalues, we can always assume that $\mathrm{Stab}(x)$ has the form given in (4-16) in Lemma 4.14. This proves the result since unipotent algebraic groups are always connected and the groups Z_{J_j} are connected since they are centralizers of regular elements in $\mathfrak{gl}(n_j)$. □

We can now prove the structural theorem about the group Z_i that lets us construct the morphism $\mathbf{\Gamma}_n$ in the general case.

Theorem 4.15. *Let* $x \in \Xi^i_{c_i, c_{i+1}}$ *and let* $\mathrm{Stab}(x) \subset Z_i$ *denote the isotropy group of* x *under the action of* Z_i *on* $\Xi^i_{c_i, c_{i+1}}$. *Then as an algebraic group,*

$$Z_i = \mathrm{Stab}(x) \times K,$$

where K *is a connected, Zariski closed algebraic subgroup of* Z_i.

Proof. For the purposes of this proof we denote by H the group $\mathrm{Stab}(x)$. Without loss of generality, we assume H is as given in (4-16). Let $\mathfrak{z}_i = \mathrm{Lie}(Z_i)$ and let

$\mathfrak{h} = \mathrm{Lie}(H)$. Now, by Lemma 4.14,

$$\mathfrak{h} = \bigoplus_{j=1}^{q} \mathfrak{z}_{J_j} \oplus \bigoplus_{j=q+1}^{r} \mathfrak{n}_j, \tag{4-18}$$

where $\mathfrak{z}_{J_j}$ is the Lie algebra of the abelian algebraic group Z_{J_j} and $\mathfrak{n}_j = \mathrm{Lie}(U_j)$ is a Lie subalgebra of $\mathfrak{n}^+(n_j)$, the strictly upper triangular matrices in $\mathfrak{gl}(n_j)$.

The proof takes two steps. We first find an algebraic Lie subalgebra $\mathfrak{k} \subset \mathfrak{z}_i$ such that $\mathfrak{z}_i = \mathfrak{h} \oplus \mathfrak{k}$ as Lie algebras. We then show that if $K \subset Z_i$ is the corresponding Zariski closed subgroup, then $Z_i = HK$ and $H \cap K = \{e\}$. To find $\mathfrak{k}$, consider the abelian Lie algebra $\mathfrak{z}_{J_j}$ for $q+1 \le j \le r$. Since $\mathfrak{z}_{J_j}$ is abelian, it has a Jordan decomposition as a direct sum of Lie algebras $\mathfrak{z}_{J_j} = \mathfrak{z}_{J_j}^{ss} \oplus \mathfrak{z}_{J_j}^{n}$, where $\mathfrak{z}_{J_j}^{ss}$ are the semisimple elements of $\mathfrak{z}_{J_j}$ and $\mathfrak{z}_{J_j}^{n}$ are the nilpotent elements. Now the Lie algebra $\mathfrak{n}_j$ in (4-18) is a subalgebra of $\mathfrak{z}_{J_j}^{n}$. Take $\tilde{\mathfrak{n}}_j$ such that $\mathfrak{z}_{J_j}^{n} = \mathfrak{n}_j \oplus \tilde{\mathfrak{n}}_j$. Let

$$\mathfrak{m}_j = \mathfrak{z}_{J_j}^{ss} \oplus \tilde{\mathfrak{n}}_j.$$

Note that $\mathfrak{m}_j \oplus \mathfrak{n}_j = \mathfrak{z}_{J_j}$. We claim that $\mathfrak{m}_j$ is an algebraic subalgebra of $\mathfrak{z}_{J_j}$. Indeed, $\tilde{\mathfrak{n}}_j$ is algebraic since it is a nilpotent Lie algebra; see [Tauvel and Yu 2005, page 383]. Let $\tilde{N}_j$ be the corresponding algebraic subgroup. Then $M_j = \mathbb{C}^\times \times \tilde{N}_j$ has $\mathrm{Lie}(M_j) = \mathfrak{m}_j$ since $\mathbb{C}^\times$ is the semisimple part of group Z_{J_j}; see (4-1). We then take $\mathfrak{k} = \bigoplus_{j=q+1}^{r} \mathfrak{m}_j$. This finishes the first step.

Let $K = \prod_{j=q+1}^{r} M_j$ be the Zariski closed, connected algebraic subgroup of $\prod_{j=q+1}^{r} Z_{J_j}$ that corresponds to the algebraic Lie algebra $\mathfrak{k}$. We now show that $Z_i = H \times K$. $H \cap K$ is finite by our choice of K. But also $H \cap K \subset \prod_{j=q+1}^{r} U_j$ and it is thus unipotent; see (4-16). Since any unipotent group must be connected, we have $H \cap K = \{e\}$. Now, it is clear that $Z_i = HK$, since HK is a closed, connected subgroup of Z_i of dimension $\dim Z_i$. □

Proof of Theorem 4.2. With Theorem 4.15 in hand, we can now define the general $\mathbf{\Gamma}_n$ morphism of (4-2) as we did in the strongly regular case. Now, suppose we are given Z_i-orbits $\mathbb{O}_{a_i}^i$ in Ξ_{c_i, c_i+1}^i, with $\mathbb{O}_{a_i}^i = K_{a_i} \cdot x_{a_i} \cong K_{a_i}$ with K_{a_i} as in Theorem 4.15 for $1 \le i \le n-1$, and with $\mathbb{O}_{a_i}^i$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \le i \le n-2$. As in (4-4), we define a morphism

$$\mathbf{\Gamma}_n := \Gamma_n^{a_1, \ldots, a_{n-1}} : K_{a_1} \times \cdots \times K_{a_{n-1}} \to \mathfrak{gl}(n)_c \cap S.$$

Propositions 4.3 and 4.7, Theorem 4.6, and Remark 4.10 from the strongly regular case remain valid in this case by simply replacing the groups Z_i by the groups K_{a_i}. We recall that the main ingredient in proving Theorem 4.6 is the fact that the group Z_i acts freely on $\mathbb{O}_{a_i}^i$. The analogue of Theorem 4.9 remains valid in this case, since it is easy to show that $T_y(\mathrm{im}\,\mathbf{\Gamma}_n) = V_y$ for V_y as in (2-7). □

The following corollary of Theorem 4.2 generalizes [Kostant and Wallach 2006a, Theorem 3.14] to include elements that are not necessarily strongly regular.

Corollary 4.16. *Let $x \in \mathfrak{gl}(n)_c \cap S$. The A-orbit $A \cdot x$ of x is a smooth, irreducible subvariety of $\mathfrak{gl}(n)$ that is isomorphic as an algebraic variety to a closed subgroup $K_{a_1} \times \cdots \times K_{a_{n-1}}$ of the connected algebraic group $Z_1 \times \cdots \times Z_{n-1}$.*

5. Counting A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$

Using Theorem 4.9, we can count the number of A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ for any $c \in \mathbb{C}^{n(n+1)/2}$ and explicitly describe the orbits. We know from Theorem 4.9 and Remark 4.10 that counting the number of A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is equivalent to counting the number of Z_i-orbits in $\Xi^i_{c_i,c_{i+1}}$ on which Z_i acts freely. We show in this section that the number of such orbits is directly related to the number of degeneracies in the roots of the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$; see (1-2). The study of this problem can be reduced to studying the structure of nilpotent solution varieties $\Xi^i_{0,0}$. Thus, we begin our discussion by describing the A-orbit structure of the nilfiber $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$.

5a. ***Nilpotent solution varieties and A-orbits in the nilfiber.*** In this section, we study strongly regular matrices in the fiber $\mathfrak{gl}(n)_0$. By definition, $x \in \mathfrak{gl}(n)_0$ if and only if $x_i \in \mathfrak{gl}(i)$ is nilpotent for all i. Such matrices have been studied by Ovsienko [2003] and Parlett and Strang [2008].

We restate Definition 4.1 of the solution variety $\Xi^i_{c_i,c_{i+1}}$ in this case. Elements of $\mathfrak{gl}(i+1)$ of the form

$$X = \begin{bmatrix} 0 & 1 & \cdots & 0 & y_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & y_i \\ z_1 & \cdots & \cdots & z_i & w \end{bmatrix} \tag{5-1}$$

that are nilpotent define the nilpotent solution variety at level i, which we denote by $\Xi^i_{0,0}$. In this case, it is easy to write down elements in $\Xi^i_{0,0}$. For example, we can take all of the z_j, y_j, and w to be 0. However, such an element is not regular, and so cannot be used to construct a $\boldsymbol{\Gamma}_n$ mapping that gives rise to a strongly regular orbit in $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$. To describe A-orbits in $\mathfrak{gl}(n)_0^{\mathrm{sreg}}$, we focus our attention on free Z_i-orbits in $\Xi^i_{0,0}$; see Theorem 4.9. To find such orbits, we need to compute the characteristic polynomial of X.

Proposition 5.1. *The characteristic polynomial of the matrix in* (5-1) *is*

$$\det(X - t) = (-1)^i\big[-t^{i+1} + wt^i + \textstyle\sum_{l=0}^{i-1}\sum_{j=1}^{i-l} z_j y_{j+l} t^{i-1-l}\big]. \tag{5-2}$$

Proof. We compute the characteristic polynomial of the matrix in (5-1) using the Schur complement formula for the determinant; see [Horn and Johnson 1985, pages 21 and 22]. In the notation of that reference, $\alpha = \{1, \dots, n-1\}$ and $\alpha' = \{n\}$. Let $J = X_i$ denote the principal nilpotent Jordan block. Then the formula gives

$$\det(X - t) = \det(J - t)\,(w - t) - \underline{z}\,\mathrm{adj}(J - t)\,\underline{y}, \tag{5-3}$$

where $\mathrm{adj}(J - t) \in \mathfrak{gl}(i)$ denotes the classical adjoint matrix, $\underline{z} = [z_1, \dots, z_i]$ is a row vector, and $\underline{y} = [y_1, \dots, y_i]^T$ is a column vector. We easily compute that $\det(J - t) = (-1)^i t^i$. It is not difficult to see that

$$\mathrm{adj}(J - t) = (-1)^{i-1} \begin{bmatrix} t^{i-1} & t^{i-2} & \cdots & \cdots & t & 1 \\ 0 & t^{i-1} & t^{i-2} & \cdots & \cdots & t \\ \vdots & 0 & t^{i-1} & \ddots & & \vdots \\ & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & t^{i-2} \\ 0 & \cdots & & \cdots & 0 & t^{i-1} \end{bmatrix}.$$

Now, we compute that the coefficient of t^{i-1-l} for $0 \le l \le i-1$ in the product $\underline{z}\,\mathrm{adj}(J - t)\,\underline{y}^T$ is $(-1)^{i-1} \sum_{j=1}^{i-l} z_j y_{j+l}$. Summing up these terms for $0 \le l \le i-1$ and using (5-3), we obtain the polynomial in (5-2). $\square$

For the matrix in (5-1) to be nilpotent, we require that all of the coefficients of the polynomial in (5-2) (excluding the leading coefficient) vanish, that is

$$\begin{aligned} z_1 y_i &= 0, \\ z_1 y_{i-1} + z_2 y_i &= 0, \\ &\vdots \\ z_1 y_1 + \cdots + z_i y_i &= 0. \end{aligned} \tag{5-4}$$

We claim that $\Xi^i_{0,0}$ has exactly two free Z_i-orbits. These correspond to choosing either $z_1 \in \mathbb{C}^\times$ and $y_i = 0$, or $y_i \in \mathbb{C}^\times$ and $z_1 = 0$ in the first equation of (5-4). We claim that any point in $\Xi^i_{0,0}$ with $z_1 \neq 0$ is in

$$\mathbb{O}^i_L = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ z_1 & \cdots & \cdots & z_i & 0 \end{bmatrix}, \tag{5-5}$$

with $z_j \in \mathbb{C}$ for $2 \le j \le i$. Any point in $\Xi^i_{0,0}$ with $y_i \in \mathbb{C}^\times$ is in

$$\mathbb{O}^i_U = \begin{bmatrix} 0 & 1 & \cdots & 0 & y_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & y_i \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{5-6}$$

with $y_j \in \mathbb{C}$ for $1 \le j \le i-1$. To verify this claim, note that if $z_1 \ne 0$ and $y_i = 0$, then $y_1 = 0$ and $y_2 = 0, \ldots, y_{i-1} = 0$ by successive use of equations (5-4). The case $y_i \ne 0$ and $z_1 = 0$ is similar. An easy computation in linear algebra, as in the proof of Lemma 4.14 gives that Z_i acts freely on $\mathbb{O}^i_U$ and $\mathbb{O}^i_L$. We think of $\mathbb{O}^i_U$ as the "upper orbit" in $\Xi^i_{0,0}$ and $\mathbb{O}^i_L$ as the "lower orbit". Both orbits consist of regular elements of $\mathfrak{gl}(i+1)$ by Theorem 4.13.

Now, suppose that both $z_1 = 0 = y_i$ in (5-4). It is easy to see that such an element has a nontrivial isotropy group in Z_i containing the one-dimensional subgroup of matrices consisting of identity matrices with an element $c \in \mathbb{C}^\times$ inserted in the upper right corner. It does not belong to a Z_i-orbit of dimension i.

Thus, to analyze $\mathfrak{gl}(n)_0^{\text{sreg}}$, we consider only the Z_i-orbits $\mathbb{O}^i_U, \mathbb{O}^i_L$. We can construct 2^{n-1} morphisms $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\ldots,a_{n-1}}$, where $\mathbb{O}^i_{a_i} = \mathbb{O}^i_U, \mathbb{O}^i_L$ for $1 \le i \le n-1$.

The following result follows immediately from Theorems 4.9 and 4.12 and Remark 4.10.

Theorem 5.2. *The nilfiber $\mathfrak{gl}(n)_0^{\text{sreg}}$ contains 2^{n-1} A-orbits. On $\mathfrak{gl}(n)_0^{\text{sreg}}$, the orbits of A are orbits of a free action of the algebraic group $(\mathbb{C}^\times)^{n-1} \times \mathbb{C}^{n(n-1)/2-n+1}$.*

The nilfiber has much more structure than Theorem 5.2 indicates, which we can see by considering an example of an A-orbit given as the image of a morphism $\mathbf{\Gamma}_n$ with $\mathbb{O}^i_{a_i} = \mathbb{O}^i_U,\ \mathbb{O}^i_L$ and its closure. Closure here means either closure in the Zariski topology in $\mathfrak{gl}(n)$ or in the Euclidean topology, since A-orbits are constructible sets these two different types of closure agree; see [Kostant and Wallach 2006a, Theorem 3.7]. We will abbreviate from now on

$$\mathbb{O}^i_{a_i} = a_i, \quad \mathbb{O}^i_L = L, \quad \mathbb{O}^i_U = U.$$

Example 5.3. Let us take our A-orbit in $\mathfrak{gl}(4)_0^{\text{sreg}}$ to be the image of $\Gamma_4^{a_1,a_2,a_3}$ with $a_1 = L$, $a_2 = L$ and $a_3 = U$. For coordinates, let us take

$$\begin{array}{lll} z_1 \in \mathbb{C}^\times & & \text{for } \mathbb{O}^1_L, \\ z_2 \in \mathbb{C}^\times, & z_3 \in \mathbb{C} & \text{for } \mathbb{O}^2_L, \\ y_1, y_2 \in \mathbb{C}, & y_3 \in \mathbb{C}^\times & \text{for } \mathbb{O}^3_U. \end{array}$$

In these coordinates, we compute that $\operatorname{im}\Gamma_4^{L,L,U}$ is

$$\operatorname{im}\Gamma_4^{L,L,U} = \begin{bmatrix} 0 & 0 & 0 & y_3/(z_1 z_2) \\ z_1 & 0 & 0 & y_2/z_2 - y_3 z_3/z_2^2 \\ z_1 z_3 & z_2 & 0 & y_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5-7}$$

We compute the closure as

$$\overline{\operatorname{im}\Gamma_4^{L,L,U}} = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & a_3 \\ a_4 & a_5 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5-8}$$

with $a_i \in \mathbb{C}$ for $1 \le i \le 6$. It is a nilradical of a Borel subalgebra that contains the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(4)$. The easiest way to see this is to note that the strictly lower triangular matrices in $\mathfrak{gl}(4)$ are conjugate to it by the permutation $\tau = (1432)$.

This example illustrates that the A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ are essentially parametrized by prescribing whether or not the $i \times i$ cutoff of an element $x \in \mathfrak{gl}(n)_0$ has zeroes in its i-th column or zeroes in its i-th row. This is because for an $x \in \mathfrak{gl}(n)_0$ to be in the image of a morphism $\boldsymbol{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$ with $a_i = L, U$, the i-th row or the i-th column of x_i must entirely consist of zeroes for each i by Proposition 4.3.

Contrast this with the following example of a matrix $x \in \mathfrak{gl}(n)_0$ each of whose cutoffs is regular, but that is not itself strongly regular.

Example 5.4. Consider $x \in \mathfrak{gl}(4)_0$ defined by

$$x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ y_1 & 0 & 0 & 0 \end{bmatrix}, \tag{5-9}$$

where $x_2 \in \mathbb{C}^\times$, $y_1 \in \mathbb{C}^\times$ and $x_3 \in \mathbb{C}$. Both the 4-th column and row of this matrix have nonzero entries. Thus, this matrix cannot be in the image of a morphism $\boldsymbol{\Gamma}_n$ with $a_i = L, U$ and is not strongly regular. However, one can easily check that each cutoff of this matrix is regular, so that $x \in \mathfrak{gl}(4)_0 \cap S$. Thus, $\mathfrak{gl}(4)_0^{\text{sreg}}$ is a proper subset of $\mathfrak{gl}(4)_0 \cap S$. (One can also see that this matrix is not strongly regular directly by observing that $\mathfrak{z}_{\mathfrak{gl}(3)}(x_3) \cap \mathfrak{z}_{\mathfrak{gl}(4)}(x) \neq 0$.)

Example 5.3 demonstrates that although the A-orbits $\operatorname{im}\boldsymbol{\Gamma}_n$ may be complicated, their closures are relatively simple. In this example, the closure is a nilradical of a Borel subalgebra that contains the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(n)$. This is in fact the case in general.

Theorem 5.5. *Let $x \in \mathfrak{gl}(n)_0^{\text{sreg}}$ and let $A \cdot x$ denote the A-orbit of x. Then $\overline{A \cdot x}$ is a nilradical of a Borel subalgebra in $\mathfrak{gl}(n)$ that contains the standard Cartan subalgebra of diagonal matrices. More explicitly, if the A-orbit is given by $\mathbf{\Gamma}_n = \Gamma_n^{a_1,a_2,\dots,a_{n-1}}$, where $a_i = U$ or L for $1 \le i \le n-1$, then $\overline{A \cdot x}$ is the set of matrices of the form*

$$\mathfrak{n}_{a_1,\dots,a_{n-1}} := \left\{ x : x_{i+1} = \begin{bmatrix} & b_1 \\ x_i & \vdots \\ & b_i \\ 0 & 0 \end{bmatrix} \right\} \qquad \text{if } a_i = U, \text{ or}$$

$$\mathfrak{n}_{a_1,\dots,a_{n-1}} := \left\{ x : x_{i+1} = \begin{bmatrix} & x_i & & 0 \\ b_1 & \cdots & b_i & 0 \end{bmatrix} \right\} \qquad \text{if } a_i = L$$

with $b_j \in \mathbb{C}$.

Proof. Let $x \in \mathfrak{gl}(n)_0^{\text{sreg}}$. By Gerstenhaber's theorem [1958], it suffices to show the second statement of the theorem. Then $\overline{A \cdot x}$ is a linear space consisting of nilpotent matrices of dimension $n(n-1)/2$ and is clearly normalized by the diagonal matrices in $\mathfrak{gl}(n)$.

Suppose that $A \cdot x = \operatorname{im} \mathbf{\Gamma}_n$ with $a_i = U,\ L$. Since $A \cdot x$ is an irreducible variety of dimension $n(n-1)/2$, $\overline{A \cdot x} \subset \mathfrak{n}_{a_1,\dots,a_{n-1}}$ is an irreducible, closed subvariety of dimension $n(n-1)/2 = \dim \mathfrak{n}_{a_1,\dots,a_{n-1}}$, and therefore $\overline{A \cdot x} = \mathfrak{n}_{a_1,\dots,a_{n-1}}$. □

Remark 5.6. The set of strictly lower triangular matrices $\mathfrak{n}^-$ is the closure of the A-orbit $\Gamma_n^{L,\dots,L}$, and the set of strictly upper triangular matrices $\mathfrak{n}^+$ is the closure of the A-orbit $\Gamma_n^{U,\dots,U}$.

By Theorem 5.5, the A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ give rise to 2^{n-1} Borel subalgebras of $\mathfrak{gl}(n)$ that contain the diagonal matrices. Moreover, each of the nilradicals $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ is conjugate to the strictly lower triangular matrices by a unique permutation in $\mathcal{S}_n$, the symmetric group on n letters. The A-orbits in $\mathfrak{gl}(n)_0^{\text{sreg}}$ thus determine 2^{n-1} permutations. We now describe these permutations.

Theorem 5.7. *Let $\mathfrak{n}^-$ denote the strictly lower triangular matrices in $\mathfrak{gl}(n)$ and let $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ be as in Theorem 5.5. Then $\mathfrak{n}_{a_1,\dots,a_{n-1}}$ is obtained from $\mathfrak{n}^-$ by conjugating by a permutation $\sigma = \tau_1\tau_2 \cdots \tau_{n-1}$, where $\tau_i \in \mathcal{S}_{i+1}$ is either the long element $w_{i,0}$ of $\mathcal{S}_{i+1}$ or the identity permutation, id_i. The τ_i are determined by the values of a_i as follows. Let $a_n = L$. Starting with $i = n-1$, we compare a_i and a_{i+1}. If $a_i = a_{i+1}$, then $\tau_i = id_i$, but if $a_i \neq a_{i+1}$, then $\tau_i = w_{0,i}$.*

The same procedure beginning with $a_n = U$ produces a permutation that conjugates the strictly upper triangular matrices $\mathfrak{n}^+$ into $\mathfrak{n}_{a_1,\dots,a_{n-1}}$.

Before proving Theorem 5.7, let us see it in action in Example 5.3. In that case the nilradical in (5-8) is $\mathfrak{n}_{L,L,U}$. Thus, according to Theorem 5.7, $\sigma = (13)(14)(23)$, the product of the long elements for $\mathcal{S}_3$ and $\mathcal{S}_4$. Notice that $\sigma = (1432)$, which is precisely the permutation that we observed conjugates the strictly lower triangular matrices in $\mathfrak{gl}(4)$ into $\mathfrak{n}_{L,L,U}$ in Example 5.3.

Proof of Theorem 5.7. Let $\pi_i : \mathfrak{gl}(n) \to \mathfrak{gl}(i)$ be the projection $\pi_i(x) = x_i$. For any subset $S \subset \mathfrak{gl}(n)$, we will denote by S_i the image $\pi_i(S)$.

Suppose that $L = a_n = a_{n-1} = \cdots = a_{i+1}$, but $a_i = U$. Conjugating $\mathfrak{n}^-$ by $\tau_i = w_{0,i}$ produces the nilradical $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ with $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1} = \mathfrak{n}^+_{i+1}$. Thus, $(\mathfrak{n}_{a_1,\ldots,a_{n-1}})_{i+1}$ and $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1}$ now have the same $(i+1)$-st columns. We also note that the components of $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ and $\mathfrak{n}_{a_1,\ldots,a_{n-1}}$ in $\mathfrak{gl}(i+1)^{\perp}$ also agree, since τ_i permutes the strictly lower triangular entries of the rows below the $(i+1)$-st row of $\mathfrak{n}^-$ amongst themselves. Now, we start the procedure again with $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1}$ and $a_i = U$ and use induction. We note that conjugating $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ by a permutation in $\mathcal{S}_k$ with $k \le i+1$ leaves the component of $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ in $\mathfrak{gl}(i+1)^{\perp}$ unchanged. This proves the theorem. □

Remark 5.8. A related result is [Parlett and Strang 2008, Lemma 1, page 1736].

5b. ***General solution varieties $\Xi^i_{c_i,c_{i+1}}$ and counting A-orbits in $\mathfrak{gl}(n)^{\mathrm{sreg}}_c$.*** Now, we use our understanding of the nilpotent case to count A-orbits in the general case. Recall the definition of the solution variety $\Xi^i_{c_i,c_{i+1}}$ in Section 4a. We also recall some notation. Given $c \in \mathbb{C}^{n(n+1)/2}$, we write $c = (c_1, \ldots, c_i, \ldots, c_n)$ with $c_i = (z_1, \ldots, z_i) \in \mathbb{C}^i$ and define a corresponding monic polynomial $p_{c_i}(t)$ with coefficients given by c_i; see (1-2). Recall that $J = J_1 \oplus \cdots \oplus J_r$, where $J_k \in \mathfrak{gl}(n_k)$, denotes the regular Jordan form that is the $i \times i$ cutoff of the matrix in (4-1). We now describe the Z_i-orbit structure of the variety $\Xi^i_{c_i,c_{i+1}}$ for any $c_i \in \mathbb{C}^i$ and $c_{i+1} \in \mathbb{C}^{i+1}$.

As in the nilpotent case, to understand $\Xi^i_{c_i,c_{i+1}}$ we must compute the characteristic polynomial of the matrix in (4-1).

Proposition 5.9. *The characteristic polynomial of the matrix in* (4-1) *is*

$$\begin{aligned}(w-t)\prod_{k=1}^{r}(\lambda_k-t)^{n_k}\\ +\sum_{j=1}^{r}\Bigl((-1)^{n_j}\prod_{k=1,k\neq j}^{r}(\lambda_k-t)^{n_k}\sum_{l=0}^{n_j-1}\sum_{j'=1}^{n_j-l} z_{j,j'}y_{j,j'+l}(t-\lambda_j)^{n_j-1-l}\Bigr).\end{aligned} \tag{5-10}$$

The proof of this proposition reduces to the case where J is a single Jordan block of eigenvalue λ. The case of a single Jordan block follows easily from the nilpotent case in Proposition 5.1 by a simple change of variables.

We need to understand the conditions that w, $z_{i,j}$, and $y_{i,j}$ must satisfy so that polynomial in (5-10) is equal to the monic polynomial $p_{c_{i+1}}(t)$. The first is easily determined by considering the trace of the matrix in (4-1). The values of the $z_{i,j}$ and the $y_{i,j}$ are directly related to the number of roots in common between the polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Suppose that the polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have j roots in common, where $1 \le j \le r$. Then we claim that $\Xi^i_{c_i,c_{i+1}}$ has precisely 2^j free Z_i-orbits. Consider the Jordan block corresponding to the eigenvalue λ_k. First, suppose that λ_k is a root of $p_{c_{i+1}}(t)$. Then Proposition 5.9 implies

$$z_{k,1} y_{k,n_k} = 0. \tag{5-11}$$

However, if λ_k is not a root of $p_{c_{i+1}}(t)$, then Proposition 5.9 gives

$$z_{k,1} y_{k,n_k} \in \mathbb{C}^\times. \tag{5-12}$$

As in the nilpotent case, (5-11) gives rise to two separate cases.

$$z_{k,1} \in \mathbb{C}^\times \quad \text{and} \quad y_{k,n_k} = 0 \tag{5-13}$$

and

$$y_{k,n_k} \in \mathbb{C}^\times \quad \text{and} \quad z_{k,1} = 0. \tag{5-14}$$

In case (5-13), we can argue using (5-10) that the coordinates $y_{k,i}$ for $1 \le i \le n_k$ can be solved uniquely as regular functions of $z_{k,1} \in \mathbb{C}^\times$ and $z_{k,2}, \dots, z_{k,n_k} \in \mathbb{C}$. In case (5-14), we can solve for $z_{k,i}$ as regular functions of $y_{k,n_k} \in \mathbb{C}^\times$ and $y_{k,i} \in \mathbb{C}$ for $1 \le i \le n_k - 1$. In the case of (5-12), we can take either the $z_{k,i}$ as coordinates that determine the $y_{k,i}$ or vice versa. For concreteness, we take $y_{k,i} = p_i(z_{k,1}, \dots, z_{k,n_k})$ to be regular functions of $z_{k,1} \in \mathbb{C}^\times$ and $z_{k,2}, \dots, z_{k,n_k} \in \mathbb{C}$.

Remark 5.10. The solutions in the cases of (5-11) and (5-12) are obtained by setting the derivatives of the polynomial in (5-10) up to order $n_p - 1$ evaluated at λ_p equal to the corresponding derivatives of the polynomial $p_{c_{i+1}}(t)$ evaluated at λ_p for $1 \le p \le r$. This produces r systems of linear equations. Each system involves only the coordinates $z_{p,k}$ and $y_{p,k}$ from the p-th Jordan block. This follows directly from the fact that the eigenvalues λ_s are all distinct. Each system can then be solved inductively using the fact that the coefficient of $(-1)^{n_p}(t-\lambda_p)^q \prod_{k=1,\, k\neq p}^r (\lambda_k - t)^{n_r}$ is given by the $(n-q)$-th row of the matrix product

$$\begin{bmatrix} z_{p,1} & z_{p,2} & \cdots & z_{p,n_p} \\ 0 & z_{p,1} & \ddots & \vdots \\ \vdots & & \ddots & z_{p,2} \\ 0 & \cdots & 0 & z_{p,1} \end{bmatrix} \cdot \begin{bmatrix} y_{p,1} \\ \vdots \\ \vdots \\ y_{p,n_p} \end{bmatrix}. \tag{5-15}$$

Recall that Z_i is the direct product $Z_i = Z_{J_1} \times \cdots \times Z_{J_r}$, with Z_{J_s} the centralizer of J_s. The adjoint action of Z_i on $\Xi^i_{c_i,c_{i+1}}$ is a diagonal action in which Z_{J_s} acts only on the columns and rows of an $x \in \Xi^i_{c_i,c_{i+1}}$ containing J_s. This observation allowed us to decompose a Z_i-orbit $\mathbb{O}$ into the product $\mathbb{O}_k \subset \mathbb{C}^{2n_k}$ of Z_{J_k}-orbits as in (4-15). If λ_k is a root of $p_{c_{i+1}}(t)$, then (5-11) gives rise to two free Z_{J_k}-orbits, an "upper" orbit $\mathbb{O}_{k,U}$ in the case of (5-14) and a "lower" orbit $\mathbb{O}_{k,L}$ in the case of (5-13). This is proved similarly to the nilpotent case. If on the other hand, λ_k is not a root of $p_{c_{i+1}}(t)$, and we have (5-12), then the vector

$$(5\text{-}16)\quad ([z_{k,1},\dots,z_{k,n_k}],[p_1(z_{k,1},\dots,z_{k,n_k}),\dots,p_k(z_{k,1},\dots,z_{k,n_k})]^T)\in\mathbb{C}^{2n_k}$$

is a free Z_{J_k}-orbit under the action of Z_{J_k} defined in (4-14). Thus, using the orbits $\mathbb{O}_{k,U}$ and $\mathbb{O}_{k,L}$ for $1 \le k \le j$, we can construct 2^j free Z_i-orbits in $\Xi^i_{c_i,c_{i+1}}$ by (4-15).

Now, using Theorem 4.13, we can construct $2^{\sum_{i=1}^{n-1} j_i}$ $\Gamma_n^{a_1,a_2,\dots,a_{n-1}}$ morphisms into $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$, where j_i is the number of roots in common to the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. The following result follows immediately from Theorem 4.9 and Theorem 4.12 and Remark 4.10.

Theorem 5.11. *Let $c = (c_1, c_2, \dots, c_i, c_{i+1}, \dots, c_n) \in \mathbb{C}^{n(n+1)/2}$. Suppose there are $0 \le j_i \le i$ roots in common between the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Then the number of A-orbits in $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ is exactly $2^{\sum_{i=1}^{n-1} j_i}$. Further, on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ the orbits of A are the orbits of a free algebraic action of the commutative, connected algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$.*

Remark 5.12. A similar result is obtained in [Bielawski and Pidstrygach 2008]. See Remark 1.3 in the introduction.

Theorem 5.11 lets us identify exactly where the action of the group A is transitive on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$. Let Θ_n be the set of $c \in \mathbb{C}^{n(n+1)/2}$ such that the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common. From [Kostant and Wallach 2006a, Remark 2.16], it follows that $\Theta_n \subset \mathbb{C}^{n(n+1)/2}$ is Zariski principal open.

Corollary 5.13. *The action of A is transitive on $\mathfrak{gl}(n)_c^{\mathrm{sreg}}$ if and only if $c \in \Theta_n$.*

Remark 5.14. We will see in the next section that $\mathfrak{gl}(n)_c^{\mathrm{sreg}} = \mathfrak{gl}(n)_c$ for $c \in \Theta_n$. Thus, the fiber $\mathfrak{gl}(n)_c$ consists entirely of strongly regular elements.

Corollary 5.13 allows us to enlarge the set of generic matrices $\mathfrak{gl}(n)_\Omega$ studied by Kostant and Wallach.

5c. *The new set of generic matrices $\mathfrak{gl}(n)_\Theta$.* We can expand the set of matrices $\mathfrak{gl}(n)_\Omega$ studied by Kostant and Wallach by relaxing the condition that each cutoff is regular semisimple. More precisely, let $\sigma(x_i)$ denote the spectrum of $x_i \in \mathfrak{gl}(i)$, where x_i is viewed as an $i \times i$ matrix. We define a Zariski open subset of elements of $\mathfrak{gl}(n)$ by $\mathfrak{gl}(n)_\Theta = \{x \in \mathfrak{gl}(n) \mid \sigma(x_{i-1}) \cap \sigma(x_i) = \varnothing, 2 \le i \le n\}$. Clearly, $\mathfrak{gl}(n)_\Theta = \bigcup_{c\in\Theta_n} \mathfrak{gl}(n)_c$.

Theorem 5.15. *The elements of $\mathfrak{gl}(n)_\Theta$ are strongly regular and hence $\mathfrak{gl}(n)_c^{\mathrm{sreg}} = \mathfrak{gl}(n)_c$ for $c \in \Theta_n$. Moreover, $\mathfrak{gl}(n)_\Theta$ is the maximal subset of $\mathfrak{gl}(n)$ for which the action of A is transitive on the fibers of Φ.*

Proof. If $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ are relatively prime polynomials, then we claim $\Xi^i_{c_i,c_{i+1}}$ is exactly one free Z_i-orbit. Indeed, in this case we only have the conditions (5-12) for $1 \le k \le r$. Thus, we can apply our observation in (5-16) to see that $\Xi^i_{c_i,c_{i+1}}$ is one free Z_i-orbit and hence consists of regular elements of $\mathfrak{gl}(i+1)$ by Theorem 4.13. Given $x \in \mathfrak{gl}(n)_c$ with $c \in \Theta_n$, we claim that $x \in \operatorname{im} \Gamma_n^{a_1,a_2,\ldots,a_{n-1}}$ with $a_i = \Xi^i_{c_i,c_{i+1}}$ for $1 \le i \le n-1$. Indeed, $x_2 \in \Xi^1_{c_1,c_2}$ and is therefore regular. Thus, by Remark 4.4, there exists a $g_2 \in GL(2)$ such that $(\mathrm{Ad}(g_2)\cdot x)_3 = (\mathrm{Ad}(g_2)\cdot x_3) \in \Xi^2_{c_2,c_3}$. Now, suppose $x_{i+1} \in \mathrm{Ad}(GL(i))\cdot \Xi^i_{c_i,c_{i+1}}$. Thus, $x_{i+1} \in \mathfrak{gl}(i+1)$ is regular and Remark 4.4 provides a $g_{i+1} \in GL(i+1)$ such that $(\mathrm{Ad}(g_{i+1})\cdot x)_{i+2} = \mathrm{Ad}(g_{i+1})\cdot x_{i+2} \in \Xi^{i+1}_{c_{i+1},c_{i+2}}$. By induction, $x_{j+1} \in \mathrm{Ad}(GL(j))\cdot \Xi^j_{c_j,c_{j+1}}$ for any j in $1 \le j \le n-1$. Proposition 4.3 implies that $x \in \operatorname{im} \mathbf{\Gamma}_n$. Thus, $\mathfrak{gl}(n)_\Theta \subset \mathfrak{gl}(n)^{\mathrm{sreg}}$ by Theorem 4.9. The rest of the theorem follows from Corollary 5.13. □

Remark 5.16. The strictly upper triangular part of a matrix $x \in \mathfrak{gl}(n)_c$ where $c \in \Theta_n$ is determined by its strictly lower triangular part. This follows from the definition of the morphisms $\mathbf{\Gamma}_n$ and the fact that all of the $y_{k,i}$ can be solved uniquely as regular functions of the $z_{k,i}$ for $1 \le i \le n_k$ and $1 \le k \le r$.

Because elements of $\mathfrak{gl}(n)_\Theta$ are strongly regular, we have the following:

Corollary 5.17. *Let $x \in \mathfrak{gl}(n)_\Theta$. Then $x_i \in \mathfrak{gl}(i)$ is regular for all i.*

Using Corollary 5.13 and Theorem 5.11, we can directly generalize [Kostant and Wallach 2006a, Theorem 3.23] for the case of Θ_n.

Corollary 5.18. *For $c \in \Theta_n \subset \mathbb{C}^{n(n+1)/2}$, we have $\mathfrak{gl}(n)_c \cong Z_1 \times \cdots \times Z_{n-1}$ as algebraic varieties.*

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MARK COLARUSSO
DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE
UNIVERSITÉ LAVAL
1045 AV DE LA MÉDECINE
QUÉBEC, QC G1V 0A6
CANADA

mark.colarusso.1@ulaval.ca
http://archimede.mat.ulaval.ca/pages/markcola/

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ON MASLOV CLASS RIGIDITY FOR COISOTROPIC SUBMANIFOLDS

VIKTOR L. GINZBURG

We define the Maslov index of a loop tangent to the characteristic foliation of a coisotropic submanifold as the mean Conley–Zehnder index of a path in the group of linear symplectic transformations, incorporating the "rotation" of the tangent space of the leaf — this is the standard Lagrangian counterpart — and the holonomy of the characteristic foliation. We also show that, with this definition, the Maslov class rigidity extends to the class of the so-called stable coisotropic submanifolds including Lagrangian tori and stable hypersurfaces.

1. Introduction and main results

1.1. ***Introduction.*** As the title indicates, the main theme of the paper is the Maslov class rigidity for coisotropic submanifolds. To be more specific, we define the Maslov index of a loop tangent to the characteristic foliation in a coisotropic submanifold and show that a displaceable, stable coisotropic submanifold carries a loop with Maslov index in the range $[1,\ 2n+1-k]$, where $2n$ is the dimension of the ambient manifold and k is the codimension of the coisotropic submanifold.

The study of symplectic topology of coisotropic submanifolds can be traced back to [Moser 1978] followed by [Banyaga 1980; Ekeland and Hofer 1989; Hofer 1990] and by the work of Bolle [1996; 1998]. Recently, the field has entered a particularly active phase; see [Albers and Frauenfelder 2010; 2008; Dragnev 2008; Ginzburg 2007; Gürel 2010; $\geq$ 2011; Kang 2009; Kerman 2008; Tonnelier 2010; Usher 2009; Ziltener 2010; 2009]. Most of these papers, with the exception of [Ziltener 2009], concern such questions as generalizations to coisotropic submanifolds of the Lagrangian intersection property or of the existence of closed characteristics on stable hypersurfaces. The present work, which can be thought of as a follow-up to [Ginzburg 2007], focuses mainly on the coisotropic version of the Maslov class rigidity, also considered in [Ziltener 2009].

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The aspect of the Maslov class rigidity we are concerned with here is the fact that the Maslov class of a closed displaceable Lagrangian submanifold automatically satisfies certain restrictions. Namely, the minimal Maslov number of such a submanifold lies between 1 and $n+1$. This phenomenon was originally studied in [Polterovich 1991a; 1991b; Viterbo 1990] and there are two methods of proving results of this type. One of these methods uses the holomorphic curves technique (see [Audin et al. 1994; Polterovich 1991a; 1991b]) and at this moment it is not known how to directly apply it to coisotropic submanifolds due to the lack of Fredholm properties for the Cauchy–Riemann problem with coisotropic boundary conditions. The second approach, originating from [Viterbo 1990], relies on Hamiltonian Floer homology (or its equivalent) and in combination with certain estimates from [Bolle 1998] can be easily adapted to the coisotropic setting; see, for example, [Ginzburg 2007]. Here, we heavily draw from the modern interpretation of this method given in [Kerman 2009; Kerman and Şirikçi 2010].

The Maslov index of a loop tangent to the characteristic foliation is the mean Conley–Zehnder index of a certain path in $\mathrm{Sp}(2n)$ associated with the loop and comprising the "rotation" of the tangent space of the leaf, as the standard Lagrangian counterpart, and the holonomy of the characteristic foliation. Hence, the index can be an arbitrary real number. This definition, which can also be found in [Ziltener 2009], where it is treated in great detail, is of independent interest. Then, the proof of the Maslov class rigidity for coisotropic submanifolds follows the path of [Kerman 2009; Kerman and Şirikçi 2010; Viterbo 1990]. The main new element of the proof is that we circumvent relating the Conley–Zehnder and Morse indices as in [Duistermaat 1976; Viterbo 1990]; instead we use the explicit expression for the geodesic flow of a metric, capitalizing on the fact that the submanifolds in question are stable and hence admit a leaf-wise flat metric.

1.2. ***Coisotropic Maslov index.*** Let M be a coisotropic submanifold of a symplectic manifold (W^{2n}, ω). Denote by $\mathcal{F}$ the characteristic foliation of M; see Section 2.1 for the definition. The normal bundle $T^\perp M$ to M is canonically isomorphic to the (leaf-wise) cotangent bundle $T^*\mathcal{F}$ to $\mathcal{F}$ and the direct sum $T\mathcal{F}\oplus T^\perp M$ is a symplectic vector bundle over M. We have a symplectic vector bundle decomposition

$$TW|_M = (T\mathcal{F}\oplus T^\perp M)\oplus T^\perp\mathcal{F}, \tag{1-1}$$

where $T^\perp\mathcal{F}$ is the normal bundle to $\mathcal{F}$ in M. Note that $T^\perp\mathcal{F}$ carries a symplectic leaf-wise flat connection.

Consider a loop $\gamma : S^1 \to M$ tangent to $\mathcal{F}$, contractible in W and equipped with a capping $u : D^2 \to W$. The capping u gives rise to a symplectic trivialization ζ, unique up to homotopy, of the pull-back bundle γ^*TW. Let us assume first that

$T\mathcal{F}$ is orientable along γ (i.e., the pull-back $\gamma^*T\mathcal{F}$ is orientable), and hence trivial, and fix a trivialization ξ of this vector bundle. Then the pull-back $\gamma^*(T\mathcal{F}\oplus T^\perp M)$ receives a symplectic trivialization $\xi\oplus\xi^*$. This trivialization can be viewed as a family of symplectic maps $\Xi(t): T_{\gamma(0)}\mathcal{F}\oplus T^\perp_{\gamma(0)}M \to T_{\gamma(t)}\mathcal{F}\oplus T^\perp_{\gamma(t)}M$ parametrized by $t\in S^1$. Combining the family $\Xi(t)$ with the holonomy $\Gamma(t): T^\perp_{\gamma(0)}\mathcal{F}\to T^\perp_{\gamma(t)}\mathcal{F}$ along γ, we obtain a family of symplectic maps $\Xi(t)\oplus\Gamma(t): T_{\gamma(0)}W \to T_{\gamma(t)}W$, which, using the trivialization ζ, we can regard as a path $\Phi:[0,\ 1]\to \mathrm{Sp}(2n)$.

Definition 1.1. The *coisotropic Maslov index* $\mu(\gamma, u)$ of the capped loop (γ, u) is the negative mean Conley–Zehnder index $-\Delta(\Phi)\in\mathbb{R}$. (We refer the reader to [Long 2002; Salamon and Zehnder 1992] for a detailed discussion of the mean index; here we use the notation and conventions from [Ginzburg and Gürel 2009]; see Section 2.2.) When $T\mathcal{F}$ is not orientable along γ, we set $\mu(\gamma, u) := \mu(\gamma^2, u^2)/2$, where (γ^2, u^2) stands for the double cover of (γ, u).

The standard argument shows that the index $\mu(\gamma, u)$ is well defined, that is, independent of the choice of the trivializations ξ and ζ. It is also independent of the choice of splitting (1-1): the normal bundle $T^\perp\mathcal{F}$ is unambiguously defined only as the quotient $TW/T\mathcal{F}$ while the splitting requires a choice of the complement to $T\mathcal{F}$ in TW. To see that $\Delta(\Phi)$ is independent of this choice, we argue as follows; see the proof of [Ginzburg and Gürel 2009, Lemma 2.6]. Observe that the path $\tilde{\Phi}$ resulting from a different splitting is homotopic to the concatenation of the path Φ with a path Ψ of the form $\Psi(t) = I + A(t)$, where I is the identity map and $A(t): T^\perp\mathcal{F}\to(T\mathcal{F}\oplus T^\perp M)$. Thus, all eigenvalues of $\Psi(t)$ are equal to one and, as a consequence, $\Delta(\Psi)=0$. Hence, by the additivity and homotopy invariance of the mean index [Ginzburg and Gürel 2009; Long 2002; Salamon and Zehnder 1992], we have $\Delta(\tilde{\Phi}) = \Delta(\Phi)$.

