Pacific Journal of Mathematics

PRINCIPAL CURVATURES OF FIBERS AND HEEGAARD SURFACES

William Breslin

Volume 250 No. 1 March 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 250, No. 1, 2011

PRINCIPAL CURVATURES OF FIBERS AND HEEGAARD SURFACES

WILLIAM BRESLIN

We study principal curvatures of fibers and Heegaard surfaces smoothly embedded in hyperbolic 3-manifolds. It is well known that a fiber or a Heegaard surface in a hyperbolic 3-manifold cannot have principal curvatures everywhere less than one in absolute value. We show that given an upper bound on the genus of a minimally embedded fiber or Heegaard surface and a lower bound on the injectivity radius of the hyperbolic 3-manifold, there exists a $\delta > 0$ such that the fiber or Heegaard surface must contain a point at which one of the principal curvatures exceeds $1 + \delta$ in absolute value.

1. Introduction

The principal curvatures of a surface or lamination smoothly embedded in a hyperbolic 3-manifold are related to the topology of the surface and the 3-manifold. For example in [Breslin 2010] we show that incompressible surfaces and strongly irreducible Heegaard surfaces embedded in hyperbolic 3-manifolds can always be isotoped to a surface with principal curvatures bounded in absolute value by a fixed constant that does not depend on the surface or the 3-manifold. In [Breslin 2009] we show that laminations in hyperbolic 3-manifolds with principal curvatures everywhere close to zero have boundary leaves with noncyclic fundamental group and that laminations in hyperbolic 3-manifolds with principal curvatures everywhere in the interval $(-1, 1)$ have boundary leaves with nontrivial fundamental group.

This note was motivated by a question about surfaces with principal curvatures near the interval $(-1, 1)$. It is well known that a closed orientable surface smoothly embedded in a finite-volume complete hyperbolic 3-manifold with principal curvatures everywhere in the interval $(-1, 1)$ is incompressible and lifts to a quasiplane in $\mathbb{H}^3$ (see [Thurston 1979] or [Leininger 2006] for a proof). Thus Heegaard surfaces and fibers in hyperbolic 3-manifolds cannot have principal curvatures everywhere in the interval $(-1, 1)$. We are interested in finding obstructions to isotoping Heegaard surfaces and fibers in hyperbolic 3-manifolds to have principal

This work was partially supported by the NSF RTG grant 0602191.
MSC2000: 57M50.
Keywords: hyperbolic manifold, Heegaard surface, fiber, principal curvatures.

curvatures close to the interval $(-1, 1)$. See [Rubinstein 2005] or [Krasnov and Schlenker 2007] for more on surfaces in hyperbolic 3-manifolds with principal curvatures in the interval $(-1, 1)$.

It follows from work of Freedman, Hass, and Scott [Freedman et al. 1983] that an incompressible surface in a closed Riemannian 3-manifold can be isotoped to a minimal surface. It follows from work of Pitts-Rubinstein that a strongly irreducible Heegaard surface in a closed Riemannian 3-manifold can be isotoped to either a minimal surface or the boundary of a regular neighborhood of a minimal surface (see [Rubinstein 2005] for a sketch of the proof). We show that given an upper bound on the genus of a minimally embedded fiber or Heegaard surface and a lower bound on the injectivity radius of the hyperbolic 3-manifold, there exists a $\delta > 0$ such that the fiber or Heegaard surface must contain a point at which one of the principal curvatures is greater than $1+\delta$ in absolute value.

Theorem 1. *For each $g \geq 2$, $\epsilon > 0$, there exists $\delta := \delta(g, \epsilon)$ such that if S is a genus g minimally embedded fiber in a closed hyperbolic mapping torus M with $\operatorname{inj}(M) > \epsilon$, then S contains a point at which one of the principal curvatures is at least $1+\delta$ in absolute value.*

Theorem 2. *For each $g \geq 2$, $\epsilon > 0$, there exists $\delta := \delta(g, \epsilon)$ such that if S is a genus g minimally embedded Heegaard surface in a closed hyperbolic 3-manifold M with $\operatorname{inj}(M) > \epsilon$, then S contains a point at which one of the principal curvatures is at least $1+\delta$ in absolute value.*

The proofs of Theorem 1 and Theorem 2 both use geometric limit arguments. Assuming that no such $\delta > 0$ exists, we consider a sequence of hyperbolic 3-manifolds as in the statement with minimally embedded fibers or Heegaard surfaces whose principal curvatures are closer and closer to the interval $[-1, 1]$. After possibly passing to a subsequence, the sequence of manifolds converges geometrically to a hyperbolic 3-manifold M and the surfaces converge to an incompressible surface S in M with principal curvatures everywhere in the interval $[-1, 1]$. This implies that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$. In either case, we show that the cover of M corresponding to the image of $\pi_1(S)$ in $\pi_1(M)$ has a doubly degenerate hyperbolic structure contradicting that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.

