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SELF-IMPROVING PROPERTIES OF INEQUALITIES OF POINCARÉ TYPE ON s-JOHN DOMAINS

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We derive weak- and strong-type global Poincaré estimates over s-John domains in spaces of homogeneous type. The results show that Poincaré inequalities over quasimetric balls with given exponents and weights are self-improving in the sense that they imply global inequalities of a similar kind, but with improved exponents and larger classes of weights. The main theorems are applications of a geometric construction for s-John domains together with self-improving results in more general settings, both derived in our companion paper *J. Funct. Anal.* 255 (2008), 2977–3007. We have reduced our assumption on the principal measure μ to be just reverse doubling on the domain instead of the usual assumption of doubling. While the primary case considered in the literature is $p \le q$, we will also study the case $1 \le q < p$.

0. Introduction

This is a companion paper to [Chua and Wheeden 2008], where we established the self-improving nature of Poincaré inequalities over domains in general measure spaces. The self-improving nature of Poincaré estimates was observed initially by Saloff-Coste [1992] in the setting of Riemannian manifolds and has been extensively studied in other general settings; see examples in [Chua and Wheeden 2008]. The main goal of this paper is to apply our previous results to derive global Poincaré estimates on s-John domains (see Definition 1.2) in spaces of homogeneous type for reverse doubling measures (see Definition 1.4) instead of the usual doubling measures; see [Franchi et al. 1998; 2003].

The notion of an s-John domain was introduced by Smith and Stegenga [1990], while the terminology John domain introduced by Martio and Sarvas [1979]. John domains are the same as s-John domains in case $s = 1$. In spaces of homogeneous type with the segment (geodesic) property, John domains are the same as Boman domains; see [Buckley et al. 1996]. It is easy to see that bounded Lipschitz domains

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(including all bounded domains with smooth boundaries) and bounded domains that satisfy the interior cone condition are John domains. When $s > 1$, the notion of an s-John domain is a generalization of that of a John domain, a weakening of requirements relative to the case $s = 1$ in order to accommodate domains with rougher boundaries. Some examples of s-John domains in case $s > 1$ are given in [Hajłasz and Koskela 1998]. There have been many studies concerning (bounded) John domains; see for example [Buckley and Koskela 1995; Chua 2001; Acosta et al. 2006] and references therein. Results for John domains have also been generalized in [Hurri-Syrjänen 2004; Väisälä 1994; Chua 2009] to "unbounded John domains" or "generalized John domains". On the other hand, for bounded convex domains, sharp estimates have been obtained in [Chua and Wheeden 2006; Chua and Duan 2009; Chua and Wheeden 2010]

One of our main goals in this paper is to extend the following Poincaré estimate for s-John domains stated in [Kilpeläinen and Malý 2000, Theorem 2.3].

Theorem A. *Suppose that $\Omega \subset \mathbb{R}^n$ is an s-John domain. Let a, b, p, q be real numbers that satisfy*

$$a \geq 0, \quad b \geq 1-n, \quad 1 \leq p < q < \infty, \quad 1/q \geq 1/p - 1/n$$

and

$$\frac{1}{q} \geq \frac{s(n+b-1)-p+1}{(n+a)p}. \tag{0-1}$$

Then there is a constant $C = C(n, a, b, p, q, \Omega) > 0$ such that

$$\|f - f_{\Omega,\rho^a dx}\|_{L^q_{\rho^a dx}(\Omega)} \leq C\|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} \quad \text{for all } f \in C^1(\Omega), \tag{0-2}$$

where $\rho(x) = \operatorname{dist}(x, \Omega^c)$ and $f_{\Omega,\rho^a dx} = \int_\Omega f(x)\rho(x)^a dx \,\big/ \int_\Omega \rho(x)^a dx$.

The assumption that $f \in C^1(\Omega)$ in Theorem A does not automatically imply that the norm on the right side of (0-2) or the average $f_{\Omega,\rho^a dx}$ on the left side is finite. However, as we shall see in Theorem 1.12, (0-2) holds under the weaker hypothesis that $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, that is, it holds for all f that are locally Lipschitz continuous on Ω, provided the average on the left side is replaced by the average $|B'|^{-1}\int_{B'} f(x)dx$ over a "central" ball $B' \subset \Omega$, which is always finite for such f. If $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and the right side of (0-2) is finite, it follows that $f \in L^q_{\rho^a dx}(\Omega)$, and then $f_{\Omega,\rho^a dx}$ is finite and it is possible to replace the average over the central ball by this average in (0-2). The inequality (0-2) was also proved in [Hajłasz and Koskela 1998] except that when $p > 1$, they required strict inequality in (0-1). The necessity of the conditions $1/q \geq 1/p - 1/n$ and $1 + (n+a)/q - (n+b)/p \geq 0$ is easy to see as usual by considering Lipschitz functions that vanish outside balls in Ω; see [Chua and Duan 2009, Final Remark]. Condition (0-1) is sharp too as can be seen by considering mushroom-like domains; see [Hajłasz and Koskela 1998] for details. On the other hand, for special s-John domains such as s-cusp domains, condition (0-1) can be relaxed; see Theorem 1.14 for an estimate of this kind.

In this paper, we will apply results from [Chua and Wheeden 2008], where we use a different approach from those in [Hajłasz and Koskela 1998] and [Kilpeläinen and Malý 2000] to obtain self-improving properties of Poincaré-type inequalities in measure spaces. The approach modifies one used in [Franchi et al. 2003]. We now apply the outcome to derive global Poincaré inequalities on s-John domains Ω (including 1-John domains) in spaces of homogeneous type and for measures that are doubling, δ-doubling or just reverse doubling on Ω; see Definition 1.4. The notions of δ-doubling and doubling on Ω are equivalent on 1-John domains. We note that power-type weights of the form $\operatorname{dist}(x, \Omega_0)^a$, with $a \geq 0$ and $\Omega_0 \subset \Omega^c$, are examples of δ-doubling measures. We are also able to prove Theorem A without the assumption $b \geq 1-n$. Moreover, we will consider the case $1 \leq q \leq p$.

1. Definitions and main results

Definition 1.1. A pair $\langle H, d\rangle$ is a *quasimetric space* if d is a *quasimetric* on the set H, that is, if there exists a constant κ such that for all $x, y, z \in H$,

(1) $d(y, x) = d(x, y)$ is positive if $x \neq y$ and vanishes if $x = y$, and

(2) $d(x, y) \leq \kappa(d(x, z) + d(y, z))$.

For a quasimetric space $\langle H, d\rangle$, any $x \in H$ and $r > 0$, we write

$$B(x, r) = \{y \in H : d(x, y) < r\}$$

and call $B(x, r)$ the ball with center x and radius r. If $B = B(x, r)$ is a ball and c is a positive constant, we use cB to denote $B(x, cr)$. If B is a ball, we use $r(B)$ and x_B to denote the radius and center of B.

Definition 1.2. Let $\langle H, d\rangle$ be a quasimetric space. Fix $\Omega \subset H$, and for $x \in H$, set

$$d(x) = \operatorname{dist}(x, \Omega^c) = \inf_{y \in \Omega^c} d(x, y).$$

Let ϕ be a strictly increasing function on $[0, \infty)$ such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. We say that Ω is a ϕ-John domain with *central point* (or *center*) $x' \in \Omega$ if for all $x \in \Omega$ with $x \neq x'$, there is a curve $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x$, $\gamma(l) = x'$,

$$d(\gamma(b), \gamma(a)) \leq b - a \qquad \text{for all } [a, b] \subset [0, l], \text{ and} \tag{1-1}$$

$$d(\gamma(t)) > \phi(t) \qquad \text{for all } t \in [0, l]. \tag{1-2}$$

If Ω is a ϕ-John domain for the function $\phi = \phi_s$ defined by $\phi_s(t) = c_s t^s$ for $t \leq 1$ and $\phi_s(t) = c_s t$ for $t > 1$, with $s \geq 1$, we say Ω is an s-John domain. We may assume that $0 < c_s < 1$. This definition is essentially the same as those by Smith and Stegenga [1990] and Hajłasz and Koskela [1998], who instead assume that $\phi_s(t) = c_0 t^s$ for some $c_0 > 0$ and all $t \geq 0$. For any $M > 1$, we will write $\mathcal{J}_M(t) = t/M$. As M varies, the class of $\mathcal{J}_M$-John domains is the same as the class of 1-John domains. If Ω is a $\mathcal{J}_M$-John domain for some M, then we will refer to M as the 1-John constant of Ω.

Note that (1-2) implies that $d(x) > 0$ for all $x \in \Omega$.

Definition 1.3. Let $\langle H, d\rangle$ be a quasimetric space. Given $\Omega \subset H$ and $\delta > 0$, we say that a ball $B(x, r)$ is a δ-ball if $x \in \Omega$ and $0 < r \leq \delta d(x)$. Balls of the form $B(x, r)$ with $x \in \Omega$ and $r = \delta d(x)$ will be called δ-Whitney balls.

Some useful properties of δ-balls are listed in Observation 2.1 in the next section. See also [Sawyer and Wheeden 2006], where such balls play a role in proving regularity of solutions of subelliptic equations.

For technical reasons (see, for example, the proof of Observation 2.1), whenever we consider δ-balls, we will always assume that $0 < \delta < 1/(2\kappa^2)$, where κ is the quasimetric constant in Definition 1.1. We note now that the weaker restriction $0 < \delta < 1/\kappa$ guarantees that every δ-ball is contained in Ω. In fact, let $x \in \Omega$ and $B(x, r)$ be a δ-ball with $\kappa\delta < 1$. If $y \in B(x, r)$, then

$$d(x) \leq \kappa(d(x, y) + d(y)) < \kappa(r + d(y)) \leq \kappa(\delta d(x) + d(y)).$$

Hence, $d(y) > [(1/\kappa) - \delta]d(x)$. In particular, $d(y) > 0$ and therefore $y \in \Omega$.

We next define what we mean by δ-doubling, doubling and reverse doubling.

