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PRINCIPAL CURVATURES OF FIBERS AND HEEGAARD SURFACES

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We study principal curvatures of fibers and Heegaard surfaces smoothly embedded in hyperbolic 3-manifolds. It is well known that a fiber or a Heegaard surface in a hyperbolic 3-manifold cannot have principal curvatures everywhere less than one in absolute value. We show that given an upper bound on the genus of a minimally embedded fiber or Heegaard surface and a lower bound on the injectivity radius of the hyperbolic 3-manifold, there exists a $\delta > 0$ such that the fiber or Heegaard surface must contain a point at which one of the principal curvatures exceeds $1 + \delta$ in absolute value.

1. Introduction

The principal curvatures of a surface or lamination smoothly embedded in a hyperbolic 3-manifold are related to the topology of the surface and the 3-manifold. For example in [Breslin 2010] we show that incompressible surfaces and strongly irreducible Heegaard surfaces embedded in hyperbolic 3-manifolds can always be isotoped to a surface with principal curvatures bounded in absolute value by a fixed constant that does not depend on the surface or the 3-manifold. In [Breslin 2009] we show that laminations in hyperbolic 3-manifolds with principal curvatures everywhere close to zero have boundary leaves with noncyclic fundamental group and that laminations in hyperbolic 3-manifolds with principal curvatures everywhere in the interval $(-1, 1)$ have boundary leaves with nontrivial fundamental group.

This note was motivated by a question about surfaces with principal curvatures near the interval $(-1, 1)$. It is well known that a closed orientable surface smoothly embedded in a finite-volume complete hyperbolic 3-manifold with principal curvatures everywhere in the interval $(-1, 1)$ is incompressible and lifts to a quasiplane in $\mathbb{H}^3$ (see [Thurston 1979] or [Leininger 2006] for a proof). Thus Heegaard surfaces and fibers in hyperbolic 3-manifolds cannot have principal curvatures everywhere in the interval $(-1, 1)$. We are interested in finding obstructions to isotoping Heegaard surfaces and fibers in hyperbolic 3-manifolds to have principal

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curvatures close to the interval $(-1, 1)$. See [Rubinstein 2005] or [Krasnov and Schlenker 2007] for more on surfaces in hyperbolic 3-manifolds with principal curvatures in the interval $(-1, 1)$.

It follows from work of Freedman, Hass, and Scott [Freedman et al. 1983] that an incompressible surface in a closed Riemannian 3-manifold can be isotoped to a minimal surface. It follows from work of Pitts-Rubinstein that a strongly irreducible Heegaard surface in a closed Riemannian 3-manifold can be isotoped to either a minimal surface or the boundary of a regular neighborhood of a minimal surface (see [Rubinstein 2005] for a sketch of the proof). We show that given an upper bound on the genus of a minimally embedded fiber or Heegaard surface and a lower bound on the injectivity radius of the hyperbolic 3-manifold, there exists a $\delta > 0$ such that the fiber or Heegaard surface must contain a point at which one of the principal curvatures is greater than $1 + \delta$ in absolute value.

Theorem 1. *For each $g \geq 2$, $\epsilon > 0$, there exists $\delta := \delta(g, \epsilon)$ such that if S is a genus g minimally embedded fiber in a closed hyperbolic mapping torus M with* $\mathrm{inj}(M) > \epsilon$, *then S contains a point at which one of the principal curvatures is at least* $1 + \delta$ *in absolute value.*

Theorem 2. *For each $g \geq 2$, $\epsilon > 0$, there exists $\delta := \delta(g, \epsilon)$ such that if S is a genus g minimally embedded Heegaard surface in a closed hyperbolic* 3*-manifold M with* $\mathrm{inj}(M) > \epsilon$, *then S contains a point at which one of the principal curvatures is at least* $1 + \delta$ *in absolute value.*

The proofs of Theorem 1 and Theorem 2 both use geometric limit arguments. Assuming that no such $\delta > 0$ exists, we consider a sequence of hyperbolic 3-manifolds as in the statement with minimally embedded fibers or Heegaard surfaces whose principal curvatures are closer and closer to the interval $[-1, 1]$. After possibly passing to a subsequence, the sequence of manifolds converges geometrically to a hyperbolic 3-manifold M and the surfaces converge to an incompressible surface S in M with principal curvatures everywhere in the interval $[-1, 1]$. This implies that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$. In either case, we show that the cover of M corresponding to the image of $\pi_1(S)$ in $\pi_1(M)$ has a doubly degenerate hyperbolic structure contradicting that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.