It is worth emphasizing that, in contrast with the ordinary Lagrangian Maslov index, the coisotropic Maslov index is not, in general, an integer and that this index is different from the one considered in [Oh 2003]. The negative sign in the definition of the coisotropic Maslov index is, of course, a matter of conventions: this is the price we have to pay to match the sign of the standard Maslov index for Lagrangian submanifolds (Example 1.2) while using the conventions from [Ginzburg and Gürel 2009]; see Section 2.2.

It is easy to see that the coisotropic Maslov index has the following properties:

- Homotopy invariance: $\mu(\gamma, u)$ is invariant, in the obvious sense, under a homotopy of γ in a leaf of $\mathcal{F}$. In particular, $\mu(\gamma, u) = 0$ when u is homotopic (rel boundary) to a disc in the leaf of $\mathcal{F}$ containing γ.
- Recapping: $\mu(\gamma, u\#v) = \mu(\gamma, u) - 2\,\langle c_1(TW), v\rangle$, where the capping $u\#v$ is obtained by attaching the sphere $v\in\pi_2(W)$ to u. In particular, $\mu(\gamma) := \mu(\gamma, u)$ is independent of u when $c_1(TW)\,|_{\pi_2(W)} = 0$.

- Homogeneity: $\mu(\gamma^k, u^k) = k\mu(\gamma, u)$, where (γ^k, u^k) stands for the k-fold cover of (γ, u). Moreover, when $c_1(TW)|_{\pi_2(W)} = 0$, the Maslov index gives rise to a homogeneous quasimorphism $\pi_1(F) \to \mathbb{R}$ for any leaf F of $\mathcal{F}$.

Example 1.2. When M is a Lagrangian submanifold of W, the foliation $\mathcal{F}$ has only one leaf, the manifold M itself, and the coisotropic Maslov index coincides with the ordinary Maslov index. Indeed, in this case, Definition 1.1 turns into one of the definitions of this index.

Example 1.3. When u is contained in M, the index $\mu(\gamma, u)$ is equal to the mean index of the holonomy along γ with respect to a symplectic trivialization of $T^{\perp}\mathcal{F}$ associated with u. For instance, when M is a regular level of a Hamiltonian and γ is a periodic orbit (and again u is contained in M), the Maslov index $\mu(\gamma, u)$ is equal to the mean index of γ in M.

Example 1.4. When all leaves of $\mathcal{F}$ are closed and form a fibration, the path Φ is a loop and $\mu(\gamma, u)$ is equal to the Maslov index of this loop. (In particular, then $\mu(\gamma, u)$ is an integer.) In this setting, the coisotropic Maslov index is introduced and investigated by Ziltener [2009]. Furthermore, one can express the coisotropic Maslov index via the Lagrangian Maslov index in the graph of $\mathcal{F}$; see [Ziltener 2009; 2010] for details.

Now we are in a position to state the main result of the paper. A much more detailed discussion of the coisotropic Maslov index can be found in [Ziltener 2009].

1.3. *Rigidity of the coisotropic Maslov index.* Let W be a symplectically aspherical manifold, which we assume to be either closed or geometrically bounded and wide (e.g., convex at infinity) in the sense of [Gürel 2008].

Theorem 1.5. Let W^{2n} be as above and let $M^{2n-k} \subset W$ be a closed, stable, displaceable coisotropic submanifold. (See Section 2.1 for the definitions.) Then, for any $\delta > 0$, there exists a loop η tangent to $\mathcal{F}$ and contractible in W and such that

$$1 \leq \mu(\eta) \leq 2n + 1 - k, \tag{1-2}$$

$$0 < \mathrm{Area}(\eta) \leq e(M) + \delta, \tag{1-3}$$

where $\mathrm{Area}(\eta)$ is the symplectic area bounded by η and $e(M)$ is the displacement energy of M.

Example 1.6. As in Example 1.2, assume that M is a stable Lagrangian submanifold (and hence a torus). Then $k = n$ and the theorem reduces to a particular case of the standard Lagrangian Maslov class rigidity. This version of rigidity is established in [Viterbo 1990] for $W = \mathbb{R}^{2n}$ and in [Kerman 2009; Kerman and Şirikçi 2010] for closed ambient manifolds; see also [Audin et al. 1994; Polterovich 1991a; 1991b] for generalizations.

Example 1.7. Assume that M is a stable, displaceable, simply connected hypersurface. Then, by (1-2) and Example 1.3, M carries a closed characteristic η with $1 \leq \Delta(\gamma) \leq 2n$. This is apparently a new observation. However, if we replace the upper bound by $2n + 1$, the assertion becomes an easy consequence of the properties of the mean index and, for instance, the displacement or symplectic homology proof of the almost existence theorem; see, for example, [Floer et al. 1994; Ginzburg 2005; Gürel 2008; Hofer and Zehnder 1994] and references therein.

Remark 1.8. A word on the hypotheses of the theorem is due now. The assumption that W be symplectically aspherical is imposed here only for the sake of simplicity and can be significantly relaxed along the lines of [Kerman 2008; Usher 2009]. Hypothetically, a combination of our argument with the reasoning from these works should lead to a generalization of the theorem to the case where we only require the subgroup $\langle \omega, \pi_2(M) \rangle \subset \mathbb{R}$ to be discrete as in [Usher 2009, Theorem 1.6] or, at least, where W is monotone or negative monotone; see [Kerman 2008]. (In such a generalization, the geodesic η is, of course, equipped with capping.)

The condition that M is stable cannot be entirely omitted due to the counterexamples to the Hamiltonian Seifert conjecture showing that there exist hypersurfaces in $\mathbb{R}^{2n}$ (C^2 when $2n = 4$) without closed characteristics; see [Ginzburg 1999; Ginzburg and Gürel 2003] and references therein. However, this condition can possibly be relaxed as in [Usher 2009, Section 7].

Finally note that the existence of a loop η satisfying (1-3) is established in [Ginzburg 2007, Theorem 2.7], where the second inequality (with $\delta = 0$) is proved under the additional hypothesis that M has restricted contact type. Thus, even when only the area bounds are concerned, Theorem 1.5 is a generalization (up to the issue of δ) of the results from [Ginzburg 2007], which became possible due to incorporating a technique from [Kerman 2009; Kerman and Şirikçi 2010] into the proof.

Remark 1.9. It is tempting to conjecture that the Maslov class of M is still nonzero even when the stability assumption in Theorem 1.5 is dropped and all leaves of $\mathscr{F}$ may be contractible. However, it is not entirely clear how to define this Maslov class and what cohomology space this class should lie in. The situation contrasts sharply with a similar question for the Liouville class of M, which can always be defined, when W is exact, as the class $[\lambda|_{\mathscr{F}}]$ of a global primitive λ of ω in the tangential de Rham cohomology $\mathrm{H}^1(\mathscr{F})$; see [Ginzburg 2007, Section 1.2].

2. Preliminaries

We start this section by recalling the relevant definitions and basic results concerning coisotropic submanifolds. In Section 2.2, we set our conventions and notation.

2.1. ***Stable coisotropic submanifolds.*** Let, as above, (W^{2n}, ω) be a symplectic manifold and let $M \subset W$ be a closed, coisotropic submanifold of codimension k. Set $\omega_M = \omega|_M$. Then, as is well known, the distribution $\ker \omega_M$ has dimension k and is integrable. Denote by $\mathscr{F}$ the characteristic foliation on M, that is, the k-dimensional foliation whose leaves are tangent to the distribution $\ker \omega_M$.

Definition 2.1. The coisotropic submanifold M is said to be *stable* if there exist one-forms $\alpha_1, \dots, \alpha_k$ on M such that $\ker d\alpha_i \supset \ker \omega_M$ for all $i = 1, \dots, k$ and

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \omega_M^{n-k} \neq 0 \tag{2-1}$$

anywhere on M. We say that M has *contact type* if the forms α_i can be taken to be primitives of ω_M. Furthermore, M has *restricted* contact type if the forms α_i extend to global primitives of ω on W.

Stable and contact type coisotropic submanifolds were introduced by Bolle [1996; 1998] and considered in a more general setting in [Ginzburg 2007] and also by Kerman [2008] and Usher [2009]. We refer the reader to [Ginzburg 2007] for a discussion of the requirements of Definition 2.1 and examples. Here we only note that although Definition 2.1 is natural, it is quite restrictive. For example, a stable Lagrangian submanifold is necessarily a torus and a stable coisotropic submanifold is automatically orientable.

Assume henceforth that M is stable. Then the normal bundle $T^\perp M$ to M in W is trivial, since it is isomorphic to $T^*\mathscr{F}$ and the latter bundle is trivial due to (2-1). From now on, we fix the trivialization $T^\perp M = T^*\mathscr{F} \cong M \times \mathbb{R}^k$ given by the forms α_i and identify a small neighborhood of M in W with a neighborhood of M in $T^*\mathscr{F} = M \times \mathbb{R}^k$. We will use the same symbols ω_M and α_i for differential forms on M and for their pullbacks to $M \times \mathbb{R}^k$. (Thus we suppress the pullback notation π^*, where $\pi : M \times \mathbb{R}^k \to M$ is the natural projection, unless its presence is essential.) As a consequence of the Weinstein symplectic neighborhood theorem, we have:

Proposition 2.2 [Bolle 1996; 1998]. *Let M be a closed, stable coisotropic submanifold of (W^{2n}, ω) with* $\operatorname{codim} M = k$. *Then, for a sufficiently small $r > 0$, there exists a neighborhood of M in W, which is symplectomorphic to*

$$U_r = \{(q, p) \in M \times \mathbb{R}^k \mid |p| < r\},$$

equipped with the symplectic form $\omega = \omega_M + \sum_{j=1}^k d(p_j\alpha_j)$. Here $(p_1, \dots, p_k)$ are the coordinates on $\mathbb{R}^k$ and $|p|$ is the Euclidean norm of p.

Thus, a neighborhood of M in W is foliated by a family of coisotropic submanifolds $M_p = M \times \{p\}$ with $p \in B_r^k$, where B_r^k is the ball of radius r centered at the origin in $\mathbb{R}^k$. Moreover, a leaf of the characteristic foliation on M_p projects onto a leaf of the characteristic foliation on M.

Proposition 2.3 [Bolle 1996; 1998; Ginzburg 2007]. *Let M be a stable coisotropic submanifold.*

(i) *The leaf-wise metric $(\alpha_1)^2 + \cdots + (\alpha_k)^2$ on $\mathcal{F}$ is leaf-wise flat.*

(ii) *The Hamiltonian flow of $\rho = (p_1^2 + \cdots + p_k^2)/2 = |p|^2/2$ is the leaf-wise geodesic flow of this metric.*

We conclude this section by pointing out that the metric ρ extends to a true metric on M such that the leaves of $\mathcal{F}$ are totally geodesic submanifolds and that the existence of such a metric is equivalent to the stability of M when M is a hypersurface; see [Sullivan 1978] and [Usher 2009, Section 7].

2.2. ***Conventions and notation.*** In this section we specify conventions and notation used throughout the paper.

2.2.1. *Action functional and the Hamilton equation.* Let (W^{2n}, ω) be a symplectically aspherical manifold, that is, $\omega|_{\pi_2(W)} = c_1|_{\pi_2(W)} = 0$. Denote by ΛW the space of smooth contractible loops $\gamma : S^1 \to W$ and consider a time-dependent Hamiltonian $H : S^1 \times W \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. Setting $H_t = H(t, \cdot)$ for $t \in S^1$, we define the action functional $\mathcal{A}_H : \Lambda W \to \mathbb{R}$ by

$$\mathcal{A}_H(\gamma) = \mathcal{A}(\gamma) + \int_{S^1} H_t(\gamma(t))\, dt,$$

where $\mathcal{A}(\gamma) = -\operatorname{Area}(\gamma)$ is the negative symplectic area bounded by γ. In other words,

$$\mathcal{A}(\gamma) = -\int_u \omega,$$

where $u : D^2 \to W$ is a capping of γ, that is, $u|_{S^1} = \gamma$. The least action principle asserts that the critical points of $\mathcal{A}_H$ are exactly the contractible one-periodic orbits of the time-dependent Hamiltonian flow φ_H^t of H, where the Hamiltonian vector field X_H of H is defined by the Hamilton equation $i_{X_H}\omega = -dH$.

2.2.2. *Conley–Zehnder index.* We consider a finite-dimensional symplectic vector space V and denote by $\operatorname{Sp}(V)$ the group of linear symplectic transformations of V, setting $\operatorname{Sp}(2n) = \operatorname{Sp}(\mathbb{R}^{2n})$ as usual. We let $\Delta(\Phi)$ stand for the mean index of a path $\Phi : [0, T] \to \operatorname{Sp}(V)$ and, when Φ is nondegenerate (i.e., $\Phi(T)$ has no eigenvalues equal to one), we denote by $\mu_{\mathrm{CZ}}(\Phi)$ the Conley–Zehnder index of Φ. We refer the reader to [Long 2002; Salamon 1999; Salamon and Zehnder 1992] and also [Ginzburg and Gürel 2009] for the definitions and a detailed discussion of these notions. In this paper, we normalize these indices as in [Ginzburg and Gürel 2009]. This normalization is different from the ones in [Long 2002; Salamon 1999; Salamon and Zehnder 1992]. For instance, our $\mu_{\mathrm{CZ}}(\Phi)$ is the negative of the Conley–Zehnder index as defined in [Salamon 1999]. For the flow $\Phi(t)$

with $0 \le t \le 1$ generated by a nondegenerate quadratic Hamiltonian H with small eigenvalues, we have $\mu_{\mathrm{CZ}}(\Phi) = -\operatorname{sgn}(H)/2$, where $\operatorname{sgn}(H)$ is the signature of H (the number of positive squares minus the number of negative squares). In particular, when H is negative definite, we have $\mu_{\mathrm{CZ}}(\Phi) = n$ where $2n = \dim V$ and $\Delta(\Phi) > 0$. In other words, when $\mu_{\mathrm{CZ}}(\Phi)$ is interpreted as the intersection index of Φ with the discriminant $\Sigma \subset \mathrm{Sp}(V)$ formed by symplectic transformations with at least one eigenvalue equal to one, Σ is co-oriented by the Hamiltonian vector field of a negative definite Hamiltonian.

Recall also from [Salamon and Zehnder 1992] that, regardless of conventions, we have

$$|\Delta(\Phi) - \mu_{\mathrm{CZ}}(\Phi)| < n \text{ and } \Delta(\Phi) = \lim_{k\to\infty} \frac{\mu_{\mathrm{CZ}}(\Phi^k)}{k}, \tag{2-2}$$

where in the inequality we require $\Phi(T)$ to be nondegenerate and, in the limit identity, we assume that $\Phi(T)^k \notin \Sigma$ for all k and thus $\mu_{\mathrm{CZ}}(\Phi^k)$ is defined. Note that here we can replace Φ^k by the concatenation of the paths Φ, $\Phi(T)\Phi$, etc., up to $\Phi(T)^{k-1}\Phi$.

Let now x be a contractible periodic orbit of H on W^{2n}. Using a trivialization of x^*TW arising from a capping of x, we can interpret the linearized flow $d\varphi_H^t$ along x as a path Φ in $\mathrm{Sp}(2n)$. The mean index $\Delta(x)$ of x is by definition $\Delta(\Phi)$. When x is nondegenerate, we also set $\mu_{\mathrm{CZ}}(x) := \mu_{\mathrm{CZ}}(\Phi)$. Since $c_1(TW)|_{\pi_2(W)} = 0$, these indices are well defined, that is, independent of the capping. When we need to emphasize the role of H, we write $\Delta_H(x)$ and $\mu_{\mathrm{CZ}}(x, H)$. By (2-2), we have

$$|\Delta(x) - \mu_{\mathrm{CZ}}(x)| < n \text{ and } \Delta(x) = \lim_{k\to\infty} \frac{\mu_{\mathrm{CZ}}(x^k)}{k}. \tag{2-3}$$

As in (2-2), we require here x to be nondegenerate for $\mu_{\mathrm{CZ}}(x)$ to be defined, and, in the limit identity, we assume that x is strongly nondegenerate, that is, all iterated orbits x^k are nondegenerate. Finally note that with our normalizations $\Delta(x) > 0$ and $\mu_{\mathrm{CZ}}(x) = n$ when x is a nondegenerate maximum (with small Hessian) of an autonomous Hamiltonian.

2.2.3. *Floer homology.* In the definition of Floer homology, we adopt literally the conventions and notation from [Ginzburg 2007]. All Hamiltonians considered in this paper are assumed to be compactly supported. The manifold W, in addition to being symplectically aspherical, is required to be either closed or geometrically bounded and wide in the sense of [Gürel 2008]. (See, e.g., [Audin et al. 1994; Cieliebak et al. 2004; Sikorav 1994] for the precise definition and a discussion of geometrically bounded manifolds.)

Examples of geometrically bounded manifolds include symplectic manifolds which are convex at infinity (e.g., $\mathbb{R}^{2n}$ and cotangent bundles) as well as twisted cotangent bundles. Under the hypotheses that W is symplectically aspherical and geometrically bounded, the compactness theorem for Floer's connecting trajectories holds (see [Sikorav 1994]) and the filtered $\mathbb{Z}$-graded Floer homology of a compactly supported Hamiltonian on W is defined for action intervals not containing zero; see, for example, [Cieliebak et al. 2004; Ginzburg and Gürel 2004] and references therein. We use the wideness hypothesis in Section 3.2 when considering a version of the "pinned" action selector introduced in [Kerman 2009]. This requirement is not restrictive, for, to the best of the author's knowledge, no examples of geometrically bounded open manifolds that are not wide are known.

We use the notation $\mathrm{HF}_*^{(a,\,b)}(H)$ for the filtered Floer homology of H, graded by the Conley–Zehnder index. The end-points a and b are always assumed to be outside the action spectrum $\mathcal{S}(H)$ of H and, if W is open, we require that $0 \notin (a,\ b)$. When W is closed, we have a canonical isomorphism

$$\mathrm{HF}_*(H) = \mathrm{H}_{*+n}(W; \mathbb{Z}_2),$$

where as usual $\mathrm{HF}_*(H) = \mathrm{HF}_*^{(-\infty,\,\infty)}(H)$. When all periodic orbits of H with action in $(a,\ b)$ are nondegenerate, we let $\mathrm{CF}_*^{(a,\,b)}(H)$ be the vector space generated over $\mathbb{Z}_2$ by such orbits, graded by the Conley–Zehnder index. The downward Floer differential

$$\partial : \mathrm{CF}_*^{(a,\,b)}(H) \to \mathrm{CF}_{*-1}^{(a,\,b)}(H)$$

is then defined in the standard way and $\mathrm{HF}_*^{(a,\,b)}(H)$ is the homology of the resulting Floer complex. The above nondegeneracy requirement is generic (as long as $0 \notin (a,\ b)$ if W is open) and, in general, we set

$$\mathrm{HF}_*^{(a,\,b)}(H) := \mathrm{HF}_*^{(a,\,b)}(\tilde{H}),$$

where $\tilde{H}$ is a small perturbation of H having only nondegenerate orbits with action in $(a,\ b)$. Since a and b are outside $\mathcal{S}(H)$, the homology $\mathrm{HF}_*^{(a,\,b)}(\tilde{H})$ is independent of $\tilde{H}$ as long as $\tilde{H}$ is sufficiently close to H. We refer the reader to [Cieliebak et al. 2004; Ginzburg 2007; Ginzburg and Gürel 2004] for the proofs and further details on the construction and properties of the Floer homology in this setting as well as for further references.

3. Proof of the main theorem

3.1. ***Maslov index for stable coisotropic submanifolds.*** Let M be a stable coisotropic submanifold. In this section, we interpret the mean index $\Delta_\rho(x)$ of a periodic orbit x of the leaf-wise geodesic flow on M as, up to a sign, the coisotropic Maslov index of the projection γ of x to M. We also establish certain bounds, going beyond (2-3), on the Conley–Zehnder index of a small nondegenerate perturbation of x. Throughout this subsection, we will use the notation from Section 2.1. In particular, we fix a neighborhood $U = M \times B$, where $B = B_r$, of M in W. Thus, let x be a nontrivial, contractible in W closed orbit of the Hamiltonian flow of ρ and let $\gamma = \pi(x)$. Then γ is also contractible in W.

Proposition 3.1. *We have*

$$\mu(\gamma) = -\Delta_\rho(x). \tag{3-1}$$

Proof. It is convenient to first extend the decomposition (1-1) from $TW|_M$ to $TW|_U$ as follows. Recall from Section 2.1 that the submanifolds $M_p = M \times \{p\} \subset M \times B$, with $p \in B$, are coisotropic and that the characteristic foliation $\mathcal{F}_p$ of M_p projects to $\mathcal{F}$ under π. Denote by $\tilde{\mathcal{F}}$ the resulting foliation of U, obtained as the union of foliations $\mathcal{F}_p$. Let TM be the horizontal tangent bundle in $M \times B$, that is, $(TM)_{(q,p)} = T_{(q,p)} M_p$ where $(q, p) \in U = M \times B$, and likewise let TB denote the vertical bundle $\ker \pi_*$. Then the normal bundle $T^\perp \tilde{\mathcal{F}}$ to $T\tilde{\mathcal{F}}$ in TM can be realized as the subbundle $E = (\bigcap_i \ker \pi^* \alpha_i) \cap TM$. We have the symplectic decomposition

$$TW = (T\tilde{\mathcal{F}} \oplus TB) \oplus E, \tag{3-2}$$

which turns into (1-1) once restricted to M.

The linearized projection π_* gives rise to an isomorphism between the fibers $(T\tilde{\mathcal{F}})_{(q,p)}$ and $T_q\mathcal{F}$, and $E_{(q,p)}$ and $T_q^\perp\mathcal{F}$. Furthermore, $(TB)_{(q,p)}$ is naturally isomorphic to $T_0 B = T_q^\perp M$. Thus, we have a (symplectic) linear isomorphism between the decomposition (3-2) along x and (1-1) along γ. In particular, we obtain an isomorphism between the bundles x^*TW and γ^*TW giving rise to a one-to-one correspondence between trivializations of TW along x and along γ. In what follows, we fix a trivialization arising from a capping of x.

Now recall that the flow of ρ on U can be identified with the geodesic flow of the leaf-wise metric ρ on M. Thus, we need to prove that the mean index of the linearized geodesic flow $G(t)$ along x is equal to $\Delta(\Phi)$. The geodesic flow preserves the terms $T\tilde{\mathcal{F}} \oplus TB$ and E in the decomposition (3-2). Indeed, the fact that the first term is conserved is clear: the geodesic flow is tangent to the leaves. To show that the second term is conserved, it suffices to recall that, as mentioned above, the flow is tangent to the manifolds M_p due to conservation of momenta and that the restrictions $\pi^*\alpha_j|_{M_p}$ are conserved since $L_{X_\rho} \pi^*\alpha_j = dp_j$.

Next let us show that

$$G|_E = \Gamma, \tag{3-3}$$

where we identified x^*E and $\gamma^*T^\perp\mathcal{F}$. To this end, let us recall the definition of the holonomy Γ. Consider an element $[v]$ in $T^\perp_{\gamma(0)}\mathcal{F} = T_{\gamma(0)}M / T_{\gamma(0)}\mathcal{F}$ represented by a vector $v \in T_{\gamma(0)}M$. (Here and below, it is more convenient to think of E and

$T^{\perp}\mathscr{F}$ as quotient bundles rather than sub-bundles.) Let $\eta : [0, \delta) \to M$ be a smooth map with $\eta(0) = \gamma(0)$ and $\eta'(0) = v$. Let now γ be parametrized by, say, $[0, T]$ and let $\sigma : [0, T] \times [0, \delta) \to M$ be a map whose restriction to $[0, T] \times 0$ is γ, to $0 \times [0, \delta)$ is η and such that $\sigma|_{[0, T]\times s}$, for all $s \in [0, \delta)$, lies in a leaf of $\mathscr{F}$. The class $[(\partial\sigma/\partial s)(t, 0)] \in T^{\perp}_{\gamma(t)}\mathscr{F}$ is independent of the choice of σ and is the image $\Gamma(t)[v]$. Let now $w(s) \in T_{\eta(s)}\mathscr{F}$ be a smooth family of vectors tangent to $\mathscr{F}$ and such that $w(0) = \dot\gamma(0)$. Consider the parametrized surface σ defined by setting $\sigma|_{[0, T]\times s}$ to be the leaf-wise geodesic with the initial conditions $(\gamma(s), w(s))$. Then, in particular, $[(\partial\sigma/\partial s)(t, 0)]$ is independent of the choice of the curve η and the family w. On the one hand, this vector represents $G(t)[v]$ by the definition of the linearized geodesic flow and, on the other, it is $\Gamma(t)[v]$ due to the above description of the holonomy.

To complete the argument, it would be sufficient to show that $G|_{T\tilde{\mathscr{F}}\oplus TB} = \Xi$, where we identified $x^*(T\tilde{\mathscr{F}} \oplus TB)$ and $\gamma^*(T\mathscr{F} \oplus T^{\perp}M)$, but this is not true. Let us fix a basis $\xi(0) \in T_{\gamma(0)}\mathscr{F}$. Then, since the metric is flat, $G(t)\xi(0)$ is the basis $\xi(t)$ in $T_{\gamma(t)}\mathscr{F}$ obtained from $\xi(0)$ by the parallel transport along γ. Let

$$\xi^*(0) \in T^*_{\gamma(0)}\mathscr{F} = T^{\perp}_{\gamma(0)}M$$

be the basis dual to $\xi(0)$. Then $G(t)\xi^*(0) = t\xi(t) + \xi^*(t) \in T_{\gamma(t)}\mathscr{F} \oplus T^*_{\gamma(t)}\mathscr{F}$ in obvious notation. We conclude that $G(t)|_{T\tilde{\mathscr{F}}\oplus TB} = \Xi(t) + A(t)$, where

$$A(t) : T^*_{\gamma(t)}\mathscr{F} \to T_{\gamma(t)}\mathscr{F}.$$

To finish the proof, we argue as when showing in Section 1.2 that the coisotropic Maslov index is independent of the splitting (1-1). With a trivialization fixed, we can view G and $\Phi = \Xi \oplus \Gamma$ as paths in $\mathrm{Sp}(2n)$. Then, G is homotopic with fixed end-points to the concatenation of Φ and the path $\Psi(t) = I + A(t)$. All eigenvalues of $\Psi(t)$ are equal to one and therefore $\Delta(\Psi) = 0$. Thus, by the additivity and homotopy invariance of the mean index (see, e.g., [Ginzburg and Gürel 2009; Long 2002; Salamon and Zehnder 1992]), we have $\Delta(G) = \Delta(\Phi) =: -\mu(\gamma)$. □

Remark 3.2. Proposition 3.1 has the following hypothetical generalization. Assume that M admits a metric with respect to which $\mathscr{F}$ is totally geodesic. Referring the reader to [Usher 2009, Section 7] for a detailed discussion of this condition, we only mention here that it is satisfied when M is Lagrangian (for any metric on M) and when M is stable. In the latter case, $\mathscr{F}$ is totally geodesic with respect to ρ. Then, conjecturally, the mean Conley–Zehnder index of x is equal, up to a sign, to the sum of the mean Morse index of γ and $\mu(\gamma)$. When M is stable, the mean Morse index is zero since ρ is flat, and this conjecture reduces to Proposition 3.1. When M is Lagrangian and x is nondegenerate, the conjecture essentially reduces to a well known relation between the Conley–Zehnder, Morse, and Maslov indices. The latter is proved in [Viterbo 1990] using the results from [Duistermaat 1976] in the context of the finite-dimensional reduction. A proof relying on the Floer theory version of the Conley–Zehnder index can be found in, for example, [Weber 2002]; see also [Kerman and Şirikçi 2010] for a simple argument.

The next proposition is a substitute for the relation between the Conley–Zehnder and Maslov indices.

Proposition 3.3. *Let K be a small perturbation of ρ and $\tilde{x}$ be a nondegenerate periodic orbit of K close to a nontrivial, contractible periodic orbit x of ρ. Then*

$$\Delta_\rho(x) - n \le \mu_{\mathrm{CZ}}(\tilde{x}) \le \Delta_\rho(x) + (n-k) \tag{3-4}$$

Proof. Note that by the continuity of Δ and (2-3) we automatically have

$$\Delta_\rho(x) - n \le \mu_{\mathrm{CZ}}(\tilde{x}) \le \Delta_\rho(x) + n,$$

regardless of the nature of the flow of ρ. Hence only the second inequality in (3-4) requires a proof.

By arguing as in the proof of Proposition 3.1, it is not hard to reduce the proposition to the following linear algebra result. Namely, consider a finite-dimensional symplectic vector space V split as a symplectic direct sum

$$V = (L \oplus L^*) \oplus E,$$

where E and $(L \oplus L^*)$ are symplectic spaces, and L and L^* are Lagrangian in $L \oplus L^*$; see (1-1) and (3-2). Set $\dim V = 2n$ and $\dim L = k$. Consider a path $G : [0,\ 1] \to \mathrm{Sp}(V)$ of the form $G = A \oplus \Gamma$, where Γ is a path in $\mathrm{Sp}(E)$ beginning at I and A is the block-diagonal path

$$A = \begin{bmatrix} I & tI \\ 0 & I \end{bmatrix},$$

in $\mathrm{Sp}(L \oplus L^*)$.

Lemma 3.4. *Let $\tilde{G} : [0,\ 1] \to \mathrm{Sp}(V)$ be a small nondegenerate perturbation of G, also beginning at I. Then*

$$\Delta(G) - n \le \mu_{\mathrm{CZ}}(\tilde{G}) \le \Delta(G) + (n-k). \tag{3-5}$$

Proof of the lemma. Again, by (2-2), we have

$$\Delta(G) - n \le \mu_{\mathrm{CZ}}(\tilde{G}) \le \Delta(G) + n,$$

for any path G. Hence, only the second inequality in (3-5) requires a proof.

Next observe that, once the end-point $\Gamma(1)$ is fixed, the path Γ is immaterial for the assertion of the lemma. In other words, if the lemma holds for one path with a given end-point, it also holds for every path with the same end-point. This follows from the facts that a homotopy of G can be traced by a homotopy of $\tilde{G}$

(both with fixed end-points) and that μ_{CZ} and Δ are invariant under such homotopy and change in the same way when a loop is attached to a path.

As the first step of the proof, let us assume that all eigenvalues of $\Gamma(1)$ are equal to one. Then $\Gamma(1)$ is in the image of the exponential mapping exp for $\mathrm{Sp}(E)$. Indeed, $\Gamma(1)$ is conjugate to a symplectic linear map which can be chosen to be arbitrarily close to I; see, for example, [Ginzburg 2010, Lemma 5.5]. Since exp is onto a neighborhood of the identity and commutes with conjugation, $\Gamma(1)$ is in the image of exp. Since 0 is a regular point of exp and the set of regular points is open, we can write $\Gamma(1) = \exp(Q)$, where Q is a regular point of exp and all eigenvalues of Q are equal to zero.

Here we identify the Lie algebra of the symplectic group with the space of quadratic Hamiltonians. As is customary in symplectic geometry, the eigenvalues of Q are, by definition, the eigenvalues of the linear Hamiltonian vector field X_Q generated by Q. Also note that if we identified $\mathrm{Sp}(E)$ with $\mathrm{Sp}(2(n-k))$ and used the matrix exponential map, we would write $X_Q = JQ$ and $\Gamma(1) = \exp(JQ)$.

We have $A(1) = \exp(\rho)$ in $\mathrm{Sp}(L \oplus L^*)$, where ρ is a positive definite form on L^* and zero on L. Arguing as above, it is not hard to show that ρ is a regular point of exp for $\mathrm{Sp}(L \oplus L^*)$ and that, moreover, $\rho + Q$ is a regular point of the exponential mapping for $\mathrm{Sp}(V)$. Now we have $\tilde{G}(1) = \exp(K)$ in $\mathrm{Sp}(V)$, where the quadratic form K is close to $\rho + Q$. In particular, K is also positive definite on L^* and all eigenvalues of K are close to those of $\rho + Q$, that is, close to zero. As has been pointed out above, we can set $\tilde{G}(t) = \exp(tK)$ and $\Gamma(t) = \exp(tQ)$. As a consequence, with our conventions,

$$\mu_{\mathrm{CZ}}(\tilde{G}) = -\operatorname{sgn}(K)/2 \le n - k,$$

where $\operatorname{sgn}(K)$ stands for the signature of K (i.e., the number of positive eigenvalues minus the number of negative eigenvalues); see [Salamon 1999, Section 2.4]. In addition, $\Delta(G) = 0$, and we obtain the second inequality of (3-5) in this case. To summarize, we have proved (3-5) when all eigenvalues of $\Gamma(1)$ are equal to one.

To treat the general case, consider the symplectic direct sum decomposition $E = E_0 \oplus E_1$, where E_0 is spanned by the generalized eigenvectors of $\Gamma(1)$ with eigenvalue one and E_1 is the symplectic orthogonal complement of E_0 in E. Clearly, $\Gamma(1)$ preserves this decomposition and, after altering if necessary the path Γ, we may assume that so do all maps $\Gamma(t)$. When $\tilde{G}(1)$ is sufficiently close to $G(1)$, we have the decomposition $V = V_0 \oplus V_1$ preserved by $\tilde{G}(1)$, where V_0 is close to $(L \oplus L^*) \oplus E_0$ and V_1 is close to E_1. Applying a time-dependent, close to the identity conjugation to $\tilde{G}(t)$, we reduce the problem to the case where $V_0 = (L \oplus L^*) \oplus E_0$ and $V_1 = E_1$. Consider now the paths G and $\tilde{G}$. Both paths begin and end in $\mathrm{Sp}(V_0) \times \mathrm{Sp}(V_1)$, the first path is contained in this subgroup, and the path $\tilde{G}$ is close to G. In particular, $\tilde{G}$ is in a tubular neighborhood of the subgroup.

Projecting $\tilde{G}$ to $\mathrm{Sp}(V_0) \times \mathrm{Sp}(V_1)$, we can further reduce the question to the case where $\tilde{G}$ is a path in $\mathrm{Sp}(V_0) \times \mathrm{Sp}(V_1)$, just as G is. Denote by $G = (G_0, G_1)$ and $\tilde{G} = (\tilde{G}_0, \tilde{G}_1)$ the corresponding decompositions of the paths. The E_0-component of $G_0(1)$ is the map $\Gamma(1)|_{E_0}$ with all eigenvalues equal to one, and hence (3-5) has already been proved for G_0:

$$\Delta(G_0) - \dim V_0/2 \le \mu_{\mathrm{CZ}}(\tilde{G}_0) \le \Delta(G_0) + (\dim V_0/2 - k).$$

On the other hand, the path $\tilde{G}_1$ is a small perturbation of the path $\Gamma|_{E_1}$. Thus, we have

$$\Delta(G_1) - \dim V_1/2 \le \mu_{\mathrm{CZ}}(\tilde{G}_1) \le \Delta(G_1) + \dim V_1/2.$$

Recall that $\Delta(G) = \Delta(G_0) + \Delta(G_1)$ and $\mu_{\mathrm{CZ}}(\tilde{G}) = \mu_{\mathrm{CZ}}(\tilde{G}_0) + \mu_{\mathrm{CZ}}(\tilde{G}_1)$ and that $\dim V_0 + \dim V_1 = \dim V = 2n$. Thus, adding up these inequalities, we obtain (3-5), which completes the proof of the lemma and hence the proof of the proposition. □

3.2. ***Action selector for "pinned" Hamiltonians, following E. Kerman.*** Our goal in this section is to describe a construction of an action selector for "pinned" Hamiltonians, which was introduced in [Kerman 2009; Kerman and Şirikçi 2010]. Although the class of Hamiltonians and manifolds we work with is somewhat different from those in the references just given, the action selector is essentially the same as the one considered there. As far as the proofs are concerned, we adopt here the line of reasoning from [Ginzburg 2007] rather than following the Hofer-geometric approach from [Kerman 2009]. Since the arguments are quite standard, for the sake of brevity, we just outline the proofs.

Let M^{2n-k} be a closed submanifold, not necessarily coisotropic, of a symplectic manifold W^{2n}. As before, we require W to be symplectically aspherical and either closed or a geometrically bounded and wide. We assume that M is displaceable and fix a displaceable open set U containing M. Denote by $\mathcal{H}$ the collection of nonnegative, autonomous Hamiltonians $H : W \to \mathbb{R}$ supported in U, constant on a small tubular neighborhood of M and attaining the absolute maximum $C := \max H$, depending on H, on this neighborhood. Let us require furthermore that $C > e(U)$, where $e(U)$ is the displacement energy of U.

It is easy to see that $\mathrm{HF}_n^{(C-\delta, C+\delta)}(H) = \mathbb{Z}_2$ once $H \in \mathcal{H}$ and $\delta > 0$ is sufficiently small. In fact, $\mathrm{HF}_*^{(C-\delta, C+\delta)}(H) = \mathrm{H}_{*+n-k}(M; \mathbb{Z}_2)$. Furthermore, when $a > C$ is large enough (namely, if $a > C + e(U)$), the inclusion map

$$i_a : \mathbb{Z}_2 \cong \mathrm{HF}_n^{(C-\delta, C+\delta)}(H) \to \mathrm{HF}_n^{(C-\delta, a)}(H)$$

is zero. The proof of this fact is, for example, contained in the proof of [Ginzburg 2007, Proposition 4.1]; see also [Kerman 2009] for the case of closed manifolds. This is the main point of the argument where we need to assume that W is wide

[Gürel 2008], unless W is closed. For $H \in \mathcal{H}$, set

$$c(H) = \inf\{a > C \mid i_a = 0\}.$$

(Strictly speaking, here we have to require $a > C + \delta$ and then also take infimum over all sufficiently small $\delta > 0$.) This is a version of the action selector for "pinned" Hamiltonians, introduced in [Kerman 2009].

Alternatively and more explicitly, the action selector c can be defined as follows. Let $\tilde{H}$ be a C^2-small, nondegenerate perturbation of H, also supported in U (or, to be more precise, in $S^1 \times U$) and such that $\tilde{H} \geq H$. Let us also assume that $\tilde{H}$ is autonomous on a small neighborhood of M and that $\max \tilde{H} = C = \max H$ is attained at $p \in M$. (In what follows, we will have p fixed and independent of $\tilde{H}$.) Then p, viewed as an element of degree n in the Floer complex $\mathrm{CF}_*^{(C-\delta,\infty)}(\tilde{H})$, is exact and there exists a chain in $\mathrm{CF}_{n+1}^{(C-\delta,\infty)}(\tilde{H})$ mapped to p by the Floer differential; see the proof of [Ginzburg 2007, Proposition 4.1]. Let us consider all such chains and, within every chain, pick an orbit with the largest action and then among the resulting orbits we choose an orbit $\tilde{x}$ with the least action. In other words, to obtain $\tilde{x}$, we first maximize the action within every chain and then minimize the result among all chains which are primitives of p. Clearly, the orbit $\tilde{x}$ is in general not unique, but the action $\mathcal{A}_{\tilde{H}}(\tilde{x})$ is defined unambiguously.

Let us now set $c(\tilde{H}) = \mathcal{A}_{\tilde{H}}(\tilde{x})$. Then $c(H)$ is the infimum or the limit (in the obvious sense) of $c(\tilde{H})$ over all such perturbations $\tilde{H}$ of H. (It is clear that $c(H)$ is less than or equal to the limit; the fact that $c(H)$ is greater than or equal to the limit is a consequence of the definition of the Floer homology for degenerate Hamiltonians such as H.)

It follows from this description that there exists an orbit x of H, referred to in what follows as a *special one-periodic orbit* of H, obtained as a limit point of the orbits $\tilde{x}$ in the space of loops as $\tilde{H} \to H$, such that

$$C < \mathcal{A}_H(x) = c(H) < C + e(U) \quad \text{and} \quad 1 \leq \Delta(x) \leq 2n + 1. \tag{3-6}$$

Here the upper bound on the action is established by a variant of the standard argument relating action change and the displacement energy; see, e.g., [Ginzburg 2005; Gürel 2008; Hofer and Zehnder 1994; Kerman 2009] and references therein. The lower bound on action is clear for $\tilde{H}$ and $\tilde{x}$. By continuity of the action (with a little extra argument showing that the inequalities are strict) it also holds for H and x . The bounds for the index follow from the continuity of the mean index and (2-3). Note that, in general, the special orbit x is not unique.

Remark 3.5. There appears to be no reason to expect the orbit $\tilde{x}$ to be necessarily connected to p by a Floer downward trajectory. However, there exists an orbit $\hat{x}$ of $\tilde{H}$ with this property and such that $C < \mathcal{A}_{\tilde{H}}(\hat{x}) \leq \mathcal{A}_{\tilde{H}}(\tilde{x})$. This is an immediate consequence of the definition of $\tilde{x}$. Carefully passing to the limit as $\tilde{H} \to H$ we obtain an orbit x' of H such that $C < \mathcal{A}_H(x') \leq \mathcal{A}_H(x)$ and x' is connected to M by a Floer downward trajectory. See [Ginzburg 2007] and, in particular, the proofs of Propositions 4.1 and 5.1 therein for the proofs of these facts; note also that $\hat{x}$ is denoted by γ in [Ginzburg 2007, Proposition 4.1]. The existence of the orbit x' is essential for showing that, in (1-3), the area bounded by η is strictly positive.

We refer the reader to [Kerman 2009] for a detailed investigation of the properties of the action selector c. One of these is particularly important for our argument.

Proposition 3.6 [Kerman 2009]. *The action selector* c *is Lipschitz, with Lipschitz constant equal to one, on* $\mathcal{H}$ *equipped with the* sup-*norm.*

As an immediate consequence of the proposition, the selector c extends from $\mathcal{H}$ to the C^0-closure of $\mathcal{H}$ in the space of continuous functions supported in U and this extension is again Lipschitz with Lipschitz constant equal to one. For the sake of completeness, we touch upon a proof of the proposition.

Outline of the proof. Let H and K be two Hamiltonians in $\mathcal{H}$. Consider the perturbations $\tilde{H}$ and $\tilde{K}$ as above. Clearly, it suffices to show that

$$|\,\mathrm{c}(\tilde{H}) - \mathrm{c}(\tilde{K})| \leq \|\tilde{H} - \tilde{K}\|_{\mathrm{H}}, \tag{3-7}$$

where

$$\|F\|_{\mathrm{H}} := \int_0^1 (\max_W F_t - \min_W F_t)\, dt$$

stands for the Hofer norm of F.

Denote by $\tilde{x}$ again a least action primitive of p in $\mathrm{CF}_*^{(C-\delta,\infty)}(\tilde{H})$ described above. In particular, $\mathrm{c}(\tilde{H}) = \mathcal{A}_{\tilde{H}}(\tilde{x})$. It is not hard to see that under the linear homotopy from $\tilde{H}$ to $\tilde{K}$, the orbit $\tilde{x}$ is mapped to a primitive $\tilde{y} = \sum \tilde{y}_i$ of p in the complex $\mathrm{CF}_*^{(C-\delta,\infty)}(\tilde{K})$, but not necessarily to a least action primitive. In any case, $\mathrm{c}(\tilde{K}) \leq \mathcal{A}_{\tilde{K}}(\tilde{y}) := \max \mathcal{A}_{\tilde{K}}(\tilde{y}_i)$. Meanwhile, a standard calculation yields

$$\mathcal{A}_{\tilde{K}}(\tilde{y}) - \mathcal{A}_{\tilde{H}}(\tilde{x}) \leq \|\tilde{H} - \tilde{K}\|_{\mathrm{H}}.$$

Hence, we also have $\mathrm{c}(\tilde{K}) - \mathrm{c}(\tilde{H}) \leq \|\tilde{H} - \tilde{K}\|_{\mathrm{H}}$. A similar argument, but using the homotopy from $\tilde{K}$ to $\tilde{H}$, shows that $\mathrm{c}(\tilde{H}) - \mathrm{c}(\tilde{K}) \leq \|\tilde{H} - \tilde{K}\|_{\mathrm{H}}$, and (3-7) follows. □

Remark 3.7. It is worth pointing out that the main advantage of using the action selector for pinned Hamiltonians in the proof of the main theorem over the ordinary action selector is that the former enables us to determine the location of the special orbit x via Lemma 3.8 without additional requirements on M such as that M has restricted contact type. This results in sharper index and energy bounds that we would have otherwise; see [Ginzburg 2007].

3.3. ***Proof of Theorem 1.5.*** Throughout the proof, as in Section 2.1, a neighborhood of M in W is identified with a neighborhood of M in $M \times \mathbb{R}^k$ equipped with the symplectic form $\omega = \omega_M + \sum_{j=1}^k d(p_j \alpha_j)$. Using this identification, we denote by U_R or just U, with $R > 0$ sufficiently small, the neighborhood of M in W corresponding to $M \times B_R^k$. (Thus, $U_R = \{\rho < R^2/2\}$.) Also set $|p| := \sqrt{2\rho}$.

The proof of the theorem relies on a method, by now quite standard, developed in [Viterbo 1990]. The first, albeit technical, step is to specify the class of "test" Hamiltonians.