Definition 1.4. Let $\langle H, d\rangle$ be a quasimetric space. A nonnegative finite functional σ defined on balls in H, that is, $\sigma : \{B : B \text{ is a ball in } H\} \to [0, \infty)$, will be called a *ball set* function (or a set function on balls). In practice, given $\Omega \subset H$, we will only consider balls B with $x_B \in \Omega$ and $r(B) \leq \operatorname{diam}(\Omega)$, where $\operatorname{diam}(\Omega)$ is defined using the quasimetric d. Given $\Omega \subset H$, $0 < \delta < 1/(2\kappa^2)$, and a ball set function σ, we say that σ is *δ-doubling* on Ω if there are positive constants A_σ and D_σ such that for all δ-balls $B(x, r)$ in Ω,

$$\frac{\sigma(B(x, \tilde{r}))}{\sigma(B(x, r))} \leq A_\sigma \left(\frac{\tilde{r}}{r}\right)^{D_\sigma} \quad \text{for all } 0 < r < \tilde{r} \leq \operatorname{diam}(\Omega).$$

If this inequality holds for all balls with center in Ω and $\tilde{r} \leq \operatorname{diam}(\Omega)$, we say that σ is *doubling* on Ω. If σ is also a measure[1] on Ω, we say that σ is a *δ-doubling* measure or *doubling* measure on Ω. Note that this definition is equivalent to the one in [Chua and Wheeden 2008, Definition 1.7]. In case σ is a ball set function or measure and there is a constant C such that $\sigma(2B) \leq C\sigma(B)$ for all balls $B \subset H$,

[1]Except in Theorem B below, we will assume that all measures are defined on a fixed σ-algebra that contains all balls.

we say simply that σ is doubling instead of doubling on H. Moreover, we say that σ is *reverse doubling* on Ω if there exist A, $D > 0$ such that

$$\frac{\sigma(B(x,r))}{\sigma(B(x,\tilde{r}))} \le A\Big(\frac{r}{\tilde{r}}\Big)^D \quad \text{for all } x \in \Omega, \text{ with } 0 < r < \tilde{r} \le \operatorname{diam}(\Omega). \tag{1-3}$$

Björn and Shanmugalingam [2007] gave a similar definition of doubling on Ω. Some properties of δ-doubling ball set functions are given in Proposition 2.2.

If $B(x,r) \setminus B(x,r') \ne \varnothing$ for all $0 < r' < r$, $x \in H$, we say the quasimetric space satisfies the *nonempty annuli property* in H. Similarly, we say that a set $\Omega \subset H$ has the nonempty annuli property if $(\Omega \cap B(x,r)) \setminus B(x,r') \ne \varnothing$ for all $0 < r' < r$ and $x \in \Omega$ for which Ω is not a subset of $B(x,r')$. A doubling measure on Ω satisfies a reverse condition of the same type provided Ω has the nonempty annuli property; this is similar to a fact from [Wheeden 1993, page 269].

We say that a family of balls (or cubes in the usual Euclidean case) has *bounded intercepts* if there exists a constant N such that each ball in the family intersects at most N other balls in the family. Such a family also has bounded overlaps in the pointwise sense since no point belongs to more than $N+1$ balls in the family.

Given an s-John domain with central point x' and a number $M > 1$, we distinguish two types of points x, depending on whether or not x can be connected to x' by a curve satisfying the $\mathcal{J}_M$-John condition:

Definition 1.5. Let $M > 1$ and Ω be an s-John domain with central point x'. Let Ω_g^M be the set of points x in Ω such that there is $\gamma_x : [0, l_x] \to \Omega$ such that $\gamma_x(0) = x$ and $\gamma_x(l_x) = x'$, and

$$\begin{aligned} d(\gamma_x(t_1), \gamma_x(t_2)) &\le |t_1 - t_2| \quad && \text{for } t_1, t_2 \in [0, l_x], \\ d(\gamma_x(t)) &> \mathcal{J}_M(t) && \text{for all } t \in [0, l_x]. \end{aligned}$$

We will say points in Ω_g^M are *M-good* points of Ω, and points in $\Omega \setminus \Omega_g^M = \Omega_\flat^M$ are *M-bad* points of Ω. Note that if $\Omega_g^M = \Omega$, then Ω is a 1-John domain.

A nonempty subset Ω_0 of Ω^c will be said to *confine the M-bad points of* Ω if there exists a constant $\overline{M} > 0$ such that

$$\sup_{x \in \Omega_\flat^M} \sup_{t \in [0, l_x]} d(\gamma_x(t), \Omega_0)/d(\gamma_x(t)) \le \overline{M}. \tag{1-4}$$

Note that (1-4) is the same as $d(B, \Omega_0) \le C(\kappa, \delta)\overline{M} r(B)$ for all $x \in \Omega_\flat^M$ and all δ-Whitney balls B with center along the s-John curve that connects x to x'.

Similar definitions can be given for ϕ-John domains.

In case Ω is a 1-John domain, there exists $M > 1$ such that $\Omega_g^M = \Omega$, and hence any nonempty set $\Omega_0 \subset \Omega^c$ confines the M-bad points of Ω. For any s-John domain, the choice $\Omega_0 = \Omega^c$ confines all the M-bad points of Ω, and

$d(B, \Omega^c) \le d(x_B) = \delta^{-1} r(B)$ for any δ-Whitney ball B. Moreover, in case $\Omega \subset \mathbb{R}^n$ with the usual Euclidean metric, we will show in the proof of Theorem 1.12 that if there exists $\varepsilon > 0$ such that $\Omega_0 \supset \partial\Omega \cap (\bigcup_{x \in \Omega_b^M} B(x, \varepsilon))$, then Ω_0 confines the M'-bad points of Ω for some $M' \ge M$.

If Ω is an s-John domain and c is a positive constant, then any point $x \in \Omega$ with $d(x) \ge c$ is an M-good point for suitably large M depending only on c, κ, s and c_s; the simple proof is given at the beginning of the proof of Theorem 1.12.

Before we state our first main theorem, we need to describe some chains of balls. We say a measure μ satisfies the *ratio condition* (R) on Ω if there are constants $0 < \theta_1 < \theta_2 < 1$ and $\alpha \ge 2$ such that for each $x \in \Omega$, there exists a strictly decreasing sequence $\{r_j^x\}_{j \in N}$ of positive real numbers such that

$$r_j^x \to 0, \quad r_1^x = \operatorname{diam}(\Omega), \quad r_j^x/\alpha < r_{j+1}^x < r_j^x$$

and

$$\theta_1 \le \frac{\mu(B(x, r_{j+1}^x))}{\mu(B(x, r_j^x))} \le \theta_2 \quad \text{for all } j. \tag{1-5}$$

It follows from (1-5) that $\mu(B(x, r_j^x)) \to 0$, and then the fact that $r_j^x \to 0$ is automatic since r_j^x decreases and we always assume that balls have positive μ-measure. See parts (1) and (2) of Remark 1.7 for further comments about (1-5).

Next, given any $\delta < 1/(2\kappa^2)$ and $1 \le \tau < 1/(2\delta\kappa^2)$, Proposition 2.3(c) implies that for any ϕ-John domain Ω, there is a sequence of δ-balls $\{Q_i^x\}_{i=1}^\infty$ with centers along the curve γ from x to x' guaranteed by the ϕ-John condition, such that $Q_1^x = B(x', \delta d(x'))$ and $\{Q_i^x\}$ has the intersection property

$$Q_i^x \cap Q_{i+1}^x \quad \text{contains a } \delta\text{-ball } Q_i' \text{ with } Q_i^x \cup Q_{i+1}^x \subset N Q_i'$$

for some positive constant N independent of x and i. Moreover, Q_i^x is centered at x for large i; in fact, for balls $B_j^x = B(x, r_j^x)$ as in (1-5), there exist $K_x, K_x' \in \mathbb{N}$ such that $\tau Q_{i+K_x}^x = B_{i+K_x'}^x$ for $i \ge 0$, B_j^x is a $\tau\delta$-ball if $j \ge K_x$, and Q_i^x is not centered at x if $i \le K_x$. We associate with each ball $B_j^x = B(x, r_j^x)$ for $j \ge 1$ the following special subcollection of $\{Q_i^x\}$:

$$\mathscr{C}(B_j^x) = \mathscr{C}_\phi(B_j^x) = \{Q_i^x : \tau Q_i^x \subset B_j^x \text{ and } \tau Q_i^x \not\subset B_{j+1}^x\}. \tag{1-6}$$

In case $j \ge K_x$, the set $\mathscr{C}(B_j^x)$ consists of just the single ball $\tau^{-1} B_j^x = Q_j^x$.

Our first self-improving result for s-John domains will be a consequence of a general weak-type theorem [Chua and Wheeden 2008, Theorem 1.2]; this theorem, which we now recall, is measure-theoretic and does not require the underlying structure of an s-John domain or even of a quasimetric space. In it, the sets Q_i^x and B_j^x are merely measurable sets, generally unrelated to the balls of (1-5) and (1-6).

Theorem B. *Let σ and μ be measures on a σ-algebra Σ of subsets of X. Let Ω be a measurable subset of X and f be a fixed measurable function such that the following assumptions hold for some constants satisfying*

$$\begin{aligned} &0 < p_0, q < \infty, \quad 0 < \theta_1 < \theta_2 < 1, \\ &0 < A_1, A_2 < \infty, \quad 0 < \theta < 1, \quad C_\sigma \geq 1, \quad \wp \geq 1. \end{aligned}$$

(1) *For each $x \in \Omega$, there is a sequence of measurable sets $\{Q_i^x\}_{i=1}^\infty$, depending on x, and a fixed set $B' \subset X$ such that $Q_1^x = B'$,*

$$0 < \sigma(Q_i^x \cup Q_{i+1}^x) \leq C_\sigma \sigma(Q_i^x \cap Q_{i+1}^x) < \infty \quad \textit{for } i = 1, 2, \dots, \tag{1-7}$$

and

$$\left(\frac{1}{\sigma(Q_i^x)} \int_{Q_i^x} |f - f_{Q_i^x}|^{p_0} d\sigma \right)^{1/p_0} \leq a(Q_i^x), \tag{1-8}$$

where $\{f_{Q_i^x}\}$ is a sequence of constants that converges to $f(x)$ and $\{a(Q_i^x)\}$ is a sequence of nonnegative numbers.