2. Preliminaries

Let M be a hyperbolic 3-manifold with no cusps and finitely generated fundamental group. By a result of Scott, M has a *compact core* which is a compact submanifold C of M whose inclusion into M is a homotopy equivalence. The connected components of $M \setminus C$ are called the *ends* of M. It follows from the positive solution of the tameness conjecture by Agol [2004] and by Calegari and Gabai [2006] that an end of M is homeomorphic to $\Sigma \times [0, \infty)$ where Σ is a closed orientable surface. The convex core, $CC(M)$, of M is the smallest convex submanifold of M whose inclusion is a homotopy equivalence. An end E of M is *convex-cocompact* if $E \cap CC(M)$ is compact and E is *degenerate* otherwise. Given a closed orientable surface Σ of genus greater than one, a hyperbolic structure on $\Sigma \times \mathbb{R}$ such that both ends are degenerate is called *doubly degenerate*.

A sequence of pointed hyperbolic n-manifolds (M_i, p_i) *converges geometrically* to the pointed hyperbolic n-manifold (M, p) if for every sufficiently large R and each $\epsilon > 0$, there exists i_0 such that for every $i \geq i_0$, there is a $(1+\epsilon)$-bilipschitz pointed diffeomorphism $\kappa_i : (B(p, R), p) \to M_i$, where $B(p, R) \subset M$ is the ball of radius R centered at p and $B(p_i, R) \subset M_i$ is the ball of radius R centered at p_i. We call the maps κ_i *almost isometries*.

We will use the fact that minimal surfaces have bounded diameter in the presence of a lower bound on injectivity radius. See [Rubinstein 2005] or [Souto 2007] for more on minimal surfaces in hyperbolic 3-manifolds.

Lemma 1. *Let S be a connected minimal surface in a complete hyperbolic* 3-*manifold M with* $\mathrm{inj}(M) \geq \epsilon$. *Then the diameter of S is at most* $4|\chi(F)|/\epsilon + 2\epsilon$.

We will also use the following Lemma in the proofs of Theorems 1 and 2.

Lemma 2. *If S is a closed orientable surface smoothly immersed with principal curvatures everywhere in the interval* $[-1, 1]$ *in a complete hyperbolic* 3-*manifold M with no cusps, then the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.*

Proof. Let $\tilde{S}$ be a lift of S to $\mathbb{H}^3$. Assume that $\tilde{S}$ is not a horosphere, as otherwise we are done. Thus the principal curvatures of S cannot be everywhere equal to 1 or everywhere equal to -1. If the principal curvatures at every point of S are -1 and 1, then there is a pair of line fields defined on the entire surface, implying that S is a torus. Since closed surfaces in M with all principal curvatures in $[-1, 1]$ are incompressible and M has no cusps, S cannot be a torus. Thus there is a point p in $\tilde{S}$ at which one of the principal curvatures is in $(-1, 1)$. Assume that the other principal curvature at p is in $[-1, 1)$. Let H be a horosphere tangent to $\tilde{S}$ at p. Use an upper half space model of $\mathbb{H}^3$ in which H is a horizontal plane and $\tilde{S}$ is below H. Let l be a simple loop in $\tilde{S}$ which contains p such that the principal curvatures at each point on l are in $[-1, 1)$ with at least principal curvature in $(-1, 1)$. At each point x in l, let H_x be the horosphere above $\tilde{S}$ tangent to $\tilde{S}$ at x. For each x in l, let $c_x \in \partial\mathbb{H}^3$ be the center of the horosphere H_x. The set of points $C = \{c_x | x \in l\}$ forms a closed curve in $\partial\mathbb{H}^3$. Since the principal curvatures of $\tilde{S}$ are everywhere in the interval $[-1, 1]$, $\tilde{S}$ cannot transversely intersect any of the horospheres H_x. Thus, the limit set of $\tilde{S}$ cannot cross the closed curve C, so that the limit set of $\tilde{S}$ is a proper subset of $\partial\mathbb{H}^3$. □

It is well-known that the limit set of a lift to $\mathbb{H}^3$ of a fiber Σ in a doubly degenerate hyperbolic $\Sigma \times \mathbb{R}$ is the entire boundary $\partial\mathbb{H}^3$. By Lemma 2, such a fiber Σ cannot be smoothly embedded with principal curvatures everywhere in the interval $[-1, 1]$.

3. Principal curvatures of fibers

In the proof of Theorem 1, we will use the following fact about geometric limits of hyperbolic mapping tori.