3.3.1. *The Hamiltonians.* Fix two real constants $r > 0$ and $\epsilon > 0$ with $\epsilon < r < R$ and a constant $C > e(U)$. Let $H : [0,\ R] \to \mathbb{R}$ be a smooth, nonnegative, (nonstrictly) decreasing function such that

- on $[0,\ \epsilon]$ the function H is a positive constant C,
- on $[\epsilon,\ 2\epsilon]$ the function H is concave (i.e., $H'' \leq 0$),
- on $[2\epsilon,\ r-\epsilon]$ the function H is linear decreasing from $C - \epsilon$ to ϵ,
- on $[r-\epsilon,\ r]$ the function H is convex (i.e., $H'' \geq 0$),
- on $[r,\ R]$ the function H is identically zero.

Abusing notation, we also denote by H the function equal to $H(|p|)$ on U and equal to zero outside U. Let us fix the value of the parameter r, which is not essential for what follows. The parameters C and ϵ will vary and we consider the family of functions $H = H_{C,\epsilon}$ parametrized by C and ϵ and depending smoothly on these parameters.

Clearly, $H \in \mathcal{H}$ for any choice of ϵ and C. As $\epsilon \to 0$, the functions $H_{C,\epsilon}$ converge uniformly to the continuous functions $H_{C,0}$ equal to C on M, zero outside U_r, and depending linearly of $|p|$ on U_r. It is clear that the limit functions $H_{C,0}$ are continuous in C. Thus, by Proposition 3.6, $\mathrm{c}(H_{C,\epsilon})$ is a continuous function of C and ϵ including the limit value $\epsilon = 0$. Moreover, the function $C \mapsto \mathrm{c}(H_{C,0})$ is Lipschitz with Lipschitz constant equal to one.

Denote by X the Hamiltonian vector field of the function $|p|$ on $U \setminus M$. By Proposition 2.3, the integral curves of X project to the geodesics of the leaf-wise metric ρ on M, parametrized by arc length. The Hamiltonian vector field of H is

$$X_H = H'X,$$

where H' stands for the derivative of H with respect to $|p|$. Note that even though X is defined only on $U \setminus M$, the vector field X_H is defined everywhere, for H is constant near M and outside U_r. Thus, nontrivial one-periodic orbits of X_H lie on the levels $|p| = const$ with $H'(|p|)$ in the length spectrum $\mathcal{S}$ of the metric ρ. (Recall that, by definition, $\mathcal{S}$ is formed by the lengths of nontrivial closed leaf-wise geodesics of ρ. Here, we may restrict our attention only to the geodesics contractible in W.) Observe that the "coordinates" p_i are constant along the orbits of the flow of X_H. In other words, every trajectory starting in U lies on a coisotropic submanifold $M \times p \subset U$. This is a particular case of conservation of momentum.

Let x be a nontrivial one-periodic orbit of H. A direct calculation relying on Proposition 2.2 shows that

$$\begin{aligned}\mathcal{A}_H(x) &= H(x) + \mathcal{A}(x)\\ &= H(x) + \mathcal{A}(\pi(x)) - |p(x)| l(\pi(x)),\end{aligned}$$

where l and $\mathcal{A}$ stand for the length of the curve and, respectively, the negative symplectic area bounded by the curve.

Assume that the slope of H (on the interval $[2\epsilon,\ r-\epsilon]$) is outside $\mathcal{S}$. (This is a generic condition.) Then the orbit x lies on the level where $|p(x)|$ is either in the range $[\epsilon,\ 2\epsilon]$ or in the range $[r-\epsilon, r]$. Let now x be a special one-periodic orbit from Section 3.2 such that, in particular, (3-6) holds. The key to the proof is the following lemma, which specifies the location of x for, at least, some sequence of the Hamiltonians H.

Lemma 3.8. *There exists a sequence $C_j \to \infty$ such that the slopes of all functions $H_{C_j,\epsilon}$, with $\epsilon > 0$ sufficiently small, are outside $\mathcal{S}$ and $|p(x)| \in [\epsilon,\ 2\epsilon]$.*

In the Lagrangian case this observation can be traced back to the original work of Viterbo [1990]. Here we follow the treatment from [Kerman 2009] with several modifications resulting from our somewhat different conventions and more importantly from the fact that M is now coisotropic.

Proof of Lemma 3.8. The slope of the function $H_{C,0}$ is C/r. This slope is in $\mathcal{S}$ if and only if $C \in r\mathcal{S}$ in the obvious notation. The set $\mathcal{S}$ (and hence $r\mathcal{S}$) is closed, and the slope of $H_{C,\epsilon}$ is close to the slope of $H_{C,0}$ when $\epsilon > 0$ is small. As a consequence, the slope of $H_{C,\epsilon}$ is outside $\mathcal{S}$ whenever $C \notin r\mathcal{S}$ and $\epsilon > 0$ is small.

Pick $C \notin r\mathcal{S}$ and a positive sequence $\epsilon_i \to 0$. Without loss of generality, we may require all ϵ_i to be sufficiently close to zero to ensure that the slope of $H_i := H_{C,\epsilon_i}$ is not in $\mathcal{S}$. Let x_i be a special orbit of H_i. Since the norms of the differentials dH_i are bounded from above, the norms of the derivatives $\dot{x}_i$ are point-wise bounded. By the Arzela–Ascoli theorem, we may assume, after passing if necessary to a subsequence, that the orbits x_i converge to a curve y lying on a level $|p| = const$ including possibly the submanifold M. It is clear that y is smooth and projects to a closed, leaf-wise geodesic on M. Furthermore,

$$\mathcal{A}_{H_i}(x_i) = \mathrm{c}(H_i) \to H_{C,0}(y) + \mathcal{A}(y) = \mathrm{c}(H_{C,0}),$$

by the continuity of the action functional and of the action selector c.

If $|p(x_i)|$ is in the range $[r-\epsilon_i,\, r]$ for all i, the orbit y is on the level $|p|=r$ and $H_{C,0}(y)=0$. Thus, we then have

$$\mathrm{c}(H_{C,0}) = \mathcal{A}(y) \in \Sigma, \tag{3-8}$$

where Σ is the action spectrum or, to be more precise, the symplectic area spectrum of the level $|p|=r$, that is, the collection of symplectic areas bounded by contractible closed characteristics on this level.

Arguing by contradiction, assume now that the lemma fails, that is, for every sufficiently large C, say $C>a$, which is not in $r\mathcal{S}$, there exists such a sequence ϵ_i with $|p(x_i)|$ in the range $[r-\epsilon_i,\, r]$. Consider the function $f(C) := \mathrm{c}(H_{C,0})$ on the interval $[a,\,\infty)$. By (3-8), f sends the set $[a,\,\infty)\setminus r\mathcal{S}$ to Σ. Recall that $r\mathcal{S}$ is not only closed, but also has zero measure; see [Ginzburg 2007, Lemma 6.6]. By Proposition 3.6, f is a Lipschitz function and, as is well known [Hofer and Zehnder 1994], Σ has measure zero. To summarize, f is a Lipschitz function sending a full measure set to a zero measure set. Such a function is necessarily constant. This is impossible, for $f(C)\geq C$ by (3-6). □

Let us fix one of the constants $C=C_j$ from Lemma 3.8 and let $H_i = H_{C_j,\epsilon_i}$. Denote by x_i, or just x, its one-periodic orbit such as in the lemma. (For the proof of the theorem we do not need the entire double sequence, but only one family of Hamiltonians H_{C_j,ϵ_i} parametrized by ϵ_i.) Clearly, $\gamma_i = \pi(x_i)$ is a leaf-wise geodesic on M. Since the slopes of Hamiltonians H_i are bounded from above (by, say, $2C_j/r$), it is easy to prove using the Arzela–Ascoli theorem that the geodesics γ_i converge as $i\to\infty$ after if necessary passing to a subsequence. Denote the limit geodesic (traversed in the opposite direction) by η. Our goal is to show that η has the required properties (1-2) and (1-3). The fact that, by Lemma 3.8, $|p(x_i)|\in[\epsilon_i,\, 2\epsilon_i]$ (i.e., x_i lies in the region where H_i is concave) will be essential for proving this.

3.3.2. *Index bounds.* Consider a perturbation $\tilde{H}$ of $H=H_i$ as in Section 3.2. This Hamiltonian has a one-periodic orbit $\tilde{x}$, a perturbation of $x=x_i$, with index $n+1$. After reparametrizing x and reversing its orientation, we can view x as a periodic orbit x^- of ρ. Likewise, $\tilde{x}$ can be viewed as a periodic orbit $\tilde{x}^-$ of a nondegenerate perturbation K of ρ. Denote by $\gamma^- = \pi(x^-)$ the geodesic $\gamma=\gamma_i$ with reversed orientation.

By Proposition 3.1, we have

$$\mu(\gamma^-) = -\Delta_\rho(x^-),$$

and thus, by Proposition 3.3,

$$-\mu(\gamma^-) - n \leq \mu_{\mathrm{CZ}}(\tilde{x}^-) \leq -\mu(\gamma^-) + (n-k).$$

It is not hard to show that $\mu_{\mathrm{CZ}}(\tilde{x}^-) = -\,\mu_{\mathrm{CZ}}(\tilde{x}) = -(n+1)$ using the fact that x is in the region where H is concave (i.e., $|p(x)| \in [\epsilon_i,\, 2\epsilon_i]$) by Lemma 3.8. As a consequence,

$$n+1 \le \mu(\gamma^-)+n \quad \text{and} \quad \mu(\gamma^-)-n+k \le n+1.$$

Hence,

$$1 \le \mu(\gamma^-) \le 2n+1-k.$$

Passing to the limit and using the continuity of the mean index, we conclude that the same holds for η, the limit of the curves γ^-. This proves (1-2).

Remark 3.9. If we had used here just the second inequality of (3-6) rather than Proposition 3.3, we would have the weaker bound $1 \le \mu(\gamma^-) \le 2n+1$.

3.3.3. *Action bounds.* By the first inequality in (3-6), we have

$$C < \mathcal{A}_H(x) = H(x) + \mathcal{A}(\gamma) - |p(x)| l(\gamma) < C + e(U). \tag{3-9}$$

Here, by the definition of H and Lemma 3.8, $|p(x)| \in [\epsilon_i,\, 2\epsilon_i]$ and $H(x) \in [C, C-\epsilon_i]$. Note that the sequence $l(\gamma)$ with $\gamma = \gamma_i$ is bounded as $i \to \infty$ due the fact that the slope of H_i is bounded. Thus, passing to the limit (for a subsequence if necessary), we have $0 \le -\mathcal{A}(\eta) \le e(U)$. Here, the negative sign comes from the fact that η is the limit of γ^-, that is, the geodesics γ with reversed orientation. Taking $r > 0$ sufficiently small, we obtain

$$0 \le \mathrm{Area}(\eta) \le e(M) + \delta,$$

for any given $\delta > 0$, where $\mathrm{Area}(\eta) = -\mathcal{A}(\eta)$ is the symplectic area bounded by η. To finish the proof, we need to ensure that the first inequality is strict: $\mathrm{Area}(\eta) > 0$. This is an immediate consequence of the non-trivial fact that, by [Ginzburg 2007, Theorem 6.1], $\mathcal{A}_H(x') - C \ge \epsilon$ for some $\epsilon > 0$ independent of i, where x' is the orbit mentioned in Remark 3.5. For then we also have $\mathcal{A}_H(x) - C \ge \epsilon$ and, by the first inequality in (3-9), $\mathrm{Area}(\gamma^-) > \epsilon/2$ when i is large enough. This concludes the proof of (1-3), and thus the proof of the theorem.

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VIKTOR L. GINZBURG

ginzburg@math.ucsc.edu
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA CRUZ, CA 95064
UNITED STATES
http://math.ucsc.edu/~ginzburg/

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DIRAC COHOMOLOGY OF WALLACH REPRESENTATIONS

JING-SONG HUANG, PAVLE PANDŽIĆ AND VICTOR PROTSAK

Let G be either the metaplectic double cover of $\mathrm{Sp}(2n, \mathbb{R})$, or $\mathrm{SO}^*(2n)$, or $\mathrm{SU}(p, q)$. Let $\mathfrak{g}$ be the complexified Lie algebra of G and let K be a maximal compact subgroup of G. Let X be one of the Wallach modules for the pair $(\mathfrak{g}, K)$. In other words, X corresponds to a discrete point in the classification of unitary lowest weight modules with scalar lowest K-type. The purpose of this paper is to calculate the Dirac cohomology of X. Our approach is based on the explicit knowledge of the K-types of X. We establish a bijection between certain K-types E_i of X and certain $\tilde{K}$-types F_i of the spin module, where $\tilde{K}$ is the spin double cover of K. The Dirac cohomology is then realized as the set of Parthasarathy–Ranga-Rao–Varadarajan components of $E_i \otimes F_i$.

1. Introduction

Let G be a connected real reductive Lie group with a Cartan involution Θ. We assume that the group of fixed points of Θ, $K = G^{\Theta}$, is a maximal compact subgroup of G. We will denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of the complexified Lie algebra of G. The classification of the unitary dual of G is a significant open problem in representation theory. An important necessary condition for unitarity (unitarizability) of a simple Harish-Chandra module X, due to Parthasarathy [1980], is the *Dirac inequality* (Proposition 3.3), a byproduct of studying the action of the Dirac operator D on $X \otimes S$ where S is a spin module for the Clifford algebra $C(\mathfrak{p})$ ([Parthasarathy 1972; Vogan 1997]; see Section 3 for details.) If X is unitary, then D is Hermitian (self-adjoint) with respect to a natural Hermitian inner product, and hence $D^2 \geq 0$ on $X \otimes S$. Writing this inequality more explicitly leads directly to the Dirac inequality.

While the Dirac inequality is necessary for unitarity, it is by no means sufficient. A careful analysis of this fact led Vogan [1997] to the notion of *Dirac cohomology*

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(see also [Huang and Pandžić 2002]). The Dirac cohomology $H_D(X)$ of X is $\operatorname{Ker} D$ divided by the intersection $\operatorname{Im} D \cap \operatorname{Ker} D$. It is a module for the spin double cover $\tilde{K}$ of K. If X is unitary, we have $H_D(X) = \operatorname{Ker} D = \operatorname{Ker} D^2$, since D is Hermitian. In particular, the unitary X for which the equation $D^2 = 0$ has nonzero solutions in $X \otimes S$ are precisely the ones with nonzero Dirac cohomology. These representations are extremal in the sense that Dirac inequality becomes equality on some of the $\tilde{K}$-types of $X \otimes S$. Once such a $\tilde{K}$-type E is fixed, there are only a few possible irreducible modules X with $H_D(X) \supseteq E$. Namely, the main result of [Huang and Pandžić 2002], conjectured by Vogan, says that such E determines the infinitesimal character of X.

The Dirac cohomology of various classes of representations turned out to be intimately related with several classical subjects of representation theory like characters and the construction of the discrete series [Huang and Pandžić 2006]. It is also related to nilpotent Lie algebra cohomology [Huang et al. 2006] and to $(\mathfrak{g}, K)$ cohomology [Huang et al. 2009; Pandžić 2004].

Modules with nonzero Dirac cohomology include finite-dimensional modules with highest weight stable under the Cartan involution [Kostant 1999; 2003; Huang et al. 2009]. Further examples are $A_{\mathfrak{q}}(\lambda)$ modules with λ stable under the Cartan involution and sufficiently regular [Huang et al. 2009]; this class of modules includes the discrete series.

Another class of modules known to have nonzero Dirac cohomology are the unitary highest (or lowest) weight modules. Their Dirac cohomology is in principle known by results in [Huang et al. 2006] and [Enright 1988]. In the first of these papers we showed that the Dirac cohomology is in some sense equal to the $\mathfrak{p}^+$ cohomology, which was in turn determined by Enright. (Recall that the K-module $\mathfrak{p}$ breaks up as $\mathfrak{p}^+ \oplus \mathfrak{p}^-$ since the pair $(\mathfrak{g}, \mathfrak{k})$ is now Hermitian symmetric.) Enright's description is quite complicated and abstract; but in [Huang et al. 2006] we showed that the Dirac cohomology can be viewed as a space of harmonic representatives of the $\mathfrak{p}^+$ cohomology, so one can hope for a more explicit and concrete description.

The goal of this paper is to determine the Dirac cohomology of certain unitary lowest weight modules as directly and explicitly as possible. The modules we consider are the Wallach representations of the symplectic, orthogonal and indefinite unitary groups. Wallach modules are named after Nolan Wallach, who constructed them in the algebraic setting [1979]; they were recovered in the analytic setting in [Vergne and Rossi 1976].[1] Together with the trivial module, Wallach modules form the discrete part in the classification of all unitary lowest weight modules with scalar lowest K-type [Enright et al. 1983].

The lowest weights of Wallach modules are of the form $ck\zeta$. Here k is an integer between 1 and $r-1$, where r is the split rank (real rank) of G. The constant c is

[1] Wallach's priority is confirmed in the introduction of Vergne and Rossi's paper.

$\frac{1}{2}$ in the symplectic case, 2 in the orthogonal case and 1 in the unitary case, while ζ is the fundamental weight of $\mathfrak{g}$ orthogonal to the roots of $\mathfrak{k}$.

The Wallach representations are also important in invariant theory and the theory of reductive dual pairs [Howe 1989; 1995]. These modules can be realized as theta lifts of the trivial representation in compact dual pair correspondences. Furthermore, they arise in the study of the geometry of nilpotent orbits [Nishiyama et al. 2001, Section 7]. They are also important in another approach to the classification of unitary lowest weight modules [Adams 1987]: they are the basic ones from which all others can be obtained by cohomological induction. Finally, they appear in mathematical physics: they are related to the generalized hydrogen atom and to generalized MICZ-Kepler problems [Meng 2008; 2007; 2010].

We note that the continuous family of unitary highest weight modules are full generalized Verma modules. As such they are also $A_{\mathfrak{q}}(\lambda)$-modules for the maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^-$ of $\mathfrak{g}$. Therefore their Dirac cohomology can be calculated using the methods and results of [Huang et al. 2009]. This is another reason we focus our consideration on Wallach representations.

In the symplectic case, together with each Wallach module V_k^+ ($k = 1, \dots, r-1$, where r is the real rank of G), we study another unitary lowest weight module V_k^-, with nonscalar lowest K-type. For example, the even half of the oscillator representation is the Wallach module V_1^+, and the odd half is V_1^-. Studying V_k^- together with V_k^+ requires no additional work, and moreover, replacing the Wallach module with $V_k = V_k^+ \oplus V_k^-$ makes our main result stated below more uniform. We provide a detailed description of Wallach representations, as well as of the modules V_k^- in the symplectic case, in Section 2.

Let us introduce some standard notation. Let $\mathfrak{t}$ be the common Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$. We choose positive roots for $\mathfrak{g}$ and $\mathfrak{k}$ with respect to $\mathfrak{t}$ in such a way that

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{k}, \mathfrak{t}) \cup \Delta(\mathfrak{p}^+), \tag{1.1}$$

where $\Delta(\mathfrak{p}^+)$ denotes the set of $\mathfrak{t}$-weights of $\mathfrak{p}^+$. As usual, we denote by ρ the half sum of roots in $\Delta^+(\mathfrak{g}, \mathfrak{t})$, by ρ_c the half sum of roots in $\Delta^+(\mathfrak{k}, \mathfrak{t})$, and by ρ_n the half-sum of roots in $\Delta(\mathfrak{p}^+)$. Let W_G and W_K be the Weyl groups of $(\mathfrak{g}, \mathfrak{t})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively. For any $\mathfrak{k}$-dominant weight $\mu \in \mathfrak{t}^*$, we denote by E_μ the irreducible finite-dimensional $\mathfrak{k}$-module with highest weight μ.

We are now ready to state our main result.

Theorem 1.2. *Let G be the metaplectic double cover of* $\mathrm{Sp}(2n, \mathbb{R})$; *or* $\mathrm{SO}^*(2n)$; *or* $\mathrm{SU}(p, q)$. *Let X be one of the Wallach modules for the corresponding pair $(\mathfrak{g}, K)$, or in the symplectic case, the direct sum of the Wallach module V_k^+ with the associated module V_k^-. Denote by $\Lambda \in \mathfrak{t}^*$ the $\mathfrak{g}$-dominant representative of the* *infinitesimal character of X. If $\Lambda_1, \dots, \Lambda_m$ are the W_G-translates of Λ which are dominant and regular for $\mathfrak{k}$, then*

$$H_D(X) = \bigoplus_{i=1}^{m} E_{\Lambda_i - \rho_c}.$$

In the rest of the introduction we explain our strategy for proving Theorem 1.2 and comment on its extensions. It will become clear from general results about Dirac cohomology that any irreducible $\tilde{K}$-submodule of $H_D(X)$ must be of the form $E_{\Lambda_i - \rho_c}$. Thus, our task will be to show that each $E_{\Lambda_i - \rho_c}$, in fact, appears with multiplicity one. In order to do this, we will first demonstrate in Proposition 3.4 that any $\tilde{K}$-submodule of $X \otimes S$ appearing in $H_D(X)$ must be the Parthasarathy–Ranga-Rao–Varadarajan (PRV) component of the tensor product of a K-type of X and a $\tilde{K}$-type of the spin module S. The rest of the paper is then devoted to case-by-case calculations showing that for each i, there is in fact a unique K-type $E_{\mu_i} \subset X$ and a unique $\tilde{K}$-type $E_{\sigma_i} \subset S$ such that $E_{\Lambda_i - \rho_c}$ is the PRV component of $E_{\mu_i} \otimes E_{\sigma_i}$.

We will determine μ_i and σ_i very explicitly from the shortest element w_i of W_G such that $w_i \Lambda = \Lambda_i$. We denote by W_X the set of all w_i for $i = 1, \dots, m$. We will prove the following version of Theorem 1.2.

Theorem 1.3. *In the setting of Theorem 1.2, let W_X be the collection of the shortest element w_i of W_G such that $w_i \Lambda = \Lambda_i$. Then*

$$H_D(X) = \bigoplus_{w \in W_X} E_{w\Lambda - \rho_c}.$$

Moreover, for each $w \in W_X$ there is a unique K-type $E_{\mu(w)}$ of X, appearing in X with multiplicity one, and a unique $\tilde{K}$-type $E_{\sigma(w)}$ of S, such that $E_{w\Lambda - \rho_c}$ is the PRV component of $E_{\mu(w)} \otimes E_{\sigma(w)}$.

In each of the cases we are considering, all ingredients of Theorem 1.3 will be made completely explicit. See Theorems 4.3, 5.2 and 7.6. We actually prefer the formulation of Theorem 1.3 because in other known cases, Dirac cohomology is usually expressed as a sum over a subset of the Weyl group.

Finally, we note that there are several other Hermitian cases which are not considered in this paper: the cases of $O(2, m)$ and the exceptional cases E III and E VII [Enright et al. 1983]. In each of these cases we have obtained a result analogous to Theorem 1.3. Note that there is only one Wallach module for $O(2, m)$, which can be constructed using the noncompact dual pair $\mathrm{Sp}(2, \mathbb{R}) \times O(p, 2)$ in $\mathrm{Sp}(2(p + 2), \mathbb{R})$. It is the theta-lift of the trivial representation of $\mathrm{Sp}(2, \mathbb{R})$. This is a special case of theta-lifting for the dual pair $\mathrm{Sp}(2k, \mathbb{R}) \times O(p, q)$ in $\mathrm{Sp}(2k(p + q), \mathbb{R})$, considered in [Zhu and Huang 1997]. There is one Wallach module for the exceptional case E III and two Wallach modules for E VII. We do not include these cases here to keep this paper to a reasonable size and we do plan to consider them together with the general cases of noncompact dual pairs listed in [Nishiyama et al. 2001, Section 3, Table 1].

2. A description of Wallach representations

Let $\mathfrak{g}_0$ be one of the simple real Lie algebras $\mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{so}^*(2n, \mathbb{R})$ or $\mathfrak{su}(p, q)$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. Since the pair $(\mathfrak{g}, \mathfrak{k})$ is Hermitian, $\mathfrak{p}$ decomposes as $\mathfrak{p}^+ \oplus \mathfrak{p}^-$ as a $\mathfrak{k}$-module, and we fix such a decomposition.

As in the introduction, let $\mathfrak{t}$ be a common Cartan subalgebra for $\mathfrak{g}$ and $\mathfrak{k}$. We choose systems of positive roots for $\mathfrak{g}$ and $\mathfrak{k}$ with respect to $\mathfrak{t}$ so that (1.1) holds. Below we will make explicit choices in each of the three cases.

Let ζ be the fundamental weight of $\mathfrak{g}$ which is orthogonal to all the roots of $\mathfrak{k}$. Furthermore, let $c = \frac{1}{2}$ if $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$, $c = 2$ if $\mathfrak{g}_0 = \mathfrak{so}^*(2n, \mathbb{R})$ and $c = 1$ if $\mathfrak{g}_0 = \mathfrak{su}(p, q)$. Finally, let r be the split rank (real rank) of $\mathfrak{g}_0$. In other words, r is the dimension of a maximal abelian subspace of $\mathfrak{p}_0$. It is well known (see [Knapp 2002, p. 107], for example) that $r = n$ if $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$, $r = [n/2]$ if $\mathfrak{g}_0 = \mathfrak{so}^*(2n, \mathbb{R})$ and $r = \min(p, q)$ if $\mathfrak{g}_0 = \mathfrak{su}(p, q)$.

For an integer k such that $1 \leq k < r$, the k-th Wallach representation is the unitary lowest weight representation with lowest weight $kc\zeta$. This definition is taken from [Enright and Willenbring 2004, 1.4] except that they consider highest and not lowest weight modules, which corresponds to exchanging the roles of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ and introducing a minus sign on the weights. The same highest weight modules are described in a slightly different fashion in [Enright et al. 1983, Section 5], where it is shown that these modules together with the trivial module form the discrete part of the classification of unitary highest weight modules with one dimensional lowest K-type. (Note that since ζ is orthogonal to all the roots of $\mathfrak{k}$, $kc\zeta$ is indeed the weight of a one-dimensional $\mathfrak{k}$-module.)

We will now describe the Wallach modules more explicitly in each of the three cases. To do this, we also review some well known structural facts.

We start with the symplectic case, $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$. In this case, both $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{C})$ have rank n. So if $\mathfrak{t}$ is a common Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$, both $\mathfrak{t}$ and its dual can be identified with $\mathbb{C}^n$. It is standard to choose the positive compact roots to be $e_i - e_j$ for $1 \leq i < j \leq n$, and the positive noncompact roots to be $e_i + e_j$ for $i < j$ and $2e_i$, $i = 1, \dots, n$. If we as usual denote by ρ and ρ_c the half sums of the positive roots for $\mathfrak{g}$ respectively for $\mathfrak{k}$, and by $\rho_n = \rho - \rho_c$

the half sum of the noncompact positive roots, then we see

$$\rho = (n, \dots, 1), \quad \rho_c = \Big(\frac{n-1}{2}, \dots, -\frac{n-1}{2}\Big), \quad \rho_n = \Big(\frac{n+1}{2}, \dots, \frac{n+1}{2}\Big).$$

(The entries of ρ_c and ρ decrease by one, while those of ρ_n are constant.)

The Weyl group W_K consists of permutations of the variables, while W_G also contains arbitrary sign changes of the variables. The fundamental chamber for $\mathfrak{g}$ is given by the inequalities $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$, while the fundamental chamber for $\mathfrak{k}$ is given by $x_1 \geq x_2 \geq \cdots \geq x_n$. (These are the closed fundamental chambers; the open ones are given by strict inequalities.)

The simple roots corresponding to our choice of positive roots are $e_i - e_{i+1}$, for $i = 1, \dots, n-1$, and $2e_n$. It follows that in this case $\zeta = (1, 1, \dots, 1)$ and consequently the lowest weight of the k-th Wallach representation is

$$\Big(\frac{k}{2}, \frac{k}{2}, \dots, \frac{k}{2}\Big).$$

This is also the (only) weight of the lowest K-type of the k-th Wallach module. The infinitesimal character is obtained by subtracting ρ from the lowest weight. We conjugate the result to the positive Weyl chamber for $\mathfrak{g}$ and obtain

$$\Lambda = \Big(n - \frac{k}{2}, n-1-\frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2}-1, \frac{k}{2}-1, \frac{k}{2}-2, \frac{k}{2}-2, \dots\Big), \tag{2.1}$$

which ends with $1, 1, 0$ if k is even and with $\frac{1}{2}, \frac{1}{2}$ if k is odd.

We will also need to describe other K-types of Wallach modules. An explicit description can be found for example in [Nishiyama et al. 2001, Corollary 6.3]. The result is that all K-types appear with multiplicity one, and their highest weights are

$$\Big(\frac{k}{2}, \frac{k}{2}, \dots, \frac{k}{2}\Big) + (d_1, d_2, \dots, d_k, 0, \dots, 0), \tag{2.2}$$

for arbitrary even integers $d_1 \geq \cdots \geq d_k \geq 0$.

The proof of this fact relies on a construction of Wallach representations via Howe duality. The relevant dual pair here is

$$\mathrm{Sp}(2n, \mathbb{R}) \times O(k) \subset \mathrm{Sp}(2nk, \mathbb{R}).$$

This dual pair construction is in turn related to invariant theory, more specifically the first and second fundamental theorems of invariant theory, as described in [Howe 1989] and [Howe 1995, Chapters 2 and 3]. Another relevant reference is [Kashiwara and Vergne 1978]. In the following we outline the setting of this approach.

Let $\mathrm{M}_{n,k}$ be the complex vector space of $n \times k$ matrices with the linear action of the orthogonal group $O(k)$ by the right matrix multiplication. This action induces an action of $O(k)$ on the algebra $\mathcal{P}(\mathrm{M}_{n,k})$ of polynomial functions on $\mathrm{M}_{n,k}$ by algebra automorphisms. Let

$$V_k^+ = \mathcal{P}(\mathrm{M}_{n,k})^{O(k)}$$

be the subalgebra of the $O(k)$-invariants in $\mathcal{P}(\mathrm{M}_{n,k})$. Then V_k^+ can be expressed as a quotient of the algebra $\mathcal{P}(S^2(\mathbb{C}^n))$ of polynomial functions by the determinantal ideal of rank k. The symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ acts on $\mathcal{P}(\mathrm{M}_{n,k})$ by polynomial coefficient differential operators. This action commutes with the action of $O(k)$, and hence V_k^+ acquires the structure of a module for $\mathfrak{sp}(2n, \mathbb{R})$ and hence also for $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Furthermore, this module is the Harish-Chandra module of an irreducible unitary representation of the metaplectic double cover $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ of $\mathrm{Sp}(2n, \mathbb{R})$.[2] This representation of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ is called the theta-lift of the trivial representation of $O(k)$. The corresponding $(\mathfrak{g}, K)$ module V_k^+ is also referred to as the theta-lift of the trivial representation of $O(k)$.

In this setting it is not too difficult to prove that the K-types of V_k^+ are indeed given by (2.2). See [Nishiyama et al. 2001, Section 6] for more details.

It will be good for our purposes to study the Wallach representation V_k^+ together with another module V_k^-, which is obtained as the theta-lift of the sign representation of $O(k)$, that is, as the isotypic component of the sign representation $O(k)$ on $\mathcal{P}(\mathrm{M}_{n,k})$. Its K-types are given again by (2.2), but now $d_1 \geq \cdots \geq d_k \geq 1$ are arbitrary odd integers. The module V_k^- is another unitary lowest weight module. However its lowest K-type is not scalar, as it has highest weight

$$\Big(\frac{k}{2}+1, \dots, \frac{k}{2}+1, \frac{k}{2} \dots, \frac{k}{2}\Big),$$

with the first k entries equal, and the last $n-k$ entries also equal. We will be able to find Dirac cohomology of V_k^- simultaneously with V_k^+, with no additional work. Moreover, replacing the Wallach module V_k^+ with the module $V_k = V_k^+ \oplus V_k^-$, will actually make our results more uniform when compared with the orthogonal and unitary case, where we do not have analogs of V_k^-. The module V_k can be described as the subalgebra of the invariants of the special orthogonal group $\mathrm{SO}(k)$ acting on $\mathcal{P}(\mathrm{M}_{n,k})$.

To conclude the discussion of the symplectic case, we mention that for $k = 1$, V_1^+ and V_1^- are respectively the even and odd oscillator (Weil, metaplectic) representations of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$.

We now consider the orthogonal case, $\mathfrak{g}_0 = \mathfrak{so}^*(2n, \mathbb{R})$. The Lie algebras $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{C})$ both have rank n, and we choose a common Cartan subalgebra $\mathfrak{t}$ in both of them. Both $\mathfrak{t}$ and $\mathfrak{t}^*$ are identified with $\mathbb{C}^n$. We choose the

[2] In particular, this means that the relevant K here is not $U(n)$ which is the maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, but the double cover of $U(n)$ which is the maximal compact subgroup of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. This is reflected by the possible presence of half integers in the expression (2.2).

positive compact roots to be $e_i - e_j$ for $1 \le i < j \le n$, and the noncompact positive roots to be $e_i + e_j$ for $1 \le i < j \le n$. Thus

$$\rho = (n-1, \dots, 0), \quad \rho_c = \Big(\frac{n-1}{2}, \dots, -\frac{n-1}{2}\Big), \quad \rho_n = \Big(\frac{n-1}{2}, \dots, \frac{n-1}{2}\Big).$$

(The entries of ρ_c and ρ decrease by one, while those of ρ_n are constant.)

The Weyl group W_K consists of permutations of the variables, while W_G also contains arbitrary sign changes of even number of the variables. The fundamental chamber for $\mathfrak{g}$ is given by the inequalities $x_1 \ge x_2 \ge \cdots \ge x_{n-1} \ge |x_n|$, while the fundamental chamber for $\mathfrak{k}$ is given by $x_1 \ge x_2 \ge \cdots \ge x_n$. (These are the closed fundamental chambers; the open ones are given by strict inequalities.)

The simple roots corresponding to our choice of positive roots are $e_i - e_{i+1}$, for $i = 1, \dots, n-1$, and $e_{n-1} + e_n$. It follows that in this case $\zeta = \big(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\big)$, so the lowest weight of the k-th Wallach representation V_k, $k = 1, 2, \dots, [\frac{n}{2}]$, is

$$(k, k, \dots, k).$$

This is also the (only) weight of the lowest K-type of V_k. The infinitesimal character is obtained by subtracting ρ from the lowest weight. We conjugate the result to the positive Weyl chamber for $\mathfrak{g}$ and obtain

$$(n-k-1, n-k-2, \dots, k+1, k, k, k-1, k-1, \dots, 1, 1, 0). \tag{2.3}$$

Note that $n-k-1 \ge k+1$, so there is at least one nonrepeated entry before k, k.

All K-types of V_k are of multiplicity one, and their highest weights are

$$(k, \dots, k) + (d_1, d_1, d_2, d_2, \dots, d_k, d_k, 0, \dots, 0), \tag{2.4}$$

for arbitrary integers $d_1 \ge \cdots \ge d_k \ge 0$. See, for example, Corollary 6.9 of [Nishiyama et al. 2001]. This follows from the fact that each V_k is the theta-lift of the trivial representation of the compact factor $\mathrm{Sp}(2k)$ of the dual pair $\mathrm{SO}^*(2n, \mathbb{R}) \times \mathrm{Sp}(2k) \subset \mathrm{Sp}(4nk, \mathbb{R})$. Each V_k is the Harish-Chandra module of a unitary representation of $\mathrm{SO}^*(2n, \mathbb{R})$.

Finally, let us consider the case $\mathfrak{g}_0 = \mathfrak{u}(p, q)$. It is slightly more convenient to work with $\mathfrak{g}_0 = \mathfrak{u}(p, q)$, which leads easily to the desired conclusion for $\mathfrak{g}_0 = \mathfrak{su}(p, q)$.

The Lie algebras

$$\mathfrak{g} = \mathfrak{u}(p, q)_{\mathbb{C}} = \mathfrak{gl}(p+q, \mathbb{C}),$$
$$\mathfrak{k} = (\mathfrak{u}(p) \times \mathfrak{u}(q))_{\mathbb{C}} = \mathfrak{gl}(p, \mathbb{C}) \times \mathfrak{gl}(q, \mathbb{C})$$

are both of rank $n = p+q$. Let $\mathfrak{t}$ be a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$. Then both $\mathfrak{t}$ and $\mathfrak{t}^*$ can be identified with $\mathbb{C}^n$. We choose the positive roots for $\mathfrak{t}$ in $\mathfrak{g}$ to be $e_i - e_j$, $1 \le i < j \le n$. Among them, the compact ones are those for which either $1 \leq i < j \leq p$ or $p+1 \leq i < j \leq n$, while the noncompact ones are those for which $1 \leq i \leq p$ and $p+1 \leq j \leq n$. Thus

$$\begin{aligned}
\rho &= \tfrac{1}{2}\big(p+q-1,\, p+q-3,\, \dots,\, -(p+q-1)\big),\\
\rho_c &= \tfrac{1}{2}\big(p-1,\, p-3,\, \dots,\, -(p-1) \mid q-1,\, q-3,\, \dots,\, -(q-1)\big),\\
\rho_n &= \tfrac{1}{2}(q,\, \dots,\, q \mid -p,\, \dots,\, -p).
\end{aligned}$$

(We will often separate the first p coordinates from the last q coordinates by a bar).

The Weyl group W_G consists of all permutations of $n = p+q$ elements, while W_K consists of those permutations that permute separately the first p elements and the last q elements. The closed fundamental chamber for $\mathfrak{g}$ consists of all $(x_1, \dots, x_{p+q})$ such that $x_1 \geq \cdots \geq x_{p+q}$, while the closed fundamental chamber for $\mathfrak{q}$ consists of all $(x_1 \dots, x_p \mid y_1, \dots, y_q)$ such that $x_1 \geq \cdots \geq x_p$ and $y_1 \geq \cdots \geq y_q$.

To describe the Wallach representations, we first note that the simple roots corresponding to our choice of positive roots are $e_i - e_{i+1}$, $i = 1, \dots, n-1$. All of these are compact except for $e_p - e_{p+1}$. It follows that $\zeta = (a, \dots, a \mid b, \dots, b)$ for some a and b such that $a - b = 1$. For $\mathfrak{g}_0 = \mathfrak{su}(p, q)$ it would also be required that $pa + qb = 0$, so it would follow that $a = p/n$ and $b = -q/n$. The lowest weights of the Wallach modules V_k, $k = 1, 2, \dots, \min(p, q) - 1$, would then be

$$\Big(\frac{kp}{n}, \dots, \frac{kp}{n} \,\Big|\, -\frac{kq}{n}, \dots, -\frac{kq}{n}\Big).$$

For $\mathfrak{g}_0 = \mathfrak{u}(p, q)$ we can however simplify the lowest weight by twisting the module by central character $(k(p-q))/2n(1, \dots, 1)$, and thus work with the lowest weight of the form

$$\Big(\frac{k}{2}, \dots, \frac{k}{2} \,\Big|\, -\frac{k}{2}, \dots, -\frac{k}{2}\Big).$$

This twisting of course does not change the modules very much, and we will call the twisted modules the Wallach modules for $\mathfrak{u}(p, q)$ and denote them by V_k, $k = 1, 2, \dots, \min(p, q) - 1$.

To obtain the infinitesimal character of V_k, we as usual subtract ρ from the lowest weight. This has the following $\mathfrak{g}$-dominant representative:

$$\begin{aligned}
\Lambda = \tfrac{1}{2}(&p+q-1-k,\, p+q-3-k,\, \dots,\, p-q+1+k,\\
&p-q-1+k,\, p-q-1+k,\, \dots,\, p-q+1-k,\, p-q+1-k,\\
&p-q-1-k,\, p-q-3-k,\, \dots,\, -p-q+1+k).
\end{aligned} \tag{2.5}$$

Here after multiplying by $\frac{1}{2}$ the first row consists of $q-k$ coordinates decreasing by $\frac{1}{2} \cdot 2 = 1$, the second row consists of k pairs of equal coordinates with the value of each pair being 1 less than the value of the previous pair, and the third row consists of $p-k$ coordinates decreasing by 1.

All K-types of V_k are of multiplicity one, and their highest weights are

$$\left(\frac{k}{2},\dots,\frac{k}{2}\,\middle|\,-\frac{k}{2},\dots,-\frac{k}{2}\right)+(d_1,\dots,d_k,0,\dots,0\,\big|\,0,\dots,0,-d_k,\dots,-d_1), \tag{2.6}$$

for arbitrary integers $d_1\ge\cdots\ge d_k\ge 0$. See for example Corollary 6.7 of [Nishiyama et al. 2001]. This follows from the fact that each V_k is the theta-lift of the trivial representation of the compact factor $U(k)$ of the dual pair

$$U(p,q)\times U(k)\subset \mathrm{Sp}(2(p+q)k,\mathbb{R}).$$

Each V_k is the Harish-Chandra module of a unitary representation of the double cover of $U(p,q)$ defined as the preimage of $U(p,q)$ in the metaplectic double cover of $\mathrm{Sp}(2(p+q)k,\mathbb{R})$.

3. Preliminaries on Dirac cohomology

In this section we review basic facts about Dirac cohomology with emphasis on the special Hermitian setting we are considering in this paper. For more details, see [Huang and Pandžić 2006, Chapter 3].

Let G be a connected real reductive Lie group with Cartan involution Θ and assume that $K=G^{\Theta}$ is a maximal compact subgroup of G. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra of G corresponding to Θ. We fix a nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$, equal to the Killing form on the semisimple part of $\mathfrak{g}$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B. The Dirac operator $D\in U(\mathfrak{g})\otimes C(\mathfrak{p})$ is defined as

$$D=\sum_i b_i\otimes d_i,$$

where b_i is a basis of $\mathfrak{p}$, and d_i is the dual basis with respect to B. It is easy to check that D does not depend on the choice of the basis b_i and moreover it is K-invariant for the diagonal action of K given by adjoint actions on both factors.

In the Hermitian setting, the $\mathfrak{k}$-module $\mathfrak{p}$ decomposes as $\mathfrak{p}^+\oplus\mathfrak{p}^-$, where $\mathfrak{p}^\pm$ are isomorphic as modules for the semisimple part of $\mathfrak{k}$, while the center of $\mathfrak{k}$ acts on them by dual characters. Let $m=\dim\mathfrak{p}^+=\dim\mathfrak{p}^-$. We now choose the basis b_i in the following way. Let $\Delta(\mathfrak{g})$ be the set of roots of the compact Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$. Let $\Delta^+(\mathfrak{g})$ denote a fixed choice of positive roots which is compatible with $\mathfrak{p}^+$, that is, the noncompact positive roots are exactly the $\mathfrak{t}$-weights $\alpha_1,\dots,\alpha_m$ of $\mathfrak{p}^+$. For each α_i we choose a root vector e_i. Let f_i be the root vector for the root $-\alpha_i$ such that $B(e_i,f_i)=1$. Then for the basis b_i of $\mathfrak{p}$ we choose $e_1,\dots e_m;\ f_1,\dots f_m$.

The dual basis is then $f_1, \dots f_m;\ e_1, \dots e_m$, and hence

$$D = \sum_{i=1}^{m} e_i \otimes f_i + f_i \otimes e_i.$$

Let X be a $(\mathfrak{g}, K)$-module. To get a module for the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, we tensor X with an irreducible (spin) module S for $C(\mathfrak{p})$. In the Hermitian case, $\mathfrak{p}$ is even-dimensional, hence there is only one irreducible $C(\mathfrak{p})$-module up to isomorphism, and it can be constructed as $S = \bigwedge \mathfrak{p}^+$.

Now $X \otimes S$ is a $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \tilde{K})$ module, where $\tilde{K}$ is the spin double cover of K, that is, the pullback of the double cover $\mathrm{Spin}(\mathfrak{p}_0) \to \mathrm{SO}(\mathfrak{p}_0)$ by the action map $K \to \mathrm{SO}(\mathfrak{p}_0)$. The action of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ on $X \otimes S$ is the obvious one, and $\tilde{K}$ acts on both factors, on X through K and on S through the spin group $\mathrm{Spin}(\mathfrak{p}_0) \subset C(\mathfrak{p})$. The copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ corresponding to $\tilde{K}$ is

$$\mathfrak{k}_\Delta = \{ Y \otimes 1 + 1 \otimes \alpha(Y) \mid Y \in \mathfrak{k} \}.$$

Here α is the complexification of the map $\mathfrak{k}_0 \to \mathfrak{so}(\mathfrak{p}_0) \cong \bigwedge^2 \mathfrak{p}_0 \hookrightarrow C(\mathfrak{p}_0)$ given by the adjoint action followed by the skew symmetrization.

Now the Dirac operator D acts on $X \otimes S$, and the Dirac cohomology of X is the $\tilde{K}$-module

$$H_D(X) = \mathrm{Ker}\, D / (\mathrm{Im}\, D \cap \mathrm{Ker}\, D).$$

If X is unitary, then it is well known that D is self-adjoint with respect to the inner product on $X \otimes S$ induced by the invariant inner product on X and the usual inner product on S (see [Wallach 1988, p. 367] or [Huang and Pandžić 2006, p. 63].) It follows that $\mathrm{Ker}\, D \cap \mathrm{Im}\, D = 0$, and hence

$$H_D(X) = \mathrm{Ker}\, D = \mathrm{Ker}\, D^2.$$

We will now summarize some earlier results. We denote by E_γ the irreducible $\tilde{K}$-module with highest weight $\gamma \in \mathfrak{t}^*$.