(2) *For each $x \in \Omega$, there is a sequence $\{B_j^x\}_{j=1}^\infty$ of measurable sets and a sequence $\{\mu^*(B_j^x)\}$ of positive numbers such that*

$$\mu(\Omega) \leq \wp \mu^*(B_1^x) \quad \textit{and} \quad A_1 \theta_1^k \leq \frac{\mu^*(B_{j+k}^x)}{\mu^*(B_j^x)} \leq A_2 \theta_2^k \quad \textit{for } j, k \in \mathbb{N}. \tag{1-9}$$

(3) *Let $\mathfrak{F} = \{B_j^x\}_{x \in \Omega, j \in \mathbb{N}}$. Assume for any $B_j^x \in \mathfrak{F}$, there is $\mathscr{C}(B_j^x) \subset \{Q_l^x\}_{l \in \mathbb{N}}$ such that $\bigcup_{j \in \mathbb{N}} \mathscr{C}(B_j^x) = \{Q_l^x\}_{l \in \mathbb{N}}$ and $\mathscr{C}(B_i^x) \cap \mathscr{C}(B_j^x) = \varnothing$ for each $x \in \Omega$ when $i \neq j$. Further, for any countable subcollection I of pairwise disjoint sets $\{B_\alpha\}$ in $\mathfrak{F}$, let*

$$A(B_\alpha) = \sum_{Q \in \mathscr{C}(B_\alpha)} a(Q)$$

and assume that

$$\sum_{B_\alpha \in I} \left(A(B_\alpha)^q \mu^*(B_\alpha) \right)^\theta \leq (C_0^q \mu(\Omega))^\theta. \tag{1-10}$$

(4) *Let the collection $\mathfrak{F}$ be a cover of Vitali type of subsets of Ω with respect to (μ, μ^*), that is, given any measurable set $E \subset \Omega$ and a collection $\mathscr{B}_E = \{B_{i(x)}^x : x \in E\}$, there is a countable, pairwise disjoint collection $\mathscr{B}'_E \subset \mathscr{B}_E$ such that*

$$\mu(E) \leq V_\mu \sum_{B_\alpha \in \mathscr{B}'_E} \mu^*(B_\alpha) \quad \textit{and} \quad V_\mu \geq 1.$$

Then

$$\sup_{t>0} t\mu\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le CC_0\big(\wp V_\mu \mu(\Omega)\big)^{1/q}, \tag{1-11}$$

where C depends on C_σ, p_0, q, A_1, A_2, θ, θ_1 and θ_2.

Note that [Chua and Wheeden 2008, Theorem 1.8] can be generalized by assuming that μ^* satisfies condition (R) instead of (1.14) there. Of course, one must change the B_j in (1.15) there accordingly.

We now revert to the context of an s-John domain and to the choice of balls made in (1-5) and (1-6). Our first self-improving result is as follows.

Theorem 1.6. *Let Ω be an s-John domain with central point x' in a quasimetric space $\langle H, d\rangle$. Let $0 < \delta < 1/(2\kappa^2)$, $1 \le \tau < 1/(2\delta\kappa^2)$ and $M > 1$. Suppose σ, μ and w are measures, σ is δ-doubling on Ω, and $a_*(B)$ is a nonnegative functional defined for all δ-balls B. Let $0 < p_0 < \infty$ and $1 \le p < \infty$, and let f and g be fixed measurable functions such that*

$$\frac{1}{\sigma(B)^{1/p_0}}\|f - f_B\|_{L^{p_0}_\sigma(B)} \le a_*(B)\|g\|_{L^p_w(\tau B)} \tag{1-12}$$

for all δ-balls B in Ω with $f_{B(x,r)} \to f(x)$ as $r \to 0$ for μ-almost all $x \in \Omega$, that is, such that (1-8) *holds with $a(B) = a_*(B)\|g\|_{L^p_w(\tau B)}$. Let Ω_0 be a nonempty subset of Ω^c that confines* (*with constant $\overline{M}$*) *the M-bad points of Ω. Set $\rho(x) = d(x, \Omega_0)$,*[2] *and for real numbers a and b, define measures μ_a and w_b by $d\mu_a = \rho^a d\mu$ and $dw_b = \rho^b dw$. Let*

$$\rho(\Omega) = \sup\{\rho(x) : x \in \Omega\}.$$

Suppose μ satisfies condition (*R*) *on Ω and there are constants η, η', β and β' with $\beta' \ge 0$ such that for all pairs of balls (B, Q) with $B = B^x_j = B(x, r^x_j)$ as in* (1-5) *and $Q \in \mathscr{C}(B)$,*

$$\mu(B)^{1/q} a_*(Q) \le C_1 r(B)^{\beta'} \tag{1-13}$$

if either $x \in \Omega^M_g$ or B is any $\tau\delta$-ball (*then $B = \tau Q$*), *and*

$$\mu(B)^{1/q} a_*(Q) \le C_1 r(B)^{\eta/q} r(Q)^{\beta - \eta'/p} \tag{1-14}$$

if $x \in \Omega^M_b$ and $r(B) \ge \tau\delta d(x)$. Let $a \ge 0$, $\eta + a \ge 0$, and

$$\varepsilon' = \beta' + \frac{a}{q} - \frac{b}{p} \ge 0, \tag{1-15}$$

$$\varepsilon = \frac{\eta + a}{q} + \min\{\chi s, \chi\} \ge 0 \quad \textit{where } \chi = \frac{s(\beta p - b - \eta') - (s-1)(p-1)}{sp}, \tag{1-16}$$

[2]See Remark 3.2 concerning the choice of $\rho(x)$ in our theorems.

and $\chi > 0$ *if* $\eta + a = 0$. *Assume further that* μ_a *satisfies the following Vitali-type condition* (*compare with condition* (4) *of Theorem B*): *given any measurable set* $E \subset \Omega$ *and a collection* $\mathcal{B}_E = \{B^x_{j(x)} : x \in E\}$, *there is a countable pairwise disjoint collection* $\mathcal{B}'_E \subset \mathcal{B}_E$ *such that*

$$\mu_a(E) \le V_a \sum_{B_\alpha \in \mathcal{B}'_E} \mu_a(B_\alpha) \quad \textit{and} \quad V_a \ge 1.$$

(i) If $p < q < \infty$, then

$$\sup_{t>0} t\mu_a\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le CC_1\left(\frac{\mu_a(\Omega)}{\mu_a(B')}\right)^{1/q} \|g\|_{L^p_{w_b}(\Omega)} \times \begin{cases} \max\{\rho(\Omega)^{\varepsilon'}, \operatorname{diam}(\Omega)^{\varepsilon}\} & \text{if } \chi \ne 0, \\ \max\{\rho(\Omega)^{\varepsilon'}, \operatorname{diam}(\Omega)^{\varepsilon}(1 + |\log \operatorname{diam}(\Omega)|)^{(p-1)/p}\} & \text{if } \chi = 0, \end{cases} \tag{1-17}$$

where $B' = B(x', \delta d(x'))$ and C depends on all parameters in the conditions but is independent of $\rho(\Omega)$ and $\operatorname{diam}(\Omega)$. If $s = 1$, neither (1-14) nor (1-16) is needed (see Remark 1.7(3)), and the weak-type constant can be chosen to have the form

$$CC_1\left(\frac{\mu_a(\Omega)}{\mu_a(B')}\right)^{1/q} \rho(\Omega)^{\varepsilon'}. \tag{1-18}$$

Here C is also independent of M, $\overline{M}$, η, η' and β.

(ii) Suppose $1 \le q \le p$ and there exist $M_1, M_2, \tilde{\eta}, \tilde{\eta}' > 0$ such that for $\lambda = \kappa + 2\kappa^2$ and all $k \in \mathbb{Z}$, the number of disjoint balls $B(x, r)$ with center $x \in \Omega^M_\flat$ and $r \ge \max\{\tau\delta d(x), \lambda^k\}$ is at most $M_1\lambda^{-\tilde{\eta}k}$, and the number of disjoint $\tau\delta$-balls B with $r(B) \ge \lambda^k$ is at most $M_2\lambda^{-\tilde{\eta}'k}$. If

$$(p-q)\tilde{\eta}/(pq) < \varepsilon \quad \text{and} \quad (p-q)\tilde{\eta}'/(pq) < \min\{\varepsilon', \beta'\}, \tag{1-19}$$

then

$$\sup_{t>0} t\mu_a\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le CC_1\left(\frac{\mu_a(\Omega)}{\mu_a(B')}\right)^{1/q} \|g\|_{L^p_{w_b}(\Omega)}, \tag{1-20}$$

where C depends on all parameters in the conditions and on $\operatorname{diam}(\Omega)$ and $\rho(\Omega)$.

Remark 1.7. (1) When Ω satisfies the nonempty annuli property, condition (1-5) will hold for $r^x_i = 2^{-i+1}\operatorname{diam}(\Omega)$ if we assume that μ is doubling on Ω since the first inequality of (1-5) will then hold because of doubling, and the second will hold since doubling implies reverse doubling; see [Chua and Wheeden 2008, Proposition 2.3].

(2) Condition (R) is implied by weaker assumptions than doubling and nonempty annuli. In fact, suppose μ is reverse doubling on Ω, and there exists $0 < \theta' < 1$ such

that for each fixed $x \in \Omega$ and $0 < r < \mathrm{diam}(\Omega)$, there exists r' with $r < r' < \mathrm{diam}(\Omega)$ and

$$\theta' \mu(B(x, r')) \le \mu(B(x, r)). \tag{1-21}$$

Then (R) holds for μ. Note that (1-21) is true for any $\theta' < 1$ if $\mu(B(x, r))$ is a right-continuous function of $r \le \mathrm{diam}(\Omega)$ for each fixed x in Ω; for Euclidean balls, this is the case whenever μ is absolutely continuous with respect to Lebesgue measure. To show that (R) holds, first choose $\alpha > 2$ such that $\theta_2 = A\alpha^{-D} < 1$, where A and D are constants in (1-3). Note that $\mu(B(x, r/\alpha))/\mu(B(x, r)) \le \theta_2$ for any $x \in \Omega$ and $0 < r \le \mathrm{diam}(\Omega)$ by (1-3). Fix any $0 < \theta_1 < \theta'\theta_2$ and define

$$r_- = \sup\{t \in [r/\alpha, r] : \mu(B(x, t)) \le \theta_2 \mu(B(x, r))\},$$
$$r_+ = \inf\{t \in [r/\alpha, r] : \mu(B(x, t)) \ge (\theta_1/\theta')\mu(B(x, r))\}.$$