2. Preliminaries

Let M be a hyperbolic 3-manifold with no cusps and finitely generated fundamental group. By a result of Scott, M has a *compact core* which is a compact submanifold C of M whose inclusion into M is a homotopy equivalence. The connected components of $M \setminus C$ are called the *ends* of M. It follows from the positive solution of the tameness conjecture by Agol [2004] and by Calegari and Gabai [2006] that an end of M is homeomorphic to $\Sigma \times [0, \infty)$ where Σ is a closed orientable surface. The convex core, $CC(M)$, of M is the smallest convex submanifold of M whose inclusion is a homotopy equivalence. An end E of M is *convex-cocompact* if $E \cap CC(M)$ is compact and E is *degenerate* otherwise. Given a closed orientable surface Σ of genus greater than one, a hyperbolic structure on $\Sigma \times \mathbb{R}$ such that both ends are degenerate is called *doubly degenerate*.

A sequence of pointed hyperbolic n-manifolds (M_i, p_i) *converges geometrically* to the pointed hyperbolic n-manifold (M, p) if for every sufficiently large R and each $\epsilon > 0$, there exists i_0 such that for every $i \geq i_0$, there is a $(1+\epsilon)$-bilipschitz pointed diffeomorphism $\kappa_i : (B(p, R), p) \to M_i$, where $B(p, R) \subset M$ is the ball of radius R centered at p and $B(p_i, R) \subset M_i$ is the ball of radius R centered at p_i. We call the maps κ_i *almost isometries*.

We will use the fact that minimal surfaces have bounded diameter in the presence of a lower bound on injectivity radius. See [Rubinstein 2005] or [Souto 2007] for more on minimal surfaces in hyperbolic 3-manifolds.

Lemma 1. *Let S be a connected minimal surface in a complete hyperbolic* 3-*manifold M with* $\operatorname{inj}(M) \geq \epsilon$. *Then the diameter of S is at most* $4|\chi(F)|/\epsilon + 2\epsilon$.

We will also use the following Lemma in the proofs of Theorems 1 and 2.

Lemma 2. *If S is a closed orientable surface smoothly immersed with principal curvatures everywhere in the interval* $[-1, 1]$ *in a complete hyperbolic* 3-*manifold M with no cusps, then the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.*

Proof. Let $\tilde{S}$ be a lift of S to $\mathbb{H}^3$. Assume that $\tilde{S}$ is not a horosphere, as otherwise we are done. Thus the principal curvatures of S cannot be everywhere equal to 1 or everywhere equal to -1. If the principal curvatures at every point of S are -1 and 1, then there is a pair of line fields defined on the entire surface, implying that S is a torus. Since closed surfaces in M with all principal curvatures in $[-1, 1]$ are incompressible and M has no cusps, S cannot be a torus. Thus there is a point p in $\tilde{S}$ at which one of the principal curvatures is in $(-1, 1)$. Assume that the other principal curvature at p is in $[-1, 1)$. Let H be a horosphere tangent to $\tilde{S}$ at p. Use an upper half space model of $\mathbb{H}^3$ in which H is a horizontal plane and $\tilde{S}$ is below H. Let l be a simple loop in $\tilde{S}$ which contains p such that the principal curvatures at each point on l are in $[-1, 1)$ with at least principal curvature in $(-1, 1)$. At each point x in l, let H_x be the horosphere above $\tilde{S}$ tangent to $\tilde{S}$ at x. For each x in l, let $c_x \in \partial\mathbb{H}^3$ be the center of the horosphere H_x. The set of points $C = \{c_x | x \in l\}$ forms a closed curve in $\partial\mathbb{H}^3$. Since the principal curvatures of $\tilde{S}$ are everywhere in the interval $[-1, 1]$, $\tilde{S}$ cannot transversely intersect any of the horospheres H_x. Thus, the limit set of $\tilde{S}$ cannot cross the closed curve C, so that the limit set of $\tilde{S}$ is a proper subset of $\partial\mathbb{H}^3$. □