Theorem 3.1 [Huang and Pandžić 2002]. *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character $\Lambda \in \mathfrak{t}^*$. Assume that the $\tilde{K}$-module E_γ appears in $X \otimes S$. Then the following are equivalent*:

(1) *the γ-isotypic component of $X \otimes S$ is contained in $H_D(X)$*;

(2) *Λ is conjugate to $\gamma + \rho_c$ under the Weyl group $W_G = W(\mathfrak{g}, \mathfrak{t})$*;

(3) $\|\Lambda\| = \|\gamma + \rho_c\|$.

The implication (1) $\Rightarrow$ (2) is true without the unitarity assumption. This is one of the main results of [Huang and Pandžić 2002], conjectured by Vogan. The implication (2) $\Rightarrow$ (3) is obvious. Let us briefly recall why (3) implies (1). By

[Parthasarathy 1972] (see [Huang and Pandžić 2006, Proposition 3.1.6]),

$$D^2 = -(\Omega_{\mathfrak{g}} + \|\rho\|^2) + (\Omega_{\mathfrak{k}_\Delta} + \|\rho_c\|^2). \tag{3.2}$$

Here $\Omega_{\mathfrak{g}}$ is the Casimir element of $U(\mathfrak{g})$ and $\Omega_{\mathfrak{k}_\Delta}$ is the Casimir element of $U(\mathfrak{k}_\Delta)$. Since $\Omega_{\mathfrak{g}}$ acts on X by the scalar $\|\Lambda\|^2 - \|\rho\|^2$ and $\Omega_{\mathfrak{k}_\Delta}$ acts on a $\tilde{K}$-type E_γ by the scalar $\|\gamma+\rho_c\|^2 - \|\rho_c\|^2$, we see that $D^2 = 0$ on the isotypic component $(X \otimes S)(\gamma)$ if and only if $\|\Lambda\| = \|\gamma + \rho_c\|$.

Another useful consequence of the fact that D is self-adjoint for unitary X, and hence $D^2 \geq 0$, is Parthasarathy's Dirac inequality:

Proposition 3.3 [Parthasarathy 1980, Lemma 2.5]. *Let X be a unitary $(\mathfrak{g}, K)$-module with infinitesimal character $\Lambda \in \mathfrak{t}^*$. Let E_γ be any $\tilde{K}$-type contained in $X \otimes S$. Then*

$$\|\Lambda\| \leq \|\gamma + \rho_c\|.$$

This simply follows from writing out the inequality $D^2 \geq 0$ on the isotypic component $(X \otimes S)(\gamma)$, using (3.2).

The following remark will be useful for our calculations.

Proposition 3.4. *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Assume that $H_D(X)$ contains a $\tilde{K}$-type*

$$E_\gamma \subset E_\mu \otimes E_\sigma \subset X \otimes S,$$

where $E_\mu \subset X$ and $E_\sigma \subset S$. Then E_γ is the PRV component of $E_\mu \otimes E_\sigma$, that is, γ is conjugate to σ plus the lowest weight of E_μ under the Weyl group W_K of $\mathfrak{k}$.

Proof. By Proposition 3.3, for any $E_{\gamma'} \subset X \otimes S$, we have

$$\|\gamma' + \rho_c\| \geq \|\Lambda\|.$$

Moreover, since $D^2 = \|\gamma' + \rho_c\|^2 - \|\Lambda\|^2$ on $E_{\gamma'}$, and $D^2 \geq 0$, the expression $\|\gamma' + \rho_c\|$ is the smallest possible when $\gamma' = \gamma$. In particular, this is true for all γ' such that $E_{\gamma'}$ is contained in $E_\mu \otimes E_\sigma$, which means that E_γ is the PRV component of $E_\mu \otimes E_\sigma$ [Parthasarathy et al. 1967; Wallach 1988, 9.1.6]. □

We can now describe the setting for calculating Dirac cohomology of an irreducible unitary $(\mathfrak{g}, K)$ module X more precisely. We are looking for $\tilde{K}$-types E_γ in $X \otimes S$ such that $\gamma + \rho_c$ is conjugate to the infinitesimal character Λ of X by some $w \in W_G$. We can assume that Λ is $\mathfrak{g}$-dominant. The following obvious fact will be useful.

Proposition 3.5. *The only possible $\tilde{K}$-types in $H_D(X)$ are of the form $E_{w\Lambda-\rho_c}$, with $w \in W_G$ such that $w\Lambda$ is dominant regular for $\mathfrak{k}$. In particular, w is in W^1, where $W^1 \subset W_G$ consists of those w that take the fundamental Weyl chamber for $\mathfrak{g}$ into the fundamental Weyl chamber for $\mathfrak{k}$.*

Each coset of W_K in W_G contains exactly one element of W^1. In particular, W^1 has $|W_G|/|W_K|$ elements.

On the other hand, it is well known (see [Wallach 1988] or [Huang and Pandžić 2006, Section 2.3], for instance) that the spin module S for $\mathfrak{k}$ decomposes as

$$S = \bigoplus_{\sigma \in W^1} E_{\sigma\rho - \rho_c}.$$

Thus we see that for a K-type E_μ of X and a $\tilde{K}$-type $E_{\sigma\rho-\rho_c}$ of S, the PRV component of $E_\mu \otimes E_{\sigma\rho-\rho_c}$ will contribute to $H_D(X)$ if and only if

$$w\Lambda - \rho_c = (\sigma\rho - \rho_c + \mu^-)'.$$

Here μ^- denotes the lowest weight of E_μ, and $(\sigma\rho - \rho_c + \mu^-)'$ denotes the $\mathfrak{k}$-dominant W_K-conjugate of $\sigma\rho - \rho_c + \mu^-$. This will be the starting point for our calculations. It will turn out that the calculation is slightly simpler if we add ρ_n to both sides of the equation:

$$w\Lambda - \rho_c + \rho_n = (\sigma\rho - \rho_c + \rho_n + \mu^-)'. \tag{3.6}$$

Note that since ρ_n is W_K-invariant, it is legitimate to put it inside the parentheses on the right side.

To end this section, let us note that for unitary lowest weight modules the Dirac cohomology never vanishes.

Proposition 3.7. *Let X be an irreducible unitary lowest weight $(\mathfrak{g}, K)$-module with lowest K-type E_μ. Then $E_\mu \otimes \mathbb{C} \cdot 1 \subset X \otimes S$ is contained in $H_D(X)$. In particular, $H_D(X)$ is not zero.*

Proof. This is straightforward: both $E_\mu \subset X$ and $\mathbb{C} \cdot 1 \subset S$ are annihilated by all $f_i \in \mathfrak{p}^-$. Thus every term of $D = \sum e_i \otimes f_i + f_i \otimes e_i$ kills $E_\mu \otimes \mathbb{C} \cdot 1$. □

4. The case of $\mathfrak{sp}(2n, \mathbb{R})$

We are going to use the facts about the structure of the pair $(\mathfrak{g}, \mathfrak{k})$ which we reviewed in Section 2. Besides that, we also need to describe the subset $W^1 \subset W_G$ consisting of those $w \in W_G$ which conjugate the fundamental chamber for $\mathfrak{g}$ into the fundamental chamber for $\mathfrak{k}$. Alternatively, W^1 can be described as the set of the minimal length representatives of the left W_K-cosets in W_G.

In the symplectic case, W^1 may be parametrized by $\mathbb{Z}_2^n$. Namely, for any choice of sign changes $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, there is a unique permutation τ of the variables such that for any $\mathfrak{g}$-dominant $(x_1, \dots x_n)$, $\tau(\epsilon_1 x_1, \dots, \epsilon_n x_n)$ is $\mathfrak{k}$-dominant. We will be slightly imprecise and identify ϵ with the corresponding element of W^1.

Recall that the modules we are interested in are $V_k = V_k^+ \oplus V_k^-$, where k in an integer such that $1 \leq k \leq n-1$. The infinitesimal character of V_k was given in (2.1). We now rewrite it as

$$\Lambda = \Big(\frac{k}{2}+n-k,\ \frac{k}{2}+n-k-1,\ \dots,\ \frac{k}{2},\ \frac{k}{2}-1,\ \frac{k}{2}-1,\ \frac{k}{2}-2,\ \frac{k}{2}-2,\ \dots\Big), \tag{4.1}$$

ending in the same way as before, by 1, 1, 0 if k is even and by $\frac{1}{2}, \frac{1}{2}$ if k is odd.

By Proposition 3.5, any w involved in (3.6) must put a minus on exactly one member of each pair of repeated coordinates. It does not matter which of the two gets a minus, since $w\Lambda$ will be the same in each case. In other words, we see that

$$w = \epsilon_1\epsilon_2 \dots \epsilon_{n-k+1}(\pm\mp)(\pm\mp)\dots(\pm\mp)(\pm),$$

where each of the pairs in parentheses can be $(+-)$ or $(-+)$, while the last sign $(\pm)$ appears when k is even. All such choices give the same $w\Lambda$, which is thus determined by the sequence $\epsilon_1\epsilon_2 \dots \epsilon_{n-k+1}$. We will therefore work only with w of the form

$$w = \epsilon_1\epsilon_2 \dots \epsilon_{n-k+1}(+-)(+-)\dots(+-)(+). \tag{4.2}$$

This means that of all possible w defining the same weight $w\Lambda$ we choose the shortest one. Namely, each time when we have $(-+)$ instead of $(+-)$, it requires an additional transposition to keep $\mathfrak{g}$-dominant weights $\mathfrak{k}$-dominant.

Theorem 4.3. *The Dirac cohomology of V_k is*

$$H_D(V_k) = \bigoplus_w E_{w\Lambda-\rho_c},$$

with the summation over all w as in (4.2). *All ingredients of the formula* (3.6) *are uniquely determined by w: σ is given by*

$$\sigma = --\cdots-\epsilon_1\epsilon_2\dots\epsilon_{n-k},$$

and the corresponding K-type in V_k is the one with highest weight of the form (2.2) *given by*

$$d_1 = \cdots = d_k = \text{ the number of pluses in the sequence } \epsilon = (\epsilon_1, \dots, \epsilon_{n-k+1}).$$

In particular, $H_D(V_k^+)$ contains all $E_{w\Lambda-\rho_c}$ with w such that the number of pluses in the sequence ϵ is even, while $H_D(V_k^-)$ contains all $E_{w\Lambda-\rho_c)}$ with w such that the number of pluses in the sequence ϵ is odd.

Proof. By (2.2), the lowest weights of the K-types of V_k appearing in the right side of (3.6) are

$$(0, \dots, 0, d_k, \dots, d_1) + \Big(\frac{k}{2}, \dots, \frac{k}{2}\Big).$$

Since $(k/2,\dots,k/2)$ is invariant under W_K, we can subtract it from both sides of (3.6). Now we can write out the new left side of (3.6) using the expression (4.1) for Λ and (4.2) for w. To do this, let

$$i_1 > i_2 > \cdots > i_r \quad \text{and} \quad j_1 < j_2 < \cdots < j_s$$

be the integers in $[0, n-k]$ such that $\epsilon_{n-k+1-i_t} = +$, $t = 1,\dots,r$, respectively $\epsilon_{n-k+1-j_u} = -$, $u = 1,\dots,s$. This means that $r+s = n-k+1$, and that

$$\{i_1,\dots,i_r, j_1,\dots,j_s\} = \{0, 1,\dots,n-k\}.$$

Note that the sequence $\epsilon_1,\dots,\epsilon_{n-k+1}$ is completely determined by is and js and vice versa.

Now we have

$$\begin{aligned}(4.4)\quad & w\Lambda - \rho_c + \rho_n - \left(\frac{k}{2},\dots,\frac{k}{2}\right)\\ &= (i_1+1,\dots,i_r+r, r,\dots,r, r-j_1, r+1-j_2,\dots,n-k-j_s).\end{aligned}$$

Note that r appears in $k-1$ places, from the $(r+1)$-st place to the $(r+k-1)$-st.

On the other hand, the right side of (3.6) after subtracting $(k/2,\dots,k/2)$ becomes

$$(4.5)\qquad \bigl(\sigma\rho - \rho_c + \rho_n + (0,\dots,0,d_k,\dots,d_1)\bigr)'.$$

(Recall that the prime means taking the $\mathfrak{k}$-dominant W_K-conjugate.)

Since the largest component of (4.4) is $i_1 + 1 \le n-k+1$, we see that for (3.6) to hold, $\sigma\rho$ cannot have entries $n, n-1,\dots,n-k+1$. Thus the starting k signs of σ are all minuses, that is,

$$(4.6)\qquad \sigma = - \cdots - \delta_1 \dots \delta_{n-k}.$$

Using similar notation as before, we see that

$$\sigma\rho = (I_1,\dots,I_a, -J_1,\dots,-J_b, -(n-k+1), -(n-k+2),\dots,-n),$$

where $I_1 > \cdots > I_a$ and $J_1 < \cdots < J_b$ are integers in $[1, n-k]$ determined by $\delta_1,\dots,\delta_{n-k}$. In particular, $a+b = n-k$,

$$\{I_1,\dots,I_a, J_1,\dots,J_b\} = \{1,\dots,n-k\},$$

and Is and Js correspond to δs via

$$\delta_{n-k+1-I_u} = +, \qquad \delta_{n-k+1-J_v} = -,$$

for all indices u and v.

It is now easily seen that (4.5) equals

$$(4.7)\qquad (I_1+1,\dots,I_a+a, -J_1+a+1,\dots,-J_b+a+b, d_k,\dots,d_1)'.$$

Thus $w\Lambda - \rho_c$ contributes to $H_D(V_k)$ if and only if (4.4) equals (4.7). Clearly, this is the case if Is are equal to is, Js are equal to js, and all ds are equal to r. In this case, $\delta_i = \epsilon_i$ for $i = 1, \dots, n-k$. Thus to prove the theorem, we must show that this is the only possibility.

Either $i_r = 0$ and $j_1 > 0$, or $i_r > 0$ and $j_1 = 0$. The proof in each of these two cases is analogous, so we will assume that $i_r = 0$ and $j_1 > 0$. Thus (4.4) is

$$(i_1+1, \dots, i_{r-1}+r-1, r, \dots, r, r-j_1, \dots, n-k-j_s), \tag{4.8}$$

where the k entries from r-th to $(r-1+k)$-th are equal to r, entries before these k are $\geq r$, and entries after the k rs are $< r$.

We claim that $a = r-1$. To see this, assume first that $a < r-1$. Since only the first a or the last k entries of (4.7) can be $\geq a+1$, and since $r > a+1$, we see that the number of entries of (4.7) that are $\geq r$ is at most $a+k$. On the other hand, the number of entries of (4.8) that are $\geq r$ is $r-1+k > a+k$. Hence (4.7) cannot be equal to (4.8) or (4.4) in this case.

Similarly, if $a > r-1$, then the first $a \geq r$ entries of (4.7) are $\geq a+1 > r$. On the other hand, the r-th entry of (4.8) is r.

So indeed $a = r-1$. Now we see that the first a terms or (4.7) are $\geq a+1 = r$, and the next b terms are all $< a+1 = r$. On the other hand, (4.8) has $r-1+k$ entries $\geq r$. So we conclude that for (4.7) to be equal to (4.8), $d_1, \dots, d_k$ must all be equal to r. Hence also Is must be equal to is and Js must be equal to js, as claimed. □

Theorem 4.3 explicitly exhibits the pairs $(\mu, \bar{\sigma}) \in \mathfrak{t}^* \times \mathfrak{t}^*$, such that E_μ is a K-type of the module $X = V_k$ and $E_{\bar{\sigma}}$ is a $\tilde{K}$-type of the spin module S, and such that the PRV component of $E_\mu \otimes E_{\bar{\sigma}}$ contributes to the Dirac cohomology of X. (Recall that both V_k and the spin module are multiplicity free as $\mathfrak{k}$-modules.)

We now make this even more explicit by exhibiting the highest weight vectors of E_μ and $E_{\bar{\sigma}}$, and also the lowest weight vectors of E_μ. These are known by [Howe 1995]. Recall that $\mathfrak{p}^+ \cong S^2\mathbb{C}^n$, the space of $n \times n$ symmetric matrices, and that up to a twist by the character $\det_n^{k/2}$, V_k can be identified as a K-module with the quotient of the symmetric algebra of $\mathfrak{p}^+$ by the k-th symmetric determinantal ideal, generated by minors of order $k+1$. For $1 \leq i \leq k$, let δ_i and δ_i' be the $i \times i$ upper left and lower right corner minors of the symmetric matrix with generators in $\mathfrak{p}^+$.

Proposition 4.9. *Let $d_1 \geq \dots \geq d_k \geq 0$, $d_{k+1} = 0$. The element*

$$\delta_1^{d_1-d_2} \dots \delta_k^{d_k-d_{k+1}} \in S(\mathfrak{p}^+)$$

has $\mathfrak{k}$-highest weight $(2d_1, \dots, 2d_k, 0, \dots, 0)$ and a nonzero image in the k-th determinantal quotient.

Conversely, every highest weight vector in the k-th determinantal quotient of $S(\mathfrak{p}^+)$ has this form. The corresponding lowest weight vectors are obtained by replacing each δ_i with δ_i'.

Up to another twist by a character of $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{C})$, the spin module is isomorphic to the exterior algebra $\Lambda(\mathfrak{p}^+)$. A weight basis of $\mathfrak{p}^+$ gives rise to the basis of the exterior algebra consisting of decomposable skew-symmetric tensors. It turns out that the highest weight vectors are decomposable. Let us introduce the following partial order on the set of pairs $A = \{(i, j) : 1 \le i \le j \le n\}$:

$$(i, j) \le (l, m) \iff i \le l \text{ and } j \le m.$$

Proposition 4.10. *For any lower-closed subset of A, the exterior product of the corresponding elements of $\mathfrak{p}^+$ is a highest weight vector of $\Lambda(\mathfrak{p}^+)$. Conversely, every highest weight vector is proportional to the exterior product of the elements of a weight basis of $\mathfrak{p}^+$ corresponding to a lower-closed subset of A.*

The set A is a lattice triangle, the upper half of the lattice $n \times n$ square $1 \le i, j \le n$. If A is represented graphically by dots in the lattice, the lower-closed subsets of A are formed by subsets that are closed under leftward and downward "slides". Each such subset is uniquely determined by its lattice boundary, consisting of alternating horizontal and vertical segments connecting the vertical axis $i = 0$ with the bisector $i = j$.

To end this section, we note that the modules we are considering include the even and odd oscillator representations; thus we recover the results of [Adams 1994].

5. The case of $\mathfrak{so}^*(2n, \mathbb{R})$

We are going to use the facts about the structure of the pair $(\mathfrak{g}, \mathfrak{k})$ which we reviewed in Section 2. Besides that, we also need to describe the subset $W^1 \subset W_G$. In the orthogonal case, $W^1 \subset W_G$ may be parametrized by $\mathbb{Z}_2^{n-1}$. Namely, for any choice of even number of sign changes $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, there is a unique permutation τ of the variables such that for any $\mathfrak{g}$-dominant $(x_1, \dots x_n)$, $\tau(\epsilon_1 x_1, \dots, \epsilon_n x_n)$ is $\mathfrak{k}$-dominant. We will be slightly imprecise and identify ϵ with the corresponding element of W^1.

Recall that the modules we are interested in are V_k, where $1 \le k \le [n/2] - 1$ is an integer and the lowest weight of V_k is $(k, k, \dots, k)$.

The infinitesimal character Λ of V_k was given in (2.3). By Proposition 3.5, any w involved in (3.6) must put a minus on exactly one member of each pair of repeated coordinates of Λ. It does not matter which of the two gets a minus, since $w\Lambda$ will be the same in each case. In other words, we see that

$$w = \epsilon_1 \epsilon_2 \dots \epsilon_{n-2k-1} (\pm\mp)(\pm\mp) \dots (\pm\mp)(\pm).$$

Here each of the pairs in parentheses can be $(+-)$ or $(-+)$, while the last sign is determined so that the total number of minuses is even. All such choices give the same $w\Lambda$, which is thus determined by the sequence $\epsilon_1\epsilon_2\dots\epsilon_{n-2k-1}$. We will therefore work only with w of the form

$$w = \epsilon_1\epsilon_2\dots\epsilon_{n-2k-1}(+-)(+-)\dots(+-)(\pm), \tag{5.1}$$

with the last sign determined as before. In other words, we choose the shortest possible w in each case.

Theorem 5.2. *The Dirac cohomology of V_k is*

$$H_D(V_k) = \bigoplus_w E_{w\Lambda-\rho_c},$$

with the summation over all w as in (5.1)*. All ingredients of the formula* (3.6) *are uniquely determined by w: σ is given by*

$$\sigma = --\cdots-\epsilon_1\epsilon_2\dots\epsilon_{n-2k-1}\pm,$$

with the last sign determined so that the total number of minuses is even. The corresponding K-type in V_k is the one with the highest weight of the form (2.4) *given by*

$$d_1 = \dots = d_k = \text{ the number of pluses in the sequence } \epsilon = (\epsilon_1, \dots, \epsilon_{n-2k-1}).$$

Proof. By (2.4), the lowest weights of the K-types of V_k appearing in the right side of (3.6) are

$$(0, \dots, 0, d_k, d_k, \dots, d_1, d_1) + (k, \dots, k).$$

Since $(k, \dots, k)$ is invariant under W_K, we can subtract it from both sides of (3.6). Now we can write out the new left side of (3.6) using the expression (2.3) for Λ and (5.1) for w. To do this, let

$$i_1 > i_2 > \dots > i_r, \quad j_1 < j_2 < \dots < j_s$$

be the integers in $[1, n-2k-1]$ such that

$$w\Lambda = (i_1+k, \dots, i_r+k, k, k-1, \dots, -k, -j_1-k, \dots, -j_s-k).$$

This means that $r+s = n-2k-1$, and that

$$\{i_1, \dots, i_r, j_1, \dots, j_s\} = \{1, 2, \dots, n-2k-1\}.$$

In terms of the sequence $\epsilon_1, \dots, \epsilon_{n-2k-1}$, the is and js are described by requiring $\epsilon_{n-2k-i_t} = +$, $t = 1, \dots, r$, respectively $\epsilon_{n-2k-j_u} = -$, $u = 1, \dots, s$. The sequence $\epsilon_1, \dots, \epsilon_{n-2k-1}$ is completely determined by the is and the js and vice versa.

Now we see that

$$(5.3)\quad w\Lambda-\rho_c+\rho_n-(k,\dots,k)$$
$$=(i_1,i_2+1,\dots,i_r+r-1,r,r,\dots,r,r+1-j_1,r+2-j_2,\dots,r+s-j_s).$$

Here r appears in $2k+1$ places, from the $(r+1)$-st place to the $(r+2k+1)$-st. Note also that r is the number of pluses in the sequence $\epsilon_1,\dots,\epsilon_{n-2k-1}$.

On the other hand, the right side of (3.6) becomes, after subtracting $(k,\dots,k)$,

$$(5.4)\qquad \big(\sigma\rho-\rho_c+\rho_n+(0,\dots,0,d_k,d_k,\dots,d_1,d_1)\big)'.$$

(Recall that the prime means taking the $\mathfrak{k}$-dominant W_K-conjugate.)

Since the largest component of (5.3) is $i_1\le n-2k-1$, we see that for (3.6) to hold, $\sigma\rho$ cannot have entries $n-1,n-2,\dots,n-2k$. Thus the starting $2k$ signs of σ are all minuses, that is,

$$(5.5)\qquad \sigma=-\cdots-\delta_1\dots\delta_{n-2k}.$$

Note that δ_{n-2k} is determined by the other δs, because the number of minuses must be even.

Using similar notation as before, we see that

$$\sigma\rho=\big(I_1,\dots,I_a,0,-J_1,\dots,-J_b,-(n-2k),-(n-2k+1),\dots,-(n-1)\big),$$

where $I_1>\cdots>I_a$ and $J_1<\cdots<J_b$ are integers in $[1,n-2k-1]$ determined by $\delta_1,\dots,\delta_{n-2k-1}$. In particular, $a+b=n-2k-1$,

$$\{I_1,\dots,I_a,J_1,\dots,J_b\}=\{1,2,\dots,n-2k-1\},$$

and Is and Js correspond to δs via

$$\delta_{n-2k-I_u}=+,\qquad \delta_{n-2k-J_v}=-,$$

for all indices u and v.

It is now easily seen that (5.4) equals

$$(5.6)\quad (I_1,I_2+1,\dots,I_a+a-1,a,a+1-J_1,$$
$$a+2-J_2,\dots,a+b-J_b,d_k,d_k,\dots,d_1,d_1)'.$$

Thus $E_{w\Lambda-\rho_c}$ contributes to $H_D(V_k)$ if and only if (5.3) equals (5.6). Clearly, this is the case if Is are equal to is, Js are equal to js, and all ds are equal to r. In this case, $\delta_i=\epsilon_i$ for $i=1,\dots,n-2k-1$. Thus to prove the theorem, we must show that this is the only possibility.

So let us assume that (5.3) equals (5.6). Either $i_r=1$ and $j_1>1$, or $i_r>1$ and $j_1=1$. The proof in each of these two cases is analogous, so we will assume that

$i_r = 1$ and $j_1 > 1$. Thus (5.3) is

$$(i_1, \dots, i_{r-1} + r - 2, r, \dots, r, r + 1 - j_1, \dots, r + s - j_s). \tag{5.7}$$

Here the first $r + 2k + 1$ entries are $\geq r$ while the last s entries are $< r$. Moreover, at least $2k + 2$ entries are equal to r - the entries starting with the r-th entry.

We claim that $a = r$. To see this, assume first that $a < r$. Then a and $a + u - J_u$ for $u = 1, \dots, b$ are all $\leq a < r$. So there are at least $b + 1 = n - 2k - a$ entries $< r$ in (5.6), and therefore at most $2k + a < 2k + r$ entries $\geq r$. But we saw that in (5.7) there are exactly $r + 2k + 1$ entries are $\geq r$, so (5.7) (i.e., (5.3)) cannot equal (5.6).

Assume now that $a > r$. Then at least the first $a + 1 > r + 1$ entries of (5.6) are $\geq a > r$, but the $(r + 1)$-st entry of (5.7) is equal to r, so again (5.7) (i.e., (5.3)) cannot equal (5.6).

So we indeed see that $a = r$ and $b = s$. We now claim that $I_r = 1$. Namely, if $I_r > 1$, then the first r entries of (5.6) are $> r$, but in (5.7) only the first $r - 1$ entries can be $> r$. So indeed $I_r = 1$ and hence $J_1 > 1$. This implies that s entries of (5.6), $r + 1 - J_1, r + 2 - J_2, \dots, r + s - J_s$, are $< r$. Since (5.7) has exactly s entries that are $< r$, namely $r + 1 - j_1, \dots, r + s - j_s$, we conclude that these entries must be equal, that is, that $J_u = j_u$, $u = 1, \dots, s$. It now follows that also $I_t = i_t,\ t = 1, \dots, r$, and finally that all ds must be equal to r, as claimed in the theorem. □

We leave it to the reader to formulate and prove analogs of Proposition 4.9 and Proposition 4.10.

6. Some combinatorics of shuffles

We now turn to some properties of shuffles, needed in the next section. An (r, s) *shuffle* is a permutation

$$i_1, \dots, i_r, j_1, \dots, j_s$$

of the numbers $1, 2, \dots, r + s$ such that

$$i_1 < i_2 < \dots < i_r, \quad j_1 < j_2 < \dots < j_s.$$

To each such shuffle we associate the $(r + s)$-tuple $L = (L_1, \dots, L_r \mid L'_1, \dots, L'_s)$, where

$$L_u = s + u - i_u \quad \text{and} \quad L'_v = v - j_v, \tag{6.1}$$

for $u = 1, \dots, r$ and $v = 1, \dots, s$. Clearly,

$$s \geq L_1 \geq \dots \geq L_r \geq 0 \geq L'_1 \geq \dots \geq L'_s \geq -r. \tag{6.2}$$

Note also that it is easy to recover the shuffle from L:

$$i_u = s + u - L_u \quad \text{and} \quad j_v = v - L'_v,$$

for $u = 1, \dots, r$ and $v = 1, \dots, s$. On the other hand, if we take an arbitrary L satisfying (6.2), and calculate is and js as above, we will not necessarily obtain a shuffle. In that case L is not attached to any shuffle.

Let $x \in \{0, 1, \dots, r\}$ be the unique index such that

$$i_x \le s \quad \text{but} \quad i_{x+1} > s.$$

(If all $i_u \le s$ we take $x = r$ and if all $i_u > s$ we take $x = 0$.) Thus to get all numbers $1, \dots, s$, besides $i_1, \dots, i_x$ we must use $j_1, \dots, j_{s-x}$. In other words,

$$j_{s-x} \le s \quad \text{and} \quad j_{s-x+1} > s.$$

There are two cases: either $i_x = s$ and $j_{s-x} < s$, or $i_x < s$ and $j_{s-x} = s$. This immediately leads to the following lemma.

Lemma 6.3. *If x is as above, then exactly one of the following cases holds*:

$$\begin{aligned} &\text{Case 1}: \ L_x = x, \ L'_{s-x} > -x \\ &\text{Case 2}: \ L_x > x, \ L'_{s-x} = -x \end{aligned}$$

In each case, $L_x \ge x \ge L_{x+1}$, while $L'_{s-x} \ge -x \ge L'_{s-x+1}$.

One can characterize x as the unique number in $\{0, \dots, r\}$ such that $L_x = x$ or $L'_{s-x} = -x$ (but not both). In this way, one can see what x is directly from L.

Here is the result we are going to need in next section.

Proposition 6.4. *Let $i_1, \dots, i_r, j_1, \dots, j_s$ be an (r, s) shuffle and let*

$$L = (L_1, \dots, L_r \mid L'_1, \dots, L'_s)$$

be attached to this shuffle as above. Let $x \in \{0, \dots, r\}$ be as above. Let M be an $(r+s)$-tuple obtained from L by replacing one or several pairs of the form $a, -a$ by $x, -x$, and then rearranging to descending order. Then M cannot be attached to a shuffle.

Proof. Assume that we are in Case 1, so $L_x = x$ and $L'_{s-x} > -x$, while $L'_{s-x+1} \le -x$. Note that after the changes we will still have $M_x = x$. If we were to replace any $(a, -a)$ with $a < x$ (and hence $-a > -x$), we would end up with $M_{s-x} = -x$. So Cases 1 and 2 happen simultaneously for M and thus by Lemma 6.3, M cannot be attached to a shuffle. So we can assume that all a are $> x$. Let us choose a to be the biggest possible, and let $u < x$ be the index such that $L_u = a$ and $L_{u+1} < a$.

It follows that

$$i_u = s + u - L_u = s - a + u, \quad \text{while} \quad i_{u+1} = s + u + 1 - L_{u+1} > s - a + u + 1.$$

This implies that

$$j_{s-a} < s-a+u, \quad \text{while} \quad j_{s-a+1} = s-a+u+1.$$

Namely, the numbers $1, 2, \ldots, s-a+u$ include $i_1, \ldots, i_u$ and not i_{u+1}, hence they must also include $j_1, \ldots, j_{s-a}$. Also, $s-a+u+1$ is not among the is, hence it must be j_{s-a+1}.

This now tells us that

$$L'_{s-a} = s-a-j_{s-a} > -u, \quad \text{while} \quad L'_{s-a+1} = s-a+1-j_{s-a+1} = -u.$$

Since none of $L'_1, \ldots, L'_{s-x}$ is changed, and $s-a+1 \le s-x$, we see that

$$M'_{s-a} = L'_{s-a} > -u, \quad \text{while} \quad M'_{s-a+1} = L'_{s-a+1} = -u.$$

On the other hand, since $L_u = a$ is replaced by x,

$$M_u = L_{u+1} < a.$$

We can now calculate $k_c = s+c-M_c$, $c = 1, \ldots, r$ and $l_d = d-M_d$, $d = 1, \ldots, s$, and check that $k_1, \ldots, k_r, l_1, \ldots, l_s$ is not a shuffle. Indeed, since L_u was the first of the Ls that got changed, we see that

$$k_{u-1} = i_{u-1} < i_u = s-a+u, \quad \text{while} \quad k_u = s+u-M_u > s-a+u,$$

so none of the k_cs equals $s-a+u$. On the other hand,

$$l_{s-a} = j_{s-a} < s-a+u, \quad \text{while} \quad l_{s-a+1} = j_{s-a+1} = s-a+u+1,$$

so none of the l_ds can be $s-a+u$ either.

The situation in Case 2 is analogous. We first use Lemma 6.3 to conclude that this time all replaced pairs $a, -a$ must satisfy $a < x$. We take the smallest (last) such a and pick index v such that $L_v = a$ while $L_{v-1} > a$. We then argue that the supposed shuffle attached to M cannot contain the number $s-a+v$. □

Remark 6.5. Using very similar reasoning as above it is not difficult to obtain the full conditions for an L satisfying (6.2) to be attached to a shuffle. We omit this since Proposition 6.4 is all we need.

7. The case of $\mathfrak{u}(p, q)$

We are going to use the facts about the structure of the pair $(\mathfrak{g}, \mathfrak{k})$ which we reviewed in Section 2. Besides that, we also need to describe the subset $W^1 \subset W_G$. In the unitary case, W^1 consists of (p, q) shuffles, that is, each $w \in W^1$ is a permutation $i_1, \ldots, i_p, j_1 \ldots, j_q$ of $1, 2, \ldots, p+q$ such that $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$.

The modules we are interested in are V_k, where $1 \le k \le \min(p, q) - 1$ is an integer, of lowest weight $\mu_0 = (k/2, \ldots, k/2 \mid -k/2, \ldots, -k/2)$. The modules V_k

were described in Section 2. In particular, the infinitesimal character Λ of V_k was given in (2.5). We rewrite Λ as $\Lambda = \Lambda_1 - \Lambda_2$, where

$$\Lambda_1 = \tfrac{1}{2}(p+q+1-k,\, p+q+1-k,\, \dots,\, p+q+1-k),$$
$$\Lambda_2 = (1,\, 2,\, \dots,\, q-k,\, q-k+1,\, q-k+1,\, \dots,\, q,\, q,\, q+1,\, q+2,\, \dots,\, q+p-k).$$

We now want to write down (3.6) in our present case. We will first modify it by subtracting our lowest K-type μ_0 from both sides. Since μ_0 is W_K-invariant, it can be put inside the parentheses on the right side. To see what the left side is, note that Λ_1 is W_G-invariant, and calculate

$$\Lambda_1 - \rho_c + \rho_n - \mu_0 = (q-k+1,\, q-k+2,\, \dots,\, q-k+p \mid 1,\, 2,\, \dots,\, q).$$

Thus, after subtracting μ_0 the left side of (3.6) becomes

$$(q-k+1, q-k+2, \dots, q-k+p \mid 1, 2, \dots, q) - w\Lambda_2. \tag{7.1}$$

We are now going to apply Proposition 3.5, which says that $w\Lambda$ must be dominant regular for $\mathfrak{k}$. We already mentioned that $w \in W^1$, that is, w is a (p,q) shuffle. Regularity of $w\Lambda$ means that the first p coordinates of $w\Lambda$ must be strictly decreasing, and so must the last q coordinates. This means that w must split apart each pair of equal coordinates in Λ. Hence we can take w to be a permutation of the form

(7.2)
$$w = \big(i_1, i_2, \dots, i_x, q-k+1, q-k+3, \dots, q+k-3, q+k-1, i_{x+k+1}, i_{x+k+2}, \dots, i_p \,\big|$$
$$j_1, j_2, \dots, j_y, q-k+2, q-k+4, \dots, q+k-2, q+k, j_{y+k+1}, j_{y+k+2}, \dots, j_q\big).$$

Here
$$\begin{aligned} 1 \le i_1 < \cdots < i_x &\le q-k, \\ 1 \le j_1 < \cdots < j_y &\le q-k \\ q+k+1 \le i_{x+k+1} < \cdots < i_p &\le p+q, \\ q+k+1 \le j_{y+k+1} < \cdots < j_q &\le p+q, \end{aligned}$$

and all i_u and j_v are different integers. In particular, it follows that $i_1, \dots, i_x$ and $j_1, \dots, j_y$ exactly exhaust all numbers $1, \dots, q-k$, and consequently $x+y = q-k$. (For each given $w\Lambda$, there are other choices for w leading to the same $w\Lambda$, and the w we choose is the shortest among them.)

It now follows that

$$w\Lambda_2 = (i_1,\, \dots,\, i_x,\, q-k+1,\, q-k+2,\, \dots,\, q,\, i_{x+k+1}-k,\, \dots,\, i_p-k \mid$$
$$j_1,\, \dots,\, j_y,\, q-k+1,\, q-k+2,\, \dots,\, q,\, j_{y+k+1}-k,\, \dots,\, j_q-k).$$

Plugging this into (7.1) we get for the left side of (3.6) the expression

(7.3)
$$(q-k+1-i_1,...,q-k+x-i_x,x,x,...,x,q+k+x+1-i_{x+k+1},...,q+p-i_p\mid$$
$$1-j_1,...,y-j_y,y-q+k,y-q+k,...,y-q+k,y+k+1-j_{y+k+1}+k,...,q-j_q+k).$$

There are k components equal to x in (7.3), and also k components equal to $y-q+k=-x$.

To analyze the right side of (3.6), we first write

$$\rho=\tfrac{1}{2}(p+q+1,\ p+q+1,\ \dots,\ p+q+1)-(1,2,\dots,p+q).$$

After a short calculation, this leads to

$$\sigma\rho-\rho_c+\rho_n=(q+1,q+2,\dots,q+p\mid 1,2,\dots,q)$$
$$-\sigma(1,2,\dots,p\mid p+1,p+2,\dots,p+q).$$

By (2.6), a general K-type of V_k has lowest weight

$$\mu_0+(0,\dots,0,d_k,\dots,d_1\mid -d_1,\dots,-d_k,0,\dots,0),$$

for some integers $d_1\ge d_2\ge\cdots\ge d_k\ge 0$. Thus, after subtracting μ_0, the right side of (3.6) becomes

$$\text{(7.4)}\quad \big((q+1,\dots,q+p\mid 1,\dots,q)-\sigma(1,\dots,p\mid p+1,\dots,p+q)$$
$$+(0,\dots,0,d_k,\dots,d_1\mid -d_1,\dots,-d_k,0,\dots,0)\big)'.$$

Now since $i_1\ge 1$, the first component of (7.3) is $\le q-k$. Since (7.3) must be equal to (7.4), the first component of $\sigma(1,\dots,p\mid p+1,\dots,p+q)$ must be $\ge k+1$. (Namely, adding some $d_u\ne 0$ to a component can only make it larger.) This further implies that $\sigma(1,\dots,p\mid p+1,\dots,p+q)$ has $1,\dots,k$ among the last q coordinates.

Similarly, since $j_q\le p+q$, the last component of (7.3) is $\ge -p+k$. Hence, the last component of $\sigma(1,\dots,p\mid p+1,\dots,p+q)$ must be $\le p+q-k$. (Namely, subtracting some $d_u\ne 0$ from a component can only make it smaller.) This further implies that $\sigma(1,\dots,p\mid p+1,\dots,p+q)$ has $p+q-k+1,\ p+q-k+2,\dots,p+q$ among its first p components. Since $\sigma\in W^1$, we conclude that

$$\sigma(1,\dots,p\mid p+1,\dots,p+q)=$$
$$(I_1,\dots,I_{p-k},\ p+q-k+1,\dots,p+q\mid 1,\dots,k,\ J_1,\dots,J_{q-k}),$$

for some $k+1\le I_1<\cdots<I_{p-k}\le p+q-k$ and $k+1\le J_1<\cdots<J_{q-k}\le p+q-k$, such that all I_u are different from all J_v; equivalently,

$$\{I_1,\dots,I_{p-k},\ J_1\dots,J_{q-k}\}=\{k+1,k+2,\dots,p+q-k\}.$$

Plugging this into (7.4), we get

$$(7.5)\quad (q+1-I_1,\ \dots,\ q+p-k-I_{p-k},\ d_k,\ \dots,\ d_1 \mid \\ -d_1,\ \dots,\ -d_k,\ k+1-J_1,\ \dots,\ q-J_{q-k})'.$$

The question now is how we can make (7.5) equal to (7.3). Finding the answer to this question is equivalent to proving the following theorem.

Theorem 7.6. *The Dirac cohomology of V_k is*

$$H_D(V_k) = \bigoplus_w E_{w\Lambda-\rho_c},$$

with the summation over all w as in (7.2). *All ingredients of the formula* (3.6) *are uniquely determined by w:*

The permutation

$$\sigma = (I_1,\ \dots,\ I_{p-k},\ p+q-k+1,\ \dots,\ p+q \mid 1,\ \dots,\ k,\ J_1,\ \dots,\ J_{q-k})$$

is given by

$$\sigma = (i_1+k,\ \dots,\ i_x+k,\ i_{x+k+1}-k,\ \dots,\ i_p-k,\ p+q-k+1,\ \dots,\ p+q \mid \\ 1,\ \dots,\ k,\ j_1+k,\ \dots,\ j_y+k,\ j_{y+k+1}-k,\ \dots,\ j_q-k),$$

where $y = q-k-x$.

The corresponding K-type in V_k is given by

$$d_1 = \cdots = d_k = x.$$

Proof. We already know that for $E_{w\Lambda-\rho_c}$ to contribute to $H_D(V_k)$, w must be as in (7.2). Moreover, it is straightforward to check that if we choose σ and $d_1, \dots, d_k$ as in the statement of the theorem, then we do get equality of (7.5) and (7.3), and hence a contribution of $E_{w\Lambda-\rho_c}$ to $H_D(V_k)$.

It thus remains to show that there is no possible other choice of $I_1, \dots, I_{p-k}$, $J_1, \dots, J_{q-k}$ and $d_1, \dots, d_k$ that would make the expressions (7.5) and (7.3) equal.

To see this, let us first relate (7.5) and (7.3) to shuffles. Let us set $r = p-k$, $s = q-k$, and define

$$i'_u = i_u \quad \text{for } u = 1, \dots, x, \quad i'_u = i_{u+k} - 2k \quad \text{for } u = x+1, \dots, r,$$
$$j'_v = j_u \quad \text{for } u = 1, \dots, y, \quad j'_v = j_{v+k} - 2k \quad \text{for } u = x+1, \dots, r.$$

Then it is easy to check that $i'_1, \dots, i'_r, j'_1, \dots, j'_s$ is an (r, s) shuffle, and the $(r+s)$-tuple L_1 associated to this shuffle as in (6.1) is exactly (7.3) with the k terms x and k terms $-x$ omitted.

We further define

$$I'_u = I_u - k, \quad u = 1, \dots, r, \quad J'_v = J_v - k, \quad v = 1, \dots, s.$$

Then obviously $I'_1, \dots, I'_r,\ J'_1, \dots, J'_s$ is an (r, s) shuffle, and the $(r+s)$-tuple L_2 associated to this shuffle as in (6.1) is exactly (7.5) with the terms $d_k, \dots, d_1$ and $-d_1, \dots, -d_k$ omitted.

Let us now suppose that (7.5) and (7.3) are equal, but not in the way stated in the theorem. This would mean that some of the d_i are not equal to x, and consequently the corresponding $-d_i$ are not equal to $-x$. These pairs d_i, $-d_i$ therefore have to appear in (7.3) with the terms x and $-x$ omitted, that is in L_1. Then there must be an equal number of pairs x, $-x$ among the entries of (7.5) with the terms $d_k, \dots, d_1$ and $-d_1, \dots, -d_k$ omitted, that is, among the entries of L_2. The rest of L_2 must agree with the rest of L_1. Thus, L_2 is obtained from L_1 by taking several pairs of the form a, $-a$ and replacing them by x, $-x$. Since both L_1 and L_2 are obtained from shuffles, this is impossible by Proposition 6.4. This proves the theorem. $\square$

We leave it to the reader to formulate and prove analogs of Proposition 4.9 and Proposition 4.10.

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JING-SONG HUANG
DEPARTMENT OF MATHEMATICS
HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
CLEAR WATER BAY, KOWLOON
HONG KONG SAR
CHINA
mahuang@ust.hk

PAVLE PANDŽIĆ
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ZAGREB
BIJENIČKA 30
10000 ZAGREB
CROATIA
pandzic@math.hr

VICTOR PROTSAK
DEPARTMENT OF MATHEMATICS
SUNY OSWEGO
7060 TOUTE 104
OSWEGO, NY 13126-3599
UNITED STATES
protsak@oswego.edu

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AN EXAMPLE OF A SINGULAR METRIC ARISING FROM THE BLOW-UP LIMIT IN THE CONTINUITY APPROACH TO KÄHLER–EINSTEIN METRICS

YALONG SHI AND XIAOHUA ZHU

A family of Kähler metrics with Calabi's symmetry on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ arises from the continuity method for finding Kähler–Einstein metrics. We study the blow-up limit of this family.