Note that $r/\alpha \le r_- < r$ by left-continuity, and also that $r/\alpha \le r_+ \le r$. If $r_+ < r_-$, then for any r' with $r_+ < r' < r_-$, we have

$$\theta_1 < \theta_1/\theta' \le \mu(B(x, r'))/\mu(B(x, r)) \le \theta_2.$$

It is impossible that $r_+ > r_-$ since otherwise there exists t with $r_- < t < r_+$, and consequently $\mu(B(x, t)) > \theta_2 \mu(B(x, r))$ and $\mu(B(x, t)) < (\theta_1/\theta')\mu(B(x, r))$, yielding the contradiction $\theta_1/\theta' > \theta_2$. We now only need to handle the case $r_+ = r_-$. But, by monotonicity of measure, in case $r_- > r/\alpha$ we have

$$\mu(B(x, r_-)) = \lim_{t \to (r_-)^-} \mu(B(x, t)) \le \theta_2 \mu(B(x, r)),$$

while in case $r_- = r/\alpha$ we have $\mu(B(x, r_-)) \le \theta_2 \mu(B(x, r))$ by (1-3) as above. On the other hand, by (1-21), there exists $r' > r_- = r_+$ (and $r' < r$ as $r_- < r$) such that

$$\theta' \mu(B(x, r')) \le \mu(B(x, r_-)).$$

But $\mu(B(x, r')) \ge (\theta_1/\theta')\mu(B(x, r))$ as $r' > r_+$. Then r_- itself has the desired properties $r/\alpha \le r_- < r$ and $\theta_1 \le \mu(B(x, r_-))/\mu(B(x, r)) \le \theta_2$. In any case we can find $r/\alpha \le r' < r$ such that

$$\theta_1 \le \frac{\mu(B(x, r'))}{\mu(B(x, r))} \le \theta_2.$$

(3) Conditions (1-14) and (1-16) are not required for 1-John domains since then $\Omega_\flat^M$ is empty if M is large. Thus we only need condition (1-13) if $s = 1$. For any $s \ge 1$, we have $r(Q) \sim r(B)$ in condition (1-13), with constants depending only on τ and M, no matter whether $x \in \Omega_g^M$ and $Q \in \mathscr{C}(B)$, or whether $x \in \Omega$ and $B = \tau Q$. Hence, if μ is δ-doubling, (1-13) is equivalent to the simpler condition

$$\mu(B)^{1/q} a_*(B) \le C_1 r(B)^{\beta'} \tag{1-22}$$

for all δ-balls B.

(4) Condition (1-13) can often be replaced by the simpler (1-22) even when μ has no doubling properties. For example, suppose that $a_*(B)$ has the special monotonicity property that given $M' > 1$, there exists $c \geq 1$ such that

$$a_*(B_1) \leq c a_*(B_2) \quad \text{if } B_1 \subset B_2 \subset M' B_1. \tag{1-23}$$

Then, whether or not μ is doubling, (1-13) follows easily if (1-22) holds with $B = M'Q$ for all δ-balls Q and an appropriate constant M' depending on M, τ, s.

As an application, we obtain results about 1-John domains of the type studied in [Drelichman and Durán 2008] and [Hurri-Syrjänen 2004]. We illustrate this now in the form of a weak-type statement; however, the analogous strong-type statement is also true by using ideas related to Theorem 1.12. Consider for simplicity the case of Euclidean balls $B \subset \mathbb{R}^n$, and let

$$p_0 = 1, \quad \beta = 1, \quad d\sigma = dx, \quad 1 < p < \infty, \quad p' = p/(p-1).$$

For nonnegative locally integrable weights w_1, w_2 such that $w_2^{-1/(p-1)}$ is locally integrable, let $d\mu = w_1 dx$ and

$$\tilde{a}_*(B) = C\frac{r(B)}{|B|}\Big(\int_B w_2^{-1/(p-1)} dx\Big)^{1/p'}.$$

It is easy to see that $\tilde{a}_*(B)$ has the special monotonicity property (1-23) since $\int_B w_2^{-1/(p-1)} dx$ is truly monotone increasing in B. On the other hand, Hölder's inequality applied to the L^1, L^1 Poincaré estimate for Euclidean balls yields the following version of (1-12) involving $\tilde{a}_*(B)$, with $\beta = p_0 = 1$ and $d\sigma = dx$:

$$\begin{aligned}\frac{1}{|B|}\int_B |f - f_B|\,dx &\leq Cr(B)\frac{1}{|B|}\int_B |\nabla f|\,dx \\ &\leq C\frac{r(B)}{|B|}\Big(\int_B |\nabla f|^p w_2 dx\Big)^{1/p}\Big(\int_B w_2^{-1/(p-1)} dx\Big)^{1/p'} \\ &= \tilde{a}_*(B)\Big(\int_B |\nabla f|^p w_2 dx\Big)^{1/p}.\end{aligned}$$

Condition (1-22) takes the form

$$\Big(\int_B w_1 dx\Big)^{1/q} r(B)^{1-n}\Big(\int_B w_2^{-1/(p-1)} dx\Big)^{1/p'} \leq Cr(B)^{\beta'} \tag{1-24}$$

for $B = M'Q$ and all δ-balls Q. If $s = 1$, (1-14) is not needed as a hypothesis in Theorem 1.6 (since $\Omega_\flat^M$ is empty for 1-John domains), and if we assume the remaining hypothesis (R) for the measure $d\mu = w_1 dx$, for example if we assume (see Remark 1.7(2)) the reverse doubling condition (1-3) and note that (1-21) is automatically true since μ is absolutely continuous with respect to the Lebesgue measure, then we obtain as a corollary of Theorem 1.6(i) that for a 1-John domain Ω and $1 < p < q < \infty$,

$$\sup_{t>0} t(w_1)_a\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le C\|\nabla f\|_{L^p_{(w_2)_b}(\Omega)}$$

for the same range of a and b as in Theorem 1.6 with $\beta = 1$ and $\eta' = \eta = n$. In fact, our hypotheses are weaker than those in [Drelichman and Durán 2008], where (1-24) with $\beta' = 0$ is assumed for *all* balls B, and where both absolute continuity and reverse doubling of μ are assumed, whereas we require (1-24) for a more restricted class of balls and can assume (R) for μ rather than absolute continuity and reverse doubling.

(5) If μ_a is doubling on Ω or if Ω has the Besicovitch covering property (for example, Euclidean space has the Besicovitch property), then μ_a will satisfy the Vitali covering condition in Theorem 1.6. See also [Sawyer and Wheeden 1992; Di Fazio et al. 2008].

(6) The exponents ε and ε' in (1-17) are nonnegative by (1-15) and (1-16). We also note for future reference that (1-16) implies $(\eta + a)/q - (\eta' + b)/p + \beta \ge 0$; in fact, this is the same as $(\eta + a)/q \ge (\eta' + b - \beta p)/p$, which follows from (1-16) when $\eta' + b - \beta p > 0$ (since $s, p \ge 1$) and is obvious when $\eta' + b - \beta p \le 0$. Moreover, if we assume (1-14) for *all* B, Q with $Q \in \mathscr{C}(B)$, then (1-13) follows in case $\beta' \le \beta + \eta/q - \eta'/p$.

(7) In (1-17) of Theorem 1.6, it is often true that $\rho(\Omega) \le C(\kappa, \overline{M})\,\mathrm{diam}(\Omega)$. This clearly occurs when $\partial\Omega \cap \Omega_0 \ne \varnothing$. It is also the case when $\Omega^M_\flat \ne \varnothing$ since if there is $x_1 \in \Omega^M_\flat$, then $d(x_1, \Omega_0) \le \overline{M} d(x_1)$ by (1-4), and hence

$$d(x, \Omega_0) \le \kappa(\mathrm{diam}(\Omega) + d(x_1, \Omega_0)) \le C(\kappa, \overline{M})\,\mathrm{diam}(\Omega) \quad \text{for all } x \in \Omega.$$

Recall that $\Omega^M_\flat$ is nonempty unless Ω is a 1-John domain. If also $\mathrm{diam}(\Omega) \le 1$ and $\beta' \ge \beta + \eta/q - \eta'/p$, then $\varepsilon' = \beta' + a/q - b/p \ge \beta + (\eta + a)/q - (\eta' + b)/p$ and so both maximums in (1-17) are the corresponding last terms that involve $\mathrm{diam}(\Omega)$ because

$$\beta + \frac{\eta + a}{q} - \frac{\eta' + b}{p} = \left(\frac{\eta + a}{q} + \chi\right) + \frac{s - 1}{sp'} \ge \varepsilon.$$

(8) By definition, $\rho(x) = \mathrm{dist}(x, \Omega_0)$ for *any* subset Ω_0 of Ω^c that confines the M-bad points of Ω. Hajłasz and Koskela [1998] and Kilpeläinen and Malý [2000] assume Ω_0 to be all of Ω^c.

(9) For particular choices of η and η', condition (1-14) is a corollary of (1-13) if μ satisfies the doubling condition $\mu(\tilde{B}) \le C(r(\tilde{B})/r(B))^{D_1}\mu(B)$ for some D_1 and all pairs $B, \tilde{B}$ of balls with $B \subset \tilde{B}$, $\tilde{B}$ centered in Ω and $r(\tilde{B}) \le \mathrm{diam}(\Omega)$. In fact,

fix a ball B centered in Ω and let Q be a δ-ball in B. Then

$$\begin{aligned}\mu(B)^{1/q}a_*(Q) &\le C\left(\frac{r(B)}{r(Q)}\right)^{D_1/q}\mu(Q)^{1/q}a_*(Q)\\ &\le C\left(\frac{r(B)}{r(Q)}\right)^{D_1/q}r(Q)^{\beta'} \quad \text{by (1-13)},\end{aligned}$$

which gives (1-14) with $\eta = D_1$ and $\eta' = p\big(\beta-\beta'+D_1/q\big)$. In particular, with this version of (1-14), condition (1-16) implies

$$\frac{D_1+a}{q} \ge \frac{s(pD_1/q+b-\beta' p)+(s-1)(p-1)}{p}.$$

While these estimates are often not sharp and the η and η' obtained in this way are often undesirable, nevertheless, in the usual Euclidean case, where $\beta = 1$, $\mu = w = 1$, $D_1 = n$ and (1-13) holds with $\beta' = 1+n/q-n/p$, they yield the same conditions as in Theorem A. In fact, the version of (1-16) given above reduces to

$$\frac{n+a}{q} \ge \frac{s(n+b-1)-p+1}{p},$$

and the restriction $\beta' \ge 0$ is the same as $1/q \ge 1/p - 1/n$. Finally, (1-15) becomes $(n+a)/q-(n+b)/p+1 \ge 0$, which follows from (1-16) as explained in part (6) of this remark.

(10) By using standard interpolation techniques, we find the weak L^q estimate (1-17) implies a strong-type inequality in which the left side of (1-17) is replaced by $\|f-f_{B'}\|_{L^{q_0}_{\mu_a}(\Omega)}$ for any q_0 with $0 < q_0 < q$; see [Chua and Wheeden 2008, Remark 1.13].