Theorem. *Let (M_i, p_i) be a sequence of pairwise distinct pointed hyperbolic mapping tori with genus g fibers and $\mathrm{inj}(M_i) > \epsilon$ for all i. Then a subsequence of (M_i, p_i) converges geometrically to a pointed hyperbolic 3-manifold (M, p) homeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a closed genus g surface and M has a doubly degenerate hyperbolic structure.*

Proof of Theorem 1. Suppose, for contradiction, that Theorem 1 does not hold. Then there exists a sequence of hyperbolic mapping tori (M_i) with $\mathrm{inj}(M_i) > \epsilon$ such that M_i has a genus g minimal surface fiber with principal curvatures less than $1 + 1/i$ in absolute value. For each i, let p_i be a point in S_i. By Theorem A the sequence (M_i, p_i) has a subsequence, say the entire sequence, which converges to a doubly degenerate pointed hyperbolic 3-manifold (M, p) homeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a genus g closed surface. By Lemma 1, the diameters of the surfaces S_i are uniformly bounded. Thus we can find a compact subset K of M homeomorphic to $\Sigma \times [-1, 1]$ such that for i large enough, say for all i, S_i is contained in $\kappa_i(K)$. The surface $S := \Sigma \times \{0\}$ in M is isotopic to $\kappa_i^{-1}(S_i)$ for each i. Since the surfaces $\kappa_i^{-1}(S_i)$ have bounded area and curvature, a subsequence converges to a smoothly immersed surface with principal curvatures in $[-1, 1]$ which is homotopic to S. Lemma 2 implies that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$, contradicting the fact that M is doubly degenerate. □

4. Principal curvatures of Heegaard surfaces

In the proof of Theorem 2, we will use the following fact about geometric limits.

Theorem. *Every sequence (M_i, p_i) of pointed hyperbolic 3-manifolds such that $\mathrm{inj}(M_i, p_i)$ is bounded away from 0 has a geometrically convergent subsequence.*

Lemma 3 [Souto 2006, Lemma 2.1]. *Let (M_i) be a sequence of hyperbolic 3-manifolds converging to a hyperbolic manifold M. Assume that there is a compact subset $K \subset M$ such that for all sufficiently large i the homomorphism $\pi_1(K) \to \pi_1(M_i)$ provided by geometric convergence is surjective. Then, if the cover of M corresponding to the image of $\pi_1(K)$ into $\pi_1(M)$ has a convex-cocompact end, so does M_i for all but finitely many i.*

Proof of Theorem 2. Suppose for contradiction that Theorem 2 does not hold. Then there exists a sequence (M_i) of closed hyperbolic 3-manifolds with $\mathrm{inj}(M_i) > \epsilon$ such that M_i has a genus g minimal Heegaard surface S_i with principal curvatures less than $1 + 1/i$ in absolute value. For each i let p_i be a point in S_i. By Theorem B the sequence (M_i, p_i) has a convergent subsequence, say the entire sequence, which converges geometrically to a pointed hyperbolic 3-manifold (M, p). By Lemma 1, the diameters of the surfaces S_i are uniformly bounded. Thus each M_i contains a compact subset K_i homeomorphic to $S_i \times [-1, 1]$ with uniformly bounded diameter. For i large enough the pull-back $\kappa_i^{-1}(K_i)$ of K_i through the almost isometries provided by geometric convergence are embedded compact subsets homeomorphic to $\Sigma \times [-1, 1]$ where Σ is a closed surface of genus g. For i large enough the surfaces $\kappa_i^{-1}(S_i)$ are all isotopic to a fixed embedded genus g surface S in M. Since the surfaces $\kappa_i^{-1}(S_i)$ have bounded area and curvature, a subsequence converges to a smoothly immersed surface with principal curvatures in $[-1, 1]$ which is homotopic to S. Thus the surface S is incompressible in M and by Lemma 2 the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.

To arrive at a contradiction we will show that the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ is doubly degenerate, implying that the limit set of a lift of S to $\mathbb{H}^3$ is all of $\partial\mathbb{H}^3$. For i large enough $\kappa_i(S)$ is isotopic to the Heegaard surface S_i in M_i, so that the homomorphism $(\kappa_i)_* : \pi_1(S) \to \pi_1(M_i)$ provided by geometric convergence is surjective. By Lemma 3, if the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ has a convex-cocompact end, so does M_i for all but finitely many i. Since each M_i is closed we have that the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ cannot have a convex-cocompact end. Thus the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ is doubly degenerate contradicting the fact that S is isotopic to a surface with principal curvatures everywhere in $[-1, 1]$. □

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