1. Introduction

Let M be a compact Kähler manifold with $c_1(M) > 0$. In algebraic geometry, M is called a Fano manifold. It is an important problem to study the existence of Kähler–Einstein metrics on such manifolds. In contrast to the $c_1 < 0$ and $c_1 = 0$ cases, there may be no Kähler–Einstein metrics on a given Fano manifold. Yau, Tian and Donaldson have conjectured that the existence of Kähler–Einstein metrics on M is equivalent to the K-polystability of M; see [Tian 1997; Donaldson 2002].

To find a Kähler–Einstein metric on M, one usually reduces the problem to solving a family of complex Monge–Ampère equations with parameter $\lambda \in [0, 1]$ via the continuity method, as Yau did in [1978]. If M does not admit a Kähler–Einstein metric, then the solutions of this family must blow up as $\lambda \to t_0$ for some $t_0 \in [0, 1]$. Since the solutions of this family give rise to a family of Kähler metrics with strictly positive Ricci curvature and the same volume, the compactness theorem of Gromov implies that this family contains a subfamily converging to a compact metric space with a length metric. The study of this limit space should be helpful in understanding the relationship between Kähler–Einstein metrics and stabilities in geometric invariant theory.

In this paper, we study a simple example, namely the blow-up of $\mathbb{C}P^2$ at one point, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, with a Calabi symmetric metric as the background metric. Note that $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is a ruled surface $\mathbb{P}(\underline{\mathbb{C}} \oplus U)$, where $\underline{\mathbb{C}}$ and U are the trivial line bundle and the universal bundle over $\mathbb{C}P^1$, respectively. It is well known that M is Fano and the automorphism group of M is not reductive [Calabi 1982].

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Therefore by Matsushima's theorem [1957], there are no Kähler–Einstein metrics on M. So if one uses the continuity method to solve the Kähler–Einstein metric equation on M with parameter $\lambda \in [0, 1]$, the parameter λ at which the equation is solvable could not reach 1. Recently, G. Székelyhidi showed that the Monge–Ampère equation is solvable if and only if the parameter λ is less than $6/7$, if one chooses a Calabi symmetric metric as a background Kähler metric [2009].[1][2] There are two distinguished divisors E_1 and E_2, respectively defined as the zero section and the infinity section of the ruled surface M. A Calabi symmetric Kähler metric g on M is defined by a convex function u in $t \in (-\infty, \infty)$ with its Kähler form ω_g given by

$$\omega_g = \sqrt{-1}\partial\bar{\partial}u \quad \text{in } \mathbb{C}^2 \setminus \{0\}, \tag{1-1}$$

where $t = \log(|z_1|^2 + |z_2|^2)$ and (z_1, z_2) are the standard coordinates on $\mathbb{C}^2 \setminus \{0\} \cong M \setminus (E_1 \cup E_2)$. Székelyhidi's result implies that the Kähler metrics g_λ arising from the solutions of Monge–Ampère equations will blow up as $\lambda \to 6/7$.

On the other hand, by a general theorem of Cheeger and Colding [1997], there exists a subsequence of metrics g_{λ_i} that converges in the Gromov–Hausdorff sense to a limit metric space g_∞ whose singular set has Hausdorff codimension at least 2. On the regular part, g_∞ is C^α-continuous. It is an interesting problem to study the geometry of the limit space.

Theorem 1.1. (1) *Among the Kähler metrics g_λ arising from the continuity method for finding Kähler–Einstein metrics, there exists a sequence converging smoothly in the Cheeger–Gromov sense to a singular Kähler metric g_∞ on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. The limit g_∞ is smooth on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus E_2$ and has conically symmetric singularities on E_2 with the same conical angle $10\pi/7$ along one direction. Moreover, g_∞ on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus E_1 \cup E_2$ is defined by a strictly increasing convex function $\psi_\infty(t)$ on $(-\infty, \infty)$, which satisfies the equation*

$$\psi'\psi'' = e^{13t/7 - 6\psi/7}. \tag{1-2}$$

(2) *The Ricci curvature of g_∞ is given by*

$$\operatorname{Ric}(g_\infty) = \sqrt{-1}\partial\bar{\partial}\left(\tfrac{1}{7}t + \tfrac{6}{7}\psi_\infty\right) \quad \text{on } \mathbb{C}^2 \setminus \{0\}. \tag{1-3}$$

In particular, the Ricci curvature of g_∞ is bounded.

By (1-2), one sees that the limit metric g_∞ is not a Kähler–Ricci soliton. This situation is quite different from the case of Kähler–Ricci flow studied in [Zhu 2007], where it was shown that the evolved Kähler metrics arising from the Kähler–Ricci

[1]Actually, Székelyhidi proved that the maximal solvable parameter λ is independent of the background metrics we choose.

[2]Chi Li [2009] has calculated the maximal solvable parameter λ for all toric Fano manifolds.

flow on a given toric Fano manifold will converge smoothly to a Kähler–Ricci soliton in the Cheeger–Gromov sense if the initial Kähler metric is toric. (See also [Koiso 1990] for the special case $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ with a Calabi symmetric metric as the initial metric.) The existence of Kähler–Ricci solitons on a toric Fano manifold was proved in [Wang and Zhu 2004]. Note that $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is a toric Fano manifold and that a Calabi symmetric metric is toric.

It is well known that the limit metric space of a sequence of 4-dimensional Riemannian manifolds with Ricci curvature bounded from below and with sectional curvature bounded in the L^2 norm can only have isolated singularities [Anderson 2005; Cheeger et al. 2002]. Theorem 1.1 gives an example of limit metric space with nonisolated singularities. Note that here the sequence of 4-dimensional Riemannian manifolds have only lower bound on their Ricci curvature (without the condition for sectional curvature).

In Section 2, we reduce the Monge–Ampère equations to a family of ordinary differential equations using Calabi's symmetry conditions. In Section 3, we use the Futaki invariant [1983] to give a simple proof to the "only if" part of Székelyhidi's result and to get some crucial estimates. The convergence problem is discussed in Section 4. Theorem 1.1 is finally proved in Section 5 by studying the structure of the singular limit metric. We remark that Theorem 1.1 still holds for the higher dimensional blow-up space $\mathbb{C}P^n \# \overline{\mathbb{C}P^n}$ according to our proof.

2. Reduction of the equation under Calabi's symmetry conditions

Let (M, g) be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$, where the Kähler class $[\omega_g]$ equals $2\pi c_1(M)$. To study the existence of Kähler–Einstein metrics on M, we use the continuity method. Consider the complex Monge–Ampère equations

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}})e^{h-\lambda\phi} \tag{2-1}$$

with parameter $\lambda \in [0, 1]$, where h is a Ricci potential of g defined by

$$\mathrm{Ric}(g) - \omega_g = \sqrt{-1}\partial\bar{\partial}h.$$

See [Yau 1978; Tian 1987]. If (2-1) is solvable at $\lambda = 1$, then the solution ϕ will define a Kähler–Einstein metric whose Kähler form given by $\omega_g + \sqrt{-1}\partial\bar{\partial}\phi$. In our case $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we choose a background Kähler metric g satisfying Calabi's symmetry conditions, namely, g is defined by a convex function u in $t \in (-\infty, \infty)$, so that

$$g_{\alpha\bar{\beta}} = \partial_\alpha\partial_{\bar{\beta}}u(t) = e^{-t}u'(t)\delta_{\alpha\beta} + e^{-2t}\bar{z}_\alpha z_\beta(u''(t) - u'(t)). \tag{2-2}$$

As Calabi pointed out [1982], g can extend across E_1 and E_2 if and only if the following hold:[3]

(1) The function $u_0(r)$ defined for all $r > 0$ by

$$u_0(r) = u_0(e^t) = u(t) - t \tag{2-3}$$

is extendable by continuity to a smooth function at $r = 0$ satisfying $u_0'(0) > 0$.

(2) The function $u_\infty(r)$ defined for all $r > 0$ by

$$u_\infty(r) = u_\infty(e^{-t}) = u(t) - 3t \tag{2-4}$$

is extendable by continuity to a smooth function at $r = 0$ satisfying $u_\infty'(0) > 0$.

Let $v(t) := -\log\det(g_{\alpha\bar{\beta}}) = 2t - \log u'(t) - \log u''(t)$. Then the Ricci curvature is

$$R_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} v(t) = e^{-t} v'(t)\delta_{\alpha\beta} + e^{-2t}\bar{z}_\alpha z_\beta (v''(t) - v'(t)). \tag{2-5}$$

Since all solutions ϕ of (2-1) are symmetric, it becomes

$$(u' + \phi')(u'' + \phi'') = e^{2t-u-\lambda\phi},$$

which we can rewrite as

$$\psi'\psi'' = e^{2t-(\lambda\psi+(1-\lambda)u)}, \tag{2-6}$$

where $\psi = u + \phi$. Note that the volume of g is computed by

$$\begin{aligned} \operatorname{Vol}(M, g) &= \int_{\mathbb{C}^2\setminus\{0\}} u''u'e^{-2t} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ &= \operatorname{Vol}(S^3) \int_{-\infty}^{\infty} u''u' dt = 4\operatorname{Vol}(S^3), \end{aligned} \tag{2-7}$$

where $\operatorname{Vol}(S^3)$ denotes the volume of the unit sphere in $\mathbb{R}^4$. So we may normalize u so that

$$\int_{-\infty}^{+\infty} e^{2t-u(t)} dt = 4. \tag{2-8}$$

3. Application of the Futaki invariant

For a convex function $\psi(t)$ on $(-\infty, \infty)$ satisfying the boundary conditions (2-3) and (2-4), we consider the integral

$$I = \int_{-\infty}^{+\infty} (2\psi'\psi'' - \psi'^2\psi'' - \psi''^2 - \psi'\psi''')dt. \tag{3-1}$$

[3] This is also clear from our proof of Proposition 5.2.

One can show that if ψ is a defining function of a Calabi symmetric metric on $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, then I is just the Futaki invariant evaluated at the holomorphic vector field $z_1\partial/\partial z_1 + z_2\partial/\partial z_2$, where z_1 and z_2 are the standard coordinates on $\mathbb{C}^2 \setminus \{0\} \simeq M \setminus (E_1 \cup E_2)$.

Now by the boundary conditions, we have

$$I_1 = \int_{-\infty}^{+\infty} 2\psi'\psi'' dt = \psi'^2\big|_{-\infty}^{+\infty} = 8,$$
$$I_2 = \int_{-\infty}^{+\infty} -\psi'^2\psi'' dt = -\tfrac{1}{3}\psi'^3\big|_{-\infty}^{+\infty} = -\tfrac{26}{3},$$
$$I_3 = \int_{-\infty}^{+\infty} -\psi''^2 - \psi'\psi''' dt = -(\psi'\psi'')\big|_{-\infty}^{+\infty} = 0.$$

These equalities imply that $I = -2/3 \neq 0$. In particular, we see that there are no Kähler–Einstein metrics on M.

Proposition 3.1. *Equation* (2-6) *is solvable only if* $\lambda < 6/7$.

Proof. According to the boundary conditions, the integral I should equal $-2/3$. But by the equation, we have

$$I = (1-\lambda)\int_{-\infty}^{+\infty} (u' - \psi')\psi'\psi'' dt = \frac{13(1-\lambda)}{3} - \frac{1-\lambda}{2}\int_{-\infty}^{+\infty} \psi'^2 u'' dt.$$

Note that $\psi'^2 < 9$, we have $-\frac{2}{3} = I > -\frac{14}{3}(1-\lambda)$. So $\lambda < 6/7$. □

We can get more information from the integral I.

Lemma 3.2. *For any fixed* t_0, *we have*

$$\lim_{\lambda\to 6/7} \int_{t_0}^{+\infty} \psi_\lambda'\psi_\lambda'' dt = 0. \tag{3-2}$$

In particular, the functions ψ_λ' *converge uniformly to the constant function* 3 *on* $[t_0, +\infty)$ *when* $\lambda \to 6/7$.

Proof. The identity $I \equiv -2/3$ is equivalent to

$$A_\lambda := \int_{-\infty}^{+\infty} u'\psi_\lambda'\psi_\lambda'' dt = \frac{26}{3} - \frac{2}{3(1-\lambda)}.$$

It follows that $\lim_{\lambda\to 6/7} A_\lambda = 4$. On the other hand, we have

$$\begin{aligned} A_\lambda &> \int_{-\infty}^{t_0} \psi_\lambda'\psi_\lambda'' dt + u'(t_0)\int_{t_0}^{+\infty} \psi_\lambda'\psi_\lambda'' dt \\ &= 4 + (u'(t_0) - 1)\int_{t_0}^{+\infty} \psi_\lambda'\psi_\lambda'' dt. \end{aligned} \tag{3-3}$$

This implies that

$$0 < \int_{t_0}^{+\infty} \psi_\lambda' \psi_\lambda'' dt < \frac{1}{u'(t_0)-1}(A_\lambda - 4) \to 0.$$

Thus

$$\tfrac{1}{2}(3^2 - (\psi_\lambda'(t_0))^2) \to 0 \quad \text{as } \lambda \to 6/7,$$

that is, $\psi_\lambda'(t_0) \to 3$ as $\lambda \to 6/7$. By the monotonicity of ψ_λ', the functions ψ_λ' converge uniformly to 3 on $[t_0, +\infty)$. □

4. Convergence

Now we analyze the behavior of ψ_λ as $\lambda \nearrow 6/7$.

Let $w_\lambda = -(2t-(1-\lambda)u-\lambda\psi_\lambda)$. Then w_λ is strictly convex. Let $p_\lambda \in M$, so that $w_\lambda(p_\lambda) = \inf_{x\in M} w_\lambda(x) = C_\lambda$. Clearly, $p_\lambda \in M\backslash(E_1 \bigcup E_2) \cong \mathbb{C}^2\backslash\{0\}$, so we may abuse the notation to identify p_λ with its coordinate in $\mathbb{C}^2\backslash\{0\}$. Let $t_\lambda = \log|p_\lambda|^2$.

Lemma 4.1. *When $\lambda \to 6/7$, we have $t_\lambda \to -\infty$.*

Proof. Suppose that there is a subsequence $\lambda_i \to 6/7$ but $t_\lambda \geq -C > -\infty$. Since $w_\lambda'(t_\lambda) = 0$, we have

$$\psi_\lambda'(-C) \leq \psi_\lambda'(t_\lambda) = \frac{2}{\lambda} - \frac{1-\lambda}{\lambda}u'(t_\lambda) \leq \frac{2}{\lambda}.$$

Then we can easily get a contradiction from this and Lemma 3.2. □

We now introduce a family of modified functions of ψ_λ by

$$\tilde{\psi}_\lambda(t) = \psi_\lambda(t+t_\lambda) - \lambda^{-1}(2t_\lambda - (1-\lambda)u(t_\lambda)).$$

Then $\tilde{\psi}_\lambda$ satisfies the equation

$$\tilde{\psi}''\tilde{\psi}' = e^{(2-(1-\lambda)u'(t_\lambda))t - \lambda\tilde{\psi} + (1-\lambda)f_\lambda(t)}, \tag{4-1}$$

where

$$f_\lambda(t) = -(u(t+t_\lambda) - u(t_\lambda) - u'(t_\lambda)t) = u_0(e^{t_\lambda}) - u_0(e^{t+t_\lambda}) + (u'(t_\lambda) - 1)t.$$

It is clear that $\lim_{\lambda\to 6/7} f_\lambda(t) = 0$ for any t.

Proposition 4.2. *There exist a sequence of convex functions $\tilde{\psi}_{\lambda_i}$, where $\lambda_i \to 6/7$, and a smooth convex function ψ_∞ defined on $(-\infty, \infty)$, such that the $\tilde{\psi}_{\lambda_i}$ converge locally uniformly and smoothly to ψ_∞, which satisfies the equation*

$$\psi''\psi' = e^{(13/7)t - (6/7)\psi} \quad \textit{for } t \in (-\infty, \infty). \tag{4-2}$$

Proof. It suffices to prove that

$$|C_\lambda| \le C.$$

In fact, if this is true, we see that all the $\tilde{\psi}_\lambda$ are uniformly bounded on any bounded intervals. As a consequence, by (4-1), the $\tilde{\psi}''_\lambda$ are also uniformly bounded on any bounded intervals. Then again by (4-1), it is easy to see that the C^k norms of the $\tilde{\psi}_\lambda$ are locally uniformly bounded. Thus there exist a sequence of convex functions $\tilde{\psi}_\lambda$ that converges locally uniformly in C^k norm to a convex function ψ_∞ defined on $(-\infty, \infty)$. On the other hand, by Lemma 4.1, the t_λ go to $-\infty$ as $\lambda \to 6/7$. Hence, by (4-1) and the fact that $f_\lambda(t) \to 0$ as $\lambda \to 6/7$, we conclude that ψ_∞ is in fact smooth and satisfies (4-2).

Now we prove the the boundedness of C_λ. By the boundary conditions, we have

$$\int_{-\infty}^{\infty} (\psi''_\lambda \psi'_\lambda)dt = \tfrac{1}{2}(\psi'^2_\lambda(\infty) - \psi'^2_\lambda(-\infty)) = 4. \tag{4-3}$$

Then by the convexity of w_λ and the fact $|w'_\lambda| \le 1$, it is easy to get a lower bound of C_λ. So we only need to obtain an upper bound. For simplicity, we write $w = w_\lambda$ and $\psi = \psi_\lambda$.

Let B_0 be the interval defined by

$$B_0 := \{t \in (-\infty, \infty) \mid C_\lambda \le w(t) \le C_\lambda + 1\}.$$

Then there exist exact two numbers s_0 and t_0 with $s_0 < t_0$ such that $w(s_0) = w(t_0) = C_\lambda + 1$. Clearly $t_\lambda \in B_0$, and it holds that

$$\psi'' \ge c_0 e^{-C_\lambda} \quad \text{on } B_0.$$

So

$$w'' \ge \lambda c_0 e^{-C_\lambda} \ge \tfrac{1}{2} c_0 e^{-C_\lambda}. \tag{4-4}$$

We want to show that

$$R := \tfrac{1}{2}(t_0 - s_0) \le \sqrt{\frac{4}{c_0}} e^{C_\lambda/2}. \tag{4-5}$$

In fact we consider the function on $\mathbb{R}$ defined by

$$v(t) = \tfrac{1}{4} c_0 e^{-C_\lambda} (|t - \tfrac{1}{2}(s_0 + t_0)|^2 - R^2) + C_\lambda + 1.$$

Then it is clear that $v(t)$ satisfies

$$v'' = \tfrac{1}{2} c_0 e^{-C_\lambda} \text{ on } B_0 \quad \text{and} \quad v(s_0) = v(t_0) = C_\lambda + 1. \tag{4-6}$$

Thus by (4-4) and (4-6), we get

$$(w - v)'' \ge 0 \text{ on } B_0 \quad \text{and} \quad w(t) = v(t) \text{ for } t = s_0 \text{ and } t = t_0.$$

It follows from the convexity that

$$w \le v \quad \text{on } B_0.$$

In particular,

$$C_\lambda \le w(\tfrac{1}{2}(s_0+t_0)) \le v(\tfrac{1}{2}(s_0+t_0)) = -\tfrac{1}{4}c_0 e^{-C_\lambda} R^2 + C_\lambda + 1.$$

This implies (4-5).

For $k \ge 1$, we choose a family of closed sets

$$B_k := \{t \in (-\infty, \infty) \mid k + C_\lambda \le w(t) \le C_\lambda + k + 1\}.$$

Then there are s_k and t_k with $s_k < t_{k-1}$, for $k \ge 1$, such that

$$B_k = [s_{k-1}, s_k] \cup [t_{k-1}, t_k].$$

By the convexity of w, it is easy to see $w'(t_0), -w'(s_0) \ge 1/(2R)$, and so

$$-w'(s), w'(t) \ge 1/(2R) \quad \text{for all } s \le s_0 \text{ and } t \ge t_0.$$

Thus

$$t_k - t_{k-1} \le 2R \quad \text{and} \quad s_k - s_{k-1} \le 2R.$$

Hence by (4-5), we get

$$s_k - s_{k-1}, t_k - t_{k-1} \le 2R \le 2\sqrt{\frac{4}{c_0}} e^{C_\lambda/2}.$$

It follows that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-w} dt &= \sum_k \int_{B_k} e^{-w} dt \\ &\le \sum_k 4\sqrt{\frac{4}{c_0}} e^{C_\lambda/2} e^{-C_\lambda - k} \qquad (4\text{-}7) \\ &= 4\sqrt{\frac{4}{c_0}} e^{-C_\lambda/2} \sum_k e^{-k} \le C e^{-C_\lambda/2}. \end{aligned}$$

This inequality and (4-3) imply that $4 \le Ce^{-C_\lambda/2}$. □

According to Proposition 4.2, we can define a Kähler metric ω_∞ on $\mathbb{C}^2 \setminus \{0\}$ by $\sqrt{-1}\partial\bar\partial\psi_\infty$. Then we have the following convergence of g_λ.

Proposition 4.3. *There exists a sequence of biholomorphic maps σ_{λ_i} on M, with $\lambda_i \to 6/7$, such that the $\sigma_{\lambda_i}^* \omega_{g_{\lambda_i}}$ converge to ω_∞ on $\mathbb{C}^2 \setminus \{0\}$ smoothly as $\lambda_i \to 6/7$. In particular, the $(M \setminus (E_1 \cup E_2), \omega_{g_{\lambda_i}})$ converge to $(\mathbb{C}^2 \setminus \{0\}, \omega_\infty)$ in the Cheeger–Gromov sense.*

Proof. Let σ_λ be the biholomorphic map on $\mathbb{C}^2\setminus\{0\}$ defined by

$$\sigma_\lambda(z_1, z_2) = (e^{t_\lambda} z_1, e^{t_\lambda} z_2).$$

Clearly this action fixes the points $\{0\}$ and ∞. Thus the action can extend to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Furthermore,

$$\sigma_\lambda^*\omega_{g_\lambda} = \sqrt{-1}\partial\bar\partial\sigma_\lambda^*\psi_\lambda = \sqrt{-1}\partial\bar\partial\tilde\psi_\lambda \quad \text{on } \mathbb{C}^2\setminus\{0\}.$$

By Proposition 4.2, we see that there exist a sequence of parameters λ_i such that $\sigma_\lambda^*\omega_{g_{\lambda_i}}$ converge locally uniformly and smoothly to ω_∞. □

5. Properties of the limit metric

Now we discuss the structure of ω_∞ near E_1 and E_2.

Lemma 5.1. *Let $a := \lim_{t\to-\infty}\psi_\infty'(t)$ and $b := \lim_{t\to\infty}\psi_\infty'(t)$. Then we have $a = 1$ and $b = 3$.*

Proof. Since $\mathrm{Ric}(\omega_\lambda) \geq \lambda\omega_\lambda$, by the Bonnet–Myers theorem, the diameters are uniformly bounded. Then by the Bishop–Gromov volume comparison theorem, we have

$$\mathrm{Vol}(B_r(x), \omega_\lambda) \geq Cr^n \quad \text{for all } x \in M \text{ and } r \leq 1.$$

This means the family of metrics ω_λ are noncollapsing. Then by a result of Cheeger and Colding [1997, Theorem 5.4], the convergent sequence ω_{λ_i} of metrics satisfy

$$\lim_{\lambda_i\to 6/7}\mathrm{Vol}(M, \omega_{\lambda_i}) = \mathrm{Vol}(M, \omega_\infty).$$

On the other hand,

$$\begin{aligned}\mathrm{Vol}(M, \omega_\lambda) &= \int_{\mathbb{C}^2\setminus\{0\}} \psi''\psi' e^{-2t} dz_1\wedge dz_2\wedge d\bar z_1\wedge d\bar z_2\\ &= \mathrm{Vol}(S^3)\int_{-\infty}^{\infty}\psi''\psi' dt = 4\,\mathrm{Vol}(S^3)\end{aligned}$$

and

$$\mathrm{Vol}(M, \omega_\infty) = \tfrac{1}{2}\,\mathrm{Vol}(S^3)(b^2 - a^2).$$

It is obvious that $a \geq 1$ and $b \leq 3$. The claim follows. □

Proposition 5.2. *The metric ω_∞ can extend to a smooth metric on $M \setminus E_2$.*

Proof. In the standard coordinates on $\mathbb{C}^2$, we can express ω_∞ as

$$\begin{aligned}\omega_\infty &= \sqrt{-1}\partial\bar\partial\psi_\infty\\ &= \sqrt{-1}\sum_{\alpha,\beta}(e^{-t}\psi_\infty'\delta_{\alpha\beta} + e^{-2t}(\psi_\infty'' - \psi_\infty')\bar z_\alpha z_\beta) dz_\alpha\wedge d\bar z_\beta,\end{aligned} \tag{5-1}$$

where $z=(z_1,z_2)\in\mathbb{C}^2\setminus\{0\}$ and $t=\log|z|^2$. We will use the coordinate transformation

$$w_1=z_1/z_2 \quad\text{and}\quad w_2=z_2$$

near $z=(z_1,z_2)=0$. In fact, this transformation blows up a neighborhood of 0 to a neighborhood of E_1 in M. Since ω_∞ is symmetric, we may consider the behavior of ω_∞ along $z=(0,z_2)$ with $|z_2|\ll 1$ under this coordinate transformation. By (5-1), it is easy to see the components of the metric at $(0,z_2)$ are given by

$$g_{1\bar{1}}=e^{-t}\psi'_\infty(t),\quad g_{2\bar{2}}=e^{-t}\psi''_\infty(t),\quad g_{1\bar{2}}=0.$$

Then, in the new coordinate system w, we have

$$\tilde{g}_{1\bar{1}}=\psi'_\infty(t),\quad \tilde{g}_{2\bar{2}}=e^{-t}\psi''_\infty(t),\quad \tilde{g}_{1\bar{2}}=w_2\bar{w}_1e^{-t}\psi'_\infty=0. \tag{5-2}$$

On the other hand, by (4-2) and Lemma 3.2, we see that for any $\alpha<1$ there is a uniform constant C_1 such that

$$\psi''_\infty(t)\le C_1e^{\alpha t}\quad\text{for all } t\le 0.$$

This implies

$$1\le\psi'_\infty(t)\le 1+C_2e^{\alpha t},$$

and so we get $|\psi_\infty-t|\le C_2$. Thus again by (4-2), we obtain

$$C_3^{-1}\le e^{-t}\psi''_\infty(t)\le C_3\quad\text{for all } t\le 0. \tag{5-3}$$

This means that

$$C^{-1}\le\tilde{g}_{2\bar{2}}\le C\quad\text{for all } t\le 0$$

and for some uniform constant C. Moreover from the argument above, one can show that $g_1(s):=\tilde{g}_{2\bar{2}}$ can extend to a continuous function on the interval $[0,1)$, where $s=e^t$. In fact, we will prove that $g_1(s)$ is C^∞ at $s=0$ in the following.

We rewrite (4-2) as

$$[\psi'^{\,2}_\infty]'_s=2e^{-(6/7)(\psi_\infty-t)}, \tag{5-4}$$

where f' and $[f]'_s$ are derivatives of f with respect to t and s, respectively. Then by (5-3), it is easy to see that $[(\psi'_\infty)^2]'_s$ is Lipschitz at $s=0$. It follows that $g_1(s)$ is also Lipschitz at $s=0$. This implies that $(\psi_\infty-t)'_s$ is Lipschitz at $s=0$. Thus by (5-4), we can repeat the arguments above to show that $(g_1)'_s(0)$ exists and $(g_1)'_s(s)$ is Lipschitz at $s=0$. Using the "bootstrap" argument, we see that $g_1(s)$ is C^∞ at $s=0$.

The argument above also proves that $g_2(s)=\psi'_\infty(t)=\tilde{g}_{1\bar{1}}$ is C^∞ at $s=e^t=0$. Note that $s=|w_2|^2$. Since the derivative of ω_∞ at $(0,0)$ along the direction of the other variable w_1 is a function in the variables w_1 and w_2, we see that ω_∞ can extend to a smooth metric on $M\setminus E_2$. $\square$

To analyze the behavior of ω_∞ near $z=\infty$, we introduce the following concept.

Definition 5.3. Let $g=\sum_{i,j} g_{i\bar{j}}dz_i\otimes d\bar{z}_j$ be a Kähler metric defined on $M^*=M\setminus D$, where D is a smooth subvariety of codimension 1. We say that the metric g has conically symmetric singularities on D along one direction with a conical angle $\alpha\pi$ if for every point $p\in D$, there exists a coordinate system $(U;w_1,\dots,w_n)$ near p such that $w(p)=(0,\dots,0)$ and in which the components $g_{i\bar{j}}$ of g on $U\setminus D$ are such that the components $(|w_1|^{2-\alpha})g_{1\bar{1}}$, $g_{1\bar{j}}$ for $j=1,\dots,n$ and $g_{l\bar{m}}$ for $l,m=2,\dots,n$ can be extended to a positive definite matrix-valued smooth function on U in the variables $|w_1|^{\alpha/2}, w_2,\bar{w}_2,\dots,w_n,\bar{w}_n$.

Remark 5.4. If $\alpha=2/k$ for some integer $k\geq 2$ in Definition 5.3, then the metric g has an orbifold structure. In fact, if $\tilde{V}$ is a branched covering of a neighborhood V of p by the map $\pi:(z_1,z_2,\dots,z_n)\mapsto(w_1=(z_1)^k, w_2=z_2,\dots,w_n=z_n)$, then π^*g can be extended to a smooth Kähler metric on $\tilde{V}$.

Theorem 5.5. (1) *The singular Kähler metric ω_∞ on $\mathbb{C}P^n\,\#\overline{\mathbb{C}P^n}$ defined by ψ_∞ has conically symmetric singularities lying on the infinity divisor E_2, with the same conical angle $10\pi/7$ along one direction.*

(2) *The Ricci curvature of ω_∞ satisfies the equation*

$$\mathrm{Ric}(\omega_\infty)=\sqrt{-1}\partial\bar{\partial}\big(\tfrac{1}{7}t+\tfrac{6}{7}\psi_\infty\big). \tag{5-5}$$

In particular, the Ricci curvature is bounded.

Proof. By Proposition 5.2, it suffices to analyze the behavior of ω_∞ near E_2. We write the homogeneous coordinates on $M\setminus E_1$ (as a subset of $\mathbb{C}P^2$) as $[Z_0,Z_1,Z_2]$, where E_2 is defined by the equation $Z_0=0$. Then we have on $M\setminus(E_1\cup E_2)$

$$z_1=\frac{Z_1}{Z_0}\quad\text{and}\quad z_2=\frac{Z_2}{Z_0}.$$

By the symmetry conditions we imposed, we may consider only the behavior of ω_∞ on the open set $U:=(M\setminus E_1)\cap\{Z_2\neq 0\}$. The affine coordinates on U are

$$w_1=\frac{Z_1}{Z_2}=\frac{z_1}{z_2}\quad\text{and}\quad w_2=\frac{Z_0}{Z_2}=\frac{1}{z_2}.$$

A direct computation shows that the components of the metric ω_∞ at $w=(0,w_2)$ are given by

$$\tilde{g}_{1\bar{1}}=\psi'_\infty(t),\quad \tilde{g}_{2\bar{2}}=e^t\psi''_\infty(t),\quad \tilde{g}_{1\bar{2}}=0, \tag{5-6}$$

where $t=\log(|z_1|^2+|z_2|^2)=\log(1/|w_2|^2)$. On the other hand, by (4-2) and the arguments in the proof of Proposition 5.2, one can show that

$$\begin{aligned}&|\psi_\infty-3t|\leq C,\\ &e^t(\psi_\infty)''(t)=O(e^{(2/7)t})\quad\text{as } t\to\infty.\end{aligned} \tag{5-7}$$

Moreover, if we set $s = e^{-(5/7)t}$ and rewrite (4-2) as

$$[\psi_\infty'^2]_s' = 2e^{-(6/7)(\psi_\infty - 3t)},$$

then we can prove that $\tilde{g}_1(s) = e^{(5/7)t}\psi_\infty''(t)$ and $\tilde{g}_2(s) = \psi_\infty - 3t$ are both C^∞ at $s = 0$. Hence we have proved that ω_∞ has a conical structure at each point in E_2 with the same conical angle $(10/7)\pi$.

By (4-2), we see that the Ricci curvature of ω_∞ satisfies (5-5). By the local formula (5-6) of ω_∞ near E_2, the Ricci curvature is bounded. □

Theorem 1.1 follows from Theorem 5.5 and Proposition 5.2.

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YALONG SHI
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
CHINA

ylshi@math.pku.edu.cn

XIAOHUA ZHU
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
CHINA

xhzhu@math.pku.edu.cn

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DETECTING WHEN A NONSINGULAR FLOW IS TRANSVERSE TO A FOLIATION

SANDRA SHIELDS

We show that any foliation transverse to a C^1 nonsingular flow ϕ on a closed 3-manifold can be detected algorithmically. We use this to describe a procedure that, for any $\delta > 0$, will determine whether or not there is a foliation whose tangent space is bounded away from the tangent space to ϕ by a distance of δ.

Introduction

An open problem in foliation theory is to determine whether a nonsingular C^1 flow ϕ on an arbitrary closed 3-manifold M has a transverse foliation. Classical results by Fried [1982] and Schwartzman [1957] state conditions for any such flow to have a transverse section, and hence a transverse foliation. Milnor [1958] and Wood [1971] found necessary and sufficiently conditions for the existence of a 2-dimensional foliation transverse to the foliation by circles of a circle bundle. Later, Naimi [1994] did the same for the foliation by circles of a Seifert fibered 3-manifold. Goodman [1986] showed that a simple topological property is, for a C^0-dense class of flows, both necessary and sufficient for the existence of a transverse foliation. However, there are flows that satisfy this property, yet do not admit a transverse foliation; for example, flows on S^3 with no periodic orbits as described in [Schweitzer 1974; Harrison 1988; Kuperberg 1994].

The subtlety of the transverse foliation problem is underscored by the Milnor–Wood result. Specifically, they showed that for circle bundles over a closed surface of positive genus, there is a foliation transverse to the fibers precisely when the Euler number of the bundle is no larger than the negative of the Euler characteristic of the surface. Since one can have a circle bundle of sufficiently small Euler number finitely covering one with a large Euler number, any property of a flow that is preserved under finite covers cannot, in general, be both necessary and sufficient for the existence of a transverse foliation.

In [Goodman and Shields 2007], we showed that when a flow ϕ has no self-return disk (that is, a disk transverse to ϕ that flows continuously into its own

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interior), a simple algorithm for modifying any branched surface transverse to ϕ will eventually produce a branched surface carrying a foliation F precisely when F is transverse to ϕ. We show in Theorem 2.2 that this algorithm also works when ϕ has self-return disks. We then find a procedure that can be used to determine whether or not any branched surface produced by our algorithm carries a foliation. In particular, we describe a process that allows us to modify any branched surface in order to produce an essential branched surface that carries a foliation if and only if the original does (Theorem 2.3). Algorithms in [Agol and Li 2003] can then be applied to determine whether or not this new branched surface carries a foliation. Hence we obtain in Theorem 2.4 an algorithmic means for detecting flows with transverse foliations.

We further show in Theorem 2.5 that any for any $\delta > 0$, one can find a positive integer K such that if the branched surface produced at the K-th stage of our algorithm does not carry a foliation transverse to ϕ, then there are no foliations that remain a bounded distance of at least δ from ϕ. If, on the other hand, this branched surface does carry a foliation, then the algorithm described in Theorem 2.4 will detect that it does.

1. Preliminaries

Throughout, M will be a closed orientable 3-manifold and $\phi : M \times \mathbb{R} \to M$ will be a C^1 nonsingular flow on M. An *orbit segment* of ϕ shall be a curve $\phi(x, t)_{t\in[a,b]}$, where $x \in M$ and $[a, b]$ is a closed interval in $\mathbb{R}$. The *forward orbit* under ϕ of a point $x = \phi(x, 0)$ in M will be the set of points $\phi(x, t)_{t>0}$; the *backward orbit* consists of the points $\phi(x, t)_{t<0}$.

The foliations we consider will be C^1 and codimension one.

Branched surface construction. The branched surfaces we associate with the flow ϕ are in the class of regular branched surfaces introduced by [Williams 1974]. In particular, each is transverse to ϕ, connected, and has a set of charts defining local orientation-preserving diffeomorphisms onto one of the models in Figure 1, such that the transition maps are smooth and preserve the transverse orientation indicated by the arrows. (Each local model projects horizontally into a vertical model of $\mathbb{R}^2$ and has a smooth structure induced by $T\mathbb{R}^2$ when we pull back the local projection.) So a branched surface W is a 2-manifold except on a dimension-one subset μ (indicted by the dashed segments) called the *branch set*. The set μ is a 1-manifold except at finitely many isolated points where it intersects itself transversely. The components of $W - \mu$ are the *sectors* of W.

Given a nonsingular flow ϕ, we construct a transverse branched surface by first choosing a finite *generating set* $\Delta = \{D_i\}_{i=1,\dots n}$ *for* ϕ, consisting of pairwise disjoint disks embedded in M that satisfy the following general position requirements:

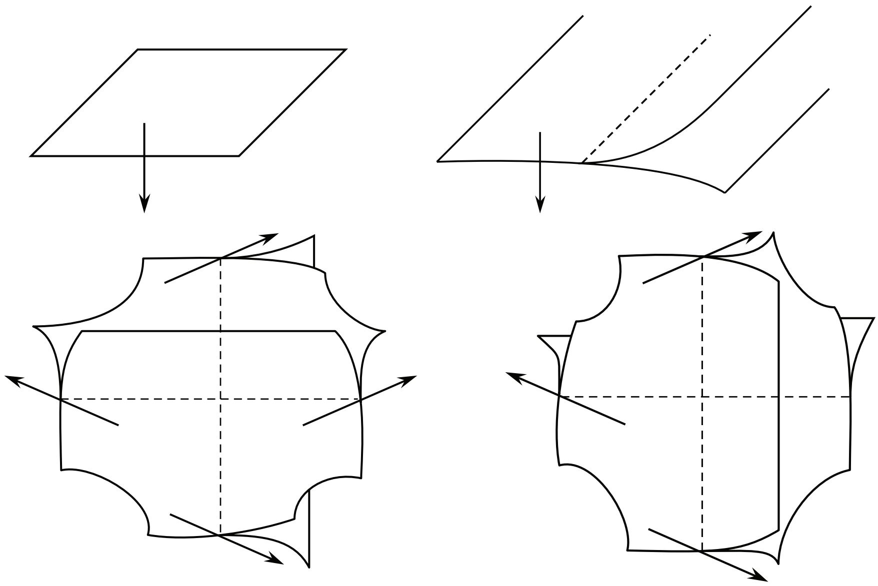


Figure 1

(i) Each D_i is transverse to ϕ.

(ii) Under ϕ, the forward and backward orbit of every point meets the interior of the generating set. In other words, the orbits all meet $\operatorname{int}\Delta = \bigcup_{i=1}^{n} \operatorname{int} D_i$.

(iii) There are only finitely many points in $\partial\Delta = \bigcup_{i=1}^{n} \partial D_i$ whose orbit, forward or backward, meets $\partial\Delta$ before meeting $\operatorname{int}\Delta$.

(iv) The forward orbit of any point in $\partial\Delta$ meets $\partial\Delta$ at most once before meeting $\operatorname{int}\Delta$.

Note that we can find such a set for any given ϕ. In particular, cover M with finitely many flow boxes for ϕ, and select a horizontal slice from each box. A slight modification of each slice can then be used to ensure that the resulting collection of disks satisfies the general position requirements above.

After choosing Δ, cut M open along the interior of each element of Δ to obtain a closed connected submanifold M^* that is transverse to ϕ (except along $\partial\Delta$) and whose boundary contains $\partial\Delta$. This can be thought of as blowing air into M to create an air pocket at each generating disk. By requirement (ii) above, the restriction of ϕ to M^* is a flow ϕ^* with the property that each orbit is homeomorphic to the unit interval [0, 1]. Form a quotient space by identifying points that lie on the same orbit of ϕ^*. That is, take the quotient $M^*/\sim$, where $x \sim y$ if x and y lie on the same interval orbit of ϕ^*. This quotient space can be embedded in M so that it is transverse to ϕ and locally modeled on Figure 1. Specifically, we can view the

Figure 2

quotient map as enlarging the components of $M - M^*$ until each interval orbit of ϕ^* is contracted to a point in M. We refer to this embedded copy of the quotient space as *the branched surface W constructed from ϕ*.

Although there are many embeddings of the quotient $M^*/\sim$ that are transverse to ϕ, the complement of each is a union of open 3-balls. So any two embeddings of $M^*/\sim$ are diffeomorphic in M; that is, there is a diffeomorphism of M that maps one onto the other. Consequently, we only distinguish between branched surfaces transverse to ϕ up to diffeomorphism of M.

The branched surface W could have many generating sets. For example, if we flow a generating disk forward or backward slightly without allowing any of its points to pass through another point of Δ, then the quotient space described above does not change.

Also, note that we can thicken W in the transverse direction to recover M^* which, for this reason, we shall henceforth call $N(W)$, *the neighborhood of W*. In particular, $N(W)$ is obtained when we replace each point x in W with the interval orbit of ϕ^* whose quotient is x. We shall refer to these interval orbits as the *fibers of $N(W)$*. See Figure 2.

Foliations carried by a branched surface. If a foliation F is transverse to ϕ, and if there exists a generating set Δ for a branched surface W where each element of Δ is contained in a leaf of F, then *F is carried by W*. In particular, when we cut M open along Δ, the foliation F becomes a foliation of $N(W)$ whose leaves (some of which are branched) are transverse to the fibers. The branched leaves are precisely those that contain a boundary component of $N(W)$, since these are the (cut-open) leaves of F containing the elements of Δ. (They can be thought of as leaves of F with air blown into them.) Figure 3 shows a local picture of such a foliation of $N(W)$.

Conversely, each foliation of $N(W)$ that is transverse to the fibers and whose branched leaves contain the boundary components of $N(W)$ corresponds to a foliation of M that is carried by W. In particular, when we collapse the components of $M - N(W)$ (that is, the air pockets) to recover (M, ϕ), each of these foliations

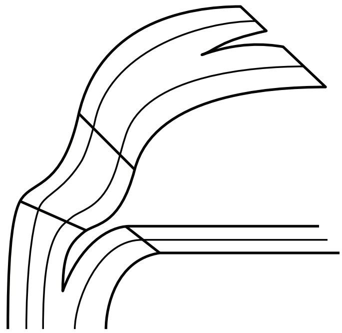
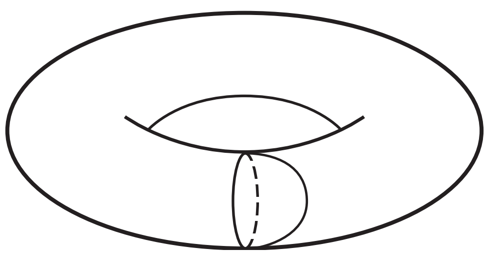


Figure 3

of $N(W)$ yields a foliation of M that is transverse to ϕ and whose leaves contain the elements of Δ. For the most part, we do not distinguish between a foliation of M carried by W and a corresponding foliation of $N(W)$.

As noted above, flowing the disks in any generating set $\Delta = \{D_i\}_{i=1,\dots n}$ for ϕ forward or backward still results in the same branched surface W, provided we do not change the relative position of any two points in $\bigcup_{i=1}^{n} D_i$ along some orbit of ϕ. It follows that W carries a foliation transverse to ϕ if and only if we can move the elements of Δ into leaves of that foliation, while preserving their relative position in the flow direction. We will use this important fact to prove Theorem 2.2.

Reeb skeletons. Given a solid torus Σ embedded in M so that $\partial\Sigma \subset W$, if $\Sigma \cap W$ carries a Reeb foliation of Σ, then we say that Σ is a *Reeb skeleton*. Such an object exists, for example, if some foliation carried by W contains a Reeb component. If a Reeb skeleton Σ contains no other Reeb skeletons, we say that Σ is *minimal*. Here is an example of a minimal Reeb skeleton:



Staircase curves. Given a nonsingular flow ϕ, let $\gamma = \tau_1 * \sigma_2 * \cdots * \tau_{k-1} * \sigma_k * \tau_k$ be a compact curve in M, where τ_1 has nonempty interior and τ_i is a positively oriented orbit segment of ϕ for any $1 \le i \le k$. If we can choose this decomposition of γ so that each *step* σ_i has nonempty interior and is contained in an element of some generating set Δ for ϕ, we say γ is a *staircase curve in* (Δ, ϕ). See Figure 4. The *horizontal length* $\|\gamma\|_{\text{hor}}$ of γ is the sum of the lengths of its steps (that is,

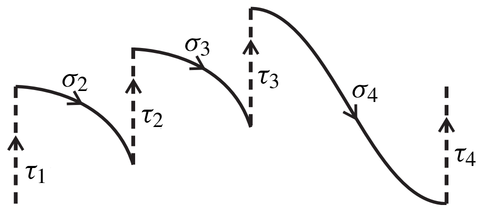


Figure 4

the lengths of the σ_i). We shall only consider staircase curves whose horizontal lengths are nonzero.