(11) In Theorem 1.6, the condition $a \ge 0$ can be replaced by assuming that (1-5), (1-13) and (1-14) hold for μ_a (instead of μ), as will be clear from the proof. When Ω is a 1-John domain in $\mathbb{R}^n$ with Euclidean distance, there exists $\varepsilon > 0$ such that if $-\varepsilon < a < 0$, the weighted Lebesgue measure $\rho(x)^a dx = \operatorname{dist}(x, \Omega^c)^a dx$ is δ-doubling (and hence doubling) on Ω; see [Hajłasz and Koskela 1998, Theorem 6 and Lemma 6]. Thus, for such a (set $a = -\varepsilon_0$ for convenience), Theorem 1.8(i) below with $d\sigma = \rho^{-\varepsilon_0}dx$, $dw = dx$ and $a = 0$ can be used to deduce [Hajłasz and Koskela 1998, Theorem 8] as it is easy to see (1-12) holds with $d\sigma = \rho^{-\varepsilon_0}dx$, $dw = dx$ and $\beta = p = p_0 = 1$ and (1-13) holds with $\mu = \sigma$ and $\beta' = (n-\varepsilon_0)/q-n+1$. Note that when $\Omega_0 = \Omega^c$, we do not need to assume $\beta' \ge 0$ since then $r(B) \sim \rho(B)$ for all δ-Whitney balls. Moreover, the argument works for $1 < p \le q$ by choosing $\beta' = (n-\varepsilon_0)/q-n/p+1$.

(12) The measure μ can be replaced in (1-5), (1-13), (1-14) and the Vitali-type condition of Theorem 1.6 by $\mu|_\Omega$ since the conclusions (for example, (1-17)) are relative to Ω.

(13) For any ball $B = B(x, r)$ with $x \in \Omega$ and $r \geq 1$, the set $\mathscr{C}(B)$ will contain a δ-ball of comparable size. Since σ is δ-doubling on Ω, there can be at most a bounded number (with bound depending on A_σ, D_σ and $\operatorname{diam}(\Omega)$) of pairwise disjoint such balls $B(x, r)$.

(14) When d is a metric, the first ball B_1^x in the ratio condition (1-5) satisfies $\Omega \subset B_1^x$ and hence the factor $(\mu_a(\Omega)/\mu_a(B'))^{1/q}$ in (1-17), (1-18) and (1-20) can be replaced by 1; see the proof of Theorem 1.6 concerning the estimate of $\wp$ in (1-9).

Next, we discuss some strong-type inequalities in the special cases when $\mu = \sigma$ and $p = q = 1$ or $s = 1$. Other estimates of strong type are given in later theorems.

Theorem 1.8. *Let Ω be an s-John domain with central point x' in a quasimetric space $\langle H, d\rangle$, and let δ, τ, M, $a_*(B)$, p_0, p, β' and B' be as in Theorem 1.6. Suppose σ and w are measures and σ is δ-doubling on Ω. Also, let f and g be as before, that is,* (1-12) *holds for all δ-balls B in Ω, but we do not assume $f_{B(x,r)} \to f(x)$ σ-almost everywhere.*

(i) *Suppose $s = 1$, $q = p_0 \geq p$, $\beta' \geq 0$ and*

$$\sigma(B)^{1/q} a_*(B) \leq C_1 r(B)^{\beta'} \tag{1-25}$$

for all balls B for which there is a concentric δ-Whitney ball $\tilde{B}$ with $\lambda^{-2}\tilde{B} \subset B \subset \tilde{B}$, where $\lambda = \kappa + 2\kappa^2$. If $a \geq 0$ and $\varepsilon' = \beta' + a/q - b/p \geq 0$, then the strong-type estimate

$$\|f - f_{B'}\|_{L^q_{\sigma_a}(\Omega)} \leq C C_1\, \rho(\Omega)^{\varepsilon'} \|g\|_{L^p_{w_b}(\Omega)} \tag{1-26}$$

holds with C depending on all relevant parameters but not on $\rho(\Omega)$ or $\operatorname{diam}(\Omega)$. *The condition $\beta' \geq 0$ is not needed when $\Omega_0 = \Omega^c$ or when $r(B) \leq c\rho(B)$ for all balls B as above.*

(ii) *Suppose $s \geq 1$, $q = p_0 = p = 1$, $\beta \in \mathbb{R}$ and* (1-14) *holds with μ replaced by σ for any pair (B, Q) of balls such that $Q \subset B$, Q satisfies $\lambda^{-2}\tilde{Q} \subset Q \subset \tilde{Q}$, where $\tilde{Q}$ is the δ-Whitney ball concentric with Q, and B is a ball centered in Ω with $r(B) \leq$* $\operatorname{diam}(\Omega)$. *If $a \geq 0$, $\beta + \eta - \eta' \geq 0$ and* (1-16) *holds, then*

$$\begin{aligned} &\|f - f_{B'}\|_{L^1_{\sigma_a}(\Omega)} \\ &\qquad \leq C C_1 \max\{\rho(\Omega)^{(\eta + a - s(\eta' + b - \beta))/s}, \rho(\Omega)^{\eta + a - \eta' - b + \beta}\} \|g\|_{L^1_{w_b}(\Omega)}, \end{aligned} \tag{1-27}$$

where C depends on all relevant parameters but not on $\rho(\Omega)$ or $\operatorname{diam}(\Omega)$. *Also,* (1-27) *holds even if Ω_0 does not confine the M-bad points provided*

$$\beta+\eta/s-\eta'\geq 0. \tag{1-28}$$

Again, one can replace the conditions for σ by the corresponding ones for $\sigma|_\Omega$.

To derive a strong-type version of (1-17) better than the one in Remark 1.7(10), we recall from [Chua and Wheeden 2008] a strong-type analogue of Theorem B.

Given $\omega>0$ and a nonnegative function g, the truncation $\tau_\omega g$ is defined by

$$\tau_\omega g(x)=\min\{g(x),2\omega\}-\min\{g(x),\omega\}=\begin{cases}\omega & \text{if } g(x)\geq 2\omega,\\ g(x)-\omega & \text{if } \omega\leq g(x)<2\omega,\\ 0 & \text{if } g(x)<\omega.\end{cases}$$

Let f be a fixed measurable function on Ω and B' be a fixed measurable set in Ω. Let $f_{B',\sigma}=\int_{B'} f\,d\sigma/\sigma(B')$. For each function $\tau_\omega|f-f_{B',\sigma}|$, $\omega>0$, and each $x\in\Omega$, we assume the existence of sequences $\{B_i^x\}$, $\{Q_i^x\}$ and $\{a(Q_i^x)\}$ with properties as in Theorem B, but as there, these sequences as well as $\mathfrak{F}$ and the sets $\mathscr{C}(B)$ may depend on $\tau_\omega|f-f_{B',\sigma}|$. For easy reference, we will denote $f^\omega=\tau_\omega|f-f_{B',\sigma}|$ and write $b(Q_i^x,f^\omega)$ instead of $a(Q_i^x)$, and $\mathfrak{F}(f^\omega)$ instead of $\mathfrak{F}$, but we will not adopt new notation to indicate that $\{B_i^x\}$ and $\{Q_i^x\}$ may vary with ω. A typical example of $b(Q,g)$ is

$$b(Q,g)=b_Y(Q,g)=r(Q)^\beta\left(\frac{1}{w(Q)}\int_Q|Yf|^p\,dw\right)^{1/p}\quad\text{for } 1\leq p<\infty,$$

where Y is a differential operator with $Y1=0$, that is, with no zero order term.

Given f and $f^\omega=\tau_\omega|f-f_{B',\sigma}|$, the analogue of (1-8) that we will assume in our strong-type analogue of Theorem B is

$$\begin{aligned}\frac{1}{\sigma(Q_i^x)^{1/p_0}}\|f^\omega-(f^\omega)_{Q_i^x,\sigma}\|_{L_\sigma^{p_0}(Q_i^x)}&\leq b(Q_i^x,f^\omega),\\ (f^\omega)_{Q_i^x,\sigma}&=\frac{1}{\sigma(Q_i^x)}\int_{Q_i^x}f^\omega\,d\sigma\end{aligned}\tag{1-29}$$

for all $\omega>0$. We will also assume an analogue of (1-10): For some constants $q>0$ and $0<\theta<1$,

$$\begin{aligned}\sum_{B_\alpha\in I}\bigl(A(B_\alpha,f^\omega)^q\mu^*(B_\alpha)\bigr)^\theta&=\sum_{B_\alpha\in I}\Bigl(\Bigl[\sum_{Q\in\mathscr{C}(B_\alpha)}b(Q,f^\omega)\Bigr]^q\mu^*(B_\alpha)\Bigr)^\theta\\ &\leq\bigl(h(\Omega,f^\omega)^q\mu(\Omega)\bigr)^\theta\end{aligned}\tag{1-30}$$

for every disjoint subcollection I of $\mathfrak{F}(f^{\omega})$ and all $\omega > 0$. Here $h(\Omega, \cdot)$ is a constant that is assumed to satisfy

$$h^*(\Omega, f)^q := \sup_{\omega>0} \sum_{k=1}^{\infty} h(\Omega, f^{2^k\omega})^q < \infty. \tag{1-31}$$

Conditions (1-30) and (1-31) are stability properties of the functional b_Y under truncation similar to ones that were introduced in [Long and Nie 1991; Maz'ja 1985] and exploited in many papers such as [Franchi et al. 1995; Franchi et al. 1998; 2003].

The following strong-type analogue of Theorem B extends both [Franchi et al. 2003, Corollary 3] and [Franchi et al. 1998, Theorem 3.1].

Theorem 1.9 [Chua and Wheeden 2008, Theorem 1.10]. *Let σ and μ be measures on a σ-algebra of subsets of X, let Ω be a measurable set, and let f be a fixed measurable function. Suppose that for each $f^{\omega} = \tau_{\omega}|f - f_{B',\sigma}|$ with $\omega > 0$, there are sets $\{Q_i^x\}$ and $\{B_i^x\}$ (possibly depending on ω and f in addition to x, but with $Q_1^x = B'$ for all x) satisfying the conditions of Theorem B, but now assuming* (1-29) *instead of* (1-8), *and* (1-30) *for all $\omega > 0$ instead of* (1-10). *If* (1-31) *is true, then the strong-type Poincaré inequality*

$$\frac{1}{\mu(\Omega)} \|f - f_{B',\sigma}\|^q_{L^q_\mu(\Omega)} \le C\wp V_\mu h^*(\Omega, f)^q + \left(\frac{8}{\sigma(B')}\|f - f_{B',\sigma}\|_{L^1_\sigma(B')}\right)^q \tag{1-32}$$

holds with C as in Theorem B.