It is well-known that the limit set of a lift to $\mathbb{H}^3$ of a fiber Σ in a doubly degenerate hyperbolic $\Sigma \times \mathbb{R}$ is the entire boundary $\partial\mathbb{H}^3$. By Lemma 2, such a fiber Σ cannot be smoothly embedded with principal curvatures everywhere in the interval $[-1, 1]$.

3. Principal curvatures of fibers

In the proof of Theorem 1, we will use the following fact about geometric limits of hyperbolic mapping tori.

Theorem. *Let (M_i, p_i) be a sequence of pairwise distinct pointed hyperbolic mapping tori with genus g fibers and $\operatorname{inj}(M_i) > \epsilon$ for all i. Then a subsequence of (M_i, p_i) converges geometrically to a pointed hyperbolic 3-manifold (M, p) homeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a closed genus g surface and M has a doubly degenerate hyperbolic structure.*

Proof of Theorem 1. Suppose, for contradiction, that Theorem 1 does not hold. Then there exists a sequence of hyperbolic mapping tori (M_i) with $\operatorname{inj}(M_i) > \epsilon$ such that M_i has a genus g minimal surface fiber with principal curvatures less than $1 + 1/i$ in absolute value. For each i, let p_i be a point in S_i. By Theorem A the sequence (M_i, p_i) has a subsequence, say the entire sequence, which converges to a doubly degenerate pointed hyperbolic 3-manifold (M, p) homeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a genus g closed surface. By Lemma 1, the diameters of the surfaces S_i are uniformly bounded. Thus we can find a compact subset K of M homeomorphic to $\Sigma \times [-1, 1]$ such that for i large enough, say for all i, S_i is contained in $\kappa_i(K)$. The surface $S := \Sigma \times \{0\}$ in M is isotopic to $\kappa_i^{-1}(S_i)$ for each i. Since the surfaces $\kappa_i^{-1}(S_i)$ have bounded area and curvature, a subsequence converges to a smoothly immersed surface with principal curvatures in $[-1, 1]$ which is homotopic to S. Lemma 2 implies that the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$, contradicting the fact that M is doubly degenerate. □

4. Principal curvatures of Heegaard surfaces

In the proof of Theorem 2, we will use the following fact about geometric limits.

Theorem. *Every sequence (M_i, p_i) of pointed hyperbolic 3-manifolds such that $\operatorname{inj}(M_i, p_i)$ is bounded away from 0 has a geometrically convergent subsequence.*

Lemma 3 [Souto 2006, Lemma 2.1]. *Let (M_i) be a sequence of hyperbolic 3-manifolds converging to a hyperbolic manifold M. Assume that there is a compact subset $K \subset M$ such that for all sufficiently large i the homomorphism $\pi_1(K) \to \pi_1(M_i)$ provided by geometric convergence is surjective. Then, if the cover of M corresponding to the image of $\pi_1(K)$ into $\pi_1(M)$ has a convex-cocompact end, so does M_i for all but finitely many i.*

Proof of Theorem 2. Suppose for contradiction that Theorem 2 does not hold. Then there exists a sequence (M_i) of closed hyperbolic 3-manifolds with $\text{inj}(M_i) > \epsilon$ such that M_i has a genus g minimal Heegaard surface S_i with principal curvatures less than $1 + 1/i$ in absolute value. For each i let p_i be a point in S_i. By Theorem B the sequence (M_i, p_i) has a convergent subsequence, say the entire sequence, which converges geometrically to a pointed hyperbolic 3-manifold (M, p). By Lemma 1, the diameters of the surfaces S_i are uniformly bounded. Thus each M_i contains a compact subset K_i homeomorphic to $S_i \times [-1, 1]$ with uniformly bounded diameter. For i large enough the pull-back $\kappa_i^{-1}(K_i)$ of K_i through the almost isometries provided by geometric convergence are embedded compact subsets homeomorphic to $\Sigma \times [-1, 1]$ where Σ is a closed surface of genus g. For i large enough the surfaces $\kappa_i^{-1}(S_i)$ are all isotopic to a fixed embedded genus g surface S in M. Since the surfaces $\kappa_i^{-1}(S_i)$ have bounded area and curvature, a subsequence converges to a smoothly immersed surface with principal curvatures in $[-1, 1]$ which is homotopic to S. Thus the surface S is incompressible in M and by Lemma 2 the limit set of a lift of S to $\mathbb{H}^3$ is a proper subset of $\partial\mathbb{H}^3$.