2. Main results

In [Goodman and Shields 2007], we described a procedure for successively modifying any branched surface transverse to a flow ϕ, which produces a sequence of branched surfaces $\{W_k\}$ all transverse to ϕ. We then showed the following:

Theorem 2.1. *Given a C^1 nonsingular flow ϕ on a closed orientable 3-manifold M that has no self-return disks, let W be a branched surface constructed from ϕ and let $\{W_k\}$ be a sequence of branched surfaces produced by applying the procedure to W. The flow ϕ is transverse to a foliation F if and only if there exists a $K > 0$ such that W_k carries F for all $k \geq K$.*

Our procedure for successively modifying W specifies a particular way to break the elements of any generating set Δ for W into smaller and smaller disks. If ϕ is transverse to a foliation F, this procedure eventually produces a generating set for a branched surface that carries F. The idea is that once these disks become sufficiently small, each slides injectively along orbit segments of ϕ into a leaf of the foliation F. Moreover, the manner in which we construct these smaller generating disks ensures that this sliding can be done without changing their relative position in the ϕ-direction. So this collection of smaller disks generates a branched surface carrying F.

The proof of Theorem 2.1 requires that we carefully control the size and spacing of the new generating disks created each time we modify Δ. However, the following algorithm for modifying Δ produces the same sequence of branched surfaces (up to diffeomorphism of M).

Given $\Delta = \{D_i\}_{i=1,\dots,n}$, let T be one-third the minimal amount of the time it takes for a point in $\bigcup_{i=1}^n D_i$ to flow back into $\bigcup_{i=1}^n D_i$. For each positive integer k, find $\varepsilon_k > 0$ with the property that flowing any disk D embedded in $\bigcup_{i=1}^n D_i$ with diameter less than ε_k forward or backward for time at most T gives a disk of diameter less than $1/k$. Cover each element of Δ by disks of diameter less than

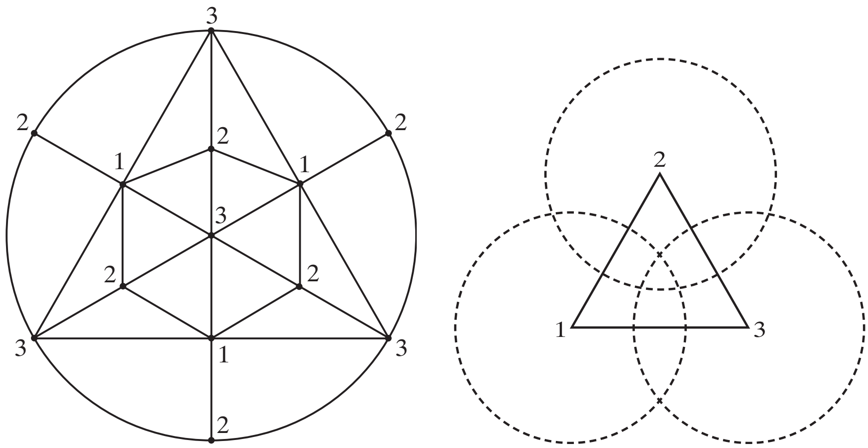


Figure 5

ε_k in the following manner: For each $D_i \in \Delta$, triangulate D_i with a graph of even valence (except along ∂D_i) so that every point in D_i is a distance of at most $\varepsilon_k/3$ from the nearest vertex. (Here we are measuring distance within D_i using the induced metric.) Cover each vertex of the graph with a disk of diameter less than ε_k so that any point $x \in D_i$ is contained in at least one and at most three disks. (Choose these disks so that their boundaries only intersect transversely.) Next, number the disks covering each $D_i \in \Delta$ 1, 2 and 3 so that no two disks of the same number meet (see Figure 5). Then lift all disks numbered 1 forward along the flow for time T and push all disks numbered 3 backward along the flow for time T. (Leave those labeled 2 fixed.) The new collection Δ_k of disks satisfies the conditions for a generating set transverse to ϕ; so Δ_k generates a branched surface W_k. If we use the same cover of Δ, but reduce the amount of time we flow its elements forward or backward, the generating set we obtain still produces the same W_k.

To prove Theorem 2.1, we showed that a flow ϕ with no self-return disks is transverse to a foliation F if and only if there exists a $K > 0$ such that W_k carries F for all $k \geq K$. We now show this to be the case, regardless of whether or not ϕ has a self-return disk.

Theorem 2.2. *Let ϕ be a C^1 nonsingular flow on a closed orientable 3-manifold M and let W be a branched surface constructed from ϕ. The flow ϕ is transverse to a foliation F if and only if iterating the modification process above finitely many times on W yields a branched surface carrying F. Specifically, ϕ is transverse to a foliation F if and only if there exists a $K > 0$ such that W_k carries F for all $k \geq K$.*

Proof. Suppose ϕ is transverse to some foliation F. Let $\Delta = \{D_i\}_{1\le i\le n}$ be a generating set for a branched surface W constructed from ϕ. If W carries F, then we're done. So suppose this is not the case. As in the proof of Theorem 2.1, construct a branched surface V using another generating set X for ϕ such that each element of X is contained in a leaf of F and $X \cap \Delta = \varnothing$. So V carries F, and when we cut M open along X to obtain $N(V)$, each element of Δ becomes embedded in the interior of $N(V)$, transverse to the fibers.

Let $\{\Delta_k\}$ be a sequence of generating sets for ϕ obtained by successively applying our modification procedure to Δ. We can change the value T used in the construction of $\{\Delta_k\}$ so that it is less than one-third the minimal amount of time it takes a point in $X \cup \Delta$ to flow back into it, without affecting the corresponding sequence $\{W_k\}$ of branched surfaces. This ensures that when we cut M open along X, each Δ_k also becomes embedded in $N(V)$, transverse to the fibers.

In the proof of Theorem 2.1, we show that if none of the branched surfaces produced by our modification process carry F, then for all k sufficiently large we can find a staircase loop γ_k in $(X \cup \Delta_k, \phi)$ that is contained in $N(V)$. In addition, we can choose these loops so that $\|\gamma_k\|_{\text{hor}} \to 0$ as $k \to \infty$. (This does not require the absence of self-return disks for ϕ.) Moreover, the sequence $\{\gamma_k\}$ corresponds to a sequence $\{\gamma_k^*\}$ of staircase loops in $(X \cup \Delta, \phi)$ contained in $N(V)$ whose horizontal lengths are also decreasing to 0. This follows from the observation that for any k, each step in γ_k has a preimage in Δ (before we flow the broken pieces of Δ forward or backward). The steps of γ_k^* consist of unions of these preimages.

Now, the projection of $\partial X \cup \partial\Delta$ along fibers of $N(V)$ onto V produces a finite graph. Furthermore, each staircase loop in $(X \cup \Delta, \phi)$ that is contained in $N(V)$ corresponds to a cycle of disks from the set $X \cup \Delta$ which, when projected, gives an (possibly self-intersecting) annulus in V. Among the generators for that annulus that are contained in its boundary and hence contained in the finite graph produced above, there exists one of minimal length. It follows that there exists a lower bound on the horizontal length of staircase loops in $(X \cup \Delta, \phi)$ contained in $N(V)$. So for all k sufficiently large, W_k carrries F. $\square$

According to Theorem 2.2, if a nonsingular flow is transverse to a foliation F, then our algorithm for successively modifying any branched surface transverse to that flow will eventually produce a branched surface that carries F. However, we still need a way to actually detect when this occurs. Our method for doing so will require the following:

Theorem 2.3. *Let ϕ be a C^1 nonsingular flow on a closed 3-manifold M and W be a branched surface constructed from ϕ. We can construct a branched surface W'' (embedded in a different manifold M'') such that W'' carries a Reebless foliation if and only if W carries a foliation.*

Proof. Let Δ be a generating set for a branched surface W transverse to ϕ. Perturb ϕ slightly, if necessary, so that inside each minimal Reeb skeleton there exists a periodic orbit that does not meet the branch set μ of W. Afterwards, if none of the periodic orbits inside the Reeb skeleton are attractors or repellors, choose one and "blow it up" so that it has a small tubular neighborhood consisting entirely of periodic orbits (which also misses μ); then perturb the flow within the tube so that it contains an attracting periodic orbit. (The new ϕ can also be used to construct W from Δ.) After all such modifications, each Reeb skeleton contains a disk, in some sector S of W, that is met by an attracting or repelling periodic orbit γ of ϕ and flows, either forward or backward, into its own interior without meeting μ. Also, there exists a corresponding self-return disk D for ϕ (or ϕ^{-1}) contained in some component of $\partial N(W)$. In other words, D projects onto our original self-return disk and is contained in some (split-open) element of Δ whose projection onto W contains S. After collapsing the complement of $N(W)$ in M, flow D slightly forward if γ is an attractor and slightly backward if γ is a repellor. Subsequently, add D to the collection Δ of generating disks for W. If γ is an attractor (repellor), then some of the original generating disks are met by forward (backward respectively) orbit segments from D back into itself. Create holes in these generating disks that are just large enough to ensure that this situation no longer occurs. See Figure 6. (As a result, our generating set no longer consists of embedded disks. However, the branched surface construction described in Section 1 can also be applied to the more general setting where Δ consists of finitely many closed planar surfaces with boundary.) These changes in Δ correspond to the insertion of a Reeb skeleton Σ through S so that the intersection of $\partial\Sigma$ with the branch set of the new W consists of finitely many meridian curves. Furthermore, all sectors branching into $\partial\Sigma$ from

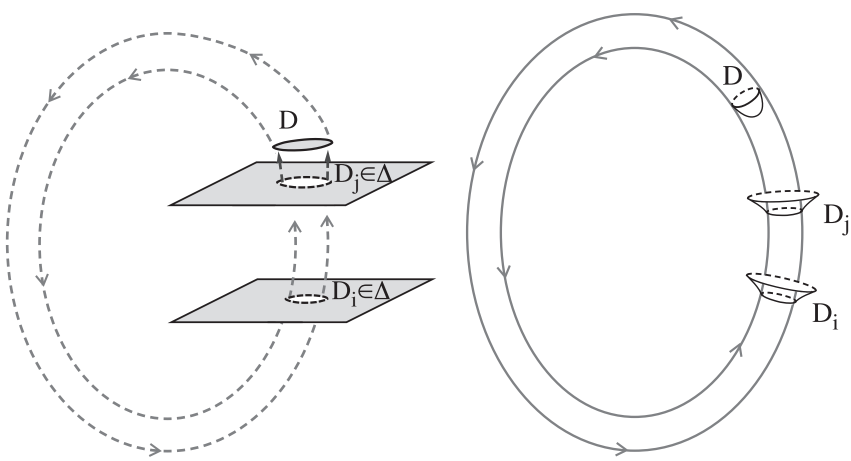


Figure 6

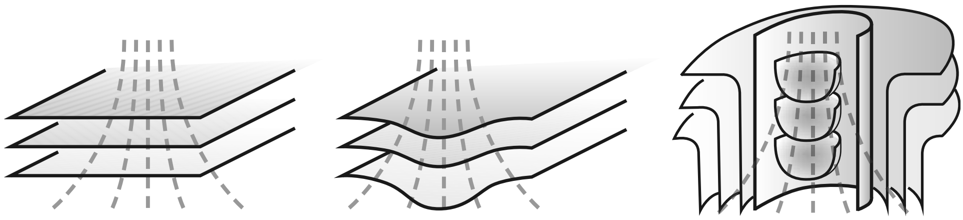


Figure 7

the exterior of Σ do so in the opposite direction than does the only sector branching into $\partial\Sigma$ from the interior of Σ. See Figure 6. So if some foliation F carried by the new W has a Reeb component carried by $\Sigma \cap W$, then F can be modified so that it has only trivial holonomy around the meridian curves of $\partial\Sigma$. This Reeb component then becomes removable in the usual sense. That is, we can modify F to eliminate this Reeb component while staying transverse to ϕ, and when we do so we get a foliation carried by the original W. Consequently, say that such a Reeb skeleton is *removable*.

Conversely, we can modify any foliation carried by the original W by inserting a Reeb component that is carried by $\Sigma \cap W$. So the modified W carries a foliation if and only if the original W carries a foliation. See Figure 7.

Continue to modify W, as above, by inserting a removable Reeb skeleton into the interior of each minimal Reeb skeleton for the original W. (These new Reeb skeletons are pairwise disjoint.) Next, excise the interior of each of the new Reeb skeletons to obtain a manifold M' with boundary. Let ϕ' and W' represent the restriction of ϕ and W, respectively, to M'. Using the identity map, glue M' to a copy of itself (on which the orientation of ϕ' has been reversed) along each of its toral boundary components $T_1, \ldots, T_N$. This produces a new manifold M'' and a new flow ϕ''. Since the flow ϕ' is transverse to $\partial M'$, the new flow is nonsingular. (It is possible that ϕ'' is not C^1 along the *seam* $\bigcup_{1\le i\le N} T_i$. Specifically, when we create M'', it is possible that some of the orbits of ϕ' in M' do not piece together smoothly with the corresponding orbits of ϕ'^{-1} in the copy of M'.)

To ensure that W' and its copy W'_c glue to give another branched surface W'', we modify W'_c slightly near each piece of its branch set contained in the seam. More precisely, the identity map used to glue each toral boundary component T_i to a copy of itself will initially yield local neighborhoods as shown in Figure 8. So we shift the location of each branching of W'_c into T_i slightly, while staying transverse to ϕ'', to obtain local neighborhoods as shown in Figure 8. We then smooth out the orbits of ϕ'' in a small neighborhood of the seam, while staying transverse to the new branched surface W'', so that ϕ'' becomes a nonsingular C^1 flow on M''.

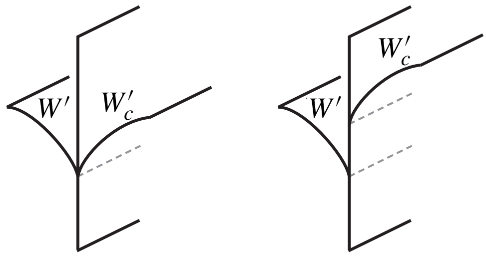


Figure 8

All sectors of W'' that branch into the same component of the seam do so in the same direction. So any smoothly embedded compact surface in W'' that intersects the seam is contained in the seam, and hence is a component of $\partial M'$. It follows that any compact surface that is smoothly embedded in W'' is also smoothly embedded in W.

As noted earlier, if the original W carries a foliation, then our modified W also carries a foliation F where T_i is a leaf contained in $\operatorname{int} N(W)$ for each $1 \leq i \leq N$. In this case, W' carries a foliation of M' where each T_i is a toral leaf in the boundary of some fiber neighborhood $N(W')$. Consequently, W'' also carries a foliation where each T_i is a leaf.

Conversely, W carries a foliation if W'' does. To see this, note that we can thicken W'' to obtain $N(W'')$ so that each T_i becomes embedded in the interior of $N(W'')$. Since for every $i \leq N$, all sectors of W'' branching into T_i do so from the same direction, we can isotope any foliation of $N(W'')$ so that each T_i is a leaf [Shields 1996]. So if W'' carries some foliation, then $N(W')$ has a foliation where each T_i is a leaf contained in $\partial N(W')$. See Figure 9. We can then glue Reeb skeletons back into W' along each T_i to get a branched surface transverse to ϕ and carrying a foliation F of M such that each T_i is a leaf bounding a Reeb component of F. In fact, the branched surface we obtain is the same modified W we obtained earlier by inserting removable Reeb skeletons into the original W. It follows that the original W will also carry a foliation.

All that remains is to show that W'' is Reebless. If not, there exists a solid torus Σ'' embedded in M'' so that $\partial\Sigma'' \subset W''$ and $\Sigma'' \cap W''$ carries a Reeb foliation of Σ''. Choose Σ'' so that it does not properly contain another solid torus with these properties. Since $\partial\Sigma''$ is compact and smoothly embedded in W'', either

$$\partial\Sigma'' \cap \bigcup_{1\leq i\leq N} T_i = \varnothing \quad \text{or} \quad \partial\Sigma'' = T_i \quad \text{for some } 1 \leq i \leq n.$$

In particular, $\partial\Sigma''$ is smoothly embedded in both W' and W. Now, recall that to create M'' we removed a tube through the interior of each Reeb skeleton for W in M to get M', and then glued M' to a copy M'_c of itself. Hence, Σ'' is not contained

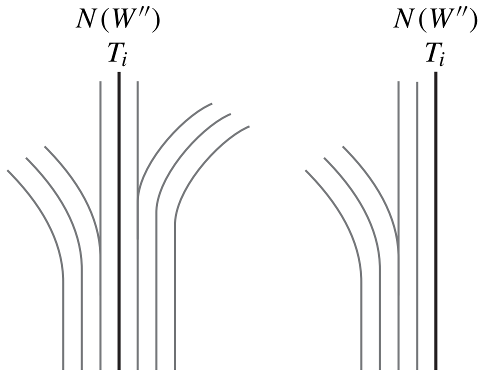


Figure 9

in M'; nor is it contained in M'_c. In other words, $T_i \subseteq \operatorname{int} \Sigma''$ for some $1 \le i \le N$. As noted above, all sectors of W'' branching into T_i do so in the same direction. So T_i is a leaf in the Reeb foliation carried by $\Sigma'' \cap W''$. However, this means that T_i bounds a Reeb component of this foliation that is properly contained in Σ'', contradicting the way we chose Σ''. It follows that any foliation carried by W'' is Reebless. □

Theorem 2.4. *Given a closed* 3*-manifold* M, *there is a procedure that detects when a* C^1 *nonsingular flow on* M *has a transverse foliation.*

Proof. Given a nonsingular flow ϕ, let Δ be a generating set for a branched surface W constructed from ϕ and let $\{W_k\}$ be a sequence of branched surfaces obtained by applying our algorithm to W. By Theorem 2.2, some W_k will carry a foliation if and only if ϕ is transverse to a foliation. So we describe a procedure for determining whether or not a given W_k carries a foliation.

For each branched surface W_k in our sequence, we can construct the corresponding Reebless branched surface W''_k and transverse flow ϕ'' by excising a finite nonempty collection τ_k of solid tori and gluing the resulting manifold with boundary to a copy of itself. Choose the set τ_k, as in the proof of Theorem 2.3, so that W''_k carries a Reebless foliation (where the boundary of each element of τ_k is a leaf) if and only if W_k carries a foliation.

Using the procedure described in [Agol and Li 2003, proof of Theorem 5.2, step 1], we can then determine whether the manifold M''_k created during the construction of W''_k is irreducible, prime or homeomorphic to $S^2 \times S^1$. If M''_k is prime, then it has no Reebless foliation, so there can be no such foliation carried by W''_k. If M''_k is homeomorphic to $S^2 \times S^1$, then the only Reebless foliation of M''_k is the trivial foliation by spheres. In this case, W''_k cannot carry a Reebless foliation in which the tori bounding the elements of τ_k are leaves.

So we can assume that M''_k is irreducible. It then follows that there are no smoothly embedded spheres in W''_k since such a sphere would be transverse to ϕ''_k and bound a 3-ball; by Pugh's generalized Poincare index theorem [Pugh 1968], this 3-ball would necessarily contain a singularity for the flow, contradicting that ϕ''_k is nonsingular.

Hence, Agol and Li's procedure of [2003, proofs of Theorems 2.8 and 3.9] can be used to determine whether or not W''_k fully carries an essential lamination. In the case that it does, the method of [Gabai 1983, proof of Theorem 5.1] can be used to extend this lamination to a Reebless foliation carried by W''_k. $\square$

We next show that if our initial generating set for ϕ is chosen carefully, then our algorithm can be used to detect whether or not there is a foliation that stays some bounded distance δ away from ϕ. To state the result more precisely, we first need some definitions.

Suppose $U = \{U_i\}_{i=1,\dots,N}$ is a covering of M by flow boxes for ϕ. For each $i \le N$, there is a homeomorphism $h_i : U_i \to I^3$, where $I = [0, 1]$ and all images of orbit segments of ϕ contained in U_i are in the vertical direction (that is, each orbit of $h_i(\phi \cap U_i)$ is of the form $(\{x_0\}X\{t\})_{0\le t\le 1}$ for some $x_0 \in I^2$). For each i, we refer to the preimage of $\partial(I^2) \times I$ under h_i as the *vertical boundary* $\partial_v U_i$ of U_i and the preimages of $I^2 \times \{0\}$ and $I^2 \times \{1\}$ as the *base* and *top*, respectively.

We say that U is a *standard covering* of M if

(1) every point of M is contained in at most three flow boxes in U,

(2) for every i and j, either $U_i \cap U_j = \varnothing$, or $\partial_v U_i$ and intersect $\partial_v U_j$ transversely along a finite number of orbit segments, and

(3) $U_i \cap U_j \cap U_k$ is connected for every i, j and k.

Theorem 2.5. *Given a C^1 nonsingular flow ϕ on a closed 3-manifold M, define $U = \{U_i\}_{i=1,\dots,N}$ to be a standard covering of M by finitely many flow boxes for ϕ such that for all $i \ne j$, the top of U_i does not intersect the top or bottom of U_j. Choose a generating set Δ for ϕ consisting of a horizontal slice from each box that does not meet the top of any box, and let $\{\Delta_k\}$ be a sequence of generating sets obtained by applying our algorithm to Δ. For any $\delta > 0$, we can find an integer $K > 0$ such that for any $k \ge K$, the branched surface generated by Δ_k carries all foliations of M that remain a bounded distance of δ from ϕ. Furthermore, K depends only on δ, ϕ, Δ and U.*

Proof. Assume $\delta > 0$ is given and choose U as in the hypotheses. Let Δ be a generating set for ϕ consisting of one horizontal slice, from each flow box, that does not meet the top of any of the flow boxes. In other words, for each $D \in \Delta$, there exists $1 \le i \le N$ and $0 \le t_0 < 1$ such that

$$D = {h_i}^{-1}(I^2 \times \{t_0\}) \quad \text{and} \quad D \cap ({h_j}^{-1}(I^2 \times \{1\})) = \varnothing$$

for all $1 \le j \le N$. Then for each $1 \le i \le N$, there exists a $t_i \in (0, 1)$ such that

$$(h_i{}^{-1}(I^2 \times [t_i, 1])) \cap \Delta = \varnothing \quad \text{and}$$
$$(h_i{}^{-1}(I^2 \times [t_i, 1])) \cap (h_j^{-1}(I^2 \times \{0\})) = \varnothing \quad \text{for all } 0 \le j \le N,$$

and

$$(h_i^{-1}(I^2 \times [t_i, 1])) \cap (h_j^{-1}(I^2 \times [t_j, 1])) = \varnothing \quad \text{for all } j \ne i.$$

Note that since U is finite, we can find some $d > 0$ such that the distance between any two components of $h_i(U_i \cap (\bigcup_{1\le j\le N} h_j^{-1}(I^2 \times [t_j, 1])))$, as well as the distance between any such component and a component of $h_i(U_i \cap (\bigcup_{1\le j\le N} h_j^{-1}(I^2 \times \{0\})))$, exceeds d for all $1 \le i \le N$.

Now suppose there exists a foliation F of M whose distance from ϕ is bounded below by δ (in that the smallest positive angle between the tangent vector to ϕ and the tangent plane to the foliation at any point exceeds δ). We can construct a branched surface V carrying F using another generating set X for ϕ, where each $C \in X$ is contained in $\bigcup_{1\le i\le N} h_i^{-1}(I^2 X[t_i, 1])$ and in a leaf of F. We can also ensure that each orbit segment of $\phi|_{U_i}$ meets $X \cap (h_i^{-1}(I^2 \times [t_i, 1]))$ for all $1 \le i \le N$. So, henceforth, we shall refer to $h_i^{-1}(I^2 \times [t_i, 1])$ as the *X-region of U_i*. Since Δ cannot intersect any of the X-regions of U, the elements of $X \cup \Delta$ are pairwise disjoint. Thus when we cut M open along X to obtain $N(V)$ (foliated by F), each element of Δ becomes embedded in the interior of $N(V)$, transverse to the fibers.

Let $\{\Delta_k\}$ be a sequence of generating sets for ϕ obtained by applying our algorithm to Δ. Recall that if we reduce the value T used in the construction of $\{\Delta_k\}$ so that it is less than one-third the minimal amount of time it takes a point in $X \cup \Delta$ to flow back into $X \cup \Delta$, we do not affect the corresponding sequence $\{W_k\}$ of branched surfaces. So we can assume that when we cut M open along X, each Δ_k also becomes embedded in $N(V)$, transverse to the fibers. (The integer K we find will not depend on X or $N(V)$. However, these objects play an important role in the proof of Theorem 2.1, which we adapt here to show that W_K carries F.)

For any $k > 0$, the branched surface W_k carries F if and only if we can flow the elements of Δ_k injectively onto disks in leaves of F without changing their relative position along orbits of ϕ (see Section 2). If we try to do so, *while staying in $N(V)$*, there are only 2 obstructions we could encounter [Goodman and Shields 2007, Lemma 2.2]. The first is the existence of a staircase loop in (Δ_k, ϕ) contained in $N(V)$. This can cause problems, for example, if all the leaves of F are compact. The other possible obstruction involves the existence of a *connecting strip*; that is, a strip embedded in the interior of $N(V)$, transverse to the fibers, with $\partial N(W)$ branching from both its ends. When such a strip is crossed with *negative index* by a staircase curve γ_k in (Δ_k, ϕ) (as in Figure 10, top), yet is crossed with *nonnegative*

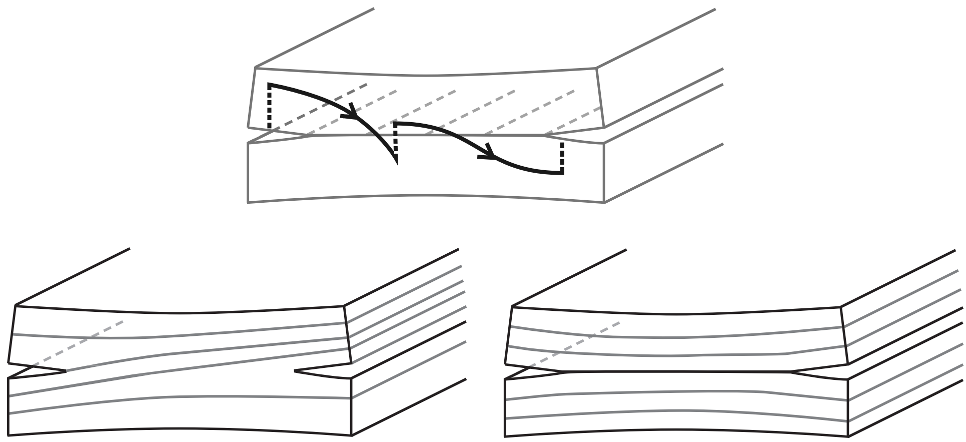


Figure 10

index by F (as in either of the bottom figures in Figure 10), we cannot move the steps of γ_k into leaves of F without either changing their relative position in the flow direction or leaving $N(V)$. (This is the only situation in which a connecting strip presents a problem.)

Both of these obstructions involve staircase curves in (Δ_k, ϕ) that miss X. As noted earlier, each such curve (or loop) γ_k corresponds to a staircase curve (or loop, respectively) γ_k^* in (Δ, ϕ) that also misses X. Also, γ_k^* crosses a connecting strip S with negative index if and only if γ_k crosses S with negative index. (For details, see [Goodman and Shields 2007, proof of Theorem 2.3, page 12].)

So we shall first consider staircase curves in (Δ, ϕ) that miss X. We show that if the horizontal length of such a curve is sufficiently small, then it cannot be a loop, nor can it cross any connecting strip with negative index that is crossed with nonnegative index by F. In other words, we find a constant η such that any staircase curve in (Δ_k, ϕ) that is involved in one of the obstructions described above corresponds to a staircase curve in (Δ, ϕ) whose horizontal length exceeds η. We then show how to find an integer K such that for every staircase curve in (Δ_K, ϕ), the horizontal length of the corresponding staircase curve in (Δ, ϕ) is less than η.

To begin, note that for any $1 \leq i \leq N$, we can project $(\partial\Delta \cap U_i) \cup (\partial_v U_i)$ onto the base of U_i to obtain a finite graph. We can use this to argue, as in the proof of Theorem 2.2, that there exists a lower bound λ_i on the horizontal length of staircase curves in (Δ, ϕ) contained in U_i that begin and end in the same component of $\Delta \cap U_i$.

For any $1 \leq i \leq N$, we can also project $U_i \cap (\bigcup_{1\leq j\leq n} \partial_v U_j)$ onto the base of U_i get another finite graph G_i. There exists a lower bound λ_i' on the lengths

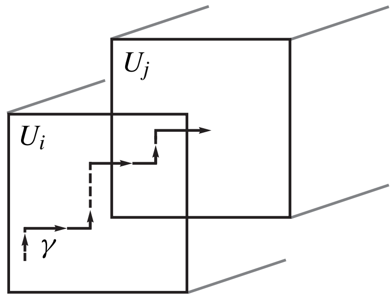


Figure 11

of paths in U_i whose projections join nonadjacent edges of G_i. Choose some $\lambda < \min(\{\lambda_i \mid 0 \le i \le N\} \cup \{\lambda'_i \mid 0 \le i \le N\})$ and let γ be a staircase curve in (Δ, ϕ) contained in $N(V)$ such that $0 < \|\gamma\|_{\text{hor}} < \lambda$.

We first show that γ cannot be a loop. If some flow box in U contains γ, then γ is not a staircase loop, since $\|\gamma\|_{\text{hor}} < \lambda$. So suppose there exist some $i, j \le N$ such that γ begins in U_i, enters U_j and then later exits U_i. In particular, choose i so that once γ exits U_i, it is no longer contained in any flow box. Choose j so that γ exits U_i while still in U_j, and so that no flow box met by γ before it enters U_j has this property. There exists a point at which γ enters U_j and remains in U_j until after its exit from U_i. Let γ' be the subcurve of γ from this point of entry into U_j to its point of exit from U_i.

For every $i \le N$, that $(h_i^{-1}(I^2 \times [t_i, 1])) \cap \Delta$ is empty means that there can be no steps of γ in the X-region of U_i. So γ cannot begin in $[h_i^{-1}(I^2 \times [t_i, 1])]$, since this would mean that the bottom of U_j intersects this X-region, contradicting the way we chose t_i. Furthermore, the way we chose X ensures that any orbit segment of ϕ that enters a X-region of U must meet X before exiting that region. So since no orbit segment in γ can flow through the X-region of U_i, the terminal point of γ' lies in $\partial_v U_i$. If the initial point of γ' lies in $\partial_v U_j$ (as in Figure 11), then projecting γ' onto the base of U_j yields a curve whose initial point and terminal point lie in adjacent edges of G_j, and whose interior does not meet G_j (since $\|\gamma\|_{\text{hor}} < \lambda'_j$). It follows that γ' is contained in U_k for some $k \ne i, j$. (Figure 12 shows the projection of γ' onto a portion of G_j.) By the way we chose i, the curve γ must then enter U_k before entering U_j, contradicting the way we chose j.

So γ enters U_j through its base, and if it subsequently exits U_j, it would have to do so through $\partial_v U_j$. However, this would mean that there exists a subcurve γ'' contained in $\gamma \cap U_j$ that begins in $\partial_v U_i$ and ends in $\partial_v U_j$. Specifically, γ'' is the portion of γ that begins at its exit from U_i and ends at its exit from U_j. Since $\|\gamma\|_{\text{hor}} < \lambda'_j$, the initial point and terminal point of the projected γ'' lie in adjacent edges of G_j, and its interior does not meet G_j. Let x be the vertex adjacent to both

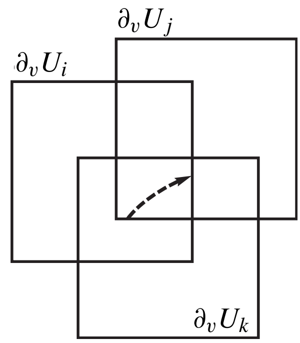


Figure 12

these edges. There exists a flow box U_l, with $l \neq i, j$, containing x. Specifically, the point at which γ enters U_j is contained in U_l. Moreover, γ cannot exit U_l before exiting U_j (by our assumption that the interior of γ'' meets no edges of G_j). So since γ exits U_i while in U_j and by the way we chose i, it enters U_l and (before entering U_j) remains there until its exit from U_i, contradicting the way we chose j. It follows that once γ enters U_j it cannot leave it. In particular, γ cannot be a loop.

Now suppose that γ crosses some connecting strip S with negative index and that F crosses S with nonnegative index. Let C' and C'' be the elements of X containing the ends of S. Specifically, the initial point $\gamma(0)$ of γ lies in C' and the terminal point $\gamma(1)$ lies in C''. Furthermore, the first step of γ intersects some fiber of $N(V)$ above C' and the last (higher) step intersects a fiber below C''. See Figure 13. Since the steps of γ are contained in Δ (which does not intersect any of the X-regions), γ must exit the X-region containing its initial point from the top, before entering the X-region containing its terminal point from the bottom.

So if γ is contained in U_i, the distance between $h_i(\gamma(0))$ and $h_i(\gamma(1))$ exceeds d. In particular, the distance in the vertical direction between the horizontal slices of I^3 containing $h_i(\gamma(0))$ and $h_i(\gamma(1))$, respectively, exceeds $d - \|h_i(\gamma)\|_{\mathrm{hor}}$ (which

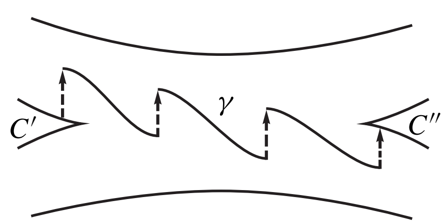


Figure 13

is possibly negative). Now, there exists a constant c such that

$$\frac{1}{c}(d(h_i(x), h_i(y)) \leq d(x, y) \leq c(d(h_i(x), h_i(y)),$$

for all $i \leq N$ and all $x, y \in M$. So the horizontal length of $h_i(\gamma)$ is less than $c\|\gamma\|_{\text{hor}}$. In particular, whenever $\|\gamma\|_{\text{hor}} < d/(2c)$, the absolute value of the smallest angle between the foliation $h_i(F|_{U_i})$ and the flow in the vertical direction must, at some point p, be less than $\arctan((2c\|\gamma\|_{\text{hor}})/(d))$ (since F crosses Σ with a nonnegative index). There also exists a constant ζ such that

$$(1/\zeta)(\angle d(h_i(v), h_i(w)) \leq \angle(v, w) \leq \zeta(\angle(h_i(v), h_i(w))$$

for all $i \leq N$ and any nonzero vectors $v, w, \in T_p(U_i)$. So, in this case, the angle between F and ϕ at $h_i^{-1}(p)$ is less than $\zeta \arctan((2c\|\gamma\|_{\text{hor}})/(d))$.

If, on the other hand, γ begins in U_i and ends in U_j (that is, one end of Σ is contained in U_i and the other is contained in U_j), then since γ enters U_j from the bottom, the lengths of both $h_i(\gamma \cap U_i)$ and $h_j(\gamma \cap U_j)$ are at least d. In particular, the distance in the vertical direction between bottom of $h_j(U_j)$ and the horizontal slice of I^3 containing $h_j(\gamma(1)$ exceeds $d - \|h_j(\gamma \cap U_j)\|_{\text{hor}}$. Likewise, the distance in the vertical direction between $h_i((h_j^{-1}(I^2 \times \{0\})) \cap U_i)$ and the horizontal slice of I^3 containing $h_i(\gamma(0))$ exceeds $d - \|h_i(\gamma \cap U_i)\|_{\text{hor}}$. Hence we can argue, as in the previous case, that whenever $\|\gamma\|_{\text{hor}} < d/(2c)$, somewhere in $U_i \cup U_j$ the angle between F and ϕ is less than $\zeta \arctan((2c\|\gamma\|_{\text{hor}})/(d))$.

Given any $\delta > 0$, we can choose an η with $0 < \eta < \min\{\lambda, d/(2c)\}$ so that $\delta > \zeta \arctan(2c\eta)/d)$. As shown above, the horizontal length of any staircase loop in (Δ, ϕ) is at least λ, and therefore exceeds η. Furthermore, since the foliation F is bounded away from ϕ by δ, the horizontal length of any staircase curve in (Δ, ϕ) that crosses a connecting strip with a different index than does F also exceeds η. So all that remains to show is that we can find an integer K such that for every staircase curve γ_K in (Δ_K, ϕ), the horizontal length of the corresponding staircase curve γ_K^* in (Δ, ϕ) is less than η.

For this, recall that to construct Δ_k, we cover each element of Δ by disks of diameter less than some number ε_k (where $\varepsilon_k \to 0$ as $k \to \infty$). We then flow some of these disks forward and some backward to obtain Δ_k. Since no two adjacent disks in same element of Δ move in the same direction, at most three consecutive steps in γ_k have preimages in the same element of Δ. Hence, if γ_k is a staircase curve in (Δ_k, ϕ), each step in the corresponding staircase curve γ_k^* in (Δ, ϕ) has length less than $3\varepsilon_k$.

Now choose K sufficiently large to ensure that $6\varepsilon_K P < \eta$, where P is the maximal number of components in $\Delta \cap U_i$ over all $i \leq N$. If γ_K^* is contained in U_i, then each of its steps is contained in a distinct element of $\Delta \cap U_i$. For suppose, to the contrary, that there exists a subcurve of γ_K^* that begins and ends in the same component of $\Delta \cap U_i$. We can choose this subcurve so that its interior does not meet any component of $\Delta \cap U_i$ more than once. This ensures that its horizontal length will then be less than $3\varepsilon_K P < \eta < \lambda < \lambda_i$, contradicting the way we chose λ_i. It follows that when γ_K^* is contained in U_i, each of its steps is contained in a distinct element of $\Delta \cap U_i$; hence $\|\gamma_K^*\|_{\mathrm{hor}} < 3\varepsilon_K P < \eta$.

If on the other hand, the initial point $\gamma_K^*(0)$ of γ_K^* lies in U_i and γ_K^* exits U_i after entering some other flow box U_j, then either γ_K^* remains in U_j or it exits U_j at some point $\gamma_K^*(s_1)$, $s_1 > 0$. In the former case, the horizontal length of γ_K^* is less than $6\varepsilon_K P < \eta$. In the latter case, this is true for the subcurve $\gamma_K^*(s)_{0 \le s \le s_1}$. But $\eta < \lambda$ and we have already shown that any staircase curve in (Δ, ϕ) with horizontal length less than λ cannot exit U_j. So this latter case cannot occur. □

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SANDRA SHIELDS
COLLEGE OF CHARLESTON
66 GEORGE STREET
CHARLESTON, SC 29424-1001
UNITED STATES
shieldss@cofc.edu

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MIXED INTERIOR AND BOUNDARY NODAL BUBBLING SOLUTIONS FOR A sinh-POISSON EQUATION

JUNCHENG WEI, LONG WEI AND FENG ZHOU

We consider here the semilinear equation $\Delta u + 2\varepsilon^2 \sinh u = 0$ posed on a bounded smooth domain Ω in $\mathbb{R}^2$ with homogeneous Neumann boundary condition, where $\varepsilon > 0$ is a small parameter. We show that for any given nonnegative integers k and l with $k+l \geq 1$, there exists a family of solutions u_ε that develops $2k$ interior and $2l$ boundary singularities for ε sufficiently small, with the property that

$$2\varepsilon^2 \sinh u_\varepsilon \rightharpoonup 8\pi \sum_{i=1}^{2k} (-1)^{i-1} \delta_{\xi_i} + 4\pi \sum_{i=1}^{2l} (-1)^{i-1} \delta_{\xi_i},$$

where $(\xi_1, \ldots, \xi_{2(k+l)})$ are critical points of some functional defined explicitly in terms of the associated Green function.

1. Introduction

The two-dimensional sinh-Poisson equation

$$\Delta u + 2\varepsilon^2 \sinh u = 0 \tag{1-1}$$

arises in various important contexts, notably as a vorticity equation in classical hydrodynamics [Gurarie and Chow 2004; Chow et al. 1998; Kuvshinov and Schep 2000; Mallier and Maslowe 1993], in physico-chemical hydrodynamics [Probstein 1994] and in the geometry of constant mean curvature surfaces [Wente 1986]. In the vorticity connection, it occurs in a remarkable manner out of natural relaxation states in the long-time computation of two-dimensional fluid motion [Mallier and Maslowe 1993] (see also the references therein). In geometry, the sinh-Poisson equation plays a very important role in the study of the construction of constant mean curvature surfaces initiated by Wente [1986]. Wente's seminal work then

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led to work by Steffen [1986], Struwe [1986] and Brezis and Coron [1984], which completed the understanding of the blow-up for constant mean curvature surfaces from a geometric point of view. Spruck [1988] was the first to study the sinh-Poisson equation from an analytic point of view. Recently, the asymptotic behavior of solutions to (1-1) was studied on a closed Riemann surface in [Ohtsuka and Suzuki 2006] and [Jost et al. 2008]. The authors applied the so-called "symmetrization method" and "Pohozaev identity", respectively, to show that there possibly exist two different types of blow-up for a family of solutions to (1-1). Conversely, Bertolucci and Pistoia [2007] tried to construct blow-up solutions to (1-1) with Dirichlet boundary conditions for $n = 2$, and proved that for ε positive and small enough, there exist at least two pairs of solutions that change sign exactly once, that concentrate in the domain and that have their nodal lines intersecting the boundary.

In [Wei et al. 2011] and [Wei 2009] the Neumann problem

$$\begin{cases} \Delta u + 2\varepsilon^2 \sinh u = 0 & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1-2}$$

was considered, where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\varepsilon > 0$ is a parameter. The authors showed a concentration phenomena of solutions to (1-2) in the domain in [Wei et al. 2011], and on the boundary in [Wei 2009].

In this paper, we continue the study of the existence of solutions to (1-2). We prove that there exists a family of solutions u_ε that concentrate positively and negatively in the domain and its boundary.

To state our results, we need to introduce some notation. First, let us define the corresponding Green function for the Neumann problem:

$$\begin{cases} -\Delta G(x, y) = \delta_y(x) - 1/|\Omega| & \text{in } \Omega, \\ \partial G/\partial \nu = 0 & \text{on } \partial\Omega, \\ \int_\Omega G(x, y)dx = 0. \end{cases} \tag{1-3}$$

The regular part of $G(x, y)$ is defined depending on whether y lies in the domain or on its boundary as

$$H(x, y) = \begin{cases} G(x, y) + \dfrac{1}{2\pi}\log|x - y| & \text{for } y \in \Omega, \\ G(x, y) + \dfrac{1}{\pi}\log|x - y| & \text{for } y \in \partial\Omega. \end{cases} \tag{1-4}$$

In this way, $H(\cdot, y)$ is of class $C^{1,\alpha}$ in $\overline{\Omega}$.

For $k + l \geq 1$ and points ξ_j for $j = 1, \dots, 2(k + l)$, with $\xi_j \in \Omega$ for $j \leq 2k$ and $\xi_j \in \partial\Omega$ for $2k + 1 \leq j \leq 2(k + l)$, we define

$$\varphi_{2(k+l)}(\xi_1, \dots, \xi_{2(k+l)}) = \sum_{j=1}^{2(k+l)} c_j^2 H(\xi_j, \xi_j) + \sum_{j\neq i} c_j c_i (-1)^{j+i} G(\xi_j, \xi_i) \tag{1-5}$$

and denote

$$\mathcal{M}_d := \Big\{\xi = (\xi_1, \dots, \xi_{2k}, \xi_{2k+1}, \dots, \xi_{2(k+l)}) \in \Omega^{2k} \times \partial\Omega^{2l} \\ \Big| \min_{j\neq i} |\xi_j - \xi_i| \geq d,\ \min_{j=1,\dots,2k} \operatorname{dist}(\xi_j, \partial\Omega) \geq d\Big\},$$

where $c_i = 8\pi$ for $i = 1, \dots, 2k$ and $c_i = 4\pi$ for $i = 2k+1, \dots, 2(k+l)$.

Definition 1.1 [Esposito et al. 2006]. We say that ξ is a C^0-*stable critical point* of $\varphi_m : \mathcal{M}_d \to \mathbb{R}$ if for any sequence of functions $\varphi_m^n : \mathcal{M}_d \to \mathbb{R}$ such that $\varphi_m^n \to \varphi_m$ uniformly on compact sets of $\mathcal{M}_d$, the function φ_m^n has a critical point ξ_n such that $\varphi_m^n(\xi_n) \to \varphi_m(\xi)$.

In particular, if ξ is a strict local minimum/maximum point of φ_m, then ξ is a C^0-stable critical point.