We will derive the following result as a corollary of Theorem 1.9 and use it to prove the strong-type estimate given below in Theorem 1.12.

Theorem 1.10. *Suppose that the conditions of Theorem 1.6*(*i*) *hold except that* (1-12) *is replaced by*

$$\frac{1}{\sigma(B)^{1/p_0}} \|f^{\omega} - f^{\omega}_{B,\sigma}\|_{L^{p_0}_\sigma(B)} \le a_*(B) \|Yf^{\omega}\|_{L^p_w(\tau B)} \tag{1-33}$$

for all $f^{\omega} = \tau_{\omega}|f - f_{B',\sigma}|$ with $\omega > 0$ (where f is a fixed function), and all δ-balls B, where Yf^{ω} is some function. Then when $q > p$, instead of (1-17), *the strong-type inequality*

$$\begin{aligned} &\frac{1}{\mu_a(\Omega)} \|f - f_{B',\sigma}\|^q_{L^q_{\mu_a}(\Omega)} \\ &\qquad \le \frac{\tilde{C}}{\mu_a(B')} \sup_{\omega>0} \sum_{k=1}^{\infty} \|Yf^{2^k\omega}\|^q_{L^p_{w_b}(\Omega)} + \frac{C}{\sigma(B')^q} \|f - f_{B',\sigma}\|^q_{L^1_\sigma(B')} \end{aligned} \tag{1-34}$$

holds, where $\tilde{C}$ is an absolute constant times those in (1-17) *and* (1-18).

Remark 1.11. In many applications, the right side of (1-34) can be reduced to a multiple of $\|Yf\|^q_{L^p_{w_b}(\Omega)}$. For example, since $q > p$, it is true for a differential operator Y on Euclidean space that

$$\sum_k \|Yf^{2^k\omega}\|^q_{L^p_{w_b}(\Omega)} \le \Big(\sum_k \|Yf^{2^k\omega}\|^p_{L^p_{w_b}(\Omega)}\Big)^{q/p} \le C\|Yf\|^q_{L^p_{w_b}(\Omega)} \quad \text{for } \omega > 0. \tag{1-35}$$

Moreover, the second term on the right side of (1-34) is often bounded by a multiple of $\|Yf\|_{L^p_{w_b}(\Omega)}$. For instance, if we assume (1-33) holds for f on the ball B' and $p_0 \ge 1$, then

$$\begin{aligned}\frac{1}{\sigma(B')}\|f - f_{B',\sigma}\|_{L^1_\sigma(B')} &\le \frac{1}{\sigma(B')^{1/p_0}}\|f - f_{B',\sigma}\|_{L^{p_0}_\sigma(B')} \\ &\le a_*(B')\|Yf\|_{L^p_w(\tau B')} \le a_*(B')\|Yf\|_{L^p_{w_b}(\Omega)}.\end{aligned}$$

Our next result, a corollary of Theorem 1.10, contains Theorem A in the special case that $\Omega_0 = \Omega^c$. We do not require that $b \ge 1-n$ and we consider more general types of distance weights than those in Theorem A. Also, we include the case $p \ge q \ge 1$.

Theorem 1.12. *Suppose that $s \ge 1$ and $\Omega \subset \mathbb{R}^n$ is an s-John domain with respect to ordinary Euclidean distance d_E. Let $0 < \delta < 1/2$ and $B' = B(x', \delta d_E(x'))$ be the δ-Whitney ball centered at the central point x' of Ω. Suppose $\varepsilon > 0$, $M > 1$ and that Ω_0 satisfies*

$$\partial\Omega \cap \Big(\bigcup_{x\in\Omega_\flat^M} B(x,\varepsilon)\Big) \subset \Omega_0 \subset \Omega^c, \tag{$*$}$$

and set $\rho(x) = d_E(x, \Omega_0)$. Let $a \ge 0$, $b \in \mathbb{R}$, and p, q satisfy $1 \le p, q < \infty$ and $1/q \ge 1/p - 1/n$. If either $q > p$ and

$$\frac{s(n+b-1) - p + 1}{(n+a)p} \le \frac{1}{q}, \tag{1-37}$$

or if $p \ge q$ and both

$$\frac{s(n+b-1) - p - n + 1}{p} < \frac{a}{q} \quad \text{and} \quad 1 + \frac{a}{q} - \frac{b}{p} > 0, \tag{1-38}$$

then there is a constant C, depending on all relevant parameters, diam(Ω) *and* $\rho(\Omega)$, *such that*

$$\|f - C(\Omega, f)\|_{L^q_{\rho^a dx}(\Omega)} \le C\|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} \quad \text{for } f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega). \tag{1-39}$$

Here $C(\Omega, f)$ can be chosen to be either

$$\frac{1}{|B'|}\int_{B'} f\,dx \quad \textit{or} \quad f_{\mathfrak{D},\rho^a dx} = \frac{1}{|\mathfrak{D}|_{\rho^a dx}}\int_{\mathfrak{D}} f\rho^a\,dx$$

for any $\mathfrak{D} \subset \Omega$ *with* $|\mathfrak{D}| > 0$. *In case* $C(\Omega, f) = f_{\mathfrak{D},\rho^a dx}$, *the constant* C *also depends on the ratio* $|\Omega|_{\rho^a dx}/|\mathfrak{D}|_{\rho^a dx}$. *Furthermore, for* $s = 1$, (1-39) *remains valid even if* $p = q$ *when* (1-37) *holds. In case* $p = q = 1$, (1-39) *holds if*

$$n + a + s(1 - b - n) \geq 0,$$

as opposed to the strict inequality required in (1-38) *when* $p = q$. *Moreover, it remains true for any nonempty set* $\Omega_0 \subset \Omega^c$ *if* $1 \leq s \leq n/(n-1)$, *that is, for such* s, *the restriction that* $\partial\Omega \cap (\bigcup_{x \in \Omega_b^M} B(x, \varepsilon)) \subset \Omega_0$ *is not needed.*

Remark 1.13. (1) The average $f_{\mathfrak{D},\rho^a dx}$ is well defined if

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} < \infty;$$

this follows as usual by first applying (1-39) with $C(\Omega, f)$ chosen to be $|B'|^{-1} \int_{B'} f\,dx$.

(2) The case $p = q = 1$ is also considered in [Hajłasz and Koskela 1998], except that $b \geq 1 - n$ is assumed there.

(3) If $s = 1$, then $\Omega_g^M = \Omega$ for some $M > 1$, and Theorems 1.12 and 1.10 are generalizations of results in [Chua 2006] and [Chua 2001], where the weights are assumed to be doubling on all of $\mathbb{R}^n$.

(4) The range of q is sharp; see [Hajłasz and Koskela 1998] for the case $q > p$.

As mentioned earlier, the q range in Theorem 1.12 can be enlarged for special s-John domains. Some results of this type are given in Section 3. In particular, for $s > 1$, we will consider the following typical s-cusp domain, which is an s-John domain:

$$D = \{(z, z') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < z < 4, |z'| < z^s\}.$$

The next result extends [Kilpeläinen and Malý 2000, Example 2.4], where the case $\mathfrak{D}_0 = \mathfrak{D}^c$ (equivalently, $\mathfrak{D}_0 = \partial\mathfrak{D}$) is mentioned.

Theorem 1.14. *Let* $\mathfrak{D}$ *be the* s*-cusp domain above, let* $\mathfrak{D}_0$ *be a subset of* $\mathfrak{D}^c$, *and let* $\rho(x) = d_E(x, \mathfrak{D}_0)$. *Suppose* $a \geq 0$, $b \in \mathbb{R}$, $1 \leq p < q$, *and*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n}. \tag{1-40}$$

(1) *If there exists* $\varepsilon > 0$ *such that* $B((0,0), \varepsilon) \cap \partial\mathfrak{D} \subset \mathfrak{D}_0$ *and*

$$\frac{1}{q} \geq \frac{s(n + b - p) + (s-1)(p-1)}{p(s(n-1) + 1 + a)}, \tag{1-41}$$

then

$$\|f - f_{\mathfrak{D},\rho^a dx}\|_{L^q_{\rho^a dx}(\mathfrak{D})} \leq C \|\nabla f\|_{L^p_{\rho^b dx}(\mathfrak{D})} \quad \text{for all } f \in Lip_{\mathrm{loc}}(\mathfrak{D}). \tag{1-42}$$

(2) *If* $\mathscr{D}_0 = \partial\mathscr{D}$ *and*

$$\frac{1}{q} \geq \frac{s(n+b-p)+(s-1)(p-1)}{p(s(n-1+a)+1)}, \tag{1-43}$$

then (1-42) *holds for all* $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathscr{D})$.

The q range in (1-43) is larger than in (1-41), and the range in (1-41) is larger than in Theorem 1.12. Results for $p \geq q$ can also be obtained by similar methods.

2. Preliminaries

In general, we will not attempt to give very precise values for constants that arise in the proofs, although we will keep track of important parameters on which constants depend. We will consistently use the notation

$$\lambda = \kappa + 2\kappa^2.$$

The constant λ arises naturally in Observation 2.1 and Proposition 2.2 and for simplicity we often use it in estimates in which better constants could be chosen.

We now recall several useful geometric facts, which require only that d be a quasimetric.

Observation 2.1 [Chua and Wheeden 2008, Observation 2.1]. (1) *If* $z \in B(x,r)$, *then*

$$B(z,r) \subset 2\kappa B(x,r) \subset \lambda B(z,r).$$

(2) *Let* B_1 *and* B_2 *be balls with* $B_1 \cap B_2 \neq \varnothing$. *Then*

(a) $B_2 \subset \lambda \max\{r(B_2)/r(B_1), 1\} B_1$.

(b) *If in addition both* B_1 *and* B_2 *are* δ*-balls with* $\delta < 1/(2\kappa^2)$, *then*

$$\lambda^{-1} d(x_{B_2}) \leq d(x_{B_1}) \leq \lambda d(x_{B_2}).$$

Thus if B_1 *and* B_2 *are intersecting* δ*-Whitney balls, then*

$$\lambda^{-1} \leq r(B_2)/r(B_1) \leq \lambda \quad \textit{and} \quad \lambda^{-2} B_1 \subset B_2 \subset \lambda^2 B_1.$$

(c) *If* $\delta < 1/(2\kappa^2)$ *and* z *is in a* δ*-ball* $B(x,r)$, *then*

$$\frac{1}{2\kappa} \leq \frac{d(x)}{d(z)} \leq 2\kappa.$$

Next, we list some facts about δ-doubling set functions on balls.