To arrive at a contradiction we will show that the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ is doubly degenerate, implying that the limit set of a lift of S to $\mathbb{H}^3$ is all of $\partial\mathbb{H}^3$. For i large enough $\kappa_i(S)$ is isotopic to the Heegaard surface S_i in M_i, so that the homomorphism $(\kappa_i)_* : \pi_1(S) \to \pi_1(M_i)$ provided by geometric convergence is surjective. By Lemma 3, if the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ has a convex-cocompact end, so does M_i for all but finitely many i. Since each M_i is closed we have that the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ cannot have a convex-cocompact end. Thus the cover of M corresponding to the image of $\pi_1(S)$ into $\pi_1(M)$ is doubly degenerate contradicting the fact that S is isotopic to a surface with principal curvatures everywhere in $[-1, 1]$. □

References

[Agol 2004] I. Agol, "Tameness of hyperbolic 3-manifolds", preprint, 2004. arXiv math/0405568

[Breslin 2009] W. Breslin, "Small curvature laminations in hyperbolic 3-manifolds", *Algebr. Geom. Topol.* **9**:2 (2009), 723–729. MR 2010b:57019 Zbl 1178.57018

[Breslin 2010] W. Breslin, "Curvature bounds for surfaces in hyperbolic 3-manifolds", *Canad. J. Math.* **62**:5 (2010), 994–1010. MR 2730352 Zbl 05799470

[Calegari and Gabai 2006] D. Calegari and D. Gabai, "Shrinkwrapping and the taming of hyperbolic 3-manifolds", *J. Amer. Math. Soc.* **19**:2 (2006), 385–446. MR 2006g:57030 Zbl 1090.57010

[Freedman et al. 1983] M. Freedman, J. Hass, and P. Scott, "Least area incompressible surfaces in 3-manifolds", *Invent. Math.* **71**:3 (1983), 609–642. MR 85e:57012 Zbl 0482.53045

[Krasnov and Schlenker 2007] K. Krasnov and J.-M. Schlenker, "Minimal surfaces and particles in 3-manifolds", *Geom. Dedicata* **126** (2007), 187–254. MR 2009c:53076 Zbl 1126.53037

[Leininger 2006] C. J. Leininger, "Small curvature surfaces in hyperbolic 3-manifolds", *J. Knot Theory Ramifications* **15**:3 (2006), 379–411. MR 2007a:57025 Zbl 1090.57012

[Rubinstein 2005] J. H. Rubinstein, "Minimal surfaces in geometric 3-manifolds", pp. 725–746 in *Global theory of minimal surfaces*, edited by D. Hoffman, Clay Math. Proc. **2**, Amer. Math. Soc., Providence, RI, 2005. MR 2006g:57038 Zbl 1119.53042

[Souto 2006] J. Souto, "Rank and topology of hyperbolic 3-manifolds, I", preprint, 2006.

[Souto 2007] J. Souto, "Geometry, Heegaard splittings and rank of the fundamental group of hyperbolic 3-manifolds", pp. 351–399 in *Workshop on Heegaard splittings* (Haifa, 2005), edited by C. Gordon and Y. Moriah, Geom. Topol. Monogr. **12**, Geom. Topol. Publ., Coventry, 2007. MR 2009k:57030 Zbl 1138.57022

[Thurston 1979] W. P. Thurston, "The geometry and topology of three-manifolds", lecture notes, Princeton University, 1979, Available at http://msri.org/publications/books/gt3m.

Received May 3, 2010.

WILLIAM BRESLIN

breslin@umich.edu
UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
530 CHURCH STREET
ANN ARBOR, MI 48109-1043
UNITED STATES
http://www-personal.umich.edu/~breslin/index.html

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

PRODUCTION

SUPPORTING INSTITUTIONS

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 250 No. 1 March 2011