Theorem 1.2 (main result). *Let k and l be nonnegative integers with $k + l \geq 1$. Assume $\xi^* \in \mathcal{M}_d$ is a C^0-stable critical point of $\varphi_{2(k+l)}$. Then for any sufficiently small $\varepsilon > 0$, there is a solution u_ε to* (1-2) *with the property that*

$$2\varepsilon^2 \int_\Omega |\sinh u_\varepsilon| dx \to 8\pi(2k + l) \quad \text{as } \varepsilon \to 0. \tag{1-6}$$

More precisely, for any sequence $\{\varepsilon_n\}_{n\geq 1}$ that tends to 0, *there is a subsequence and $2(k + l)$ points $\xi_i \in \overline{\Omega}$ for $i = 1, \dots, 2(k + l)$, with $\xi_j \in \Omega$ for $j \leq 2k$ and $\xi_j \in \partial\Omega$ for $2k+1 \leq j \leq 2(k+l)$, and positive constants μ_i for $i = 1, \dots, 2(k+l)$ such that*

$$u_\varepsilon(x) = \sum_{i=1}^{2(k+l)} (-1)^{i-1} \Big(\log \frac{1}{(\varepsilon^2\mu_i^2 + |x - \xi_i|^2)^2} + c_i H(x, \xi_i)\Big) + o(1) \tag{1-7}$$

and

$$2\varepsilon^2 \sinh u_\varepsilon \rightharpoonup 8\pi \sum_{i=1}^{2k} (-1)^{i-1} \delta_{\xi_i} + 4\pi \sum_{i=2k+1}^{2(k+l)} (-1)^{i-1} \delta_{\xi_i} \tag{1-8}$$

in the sense of measure. Moreover, the constants μ_i are given by

$$\log(8\mu_i^2) = c_i H(\xi_i, \xi_i) + \sum_{j\neq i} (-1)^{j+i} c_j G(\xi_i, \xi_j).$$

The $l = 0$ (or $k = 0$) case of this theorem was proved in [Wei et al. 2011] (or [Wei 2009]). The conditions that $\xi^* \in \mathcal{M}_d$ be a C^0-stable critical point of $\varphi_{2(k+l)}$ is perhaps not necessary. Here, we need it only because of the technique we will use. In particular, for the case $k = l = 1$ and $\Omega = B = B(0, 1)$, the unit ball in $\mathbb{R}^2$, we don't need the condition and can obtain the existence and the profile of sign-changing solutions that concentrate positively and negatively at different points $\xi_1, \xi_2 \in B$ and $\xi_3, \xi_4 \in \partial B$. More precisely:

Theorem 1.3. *Let $k = l = 1$. Then, there exists a solution u_ε to* (1-2) *that concentrates at different points $\xi_1, \xi_2 \in B$ and $\xi_3, \xi_4 \in \partial B$, according to* (1-6), (1-7) *and* (1-8) *with $k = l = 1$, as ε goes to* 0.

Del Pino and Wei [2006] considered the problem $-\Delta u + u = \lambda e^u$ under Neumann boundary conditions and built a solution with $\lambda \int_\Omega e^u$ uniformly bounded and boundary-interior concentrating, such that $\lambda e^u \rightharpoonup 8\pi \sum_{j=1}^{k} \delta_{\xi_j} + 4\pi \sum_{j=k+1}^{m} \delta_{\xi_j}$. For basic cells, they used explicit solutions of

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u dx < +\infty$$

given by

$$U_{\mu,\xi} = \log \frac{8\mu^2}{(\mu^2 + |x-\xi|^2)^2} \quad \text{for } \mu > 0 \text{ and } \xi \in \mathbb{R}^2.$$

In this paper, we will also construct solutions predicted by the theorems using these ones, but suitably scaled and projected so that it works for the nonlinearity we consider here. A special feature of our problem is presence of *mixed positive-negative boundary-interior* bubbling solutions. This is a new concentration phenomenon. To capture such solutions, we use the so-called localized energy method, which combines Lyapunov–Schmidt reduction and variational techniques. Such a scheme was been used in many works; see for instance [Dávila et al. 2005; del Pino et al. 2005; del Pino and Wei 2006] and references therein. Here we follow [del Pino and Wei 2006; Wei et al. 2011; Wei 2009], but we will overcome some of the difficulties that the mixed concentration phenomenon brings by delicate analysis.

2. Ansatz for the solution

In this section we will provide a first approximation for the solution of the problem (1-2) predicted by Theorems 1.2 and 1.3. Let us fix $k+l \geq 1$. For $i = 1, \ldots, 2(k+l)$, let $\xi_i \in \overline{\Omega}$ and let μ_i be positive numbers to be chosen later. We define

$$u_i(x) = \log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + |x-\xi_i|^2)^2}. \tag{2-1}$$

The ansatz is

$$U(x) = \sum_{i=1}^{2(k+l)} (-1)^{i-1}(u_i(x) + H_i^\varepsilon(x)) \tag{2-2}$$

where $H_i^\varepsilon(x)$ is a correction term defined as the solution of

$$\begin{cases} \Delta H_i^\varepsilon = \varepsilon^2 \dfrac{1}{|\Omega|} \displaystyle\int_\Omega e^{u_i} & \text{in } \Omega, \\ \dfrac{\partial H_i^\varepsilon}{\partial \nu} = -\dfrac{\partial u_i}{\partial \nu} & \text{on } \partial\Omega \end{cases} \tag{2-3}$$

with the property that

$$\int_\Omega H_i^\varepsilon(x)dx = -\int_\Omega u_i\, dx. \tag{2-4}$$

This function resembles the shape of the regular part of the Green's function. Indeed, the following estimate for H_i^ε holds true.

Lemma 2.1. *For any* $0 < \alpha < 1$

$$H_i^\varepsilon(x) = c_i H(x,\xi_i) - \log(8\mu_i^2) + O(\varepsilon) \tag{2-5}$$

holds uniformly in $\overline{\Omega}$, *where* H *is the regular part of the Green function defined by* (1-4).

Proof. The regular part of Green's function $H(x,\xi_i)$ satisfies

$$\begin{cases} \Delta H(x,\xi_i) = \dfrac{1}{|\Omega|} & \text{in } \Omega, \\ \dfrac{\partial H}{\partial \nu}(x,\xi_i) = \dfrac{4}{c_i}\dfrac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} & \text{on } \partial\Omega. \end{cases} \tag{2-6}$$

Now we define $z_\varepsilon(x) = H_i^\varepsilon(x) + \log(8\mu_i^2) - c_i H(x,\xi_i)$. Then

$$\begin{cases} \Delta z_\varepsilon = \varepsilon^2 \dfrac{1}{|\Omega|}\displaystyle\int_\Omega e^{u_i} - \dfrac{c_i}{|\Omega|} & \text{in } \Omega, \\ \dfrac{\partial z_\varepsilon}{\partial \nu} = 4\dfrac{(x-\xi_i)\cdot\nu(x)}{\varepsilon^2\mu_i^2 + |x-\xi_i|^2} - 4\dfrac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} & \text{on } \partial\Omega. \end{cases}$$

First, by the definition of u_i, we have

$$\begin{aligned} \varepsilon^2\int_\Omega e^{u_i} &= \varepsilon^2\int_\Omega \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2+|x-\xi_i|^2)^2} \\ &= 8\varepsilon^2\int_{\Omega/\varepsilon\mu_i}\frac{\mu_i^2}{(\varepsilon^2\mu_i^2+\varepsilon^2\mu_i^2y^2)^2}\varepsilon^2\mu_i^2 \\ &= 8\int_{\Omega/\varepsilon\mu_i}\frac{dy}{(1+y^2)^2} \\ &= 2c_i\Big(\int_0^\infty \frac{tdt}{(1+t^2)^2} + O\Big(\int_{1/\varepsilon\mu_i}^\infty \frac{tdt}{(1+t^2)^2}\Big)\Big) \\ &= c_i + O(\varepsilon^2\mu_i^2) \end{aligned} \tag{2-7}$$

Next, for $\xi_i \in \Omega$ with $i = 1,\dots,2k$, we have

$$\frac{\partial H_i^\varepsilon}{\partial \nu} = 4\frac{(x-\xi_i)\cdot\nu(x)}{\varepsilon^2\mu_i^2+|x-\xi_i|^2} = 4\frac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} + O(\varepsilon^2) \quad \text{for all } \xi_i\in\Omega,\ x\in\partial\Omega.$$

For $\xi_i \in \partial\Omega$ with $i = 2k+1, \dots, 2(k+l)$, we have

$$\lim_{\varepsilon\to 0} \frac{\partial H_i^\varepsilon}{\partial \nu} = 4\frac{(x-\xi_i)\cdot \nu(x)}{|x-\xi_i|^2} \quad \text{for all } x \neq \xi_i. \tag{2-8}$$

We claim that for any $p > 1$ there exists $C > 0$ such that

$$\Big\| \frac{\partial H_i^\varepsilon}{\partial \nu} - 4\frac{(x-\xi_i)\cdot \nu(x)}{|x-\xi_i|^2} \Big\|_{L^p(\partial\Omega)} \leq C\varepsilon^{1/p}. \tag{2-9}$$

It is not difficult to prove that the inequality

$$|(x-\xi_i)\cdot\nu(x)| \leq C|x-\xi_i|^2 \quad \text{for all } x \in \partial\Omega \tag{2-10}$$

holds for $\xi_i \in \partial\Omega$ by assuming that $\xi_i = 0$ and that near the origin $\partial\Omega$ is the graph of a function $P : (-\delta, \delta) \to \mathbb{R}$ with $P(0) = P'(0) = 0$. Now from (2-10) we obtain

$$\begin{aligned} \Big| \frac{\partial H_i^\varepsilon}{\partial \nu} - 4\frac{(x-\xi_i)\cdot \nu(x)}{|x-\xi_i|^2} \Big| &= 4\varepsilon^2\mu_i^2 \frac{|(x-\xi_i)\cdot\nu(x)|}{|x-\xi_i|^2(\varepsilon^2\mu_i^2 + |x-\xi_i|^2)} \\ &\leq \frac{C\varepsilon^2}{\varepsilon^2\mu_i^2 + |x-\xi_i|^2}. \end{aligned} \tag{2-11}$$

Thus for $\lambda > 0$ small but fixed,

$$\Big| \frac{\partial H_i^\varepsilon}{\partial \nu} - 4\frac{(x-\xi_i)\cdot \nu(x)}{|x-\xi_i|^2} \Big| \leq C\varepsilon^2 \quad \text{for all } |x-\xi_i| \geq \lambda,\ x \in \partial\Omega. \tag{2-12}$$

Letting $p > 1$ and changing variables $x - \xi_i = \varepsilon y \mu_i$, we have

$$\begin{aligned} \int_{B_\lambda(\xi_i)\cap\partial\Omega} \Big| \frac{\varepsilon^2}{\varepsilon^2\mu_i^2 + |x-\xi_i|^2} \Big|^p &= C\varepsilon \int_{B_{\lambda/\varepsilon\mu_i}(0)\cap\partial\Omega_\varepsilon} \Big| \frac{1}{1+|y|^2} \Big|^p dy \\ &= C\varepsilon \int_0^{\lambda/\varepsilon\mu_i} \frac{1}{(1+t^2)^p} dt \leq C\varepsilon. \end{aligned}$$

This, combined with (2-11) and (2-12), shows that (2-9) holds.

By elliptic regularity theory, we obtain $z_\varepsilon \in W^{1+s,p}(\Omega)$ for any $p \geq 1$, with $0 < s < 1/p$. On the other hand, from the Poincaré inequality we get

$$\Big\| z_\varepsilon - \frac{1}{|\Omega|}\int_\Omega z_\varepsilon \Big\|_{W^{1+s,p}(\Omega)} \leq C\|\nabla z_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{1/p}.$$

This implies the existence of a constant M such that

$$z_\varepsilon(x) = M + O(\varepsilon^\alpha) \quad \text{for any } \alpha \in (0, 1),$$

uniformly in $\overline{\Omega}$, where $M = \lim_{\varepsilon\to 0}|\Omega|^{-1}\int_\Omega z_\varepsilon dx$.

To obtain the result, we only need to show $M = 0$. First, by the definition of z_ε we have

$$M = \lim_{\varepsilon\to 0}\Big(\frac{1}{|\Omega|}\int_\Omega H_i^\varepsilon(x)dx + \log(8\mu_i^2) - \frac{c_i}{|\Omega|}\int_\Omega H(x,\xi_i)dx\Big). \tag{2-13}$$

The direct computation from (2-4) shows that

$$\begin{aligned}\int_\Omega H_i^\varepsilon(x) &= -\int_\Omega\Big(\log(8\mu_i^2) + \log\frac{1}{(\varepsilon^2\mu_i^2+|x-\xi_i|^2)^2}\Big)\\ &= -|\Omega|\log(8\mu_i^2) + 2\int_\Omega \log\Big(1+\frac{\varepsilon^2\mu_i^2}{|x-\xi_i|^2}\Big) - 4\int_\Omega \log\frac{1}{|x-\xi_i|}\\ &= -|\Omega|\log(8\mu_i^2) + c_i\int_\Omega H(x,\xi_i)dx + O(\varepsilon^2\log\varepsilon^{-1}),\end{aligned}$$

where the last equality is consequence of the definition of H and the property of the Green function. Therefore (2-13) implies $M = 0$. □

In $\Omega_\varepsilon = \Omega/\varepsilon$, let $v(y) = u(\varepsilon y)$; then solving problem (1-2) is equivalent to solving

$$\begin{cases}\Delta v(y) + 2\varepsilon^4\sinh v = 0 & \text{in } \Omega_\varepsilon,\\ \partial v/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{2-14}$$

We will seek a solution v of (2-14) of the form

$$v(y) = V(y) + \phi(y) \quad \text{for all } y\in\Omega_\varepsilon, \tag{2-15}$$

where

$$V(y) = \sum_{i=1}^{2(k+l)}(-1)^{i-1}(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)). \tag{2-16}$$

Problem (2-14) can be restated: Find a solution ϕ to

$$\begin{cases}\Delta\phi + W\phi + R + N(\phi) = 0 & \text{in } \Omega_\varepsilon,\\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon,\end{cases} \tag{2-17}$$

where

$$W = 2\varepsilon^4\cosh V, \tag{2-18}$$

$$N(\phi) = 2\varepsilon^4(\sinh(V+\phi) - \phi\cosh V - \sinh V) \quad \text{(the nonlinear term)}, \tag{2-19}$$

$$R = \Delta V + 2\varepsilon^4\sinh V \quad \text{(the error term)}. \tag{2-20}$$

We choose the parameters μ_i as

$$\log(8\mu_i^2) = H(\xi_i,\xi_i) + \sum_{j\neq i}(-1)^{j+i}G(\xi_i,\xi_j). \tag{2-21}$$

From Appendix A, we have for all $y \in \Omega_\varepsilon$ the estimates

$$|R(y)| \le C\varepsilon^\alpha \sum_{i=1}^{2(k+l)} \frac{1}{1+|y-\xi_i'|^3}, \tag{2-22}$$

$$W(y) = \sum_{i=1}^{2(k+l)} \frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}(1+\theta_\varepsilon(y)), \tag{2-23}$$

with

$$|\theta_\varepsilon(y)| \le C\varepsilon^\alpha + C\varepsilon \sum_{i=1}^{2(k+l)} |y-\xi_i'|, \tag{2-24}$$

where $\xi_i' = \xi_i/\varepsilon$.

3. Analysis of the linearized problem

In this section we study the solvability of the problem

$$\begin{cases} -\Delta\phi = W\phi + h + \displaystyle\sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji}\chi_i Z_{ji} + c_0 \chi Z & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{3-1}$$

with

$$\int_{\Omega_\varepsilon} \chi_i Z_{ji}\phi = 0 \quad \text{for } i = 1, \dots, 2(k+l),\ \ j = 1, J_i, \tag{3-2}$$

$$\int_{\Omega_\varepsilon} \chi Z\phi = 0, \tag{3-3}$$

where W is a function that satisfies (2-23) and (2-24), $h \in L^\infty(\Omega_\varepsilon)$, $c_0, c_{ji} \in \mathbb{R}$, the functions χ, χ_i, Z and Z_{ji} will be defined below, $J_i = 2$ for $i = 1, \dots, 2k$, and $J_i = 1$ for $i = 2k+1, \dots, 2(k+l)$.

Define z_{ji} by

$$z_{0i} = \frac{1}{\mu_i} - 2\frac{\mu_i}{\mu_i^2+|y|^2} \quad \text{and} \quad z_{ji} = \frac{y_j}{\mu_i^2+|y|^2}.$$

It is well known that any solution to

$$\Delta\phi + \frac{8\mu_i^2}{(\mu_i^2+|y|^2)^2}\phi = 0, \quad |\phi| \le C(1+|y|)^\sigma \tag{3-4}$$

is a linear combination of z_{ji} for $j = 0, 1, 2$; see [Chen and Lin 2002, Lemma 2.1].

Next, we fix a large constant R_0 and a nonnegative smooth function $\bar{\chi}:\mathbb{R}\to\mathbb{R}$ such that $\bar{\chi}(r)=1$ for $r\le R_0$, $\bar{\chi}(r)=0$ for $r>R_0+1$, and $0\le\bar{\chi}\le 1$.

For $i=1,\dots,2k$ (corresponding to the interior bubble case), we define

$$\chi_i(y)=\bar{\chi}(|y-\xi_i'|),\quad Z_{ji}(y)=z_{ji}(y-\xi_i')\quad\text{for } j=0,1,2,\ i=1,\dots,2k.$$

For $i=2k+1,\dots,2(k+l)$ (corresponding to the boundary bubble case), first we strength the boundary similarly to [del Pino and Wei 2006]. Let us concentrate on $\xi_i\in\partial\Omega$. Without loss of generality, we assume that $\xi_i=0$ and the unit outward normal at ξ_i is $(0,-1)$. Let $P(x_1)$ be the defining function for the boundary $\partial\Omega$ in a neighborhood $B_\rho(\xi_i)$, that is,

$$\Omega\cap B_\rho(\xi_i)=\{(x_1,x_2)\mid x_2>P(x_1),\,(x_1,x_2)\in B_\rho(\xi_i)\},$$

and then define $F_i:B_\rho(\xi_i)\cap\mathcal{N}\to\mathbb{R}^2$ by $F_i=(F_{i1},F_{i2})$, where

$$F_{i1}=x_1+\frac{x_2-P(x_1)}{1+|P'(x_1)|^2}P'(x_1)\quad\text{and}\quad F_{i2}=x_2-P(x_1).$$

Then we set

$$F_i^\varepsilon(y)=\varepsilon^{-1}F_i(\varepsilon y)$$

and define

$$\chi_i(y)=\bar{\chi}(F_i^\varepsilon(y)),\quad Z_{ji}(y)=z_{ji}(F_i^\varepsilon(y))\quad\text{for } j=0,1,\ i=2k+1,\dots,2(k+l).$$

It is important to observe that F_i preserves the Neumann boundary condition and

$$\Delta Z_{0i}+\frac{8\mu_i}{(\mu_i^2+|y-\xi_i'|^2)^2}Z_{0i}=O\Big(\frac{\varepsilon^\alpha}{(1+|y-\xi_i'|)^3}\Big).$$

Let $0<b<1$ and define for all $i=1,\dots,2(k+l)$,

$$Z(y)=\begin{cases}\min\{1/\mu_i-\varepsilon^b,\,Z_{0i}(y)\} & \text{if } |y-\xi_i'|<\delta/\varepsilon,\\ 1/\mu_i-\varepsilon^b & \text{if } |y-\xi_i'|\ge\delta/\varepsilon\end{cases}\tag{3-5}$$

and $\chi=\sum_{i=1}^{2(k+l)}\chi_i$.

Now let us introduce the norms

$$\|h\|_\infty=\sup_{y\in\Omega_\varepsilon}|h(y)|\quad\text{and}\quad\|h\|_*=\sup_{y\in\Omega_\varepsilon}\frac{|h(y)|}{\varepsilon^2+\sum_{i=1}^{2(k+l)}(1+|y-\xi_i'|)^{-2-\sigma}},$$

where we fix $0<\sigma<1$, reserving the precise choice for later. Our main result in this section is stated as follows:

Proposition 3.1. *Let $d>0$ and let k,l be nonnegative integers with $k+l\ge 1$. Then there exists a ε_0 such that for any $0<\varepsilon<\varepsilon_0$, any $2(k+l)$-points $(\xi_1,\dots,\xi_{2(k+l)})\in\mathcal{M}_d$ and any $h\in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi\in L^\infty(\Omega_\varepsilon)$,*

$c_0, c_{ji} \in \mathbb{R}$ *to* (3-1), *with* $i = 1, \ldots, 2(k+l)$ *and* $j = 1, J_i$. *Moreover there is a positive* C *independent of* ε *such that*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log \varepsilon|\|h\|_*,$$
$$\max\{|c_0|, |c_{ji}|\} \le C\|h\|_* \quad \textit{for } i = 1, \ldots, 2(k+l),\ j = 1, J_i.$$

We begin to prove this result by studying a linear problem

$$\begin{cases} -\Delta\phi = h + W\phi & \text{in } \Omega_\varepsilon, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3-6}$$

together with orthogonality conditions (3-2) and (3-3).

Proposition 3.2. *Let* $h \in L^\infty(\Omega_\varepsilon)$. *For fixed* $d > 0$ *there exist* $\varepsilon_0 > 0$ *and* C *such that if* $0 < \varepsilon < \varepsilon_0$, $\xi = (\xi_1, \ldots, \xi_{2(k+l)}) \in \mathcal{M}_d$ *and* $\phi \in L^\infty(\Omega_\varepsilon)$ *is a solution of* (3-6) *such that* (3-2) *and* (3-3) *hold*, *then*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1}\|h\|_*,$$

where C *is independent of* ε.

We will prove this estimate by contradiction assuming that there exist a sequence $\varepsilon \to 0$, points $(\xi_1, \ldots, \xi_{2(k+l)}) \in \mathcal{M}_d$ (we omit the dependence on ε in the notation) and functions $h, \phi \in L^\infty(\Omega_\varepsilon)$ such that

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} = 1 \quad \text{and} \quad \log \varepsilon^{-1}\|h\|_* = o(1). \tag{3-7}$$

Fix $0 < \gamma < \beta < 1/2$ and consider the function η given by

$$\eta(r) = \begin{cases} 1 & \text{if } r < \varepsilon^{-\gamma}, \\ \dfrac{\log \varepsilon^{-\beta} - \log r}{\log \varepsilon^{-\beta} - \log \varepsilon^{-\gamma}} & \text{if } \varepsilon^{-\gamma} < r < \varepsilon^{-\beta}, \\ 0 & \text{if } r > \varepsilon^{-\beta}. \end{cases} \tag{3-8}$$

Let $\tilde\eta$ be a radial smooth cut-off function on $\mathbb{R}^2$ such that $\tilde\eta(r) \equiv 1$ for $r < \varepsilon^{-\beta}$, $\tilde\eta \equiv 0$ for $r > 2\varepsilon^{-\beta}$, $|\tilde\eta'(r)| \le C\varepsilon^\beta$ and $|\tilde\eta''(r)| \le C\varepsilon^{2\beta}$. Then we set

$$\eta_{1i}(y) = \begin{cases} \eta(|y - \xi_i'|) & \text{for } i = 1, \ldots, 2k, \\ \eta(|F_i^\varepsilon(y)|) & \text{for } i = 2k+1, \ldots, 2(k+l); \end{cases}$$
$$\eta_{2i}(y) = \begin{cases} \tilde\eta(|y - \xi_i'|) & \text{for } i = 1, \ldots, 2k, \\ \tilde\eta(|F_i^\varepsilon(y)|) & \text{for } i = 2k+1, \ldots, 2(k+l); \end{cases}$$
$$a_{0i} = \frac{1}{\mu_i((4/c_i)\log \varepsilon^{\gamma-1} + H(\xi_i, \xi_i))}$$

and also

$$\widehat{Z}_{0i}(y) = Z_{0i}(y) - \mu_i^{-1} + a_{0i}G(\varepsilon y, \xi_i).$$

Now define a test function

$$\tilde{Z}_{0i} = \eta_{1i} Z_{0i} + \varepsilon(1-\eta_{1i})\eta_{2i}\widehat{Z}_{0i}.$$

Given ϕ satisfying (3-6) and the orthogonality conditions (3-2) and (3-3), let

$$\tilde{\phi} = \phi - \sum_{i=1}^{2(k+l)} d_i \tilde{Z}_{0i},$$

where the numbers d_i are chosen so that $\int_{\Omega_\varepsilon} \chi_i Z_{0i}\tilde{\phi} = 0$ for any $i = 1, \dots, 2(k+l)$, namely $d_i = \int_{\Omega_\varepsilon} \chi_i Z_{0i}\phi / \int_{\Omega_\varepsilon} \chi_i Z_{0i}^2$. Observe that

$$d_i = O(1) \quad \text{and} \quad \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} = O(1).$$

Moreover, $\tilde{\phi}$ satisfies

$$\begin{cases} -\Delta\tilde{\phi} = W\tilde{\phi} + h - \displaystyle\sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i}) & \text{in } \Omega_\varepsilon, \\ \partial\tilde{\phi}/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3-9}$$

and the orthogonality condition

$$\int_{\Omega_\varepsilon} \chi_i Z_{ji}\tilde{\phi} = 0 \quad \text{for all } i = 1, \dots, 2(k+l),\ \ j = 0, 1, J_i, \tag{3-10}$$

where $L := -\Delta - W$.

To reach a contradiction it is sufficient to establish the following:

Lemma 3.3. $\tilde{\phi} \to 0$ *uniformly in* Ω_ε.

Lemma 3.4. $d_i \to 0$ *for all* $i = 1, \dots, 2(k+l)$.

We postpone proofs of these lemmas and mention first some key steps.

Lemma 3.5. *For all* $i = 1, \dots, 2(k+l)$ *and* $R > 0$, *we have*

$$\tilde{\phi} \to 0 \quad \textit{uniformly in } \Omega_\varepsilon \cap B_R(\xi_i').$$

Proof. Assume that for some $R > 0$ and $i = 1, \dots, 2(k+l)$ there is a $c > 0$ such that $\sup_{B_R(\xi_i')} |\tilde{\phi}| \geq c > 0$ for a subsequence $\varepsilon \to 0$. Let us translate and rotate Ω_ε so that $\xi_i' = 0$ and Ω_ε approaches the upper half plane $\mathbb{R}^2_+$. By the elliptic estimate, $\tilde{\phi} \to \tilde{\phi}_0$ uniformly on compact sets and $\tilde{\phi}_0$ is a nontrivial bounded solution of (3-4). Then we conclude that $\tilde{\phi}_0$ is a linear combination of z_{ji} for $j = 0, 1, J_i$. On the other hand, we can take the limit in the orthogonality relations (3-10), observing that the limits of the functions Z_{ji} are just rotations and translations of z_{ji}, and we find that $\int_{\mathbb{R}^2_+} \chi\tilde{\phi}_0 z_{ji} = 0$. This contradicts the fact that $\tilde{\phi}_0 \not\equiv 0$. □

Lemma 3.6. $\overline{\tilde{\phi}} \equiv \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \tilde{\phi} \to 0.$

Proof. By potential theory we have

$$\tilde{\phi}(y) - \overline{\tilde{\phi}} = \int_{\Omega_\varepsilon} G(\varepsilon y, \varepsilon z)\Big(W\tilde{\phi} + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i})\Big)dz,$$

where G is the Green function defined by (1-3).

Note that since

$$\int_{\Omega_\varepsilon} W\tilde{\phi} + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i}) = 0$$

and

$$G(\varepsilon y, \varepsilon z) = -\frac{4}{c_i}\log\varepsilon - \frac{4}{c_i}\log|y-z| + H(\varepsilon y, \varepsilon z),$$

we have

$$\begin{aligned} \text{(3-11)} \quad & \tilde{\phi}(y) - \overline{\tilde{\phi}} \\ &= \frac{1}{8\pi}\int_{\Omega_\varepsilon}\Big(H(\varepsilon y, \varepsilon z) - \frac{4}{c_i}\log|y-z|\Big)\Big(W\tilde{\phi} + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i})\Big)dz. \end{aligned}$$

Since $\tilde{\phi}(y) \to 0$ uniformly on sets of the form $|y-\xi_i'| < R$, we can select a sequence $R_\varepsilon \to \infty$ such that

$$\tilde{\phi}(y) \to 0 \quad \text{uniformly for } |y-\xi_i'| < R_\varepsilon.$$

We can assume $R_\varepsilon \to \infty$ as slowly as we need.

Select a point $y_m \in \Omega_\varepsilon$ for $m = 1, \dots, 2k$ or $y_m \in \partial\Omega_\varepsilon$ for $m = 2k+1, \dots, 2(k+l)$, such that $|y_m - \xi_m'| = R_\varepsilon$. We claim that when we evaluate (3-11) at y_m, all terms in the right side of (3-11) converge to zero except for

$$\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i})dz = \frac{2\pi}{\mu_i}\delta_{mi} + o(1),$$

where δ_{mi} is Kronecker's delta.

Claim 1. $\displaystyle\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i})dz = \frac{2\pi}{\mu_i}\delta_{mi} + o(1).$

This is proved in Appendix B.

Claim 2. $\displaystyle\int_{\Omega_\varepsilon} \log|y-z| h(z)dz = o(1)$ *uniformly for* $y \in \Omega_\varepsilon$.

Proof. Observe that $\log|y-z| = O(\log\varepsilon^{-1})$ for $|y-z|>R$, where $R>0$ is fixed, and that $\int_{\Omega_\varepsilon\cap B_R(y)}\big|\log|y-z|\big|dz \le C$. Then

$$\Big|\int_{\Omega_\varepsilon}\log|y-z|h\,dz\Big| \le C\log\varepsilon^{-1}\|h\|_* = o(1). \qquad \square$$

Claim 3. $\displaystyle\int_{\Omega_\varepsilon}\log|y-z|W\tilde\phi\,dz = o(1).$

Proof. It suffices to show that $\log\varepsilon^{-1}\int_{\Omega_\varepsilon}W\tilde\phi\,dz = o(1)$. Integrating equation (3-9), we have

$$\int_{\Omega_\varepsilon} W\tilde\phi + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde Z_{0i}) = 0.$$

The claim then follows from (B-10) and (3-7). $\square$

Claim 4. $A \equiv \displaystyle\int_{\Omega_\varepsilon} H(\varepsilon y,\varepsilon z)(W\tilde\phi + h - L(\tilde Z_{0i})) = o(1)$ *uniformly for* $y\in\Omega_\varepsilon$.

This is proved in Appendix B.

We now return to the proof of Lemma 3.6. From claims above, we get

$$\tilde\phi(y_i) - \overline{\tilde\phi} = \frac{8\pi d_i}{c_i\mu_i} + o(1) \quad \text{for all } i = 1,\dots,2(k+l). \tag{3-12}$$

But the orthogonality condition (3-3) implies that

$$\sum_{i=1}^{2(k+l)} d_i a_i = 0, \quad \text{where } a_i = \int_{\Omega_\varepsilon}\chi_i Z_{0i}^2 > 0. \tag{3-13}$$

Multiplying (3-12) by $c_i a_i\mu_i$, adding and using (3-13), we find

$$\sum_{i=1}^{2(k+l)} c_i\mu_i a_i\tilde\phi(y_i) - a\overline{\tilde\phi} = o(1), \quad \text{where } a = \sum_{i=1}^{2(k+l)} c_i a_i\mu_i.$$

Since $\tilde\phi(y_i)\to 0$ and a is bounded away from zero, we get that $\overline{\tilde\phi} = o(1)$. $\square$

Proof of Lemma 3.3. Let $\check\phi = \tilde\phi(x/\varepsilon)$, with $x\in\Omega$. Then $\check\phi$ satisfies

$$\begin{cases} -\Delta\check\phi(x) = \varepsilon^{-2}\big(\check W\check\phi + h + \sum_{i=1}^{2(k+l)} d_i(\Delta\check Z_{0i} + \check W\check Z_{0i})\big) & \text{in } \Omega,\\ \partial\check\phi/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\check W(x) = W(x/\varepsilon)$, $\check Z_{0i}(x) = \tilde Z_{0i}(x/\varepsilon)$ and $\check h(x) = h(x/\varepsilon)$. For given $\delta>0$, let $E_\delta = \Omega\setminus\bigcup_{i=1}^{2(k+l)} B_\delta(\xi_i)$. Then

$$\frac{1}{\varepsilon^2}\|\check h\|_{L^\infty(E_\delta)} \le C\|h\|_* \to 0 \quad\text{and}\quad \frac{1}{\varepsilon^2}\|\check W\check\phi\|_{L^\infty(E_\delta)} \le C\varepsilon^2.$$

Furthermore, in E_δ we have $\check{Z}_{0i} \equiv 0$. Recalling $\|\check{\phi}\|_{L^\infty(\Omega)} \le 1$ and $|\Omega|^{-1} \int_\Omega \check{\phi} \to 0$, we obtain $\check{\phi} \to 0$ uniformly in E_δ and this implies

$$\tilde{\phi} \to 0 \quad \text{uniformly in } \Omega_\varepsilon \setminus \textstyle\bigcup_{i=1}^{2(k+l)} B_{\delta/\varepsilon}(\xi_i') \quad \text{for any } \delta > 0.$$

For a given $R_1 > 0$, let $A_i = B_{\delta/\varepsilon}(\xi_i') \setminus B_{R_1}(\xi_i')$. Given $\varepsilon > 0$ small enough, there exist $R_1 > 1$ independent of ε (if necessary we can choose R_1 large enough) and $\psi_i : \Omega_\varepsilon \cap A_i \to \mathbb{R}$ smooth and positive such that

$$\begin{cases} -\Delta\psi_i - W\psi_i \ge C|y-\xi_i'|^{-2-\sigma} + \varepsilon^2 & \text{in } \Omega_\varepsilon \cap A_i\ , \\ \partial\psi_i/\partial\nu \ge 0 & \text{on } \partial\Omega_\varepsilon \cap A_i, \\ \psi_i > 0 & \text{in } \Omega_\varepsilon \cap A_i, \\ \psi_i \ge c > 0 & \text{on } \partial A_i \cap \Omega_\varepsilon, \end{cases}$$

where $C, c > 0$ can be chosen independent of ε and ψ_i is bounded uniformly in $\Omega_\varepsilon \cap A_i$. Let Ψ_0 be the unique solution of

$$\Delta\Psi_0 - \varepsilon^4\Psi_0 + \varepsilon^2 = 0 \quad \text{in } \Omega_\varepsilon, \qquad \partial\Psi_0/\partial\nu = \varepsilon \quad \text{on } \partial\Omega_\varepsilon,$$

and take $\psi_{1i} = 1 - r^{-\sigma}$, where $r = |y - \xi_i'|$. Then we claim that the function

$$\psi_i(y) = \frac{4}{\sigma^2}(C\Psi_0 + \psi_{1i})$$

satisfies the requirements.

In fact, a simple calculation shows that

$$-\Delta\psi_{1i} = \sigma^2 r^{-2-\sigma}.$$

If $\xi_i' \in \Omega_\varepsilon$, we have

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = O(\varepsilon^{1+\sigma}) \quad \text{on } \partial\Omega_\varepsilon.$$

If $\xi_i' \in \partial\Omega_\varepsilon$ and $|y - \xi_i'| > R$, we have

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = \sigma\frac{(y-\xi_i')\cdot\nu_\varepsilon}{r^{2+\sigma}} \quad \text{on } \partial\Omega_\varepsilon.$$

As before, we write $\partial\Omega_\varepsilon$ near ξ_i' as the graph $\{(y_1, y_2) \mid y_2 = \varepsilon^{-1}P(\varepsilon y_1)\}$ with $P(0) = P'(0) = 0$. Then we have

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = \frac{\sigma}{r^{2+\sigma}}\frac{y_1P'(\varepsilon y_1) - P(\varepsilon y_1)}{\sqrt{1+P'(\varepsilon y_1)^2}} = \frac{\sigma}{r^{2+\sigma}}\frac{O(\varepsilon r^2)}{\sqrt{1+O(\delta^2)}} = O\Big(\frac{\varepsilon}{r^\sigma}\Big)$$

for all $R < r < \delta/\varepsilon$. Thus we see that

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = o(\varepsilon) \quad \text{on } \partial\Omega_\varepsilon.$$

Therefore, for $|y-\xi_i'|>R$ with $i=1,\dots,2(k+l)$, where R is large, we have by the definition of ψ_i and the fact that $W\le 1/(1+|y-\xi_i'|^4)$ that

$$-\Delta\psi_i - W\psi_i = \frac{C}{\sigma^2}(\varepsilon^2-\varepsilon^4\Psi_0) - \frac{4}{\sigma^2}\frac{C\Psi_0+\psi_{1i}}{1+r^4} + \frac{C}{r^{2+\sigma}} \ge \varepsilon^2 + \frac{C}{r^{2+\sigma}}.$$

And on $\partial\Omega_\varepsilon$,

$$\frac{\partial\psi_i}{\partial\nu_\varepsilon} \ge C\varepsilon.$$

This verifies the claim.

Thanks to the barrier ψ_i, we deduce that the following maximum principle holds in $\Omega_\varepsilon\cap A_i$. If $\phi\in H^1(\Omega_\varepsilon\cap A_i)$ satisfies

$$\begin{cases} -\Delta\phi - W\phi \ge 0 & \text{in } \Omega_\varepsilon\cap A_i, \\ \phi\ge 0 & \text{on } \partial\Omega_\varepsilon\cap A_i, \end{cases}$$

then $\phi\ge 0$ in $\Omega_\varepsilon\cap A_i$.

Let h be bounded and $\tilde\phi$ be a solution of (3-9) satisfying (3-10). We first claim that $\|\tilde\phi\|_{L^\infty(\Omega_\varepsilon\cap A_i)}$ can be controlled in terms of

$$\sum_{i=1}^{2(k+l)} |d_i|\,\|L(\tilde Z_{0i})\|_*, \qquad \sup_{\Omega_\varepsilon\cap\partial A_i}|\tilde\phi|, \qquad \text{and} \qquad \|h\|_*.$$

Indeed, set

$$\Phi = C\Big(\sup_{\Omega_\varepsilon\cap\partial A_i}|\tilde\phi| + \|h\|_* + \sum_{i=1}^{2(k+l)}|d_i|\,\|L(\tilde Z_{0i})\|_*\Big)\psi_i.$$

By the maximum principle above, we have $|\tilde\phi|\le\Phi$ in $\Omega_\varepsilon\cap A_i$. Since ψ_i is uniformly bounded, we get

$$|\tilde\phi| \le C\Big(\sup_{\Omega\varepsilon\cap\partial B_{R_1}(\xi_i')}|\tilde\phi| + \sup_{\Omega\varepsilon\cap\partial B_{\delta/\varepsilon}(\xi_i')}|\tilde\phi| + \|h\|_* + \sum_{i=1}^{2(k+l)}|d_i|\,\|L(\tilde Z_{0i})\|_*\Big)$$

in $\Omega_\varepsilon\cap A_i$. But $\|h\|_*=o(1)$ by the assumption, $\sup_{\Omega\varepsilon\cap\partial B_{R_1}}(\xi_i')|\tilde\phi|\to 0$ by Lemma 3.5, and $\sup_{\Omega\varepsilon\cap\partial B_{\delta/\varepsilon}(\xi_i')}|\tilde\phi|\to 0$ as shown above. At the same time, we also know $|d_i|=O(1)$ and $\|L(\tilde Z_{0i})\|_*=O(\varepsilon^{2\gamma})=o(1)$ from (B-10), this proves the result. □

Proof of Lemma 3.4. We take $\tilde Z_{0i}$ as test function to (3-9), obtaining

$$\sum_{i=1}^{2(k+l)} d_i\int_{\Omega_\varepsilon} L(\tilde Z_{0i})\tilde Z_{0i} = \int_{\Omega_\varepsilon}\tilde\phi(\Delta\tilde Z_{0i}+W\tilde Z_{0i}) + \int_{\Omega_\varepsilon} h\tilde Z_{0i}. \tag{3-14}$$

Observe that

$$\Big|\int_{\Omega_\varepsilon} \tilde{Z}_{0i} h\Big| \le \|h\|_* \|\tilde{Z}_{0i}\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1} \|h\|_* \frac{1}{\log \varepsilon^{-1}} = o(1) \frac{1}{\log \varepsilon^{-1}}, \tag{3-15}$$

and

$$\Big|\int_{\Omega_\varepsilon} \tilde{\phi}(\Delta \tilde{Z}_{0i} + W \tilde{Z}_{0i})\Big| \le \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \|L(\tilde{Z}_{0i})\|_* = o(1) \frac{1}{\log \varepsilon^{-1}}. \tag{3-16}$$

It is not difficult to show as above that

$$\Big|\int_{\Omega_\varepsilon} L(\tilde{Z}_{0i}) \tilde{Z}_{0i}\Big| \ge \frac{C}{\log \varepsilon^{-1}}. \qquad \square$$

Proof of Proposition 3.1. First we prove that for any ϕ, c_{ji}, c_0 and any solution to (3-1), we have the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1} \|h\|_*. \tag{3-17}$$

From Proposition 3.2, we obtain that

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1} \Big(\|h\|_* + \sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} |c_{ji}| + |c_0| \Big). \tag{3-18}$$

So it suffices to estimate the values of the constants a_{ji} and c_0.

To this end, we multiple (3-1) by Z_{ji} and integrate to find

$$\int_{\Omega_\varepsilon} L(\phi) Z_{ji} = \int_{\Omega_\varepsilon} h Z_{ji} + c_{ji} \int_{\Omega_\varepsilon} \psi_i Z_{ji}^2. \tag{3-19}$$

Note that $Z_{ji} = O(1/(1+|y-\xi_i|))$ for $j \ne 0$, so

$$\int_{\Omega_\varepsilon} h Z_{ji} = O(\|h\|_*) \tag{3-20}$$

and

$$\int_{\Omega_\varepsilon} L(\phi) Z_{ji} = \int_{\Omega_\varepsilon} L(Z_{ji}) \phi + \int_{\partial\Omega_\varepsilon} \frac{\partial Z_{ji}}{\partial \nu} \phi = O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-21}$$

Substituting (3-20) and (3-21) into (3-19), we obtain

$$|C_{ji}| = O(\|h\|_*) + O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-22}$$

On the other hand, multiplying (3-1) by Z we get

$$c_0 \int_{\Omega_\varepsilon} \chi Z^2 = \int_{\Omega_\varepsilon} L(\phi) Z - \int_{\Omega_\varepsilon} h Z. \tag{3-23}$$

Estimating as before, we have

$$\int_{\Omega_\varepsilon} hZ = O(\|h\|_*) \tag{3-24}$$

and

$$\int_{\Omega_\varepsilon} L(\phi)Z = \int_{\Omega_\varepsilon} L(Z)\phi = O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-25}$$

Thus it follows from (3-23)–(3-25) that

$$|c_0| = O(\|h\|_*) + O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-26}$$

From (3-22) and (3-26) we see that the desired bound holds.

Now consider the Hilbert space

$$H = \Big\{\phi \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \chi Z\phi = 0, \int_{\Omega_\varepsilon} \chi_i Z_{ji}\phi = 0 \text{ for } i = 1, \dots, 2(k+l),\ j = 1, J_i\Big\}$$

with the norm $\|\phi\|_H^2 = \int_{\Omega_\varepsilon} |\nabla\phi|^2$. Problem (3-1) is equivalent to finding $\phi \in H$ such that

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla\psi - \int_{\Omega_\varepsilon} W\phi\psi = \int_{\Omega_\varepsilon} h\psi \quad \text{for all } \psi \in H.$$

By Fredholm's alternative, this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by the a priori estimate (3-17). □

Remark. The result of Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3-1) defines a continuous linear map from $L^\infty(\Omega_\varepsilon)$, with norm $\|\cdot\|_*$, into $L^\infty(\Omega_\varepsilon)$. Moreover, the operator T is differential with respect to the variables ξ'_m. In fact, computations similar to those used in [Wei et al. 2011] yield the estimate

$$\|\partial_{\xi'_m}\phi\|_{L^\infty(\Omega_\varepsilon)} \le C(\log \varepsilon^{-1})^2 \|h\|_*. \tag{3-27}$$

4. The nonlinear problem with constraints

Let us introduce a small parameter τ and consider

$$V_1(y) = V(y) + \tau Z(y) \quad \text{for } y \in \Omega_\varepsilon, \tag{4-1}$$

where V and Z are given by (2-16) and (3-5). Then we set

$$W_1 = 2\varepsilon^4 \cosh V_1, \qquad R_1 = \Delta V_1 + 2\varepsilon^4 \sinh V_1$$

and

$$N_1(\phi_1) = 2\varepsilon^4(\sinh(V_1 + \phi_1) - \phi_1 \cosh V_1 - \sinh V_1).$$

Now we consider the following auxiliary nonlinear problem:

$$
\begin{cases}
\Delta\phi_1+W_1\phi_1+R_1+N_1(\phi_1)+\displaystyle\sum_{i=1}^{2(k+l)}\sum_{j=1}^{J_i}c_{ji}\chi_i Z_{ji}+c_0\chi Z=0 & \text{in } \Omega_\varepsilon,\\
\partial\phi_1/\partial\nu=0 & \text{on } \partial\Omega_\varepsilon,\\
\displaystyle\int_{\Omega_\varepsilon}\chi Z\phi_1=0,\quad \int_{\Omega_\varepsilon}\chi_i Z_{ji}\phi_1=0 & \\
\qquad\qquad\text{for all } i=1,\dots,2(k+l),\ j=1,J_i.
\end{cases}
\tag{4-2}
$$

Then we can follow the proofs [Wei et al. 2011, Lemma 4.1 and Theorem 4.2] to obtain the following results; we omit the details.