Proposition 2.2. (1) *If* $0 < \delta_1, \delta_2 < 1/\kappa$ *and* σ *is* δ_1*-doubling on* Ω, *then* σ *is also* δ_2*-doubling on* Ω.

(2) *Let* σ *be a measure on* Ω. *If* σ *is* δ*-doubling on balls in* Ω *and* $\sigma|_\Omega$ *is defined by* $\sigma|_\Omega(B) = \sigma(B \cap \Omega)$ *for balls* $B \subset H$, *then* $\sigma|_\Omega$ *is also* δ*-doubling since* $\sigma|_\Omega$ *and* σ *are the same on* δ*-balls.*

(3) *If Ω is a ϕ-John domain and $M > 1$, then for any x and r that satisfy $x \in \Omega_g^M$ and $\delta d(x) \le r \le \operatorname{diam}(\Omega)$, there is a δ-ball Q such that $Q \subset B(x,r)$ and $r \le c_2 r(Q)$ with c_2 depending only on κ, δ, $\operatorname{diam}(\Omega)/d(x')$ and M.*

(4) *If Ω is a 1-John domain, then the notions of δ-doubling on Ω and doubling on Ω are equivalent.*

Proof. Parts (1) and (2) are easy to show, and we will only prove (3) and (4).

Proof of (3): Let x, r be as in part (3) and $B' = B(x', \delta d(x'))$. If $B' \subset B(x,r)$, then since

$$r \le \operatorname{diam}(\Omega) = \frac{\operatorname{diam}(\Omega)}{d(x')} d(x') = \frac{1}{\delta}\frac{\operatorname{diam}(\Omega)}{d(x')} r(B'),$$

we may choose $Q = B'$ and $c_2 \ge \operatorname{diam}(\Omega)/(\delta d(x'))$. If $B' \not\subset B(x,r)$, we let $\gamma : [0,l] \to \Omega$ be a 1-John curve that connects x to x' and define

$$t_0 = \sup\{t \in [0,l] : B(\gamma(t), \delta d(\gamma(t))) \subset B(x,r)\}.$$

Clearly $0 \le t_0 \le l$.

Claim: There exist t_1 and t_2 with $0 \le t_1 \le t_0 \le t_2 \le l$ such that the balls

$$Q_1 = B(\gamma(t_1), \delta d(\gamma(t_1))) \quad \text{and} \quad Q_2 = B(\gamma(t_2), \delta d(\gamma(t_2)))$$

satisfy

$$Q_1 \subset B(x,r), \quad Q_2 \not\subset B(x,r), \quad x_{Q_2} = \gamma(t_2) \in Q_1.$$

We will prove the claim by considering 2 cases.

Case (i): $B(\gamma(t_0), \delta d(\gamma(t_0))) \subset B(x,r)$. In this case, $t_0 < l$ since we have assumed $B' \not\subset B(x,r)$. We then choose $t_1 = t_0$ and $t_2 = t_0 + \varepsilon < l$ for sufficiently small $\varepsilon > 0$ such that $\gamma(t_2) \in B(\gamma(t_0), \delta d(\gamma(t_0)))$, using the fact that $d(\gamma(t_2), \gamma(t_0)) \le |t_2 - t_0|$.

Case (ii): $B(\gamma(t_0), \delta d(\gamma(t_0))) \not\subset B(x,r)$. In this case, $t_0 > 0$ since $\gamma(0) = x$ and $\delta d(x) \le r$. We then let $t_2 = t_0$ and pick $t_1 < t_0$ such that $Q_1 \subset B(x,r)$ and $|t_1 - t_0| < \delta d(\gamma(t_0))/\lambda$. Clearly $\gamma(t_1) \in B(\gamma(t_0), \delta d(\gamma(t_0)))$, and hence by Observation 2.1(2b),

$$d(\gamma(t_1), \gamma(t_0)) \le |t_1 - t_0| < \delta d(\gamma(t_0))/\lambda \le \delta d(\gamma(t_1)).$$

Therefore $\gamma(t_2) = \gamma(t_0) \in Q_1$. This completes the proof of the claim.

With t_1 and t_2 as in the claim, set $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$. Let us show that there exists $c_1 > 0$ depending on κ, δ and M such that $d(x_1) > c_1 r$. To this end, pick $z \in Q_2$ with $z \notin B(x,r)$. Then

$$r \le d(z,x) \le \kappa(d(x_1,x) + \kappa(d(x_2,x_1) + d(x_2,z))) \le \kappa(t_1 + \kappa(\delta d(x_1) + \delta\lambda d(x_1)))$$

by Observation 2.1(2b). Also $d(x_1) = d(\gamma(t_1)) > t_1/M$, and it is now clear that $d(x_1) \geq C(M, \kappa, \delta)r$. Thus, after choosing $Q = Q_1$, we have $Q \subset B(x, r)$, $r(Q) = \delta d(x_1)$ and $r \leq c_2 r(Q)$.

Proof of (4): It is clear that if σ is doubling on Ω, then it is also δ-doubling on Ω. Next, suppose σ is δ-doubling on Ω with $\sigma(2^k B) \leq c^k \sigma(B)$ for all δ-balls B in Ω and all positive integers k. Let us show that $d(x') \sim \operatorname{diam}(\Omega)$ with constants of equivalence depending only on κ and M. Indeed, choose $x_0 \in \Omega$ such that $d(x_0, x') > C(\kappa) \operatorname{diam}(\Omega)$ and let $\gamma : [0, l] \to \Omega$ be a 1-John curve that connects x_0 to x', that is,

$$d(\gamma(s_1), \gamma(s_2)) \leq |s_1 - s_2| \quad \text{and} \quad d(\gamma(t)) > t/M \quad \text{if } t, s_1, s_2 \in [0, l].$$

Then $l \geq d(x_0, x')$ and $d(x') = d(\gamma(l)) > l/M \geq C(\kappa, M) \operatorname{diam}(\Omega)$, while the opposite inequality $d(x') \leq \operatorname{diam}(\Omega)$ is obvious.

Let $B(x, r)$ be a ball with $\delta d(x) < r \leq \operatorname{diam}(\Omega)$ and $x \in \Omega$. By part (3), we can find a δ-ball Q such that $Q \subset B(x, r)$ and $r \leq c_2 r(Q)$, with c_2 depending only on κ, δ and the 1-John constant M of Ω. Hence by Observation 2.1(2a), $B(x, r) \subset CQ$ with C depending on κ, δ and M, and by Observation 2.1(1), $2^k B(x, r) \subset C 2^k Q$. Consequently,

$$\sigma(2^k B(x, r)) \leq \sigma(C 2^k Q) \leq C(\kappa, \delta, M) c^k \sigma(Q) \leq C(\kappa, \delta, M) c^k \sigma(B(x, r)),$$

where the second inequality follows from the fact that Q is a δ-ball. This completes the proof of part (4). □

The next proposition guarantees the existence of a covering of a ϕ-John domain by balls with Whitney-like properties, as well as with extra properties that are useful for deriving weighted Poincaré estimates.

Proposition 2.3 [Chua and Wheeden 2008, Proposition 2.6]. *Let $\langle H, d\rangle$ be a quasi-metric space and $0 < \delta < 1/(2\kappa^2)$. Suppose $\Omega \subset H$, there is a δ-doubling measure μ on Ω with doubling constant D_μ, and $d(x) = d(x, \Omega^c) > 0$ for all $x \in \Omega$. Then there exists a covering $W = \{B_i\}$ of Ω by δ-balls B_i with the following properties*:

(a) $r(B_i) \leq \delta d(x_{B_i}) \leq \lambda^2 r(B_i)$, *where x_{B_i} is the center of B_i.*

(b) *For every $\tau \geq 1$ that satisfies $\tau\delta < 1/(2\kappa^2)$, there is a constant K depending only on τ, κ and D_μ such that the balls $\{\tau B_i : B_i \in W\}$ have bounded intercepts with bound K; in particular, the balls $\{\tau B_i : B_i \in W\}$ also have pointwise bounded overlaps with overlap constant K.*

(c) *Let $x' \in \Omega$ and ϕ be a strictly increasing function on $[0, \infty)$ that satisfies $\phi(0) = 0$ and $\phi(t) < t$ for all t. Then for each $x \in \Omega$ for which there is a curve $\gamma : [0, l] \to \Omega$ satisfying $\gamma(0) = x$ and $\gamma(l) = x'$ and the ϕ-John properties* (1-1) *and* (1-2), *there exists a finite chain of δ-balls $\{B_i\}_{i=0}^L \subset W$, depending* *on x and with $L = L_x$, such that $x \in B_0$, $x' \in B_L$, B_L is independent of x and satisfies $\lambda^{-2}B(x', \delta d(x')) \subset B_L \subset B(x', \delta d(x'))$, $B_i \cap B_{i+1}$ contains a δ-ball B_i' with $B_i \cup B_{i+1} \subset \lambda^4 B_i'$ for all i, and*

$$B_0 \subset \frac{\lambda^2\phi^{-1}(2\kappa\lambda^2 r(B_i)/\delta)}{r(B_i)} B_i \quad \text{for all } i. \tag{2-1}$$

Also, there is a finite chain of δ-Whitney balls $\{\mathcal{Q}_i\}_{i=0}^L$ depending on x with bounded intercepts and centers on γ such that

$$\mathcal{Q}_0 = B(x, \delta d(x)), \quad \mathcal{Q}_L = B(x', \delta d(x')), \quad \lambda^{-2}\mathcal{Q}_i \subset B_i \subset \mathcal{Q}_i,$$

and $\mathcal{Q}_i \cap \mathcal{Q}_{i+1}$ contains a δ-ball $\mathcal{Q}_i'$ with $\mathcal{Q}_i \cup \mathcal{Q}_{i+1} \subset \lambda^6\mathcal{Q}_i'$. Note that the last ball $\mathcal{Q}_L$ in the chain does not depend on x.