Lemma 4.1. *Let $k+l\ge 1$, $d>0$, $\alpha\in(0,1)$ and $\tau=O(\varepsilon^\theta)$ with $\theta>\alpha/2$. Then there exist $\varepsilon_0>0$ and $C>0$ such that for $0<\varepsilon<\varepsilon_0$ and for any $\xi_1,\dots,\xi_{2(k+l)}\in\mathcal{M}_d$, problem* (4-2) *admits a unique solution ϕ_1, c_0, c_{ji} for $i=1,\dots,2(k+l)$, $j=1,J_i$, such that*

$$\|\phi_1\|_{L^\infty(\Omega_\varepsilon)}\le C\varepsilon^\alpha. \tag{4-3}$$

Furthermore, the function $(\tau,\xi')\to\phi_1(\tau,\xi')\in C(\overline{\Omega}_\varepsilon)$ is C^1 and

$$
\begin{aligned}
\|D_{\xi'}\phi_1\|_{L^\infty(\Omega_\varepsilon)}&\le C|\log\varepsilon|^2(\varepsilon+\varepsilon^{2\theta}+\varepsilon^{2\alpha}),\\
\|D_\tau\phi_1\|_{L^\infty(\Omega_\varepsilon)}&\le C(\varepsilon^\alpha+\varepsilon^\theta)|\log\varepsilon|.
\end{aligned}
\tag{4-4}
$$

Lemma 4.2. *Let $k+l\ge 1$ and $d>0$. For any $0<\alpha<1$ there exist $\varepsilon_0>0$ and $C>0$ such that for $0<\varepsilon<\varepsilon_0$ and any $(\xi_1,\dots,\xi_{2(k+l)})\in\mathcal{M}_d$, there exists a unique τ with $|\tau|=O(\varepsilon^\alpha)$ such that problem* (4-2) *admits a unique solution ϕ, c_0, c_{ji} for $i=1,\dots,2(k+l)$, $j=1,J_i$ with $c_0=0$ and such that*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)}\le C\varepsilon^\alpha. \tag{4-5}$$

Furthermore, the function $\xi'\mapsto\phi(\xi')$ is C^1 and

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)}\le C\varepsilon^\alpha|\log\varepsilon|^2.$$

5. Variational reduction and expansion of the energy

In view of Lemmas 4.1 and 4.2, given $\xi=(\xi_1,\dots,\xi_{2(k+l)})\in\mathcal{M}_d$, we set $\phi(\xi)$ and $c_{ji}(\xi)$ to be the unique solution to (4-2) with $c_0=0$ satisfying the bounds (4-3) and (4-4). Let

$$J_\varepsilon(v)=\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla v|^2dx-2\varepsilon^4\int_{\Omega_\varepsilon}\cosh v\,dx$$

and define

$$F_\varepsilon(\xi)=J_\varepsilon(V_1(\xi')+\phi(\xi')), \tag{5-1}$$

where $\xi' = \xi/\varepsilon$ and $V_1(\xi') = V(\xi') + \tau(\xi')Z(\xi')$ with $\tau(\xi)$ given by Lemma 4.2.

Lemma 5.1. *If $\xi = (\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$ is a critical point of F_ε, then*

$$v = V_1(\xi') + \phi(\xi')$$

is a critical point of J_ε, that is, a solution to (2-14).

Proof. A direct computation gives

$$\frac{\partial F_\varepsilon}{\partial \xi_m} = \varepsilon^{-1} \frac{\partial J_\varepsilon(V_1(\xi') + \phi(\xi'))}{\partial \xi'_m} = \varepsilon^{-1} D J_\varepsilon(V_1(\xi') + \phi(\xi'))\Big(\frac{\partial V_1(\xi')}{\partial \xi'_m} + \frac{\partial \phi(\xi')}{\partial \xi'_m}\Big).$$

Since $V_1(\xi') + \phi(\xi')$ solves (4-2) with $c_0 = 0$, we have

$$\frac{\partial F_\varepsilon}{\partial \xi_m} = \varepsilon^{-1} \sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji} \int_{\Omega_\varepsilon} \chi_i Z_{ji}\Big(\frac{\partial V_1(\xi')}{\partial \xi'_m} + \frac{\partial \phi(\xi')}{\partial \xi'_m}\Big).$$

From the assumption $DF_\varepsilon(\xi) = 0$, we obtain

$$\sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji} \int_{\Omega_\varepsilon} \chi_i Z_{ji}\Big(\frac{\partial V_1(\xi')}{\partial \xi'_m} + \frac{\partial \phi(\xi')}{\partial \xi'_m}\Big) = 0 \quad \text{for all } m = 1, \dots, 2(k+l).$$

Since

$$\|\partial_{\xi'_m} \phi(\xi')\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^\alpha |\log \varepsilon|^2 \quad \text{and} \quad \partial_{\xi'_m} V(\xi') = (-1)^m Z_{jm} + o(1)$$

for $j = 1, J_i$, where $o(1)$ is in the L^∞-norm as a direct consequence of (4-1), it follows that

$$\sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji} \int_{\Omega_\varepsilon} \chi_i Z_{ji}((-1)^m Z_{jm} + o(1)) = 0 \quad \text{for all } m = 1, \dots, 2(k+l),$$

which is a strictly diagonal dominant system. This implies that $c_{ji} = 0$ for all $i = 1, \dots, 2(k+l)$, $j = 1, J_i$. □

A key step in seeking the critical points of the functional F_ε is finding its expected closeness to the functional $J_\varepsilon(V_1(\xi))$. The procedure is completely similar to that of [Wei et al. 2011, Theorem 5.2], so we omit it here.

Lemma 5.2. *The expansion*

$$F_\varepsilon(\xi) = J_\varepsilon(V) + \theta_\varepsilon(\xi)$$

holds with $|\theta_\varepsilon(\xi)| + |\nabla \theta_\varepsilon(\xi)| = o(1)$ uniformly on points in $\mathcal{M}_d$.

Now we will give an asymptotic estimate of $J_\varepsilon(V)$, where V is defined by (2-16) and J_ε is given as above.

Lemma 5.3. *Let $k+l \geq 1$, let $d > 0$, let μ_i be given by* (2-21) *and let V be the function defined in* (2-16). *Then the expansion*

$$J_\varepsilon(V) = -\frac{1}{2}\sum_{i=1}^{2(k+l)} c_i\Big(c_i H(\xi_i,\xi_i) + \sum_{j,j\neq i}(-1)^{j+i} c_j G(\xi_j,\xi_i)\Big) + 2\sum_{i=1}^{2(k+l)} c_i \log\varepsilon^{-1} + \sum_{i=1}^{2(k+l)} c_i(\log 8 - 2) + O(\varepsilon^\alpha). \tag{5-2}$$

holds uniformly on points $\xi = (\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$.

Proof. Recall the definition of $V(y) = \sum_{i=1}^{2(k+l)}(-1)^{i-1}(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y))$. We find that it satisfies

$$\begin{cases} -\Delta V = \varepsilon^4 \displaystyle\sum_{i=1}^{2(k+l)}(-1)^{i-1}\Big(e^{u_i(\varepsilon y)} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} e^{u_i(\varepsilon y)}\Big) & \text{in } \Omega_\varepsilon, \\ \partial V/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{5-3}$$

We will compute the two terms in $J_\varepsilon(V)$.

First, by (5-3) we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla V|^2 &= \int_{\Omega_\varepsilon}(-\Delta V)V \\ &= \int_{\Omega_\varepsilon}\Big(\varepsilon^4 \sum_{j=1}^{2(k+l)}(-1)^{j-1}\Big(e^{u_j(\varepsilon y)} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} e^{u_j(\varepsilon y)}\Big)\Big) \\ &\qquad \times\Big(\sum_{i=1}^{2(k+l)}(-1)^{i-1}\Big(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)\Big)\Big) \\ &= \varepsilon^4 \sum_{j,i}(-1)^{j+i}\int_{\Omega_\varepsilon}\Big(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)\Big)e^{u_j(\varepsilon y)} \\ &\quad - \frac{\varepsilon^4}{|\Omega_\varepsilon|}\Big(\sum_{j=1}^{2(k+l)}(-1)^{j-1}\int_{\Omega_\varepsilon} e^{u_j(\varepsilon y)}\Big)\Big(\int_{\Omega_\varepsilon}\sum_{i=1}^{2(k+l)}(-1)^{i-1}\Big(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)\Big)\Big) \\ &= \varepsilon^4 \sum_{j,i}(-1)^{j+i}\int_{\Omega_\varepsilon}(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y))e^{u_j(\varepsilon y)} + O(\varepsilon), \end{aligned}$$

where the last equality is due to the fact $\varepsilon^4 \sum_{j=1}^{2(k+l)}(-1)^{j-1}\int_{\Omega_\varepsilon} e^{u_j(\varepsilon y)} = O(\varepsilon^4)$, which can be easily deduced from (2-7).

For $j \neq i$, we have by a calculation similar to (2-23)

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \varepsilon^4 (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_j(\varepsilon y)} \\
&= \Big(\int_{\Omega_\varepsilon^1} + \int_{\Omega_\varepsilon^2} \Big) (\varepsilon^4 (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_j(\varepsilon y)}) \\
&= \int_{\Omega_\varepsilon^1|_{\xi_j'=0}} \frac{8}{(1+y^2)^2} \big(\log|\xi_i - \xi_j|^{-4} + c_i H(\xi_j, \xi_i) \big) + O(\varepsilon^\alpha) \\
&= c_j c_i G(\xi_j, \xi_i) + O(\varepsilon^\alpha).
\end{aligned} \tag{5-4}$$

where $\Omega_\varepsilon^1 := B_{\delta/(\varepsilon\mu_j)}(\xi_j') \cap (\Omega_\varepsilon/\mu_i)$ and $\Omega_\varepsilon^2 := (\Omega_\varepsilon/\mu_i) \setminus \Omega_\varepsilon^1$. For $j = i$, we have

$$\begin{aligned}
&\varepsilon^4 \int_{\Omega_\varepsilon} (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_i(\varepsilon y)} \\
&\quad = \int_{\Omega_\varepsilon} \frac{8\mu_i^2}{(\mu_i^2 + |y - \xi_i'|^2)^2} \Big(\log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + |\varepsilon y - \xi_i|^2)^2} + c_i H(\xi_i, \xi_i) \\
&\qquad\qquad - \log(8\mu_i^2) + O(\varepsilon^\alpha) + O(\varepsilon|y - \xi_i'|) \Big) \\
&\quad = 4c_i \log \varepsilon^{-1} + c_i (c_i H(\xi_i, \xi_i) - 2 \log 8\mu_i^2) + 2c_i(\log 8 - 1) + O(\varepsilon^\alpha).
\end{aligned}$$

So from the choice of μ_i (see (2-21)), we get

$$\begin{aligned}
\varepsilon^4 \int_{\Omega_\varepsilon} (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_i(\varepsilon y)} &= 4c_i \log \varepsilon^{-1} + 2c_i(\log 8 - 1) \\
&\quad - c_i \Big(c_i H(\xi_i, \xi_i) + 2 \sum_{m,\, m \neq i} (-1)^{m+i} c_m G(\xi_m, \xi_i) \Big) + O(\varepsilon^\alpha).
\end{aligned} \tag{5-5}$$

Combining (5-4) and (5-5), we have

$$\begin{aligned}
\frac{1}{2} \int_{\Omega_\varepsilon} |\nabla V|^2 = -\frac{1}{2} \sum_{i=1}^{2(k+l)} c_i \Big(c_i H(\xi_i, \xi_i) + \sum_{j,\, j \neq i} (-1)^{j+i} c_j G(\xi_j, \xi_i) \Big) \\
+ 2 \sum_{i=1}^{2(k+l)} c_i \log \varepsilon^{-1} + (\log 8 - 1) \sum_{i=1}^{2(k+l)} c_i + O(\varepsilon^\alpha).
\end{aligned} \tag{5-6}$$

Next, let us compute the second term in $J_\varepsilon(V)$. Let $\Omega_i^1 = B_{\delta/\varepsilon}(\xi_i') \cap (\Omega_\varepsilon/\mu_i)$. Then

$$2\varepsilon^4 \int_{\Omega_\varepsilon} \cosh V = 2\varepsilon^4 \sum_{i=1}^{2(k+l)} \int_{\Omega_i^1} \cosh V + O(\varepsilon^2).$$

Suppose first i is odd. Then

$$
\begin{aligned}
2\varepsilon^4 \int_{\Omega_i^1} \cosh V &= \varepsilon^4 \int_{\Omega_i^1} e^V + O(\varepsilon) \\
&= \int_{\Omega_i^1} \varepsilon^4 e^{u_i(\varepsilon y)} \exp\Big(H_i^\varepsilon + \sum_{m \neq i} (-1)^{m-1} (u_m + H_m^\varepsilon) \Big) + O(\varepsilon) \\
&= c_i + O(\varepsilon).
\end{aligned}
$$

Therefore

$$
2\varepsilon^4 \int_{\Omega_i^1} \cosh V = c_i + O(\varepsilon). \tag{5-7}
$$

Similarly for i even, we also have (5-7). So we obtain

$$
2\varepsilon^4 \int_{\Omega_\varepsilon} \cosh V = \sum_{i=1}^{2(k+l)} c_i + O(\varepsilon). \tag{5-8}
$$

Finally, from (5-6) and (5-8) we conclude that (5-2) holds. □

6. Proof of main theorems

Proof of Theorem 1.2. Let

$$
v(y) = V_1(\xi')(y) + \phi(\xi')(y) \quad \text{for } y \in \Omega_\varepsilon,
$$

where V_1 is given by (4-1) and ϕ is the unique solution to problem (4-2) with $c_0 = 0$, whose existence and properties are established in Lemma 4.2. According to Lemma 4.1, v is a solution to problem (2-14) if we adjust ξ so that it is a critical point of the function $F_\varepsilon(\xi)$ defined in (5-1), or equivalently, so that it is a critical point of

$$
\tilde{F}_\varepsilon(\xi) = 2\Big(2 \sum_{i=1}^{2(k+l)} c_i \log \varepsilon^{-1} + \sum_{i=1}^{2(k+l)} c_i (\log 8 - 2) - F_\varepsilon(\xi) \Big). \tag{6-1}
$$

From Lemmas 5.2 and 5.3 it follows that for $\xi \in \mathcal{M}_d$,

$$
\tilde{F}_\varepsilon(\xi) = \varphi_{2(k+l)}(\xi) + \varepsilon \Theta_\varepsilon(\xi), \tag{6-2}
$$

where Θ_ε and $\nabla_\xi \Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon \to 0$. On the other hand, $\tilde{F}_\varepsilon \to \varphi_{2(k+l)}$ uniformly on compact sets of $\mathcal{M}_d$ as ε goes to 0. Now by Definition 1.1, we deduce that if ε is small enough, there exists a critical point $\xi_\varepsilon \in \mathcal{M}_d$ of $\tilde{F}_\varepsilon$ such that $\tilde{F}_\varepsilon \to \varphi_{2(k+l)}(\xi^*)$. Moreover, up to subsequence, $\xi_\varepsilon \to \xi$ as ε tends to 0, with $\varphi_{2(k+l)}(\xi) = \varphi_{2(k+l)}(\xi^*)$. The function $u_\varepsilon(x) = v(y)$ is therefore a solution to (1-2) with the qualitative properties predicted by the theorem, as can be easily shown. □

Proof of Theorem 1.3. First, we recall here some facts about the regular part of the Green function $H(x, y)$ defined by (1-4). If $y \in \Omega$ is a point close to $\partial\Omega$, we let y^* be its uniquely determined reflection with respect to $\partial\Omega$. Now, we consider the auxiliary function

$$H^*(x, y) = -\frac{1}{2\pi} \log \frac{1}{|x-y^*|},$$

and set

$$\psi(x, y) = H(x, y) - H^*(x, y)$$

Then from the equation corresponding to $H(x, y)$ and the elliptic regularity theory, it is not difficult to verify $\psi(x, y)$ is bounded in $\overline{\Omega} \times \overline{\Omega}$ and hence one can derive the estimates

$$H(x, y) = -\frac{1}{2\pi} \log \frac{1}{|x-y^*|} + O(1) \quad \text{for all } x \in \overline{\Omega} \text{ uniformly.} \tag{6-3}$$

If $y \in \partial\Omega$, note that $H(x, y)$ satisfies

$$\begin{cases} \Delta H(x, y) = \dfrac{1}{|\Omega|} & \text{in } \Omega, \\ \dfrac{\partial H}{\partial \nu}(x, y) = \dfrac{1}{\pi} \dfrac{(x-y) \cdot \nu(x)}{|x-y|^2} & \text{on } \partial\Omega. \end{cases}$$

With this and (2-10), we obtain that $x \mapsto H(x, y) \in C^{1,\alpha}(\overline{\Omega})$. On the other hand, by the continuity of the boundary term with respect to y in $L^\infty(\partial\Omega)$, we can get $H(x, y) \in C(\overline{\Omega}, \partial\Omega)$. In particular, $H(x, x)$ is in $C(\partial\Omega)$.

Now, we prove the result. It suffices to show the existence of critical points of the function $\varphi_{2+2}(\xi_1, \dots, \xi_4)$ in $\mathcal{M}_d$. In this case,

$$\begin{aligned} \varphi_{2+2}(\xi_1, \dots, \xi_4) = 16\pi^2\big(&4H(\xi_1, \xi_1) + 4H(\xi_2, \xi_2) + H(\xi_3, \xi_3) + H(\xi_4, \xi_4) \\ &- 4G(\xi_1, \xi_2) + 2G(\xi_1, \xi_3) - 2G(\xi_1, \xi_4) \\ &- 2G(\xi_2, \xi_3) + 2G(\xi_2, \xi_4) - G(\xi_3, \xi_4)\big). \end{aligned} \tag{6-4}$$

We will look for a solution to problem (1-2) with the concentration points ξ given by

$$\xi_1 = (-\lambda, 0), \quad \xi_2 = (\lambda, 0), \quad \xi_3 = (1, 0), \quad \text{and} \quad \xi_4 = (-1, 0) \quad \text{for } \lambda \in (0, 1).$$

Using results obtained in the previous sections (or from the proof of Theorem 1.2), we reduce the problem of finding solution to (1-2) to that finding critical points of the function $\varphi_{2+2}(\lambda) : (0,1) \to \mathbb{R}$ defined by

$$
\begin{aligned}
\varphi_{2+2}(\lambda) &:= \varphi_{2+2}(\xi(\lambda)) \\
&= 16\pi^2\Big(H(\xi_3,\xi_3)+H(\xi_4,\xi_4)-\frac{4}{\pi}\log\frac{1}{2-\lambda}+O(1) \\
&\qquad -\frac{2}{\pi}\log\frac{1}{2\lambda}-\frac{4}{\pi}\log\frac{1}{1-\lambda}+\frac{4}{\pi}\log\frac{1}{1+\lambda}-\frac{1}{\pi}\log\frac{1}{2} \\
&\qquad -H(\xi_1,\xi_2)+H(\xi_1,\xi_3)-H(\xi_1,\xi_4)-H(\xi_2,\xi_3)+H(\xi_2,\xi_4)-H(\xi_3,\xi_4)\Big) \\
&= 32\pi(2\log(2-\lambda)+\log\lambda+2\log(1-\lambda)-2\log(1+\lambda))+O(1).
\end{aligned}
$$

Here, we have used the fact that $H(x,y) \in C(\bar{B}, \partial B)$ and (6-3). Now there exists a $\lambda_0 \in (0,1)$ such that $\varphi_{2+2}(\lambda_0) = \max_{\lambda\in(0,1)} \varphi_{2+2}(\lambda)$, since $\lim_{\lambda\to 0^+} \varphi_{2+2}(\lambda) = \lim_{\lambda\to 1^-} \varphi_{2+2}(\lambda) = -\infty$. Then λ_0 is a C^0-stable critical point of φ_{2+2}, and so the function $\tilde{F}_\varepsilon(\xi)$ defined by (6-1) has a critical point. This proves our result. □

Appendix A.

Proof of (2-22) *and* (2-23). By Lemma 2.1 and the fact that H is C^1 in $\overline{\Omega}$, we have

$$
\begin{aligned}
H_j^\varepsilon(\varepsilon y) &= c_j H(\varepsilon y, \xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha) \\
&= c_j H(\xi_i, \xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha) + O(\varepsilon|y-\xi_i'|).
\end{aligned}
$$

Let us fix a small constant $\delta > 0$. For $|y-\xi_i'| \le \delta/\varepsilon$,

$$
\begin{aligned}
(-1)^{i-1}H_i^\varepsilon(\varepsilon y) &+ \sum_{j\neq i}(-1)^{j-1}\Big(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_i^2+|\varepsilon y-\varepsilon\xi_j'|^2)^2}+H_j^\varepsilon(\varepsilon y)\Big) \\
&= (-1)^{i-1}\big(c_i H(\xi_i,\xi_i)-\log(8\mu_i^2)\big) \\
&\qquad + \sum_{j\neq i}(-1)^{j-1}\Big(\log\frac{8\mu_j^2}{|\xi_i-\xi_j|^4}+c_j H(\xi_i,\xi_j)-\log(8\mu_j^2)\Big) \\
&\qquad\qquad + O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|) \\
&= (-1)^{i-1}(c_i H(\xi_i,\xi_i)-\log(8\mu_i^2)) \\
&\qquad\qquad + \sum_{j\neq i}(-1)^{j-1}c_j G(\xi_i,\xi_j)+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)
\end{aligned}
$$

which is equal to $O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)$; here first equality follows because

$$
\begin{aligned}
\varepsilon^2\mu_j^2+\varepsilon^2|y-\xi_j'|^2 &= \big(|\xi_j-\xi_i|+O(|\varepsilon y-\xi_i|)\big)^2+\varepsilon^2\mu_j^2 \\
&= |\xi_j-\xi_i|^2\Big(1+O\Big(\frac{|\varepsilon y-\xi_i|^2}{|\xi_j-\xi_i|^2}\Big)+\frac{\varepsilon^2\mu_j^2}{|\xi_j-\xi_i|^2}\Big) \\
&= |\xi_j-\xi_i|^2\big(1+O(\varepsilon^2|y-\xi_i'|^2)+O(\varepsilon^2)\big).
\end{aligned}
$$

First, we estimate W. For $|y-\xi_i'|\le \delta/\varepsilon$, a direct computation shows

$$\begin{aligned}
W &= 2\varepsilon^4\cosh V\\
&= \varepsilon^4\exp\Bigl(\sum_{i=1}^{2(k+l)}(-1)^{i-1}(u_i+H_i^\varepsilon)\Bigr)+\varepsilon^4\exp\Bigl(\sum_{i=1}^{2(k+l)}(-1)^{i}(u_i+H_i^\varepsilon)\Bigr)\\
&= \varepsilon^4\Bigl(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Bigr)^{(-1)^{i-1}}\\
&\quad\times\exp\Bigl((-1)^{i-1}H_i^\varepsilon(\varepsilon y)+\sum_{j\ne i}(-1)^{j-1}\Bigl(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+\varepsilon^2|y-\xi_j'|^2)^2}+H_j^\varepsilon(\varepsilon y)\Bigr)\Bigr)\\
&\quad+\varepsilon^4\Bigl(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Bigr)^{(-1)^{i}}\\
&\qquad\times\exp\Bigl((-1)^{i}H_i^\varepsilon(\varepsilon y)+\sum_{j\ne i}(-1)^{j}\Bigl(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+\varepsilon^2|y-\xi_j'|^2)^2}+H_j^\varepsilon(\varepsilon y)\Bigr)\Bigr)\\
&= \varepsilon^4\Bigl(\Bigl(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Bigr)^{(-1)^{i-1}}+\Bigl(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Bigr)^{(-1)^{i}}\Bigr)\\
&\qquad\qquad\times\exp\Bigl[O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\Bigr]\\
&= \frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}\bigl(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\bigr)+O(\varepsilon^4).
\end{aligned}$$

Therefore

$$W(y)=\frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|))\quad\text{for all } |y-\xi_i'|<\delta/\varepsilon. \tag{A-1}$$

Similarly, for $|y-\xi_i'|<\delta/\varepsilon$ we have

$$\begin{aligned}
&2\varepsilon^4\sinh V\\
&\quad= \varepsilon^4\Bigl(\Bigl(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Bigr)^{(-1)^{i-1}}-\Bigl(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Bigr)^{(-1)^{i}}\Bigr)\\
&\qquad\qquad\times\exp\bigl(O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\bigr)\\
&\quad= (-1)^{i-1}\frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}\bigl(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\bigr)+O(\varepsilon^4).
\end{aligned}\tag{A-2}$$

On the other hand, for $|y-\xi_i'|\ge\delta/\varepsilon$, it is easy to see that $W(y)=O(\varepsilon^4)$ and $2\varepsilon^4\sinh V=O(\varepsilon^4)$. This, together with (A-1), implies (2-23) and (2-24).

Next, by our definitions,

$$\begin{aligned}\Delta V &= \sum_{i=1}^{2(k+l)} (-1)^{i-1}\big(\varepsilon^2 \Delta u_i(\varepsilon y) + \varepsilon^2 \Delta H_i^{\varepsilon}(\varepsilon y)\big)\\ &= \sum_{i=1}^{2(k+l)} (-1)^{i-1}\Big(-\varepsilon^4 e^{u_i(\varepsilon y)} + \frac{\varepsilon^4}{|\Omega|}\int_\Omega e^{u_i(x)}dx\Big)\\ &= \sum_{i=1}^{2(k+l)} (-1)^{i-1}\Big(-\frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}\Big) + \sum_{i=1}^{2(k+l)} (-1)^{i-1}\frac{\varepsilon^4}{|\Omega|}\int_\Omega e^{u_i(x)}dx.\end{aligned}$$

The last term in the above equality can be controlled by $O(\varepsilon^4)$ since from (2-7), we have

$$\varepsilon^2 \sum_{i=1}^{2(k+l)} (-1)^{i-1}\int_\Omega e^{u_i} = O(\varepsilon^2|\mu_i - \mu_j|),$$

Combining this with (A-2), we get (2-22). □

Appendix B.

Proof of Claim 1. Since $\eta'(r)$ has a jump at $r = \varepsilon^{-\gamma}$ and $r = \varepsilon^{-\beta}$ and is otherwise smooth, we see that $L(\tilde{Z}_{0i})$ is a measure.

$$\begin{aligned}L(\tilde{Z}_{0i}) &= (-\Delta - W)\big(\eta_{1i}Z_{0i} + \varepsilon(1-\eta_{1i})\eta_{2i}\hat{Z}_{0i}\big)\\ &= -(Z_{0i} - \varepsilon\eta_{2i}\hat{Z}_{0i})\big([\eta_{1i}'(\varepsilon^{-\gamma})]\mu_{\varepsilon^{-\gamma}} + [\eta_{1i}'(\varepsilon^{-\beta})]\mu_{\varepsilon^{-\beta}}\big)\\ &\quad - 2\nabla\eta_{1i}(\nabla Z_{0i} - \varepsilon\hat{Z}_{0i}\nabla\eta_{2i} - \varepsilon\eta_{2i}\nabla\hat{Z}_{0i}) - \eta_{1i}(\Delta Z_{0i} + WZ_{0i})\\ &\quad - \varepsilon(1-\eta_{1i})(\hat{Z}_{0i}\Delta\eta_{2i} + \eta_{2i}\Delta\hat{Z}_{0i} + 2\nabla\eta_{2i}\nabla\hat{Z}_{0i} + W\eta_{2i}\hat{Z}_{0i})\end{aligned}$$

where $[\eta_{1i}'(r)] = \eta_{1i}'(r^+) - \eta_{1i}'(r^-)$ denotes the jump of η_{1i}' at r, and μ_r is the 1-dimensional measure on the circle of radius r.

Let us consider first the case $m = i$:

$$\text{(B-1)}\quad \int_{\Omega_\varepsilon} \log|y_i - z| L(\tilde{Z}_{0i}) = \int_{\Omega_\varepsilon} (\log|y_i - z| - \log|\xi_i' - z|)L(\tilde{Z}_{0i})dz + \int_{\Omega_\varepsilon} \log|\xi_i' - z| L(\tilde{Z}_{0i})dz.$$

Let $r = |z - \xi_i'|$, and note that $\Delta\eta_{2i} = O(\varepsilon^{2\beta})$ and $\nabla\eta_{2i} = O(\varepsilon^{\beta})$. For $r < \varepsilon^{-\beta}$, we have

$$\text{(B-2)}\quad \begin{aligned}\eta_{1i}(\Delta Z_{0i} + WZ_{0i}) &= \eta_{1i}(\Delta Z_{0i} + e^{v_i}(1+\theta_\varepsilon)Z_{0i})\\ &\le \frac{8\mu_i^2}{(\mu_i^2+|z-\xi_i'|^2)^2} O(\varepsilon^\alpha + \varepsilon|z-\xi_i'|) + O\Big(\frac{\varepsilon^\alpha}{(1+|y-\xi_i'|)^3}\Big).\end{aligned}$$

Thus

$$
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} \eta_{1i}(\Delta Z_{0i} + W Z_{0i}) \log|z-\xi_i'|\Big| \\
&\quad \le \int_{\Omega_\varepsilon} \eta_{1i}\Big(\frac{8\mu_i^2 O(\varepsilon^\alpha + \varepsilon|z-\xi_i'|)}{(\mu_i^2+|z-\xi_i'|^2)^2} + O\Big(\frac{\varepsilon^\alpha}{(1+|y-\xi_i'|)^3}\Big)\Big) \log|z-\xi_i'| \\
&\quad \le C\int_0^{\varepsilon^{-\beta}} \Big(\frac{\varepsilon^\alpha}{(1+r)^3} + \frac{\varepsilon^\alpha+\varepsilon r}{(1+r^2)^2}\Big) r\log r\,dr \\
&\quad = O\Big((\varepsilon^\alpha + \varepsilon^{1-\beta})\log\varepsilon^{-1}\Big) \\
&\quad = o(1).
\end{aligned}
\tag{B-3}
$$

For $\varepsilon^{-\gamma} < r < \varepsilon^{-\beta}$,

$$
\begin{aligned}
\frac{1}{\mu_i} - a_{0i} G(\varepsilon z, \xi_i) &= \frac{1}{\mu_i} - \frac{4\log\varepsilon^{-1} - 4\log|z-\xi_i'| + c_i H(\varepsilon z, \xi_i)}{\mu_i[4(1-\gamma)\log\varepsilon^{-1} + c_i H(\xi_i,\xi_j)]} \\
&= \frac{\log r - \gamma\log\varepsilon^{-1} + \varepsilon r}{(1-\gamma)\mu_i\log\varepsilon^{-1}}(1+O(\varepsilon)).
\end{aligned}
\tag{B-4}
$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} (1-\eta_{1i}) W(\mu_i^{-1} - a_{0i} G) \log|z-\xi_i'|\,dz \\
&\qquad = \int_{r>\varepsilon^{-\gamma}} O\Big(\frac{\log r - \gamma\log\varepsilon^{-1} + \varepsilon r}{(1-\gamma)\mu_i\log\varepsilon^{-1}}\Big) O(r^{-4r}) \log r\,dr \\
&\qquad = O(\varepsilon^{2\gamma}\log\varepsilon^{-1})
\end{aligned}
\tag{B-5}
$$

and

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla\eta_{1i}(\nabla Z_{0i} - \varepsilon\hat{Z}_{0i}\nabla\eta_{2i} - \varepsilon\eta_{2i}\nabla\hat{Z}_{0i}) \log|z-\xi_i'|\,dz \\
&\quad = 2\pi\int_{\varepsilon^{-\gamma}}^{\varepsilon^{-\beta}} \frac{-r^{-1}}{(\beta-\gamma)\log\varepsilon^{-1}} \\
&\qquad\qquad \times\Big(O(r^{-3}) + O(\varepsilon^{1+\beta}) + O\Big(\frac{\varepsilon}{\log\varepsilon^{-1}}(r^{-1}+C)\Big)\Big) r\log r\,dr \\
&\quad = O(\varepsilon^{2\gamma}) + O(\frac{\varepsilon^{1-\beta}}{\log\varepsilon^{-1}}).
\end{aligned}
\tag{B-6}
$$

For $r > \varepsilon^{-\gamma}$,

$$
\begin{aligned}
&\hat{Z}_{0i}\Delta\eta_{2i} + \eta_{2i}\Delta\hat{Z}_{0i} + 2\nabla\eta_{2i}\nabla\hat{Z}_{0i} + W\eta_{2i}\hat{Z}_{0i} \\
&\qquad = \hat{Z}_{0i}\Delta\eta_{2i} + 2\nabla\eta_{2i}\nabla\hat{Z}_{0i} + \eta_{2i}(\Delta Z_{0i} + W Z_{0i} + a_{0i}\Delta G - W\mu_i^{-1} + W a_{2i} G).
\end{aligned}
$$

So, recalling (B-5), we have

$$
\begin{aligned}
&\varepsilon \int_{\Omega_\varepsilon} (1-\eta_{1i})(\hat{Z}_{0i}\Delta\eta_{2i} + \eta_{2i}\Delta\hat{Z}_{0i} + 2\nabla\eta_{2i}\nabla\hat{Z}_{0i} + W\eta_{2i}\hat{Z}_{0i}) \log|z-\xi_i'|\,dz \\
&= \varepsilon \int_{\varepsilon^{-\beta}}^{2\varepsilon^{-\beta}} O(\varepsilon^{2\beta}) r \log r\,dr + \varepsilon \int_{\varepsilon^{-\beta}}^{2\varepsilon^{-\beta}} O(\varepsilon^{\beta}) O\Big(r^{-3} + \frac{\varepsilon}{\log\varepsilon^{-1}}(C+r^{-1})\Big) r \log r\,dr \\
&\quad + \varepsilon \int_{\varepsilon^{-\gamma}}^{2\varepsilon^{-\beta}} \Big(O\Big(\frac{\varepsilon^\alpha + \varepsilon r}{r^4}\Big) + O\Big(\frac{\varepsilon^\alpha}{(1+r)^3}\Big) + O\Big(\frac{\varepsilon^2}{\log\varepsilon^{-1}}\Big)\Big) r \log r\,dr \\
&\quad - \varepsilon \int_{\Omega_\varepsilon} (1-\eta_{1i}) W(\mu_i^{-1} - a_{0i}G) \log|z-\xi_i'|\,dz,
\end{aligned}
$$

which is equal to $O(\varepsilon\log\varepsilon^{-1})$. A direct computation shows

$$
\begin{aligned}
\int_{\Omega_\varepsilon} [\eta_{1i}'(\varepsilon^{-\gamma})]\mu_{\varepsilon^{-\gamma}} (Z_{0i} - \varepsilon\eta_{2i}\hat{Z}_{0i}) \log|z-\xi_i'|\,dz& \\
&= \frac{-\varepsilon^\gamma}{(\beta-\gamma)\log\varepsilon^{-1}} \int_{r=\varepsilon^{-\gamma}} (Z_{0i} - \varepsilon\hat{Z}_{0i}) \log|z-\xi_i'| \\
&= \frac{-\varepsilon^\gamma}{(\beta-\gamma)\log\varepsilon^{-1}} \times \frac{1+O(\varepsilon^{2\gamma})}{\mu_i} \times 2\pi\varepsilon^{-\gamma}\log\varepsilon^{-\gamma} \\
&= \frac{-2\pi\gamma}{\mu_i(\beta-\gamma)} + O(\varepsilon^{2\gamma}).
\end{aligned}
$$

Similarly,

$$
\int_{\Omega_\varepsilon} [\eta_{1i}'(\varepsilon^{-\beta})]\mu_{\varepsilon^{-\beta}} (Z_{0i} - \varepsilon\eta_{2i}\hat{Z}_{0i}) \log|z-\xi_i'|\,dz = \frac{2\pi\beta}{\mu_i(\beta-\gamma)} + O(\varepsilon^{2\beta}).
$$

Hence

$$
\int_{\Omega_\varepsilon} L(\tilde{Z}_{0i}) \log|z-\xi_i'|\,dz = \frac{2\pi}{\mu_i} + o(1).
$$

For the first integral in the right side of (B-1), we can assume $R_\varepsilon \to +\infty$ slowly enough so that $\varepsilon^\gamma R_\varepsilon \to 0$. Then

$$
\text{(B-7)} \qquad \big|\log|y_i - z| - \log|\xi_i' - z|\big| = \Big|\log\frac{|y_i - z|}{r}\Big| \le \Big|\log\frac{|y_i - \xi_i'| + r}{r}\Big|
$$

for $r = |\xi_i' - z|$; therefore we have from (B-2)

$$
\text{(B-8)} \qquad
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} (\log|y_i - z| - \log|\xi_i' - z|)\eta_{1i}(\Delta Z_{0i} + WZ_{0i})\,dz\Big| \\
&\le C \int_0^{\varepsilon^{-\beta}} \log(R_\varepsilon r^{-1} + 1)\Big(O\Big(\frac{\varepsilon^\alpha + \varepsilon r}{(1+r^2)^2}\Big) + O\Big(\frac{\varepsilon^\alpha}{(1+r)^3}\Big)\Big) r\,dr \\
&= O(\varepsilon^\alpha(R_\varepsilon + \log\varepsilon^{-1})).
\end{aligned}
$$

On the other hand, from (B-7), for $\varepsilon^{-\gamma} \le r = |z-\xi_i'| \le \varepsilon^{-\beta}$ we have

$$\big|\log|y_i - z| - \log|\xi_i' - z|\big| \le C|y_i - \xi_i'|/\varepsilon^{-\gamma}$$

and it follows that

$$\Big|\int_{\Omega_\varepsilon} (\log|y_i - z| - \log|\xi_i' - z|)\big(L(\tilde{Z}_{0i}) + \eta_{1i}(\Delta Z_{0i} + WZ_{0i})\big)dz\Big| = O(\varepsilon^\gamma R_\varepsilon).$$

Thus, from this and (B-8), we obtain

$$\Big|\int_{\Omega_\varepsilon} \Big(\log|y_i - z| - \log|\xi_i' - z|\Big) L(\tilde{Z}_{0i})\Big| = o(1). \tag{B-9}$$

Next, we show that if $m \neq i$, then

$$\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i}) dz = o(1).$$

In fact,

$$\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i}) dz$$
$$= \int_{\Omega_\varepsilon} (\log|y_m - z| - \log|y_m - \xi_i'|) L(\tilde{Z}_{0i}) dz + \int_{\Omega_\varepsilon} \log|y_m - \xi_i'| L(\tilde{Z}_{0i}) dz.$$

We assume that $R_\varepsilon < \varepsilon^{-\gamma}/2$, so that

$$\big|\log|y_m - z| - \log|y_m - \xi_i'|\big| \le \log\Big(1 + \frac{|z-\xi_i'|}{|y_m - \xi_i'|}\Big) = O(\varepsilon|z-\xi_i'|).$$

Thus

$$\Big|\int_{\Omega_\varepsilon} (\log|y_m - z| - \log|y_m - \xi_i'|) L(\tilde{Z}_{0i}) dz\Big| = O\Big(\frac{\varepsilon^{1-\beta}}{\log \varepsilon^{-1}}\Big).$$

Finally,

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0i}) dz = O(\varepsilon^{2\gamma}). \tag{B-10}$$

This implies

$$\int_{\Omega_\varepsilon} \log|y_m - \xi_i'| L(\tilde{Z}_{0i}) dz = o(1).$$

Therefore Claim 1 holds. □

Proof of Claim 4. Let

$$\zeta(r) = \begin{cases} 1 & \text{if } r < \varepsilon^{-1/2}, \\ (\log(\delta/\varepsilon) - \log r)/(\log(\delta/\varepsilon) - \log \varepsilon^{-1/2}) & \text{if } \varepsilon^{-1/2} < r < \delta/\varepsilon, \\ 0 & \text{if } r > \delta/\varepsilon, \end{cases}$$

and set

$$\psi(z)=\sum_{i=1}^{2(k+l)} H(\varepsilon y,\xi_i)\zeta(|z-\xi_i'|).$$

Testing (3-9) by ψ and integrating by parts, we obtain

$$\int_{\Omega_\varepsilon}\Big(W\tilde{\phi}+h-\sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i})\Big)\psi+\int_{\Omega_\varepsilon}\tilde{\phi}\Delta\psi-\int_{\partial\Omega_\varepsilon}\tilde{\phi}\frac{\partial\psi}{\partial\nu}=0.$$

Thus

$$A=\int_{\Omega_\varepsilon}(H(\varepsilon y,\varepsilon z)-\psi)\Big(W\tilde{\phi}+h-\sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i})\Big)-\int_{\Omega_\varepsilon}\tilde{\phi}\Delta\psi+\int_{\partial\Omega_\varepsilon}\tilde{\phi}\frac{\partial\psi}{\partial\nu}.$$

Since H, ψ and $\tilde{\phi}$ are bounded,

$$\Big|\int_{\Omega_\varepsilon}(H(\varepsilon y,\varepsilon z)-\psi)h\,dz\Big|\le C\|h\|_*=o(1) \tag{B-11}$$

and

$$\Big|\int_{\Omega_\varepsilon}(H(\varepsilon y,\varepsilon z)-\psi)L(\tilde{Z}_{0i})\Big|\le C\Big|\int_{\Omega_\varepsilon}L(\tilde{Z}_{0i})\,dz\Big|=o(1). \tag{B-12}$$

Also, it is not difficult to show that

$$\int_{\Omega_\varepsilon}\tilde{\phi}\Delta\psi=O\Big(\frac{1}{\log(\delta/\varepsilon)}\Big)=o(1),\quad \int_{\partial\Omega_\varepsilon}\tilde{\phi}\frac{\partial\psi}{\partial\nu}=O\Big(\frac{1}{\log(\delta/\varepsilon)}\Big)=o(1). \tag{B-13}$$

For instance, the first integer in (B-13) can be estimated as

$$\Big|\int_{\Omega_\varepsilon}\tilde{\phi}\Delta\psi\Big|\le\|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)}\int_{\Omega_\varepsilon}|\Delta\psi|.$$

But $\Delta\psi$ is a measure with support on the arcs $r=\varepsilon^{-1/2}$ and $r=\delta/\varepsilon$, where $r=|z-\xi_i'|$, and

$$\int_{\Omega_\varepsilon}|\Delta\psi|=O\Big(\varepsilon^{-1/2}\frac{1}{\varepsilon^{-1/2}\log\varepsilon^{-1}}+\frac{\delta}{\varepsilon}\frac{1}{(\delta/\varepsilon)\log\varepsilon^{-1}}\Big)=O\Big(\frac{1}{\log(\delta/\varepsilon)}\Big)=o(1).$$

Note that for $|z-\xi_i'|>\delta/\varepsilon$, we have $W=O(r^{-4})$, and H and $\tilde{\phi}$ are bounded; thus

$$\int_{\Omega_\varepsilon\setminus(\bigcup_i B_{\delta/\varepsilon}(\xi_i'))}(H(\varepsilon y,\varepsilon z)-\psi)W\tilde{\phi}=o(1). \tag{B-14}$$

On the other hand, for $|z-\xi_i'|\le\delta/\varepsilon$, we have $H(\varepsilon y,\varepsilon z)-H(\varepsilon y,\xi_i)=O(\varepsilon|z-\xi_i'|)$ and $W=O((r^2+1)^{-2})$. So

$$\begin{aligned}&\Big|\int_{\Omega_\varepsilon\cap B_{\varepsilon^{-1/2}}(\xi_i')}(H(\varepsilon y,\varepsilon z)-\psi(z))W\tilde{\phi}dz\Big|\\ &\qquad=\Big|\int_{\Omega_\varepsilon\cap B_{\varepsilon^{-1/2}}(\xi_i')}(H(\varepsilon y,\varepsilon z)-H(\varepsilon y,\xi_i))W\tilde{\phi}dz\Big|\\ &\qquad\le C\varepsilon\int_0^{\varepsilon^{-1/2}}\frac{r^2}{(r^2+1)^2}dr=O(\varepsilon^{1/2})=o(1).\end{aligned}\tag{B-15}$$

In the region $\varepsilon^{-1/2}<r=|z-\xi_i'|<\delta/\varepsilon$, noting the fact that H, ζ and $\tilde{\phi}$ are bounded and that $W=O(r^{-4})$, we find

$$\Big|\int_{\Omega_\varepsilon\cap B_{\delta/\varepsilon}(\xi_i')\setminus B_{1/\sqrt{\varepsilon}}(\xi_i')}(H(\varepsilon y,\varepsilon z)-\psi(z))W\tilde{\phi}dz\Big|\le C\int_{1/\sqrt{\varepsilon}}^{\delta/\varepsilon}r^{-3}dr=o(1).\tag{B-16}$$

Therefore, Claim 4 follows from (B-10)–(B-16). □

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JUNCHENG WEI
DEPARTMENT OF MATHEMATICS
THE CHINESE UNIVERSITY OF HONG KONG
ROOM 220, LADY SHAW BUILDING
SHATIN, HONG KONG
HONG KONG
wei@math.cuhk.edu.hk

LONG WEI
INSTITUTE OF APPLIED MATHEMATICS AND ENGINEERING COMPUTATIONS
HANGZHOU DIANZI UNIVERSITY
HANGZHOU, ZHEJIANG 310018
CHINA
alongwei@gmail.com

FENG ZHOU
DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
SHANGHAI 200062
CHINA
fzhou@math.ecnu.edu.cn

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