(d) *Let x', ϕ, x and $\{\mathcal{Q}_i\}$ be as in* (*c*). *If $\mathcal{Q}_i \not\subset B(x, r)$, then $r(\mathcal{Q}_i) \geq \delta\phi(r/(2\kappa))$.*

(e) *Let x, γ and $\{\mathcal{Q}_i\}$ be as in* (c). *For all $\varepsilon > 0$, the number of* disjoint *$\mathcal{Q}_i$ having radius between ε and 2ε is at most $2\phi^{-1}(2\varepsilon/\delta)/\varepsilon$. In particular, if $\phi = \mathcal{J}_M$, the number of disjoint $\mathcal{Q}_i$ with radius between $\delta\varepsilon/(4\kappa^2 M)$ and $4\kappa\varepsilon$ is at most a constant depending only on δ, κ and M.*

Finally, the next result gives a simple extension of [Chua 1993, Theorem 1.5]:

Proposition 2.4 [Chua and Wheeden 2008, Theorem 2.9]. *Let Ω be a domain in a quasimetric space with quasimetric constant κ, and let $0 < \delta < 1/(2\kappa^2)$. Suppose Ω is covered by a countable collection W of δ-balls such that for some $N \geq 1$,*

(i) *$\sum_{B \in W} \chi_B \leq N\chi_\Omega$, and*

(ii) *there is a central ball $B_0 \in W$ that can be connected with every ball $B \in W$ by a finite chain of balls $B_0, B_1, \dots, B_{k(B)} = B$ from W such that $B \subset NB_j$ for all j and each $B_j \cap B_{j+1}$ contains a ball B_j' with $B_j \cup B_{j+1} \subset NB_j'$.*

(*Domains satisfying* (i) *and* (ii) *are often called Boman chain domains.*)

Let f be a function on Ω and f_B be an associated constant for every $B \in W$. If w is a δ-doubling measure on Ω and $1 \leq q < \infty$, then

$$\|f - f_{B_0}\|^q_{L^q_w(\Omega)} \leq C \sum_{B \in W} \|f - f_B\|^q_{L^q_w(B)}, \tag{2-2}$$

where C depends only on κ, q, N and the doubling constant of w.

Remark 2.5. It is easy to see from parts (a)–(c) of Proposition 2.3 with $\phi = \mathcal{J}_M$ that 1-John domains satisfy the Boman chain condition. The converse is also true if the domain is assumed to satisfy a segmental geodesic condition; for this fact on metric spaces, see [Buckley et al. 1996].

The proof mentioned in [Chua and Wheeden 2008] only works if w is a doubling measure on Ω. It is true that δ-doubling measures are doubling on 1-John domains.

However, a Boman chain domain may not be a 1-John domain. Thus, in order to prove Proposition 2.4, one must modify [Chua and Wheeden 2008, Lemma 2.8] by assuming that w is δ-doubling and that the family of balls consists of δ-balls. The modified lemma can be proved by considering the Hardy–Littlewood maximal function with respect to δ-balls instead of all balls.

3. Proofs of the main theorems

Proof of Theorem 1.6. Let Ω be an s-John domain, let $M > 1$ and let Ω_0 be a nonempty subset of Ω^c that confines the M-bad points of Ω. Set $\rho(x) = d(x, \Omega_0)$ and $d\mu_a = \rho^a \, d\mu$. For any ball B, let

$$\rho(B) = \sup\{\rho(x) : x \in B\}, \quad \rho^*(B) = \rho(B) + r(B), \quad \mu_a^*(B) = \rho^*(B)^a \mu(B).$$

Note that $\mu_a(B) \le \mu_a^*(B)$ when $a \ge 0$.

Let us show that ρ is essentially constant on any δ-ball B for $\delta < 1/(2\kappa)^2$. In fact, if $x, y \in B$, then

$$\rho(y) = d(y, \Omega_0) \le \kappa(d(y, x) + d(x, \Omega_0)) \le \kappa(2\kappa r(B) + \rho(x)).$$

But

$$r(B) \le \delta d(x_B) \sim d(x) \le d(x, \Omega_0) = \rho(x),$$

and we get $\rho(y) \le C\rho(x)$ by combining inequalities. It's also true that $r(B) \le \rho(B)$ for any δ-ball B. Otherwise we would have $d(x_B, \Omega_0) < r(B)$, so there would exist $z \in \Omega_0$ with $d(x_B, z) < r(B)$, and then $z \in B \cap \Omega_0$, while B must lie in Ω since it is a δ-ball. Hence, if B is a δ-ball and $\delta < 1/(2\kappa)^2$, there is a positive constant $C(\kappa) \le 1$ such that

$$C(\kappa)\rho(B) \le \rho(x) \le \rho(B) \text{ for all } x \in B, \quad \text{and} \quad r(B) \le \rho(B). \tag{3-1}$$

Let us show that

$$C(\kappa)\frac{r(B)}{r(\tilde{B})} \le \frac{\rho^*(B)}{\rho^*(\tilde{B})} \le 1 \quad \text{for all concentric balls } B \subset \tilde{B}. \tag{3-2}$$

The second inequality holds since $\rho(B) \le \rho(\tilde{B})$ and $r(B) \le r(\tilde{B})$ for such B and $\tilde{B}$. Also then $\rho(\tilde{B}) \le \kappa(\rho(B) + 2\kappa r(\tilde{B}))$ and hence there is a constant c_1 depending on κ such that

$$\rho^*(\tilde{B}) \le c_1(\rho(B) + r(\tilde{B})).$$

We now consider two cases:

Case $\rho(B) \ge r(\tilde{B})$: Then

$$\frac{\rho^*(B)}{\rho^*(\tilde{B})} \ge \frac{\rho(B)}{c_1(\rho(B) + r(\tilde{B}))} \ge \frac{1}{2c_1} \ge \frac{r(B)}{2c_1 r(\tilde{B})}.$$

Case $\rho(B) < r(\tilde{B})$: Then

$$\frac{\rho^*(B)}{\rho^*(\tilde{B})} \geq \frac{r(B)}{c_1(\rho(B) + r(\tilde{B}))} \geq \frac{r(B)}{2c_1 r(\tilde{B})}.$$

It follows that (3-2) holds.

Let $a \geq 0$. By hypothesis, μ satisfies (1-5). We will now show that under the hypothesis of Theorem 1.6, conditions (1)–(4) in Theorem B hold with μ, μ^* there replaced by μ_a, μ_a^*, and with $B' = B(x', \delta d(x'))$. Recall that $B_j^x = B(x, r_j^x)$ for $x \in \Omega$ are balls as in (1-5) and satisfy $r_1^x = \operatorname{diam}(\Omega)$, $r_j^x/\alpha < r_{j+1}^x < r_j^x$ for some $\alpha \geq 2$, and $r_j^x \to 0$. We next define $\{Q_i^x\}_{i=1}^{\infty}$ for $x \in \Omega$ by letting $\{\mathcal{Q}_i\}_{i=0}^{L}$ be as in Proposition 2.3 and defining $\{Q_i^x\}_{i=1}^{L+1}$ by

$$Q_1^x = \mathcal{Q}_L = B', \quad Q_2^x = \mathcal{Q}_{L-1}, \quad \dots, \quad Q_{L+1}^x = \mathcal{Q}_0 = B(x, \delta d(x)).$$

Note that there exists l such that $B_{l+1}^x \subset \tau\mathcal{Q}_0 \subset B_l^x$ and $B_{l+1}^x \neq \tau\mathcal{Q}_0$. We then define $Q_{L+i+1}^x = \tau^{-1} B_{l+i}^x$ for $i \geq 1$. Then since σ is δ-doubling by hypothesis, (1-7) follows from $r_j^x \sim r_{j+1}^x$ and from Proposition 2.3(c) since the balls $\mathcal{Q}_i'$ there are δ-balls. Also (1-8) holds with $a(Q) = a_*(Q)\|g\|_{L^p_w(\tau Q)}$ by (1-12), so condition (1) of Theorem B holds for $\{Q_i^x\}$ with this choice of $a(Q)$.

By (1-5) for μ and (3-2), μ_a^* satisfies the ratio estimate in (1-9) for $\{B_j^x\}$ with θ_1 be replaced by θ_1/α^a and θ_2 remaining the same. Moreover, since $B' = Q_1^x \subset \cup_i Q_i^x \subset B_1^x$, we have $\mu_a(B') \leq \mu_a(B_1^x) \leq \mu_a^*(B_1^x)$ and

$$\mu_a(\Omega) = \frac{\mu_a(\Omega)}{\mu_a^*(B_1^x)}\mu_a^*(B_1^x) \leq \frac{\mu_a(\Omega)}{\mu_a(B')}\mu_a^*(B_1^x).$$

Hence the first estimate in (1-9) holds for the pair μ_a, μ_a^* with $\wp = \mu_a(\Omega)/\mu_a(B')$, and we have verified condition (2) of Theorem B.

We will now verify condition (3). The partitioning properties follow easily and we only need to check (1-10) for (μ_a, μ_a^*). Let us show that (1-10) holds with $\theta = p/q$ for p and q as in Theorem 1.6. Let I be a collection of disjoint balls $\{B_j\}$ in $\{B_l^x : x \in \Omega, l \in \mathbb{N}\}$. Consider first those B_j that are $\tau\delta$-balls, so that $A(B_j) = a(Q_j)$ where $B_j = \tau Q_j$. Since $\rho(B_j) \geq r(B_j)$ by (3-1), we have

$$\begin{aligned} A(B_j)\mu_a^*(B_j)^{1/q} &= a(Q_j)\mu_a^*(B_j)^{1/q} = a(Q_j)\mu(B_j)^{1/q}\rho^*(B_j)^{a/q} \\ &\leq a_*(Q_j)\|g\|_{L^p_w(\tau Q_j)}\mu(B_j)^{1/q}(2\rho(B_j))^{a/q} \\ &\leq C a_*(Q_j)\|g\|_{L^p_{w_b}(\tau Q_j)}\mu(B_j)^{1/q}\rho(B_j)^{a/q-b/p} \end{aligned}$$

since $\rho(z) \sim \rho(B_j)$ for all $z \in B_j$ by (3-1). Here C and the constants of equivalence depend at most on p, q, a, b, κ and τ. For the rest of the proof, C and various constants of equivalence are positive and may depend on these parameters and many others, but not on the constant C_1 in (1-13) and (1-14). Continuing the estimate above, we obtain

$$
\begin{aligned}
&\le CC_1 \, \|g\|_{L^p_{w_b}(\tau Q_j)} \rho(B_j)^{a/q-b/p} r(B_j)^{\beta'} && \text{by (1-13)} \\
&\le CC_1 \, \|g\|_{L^p_{w_b}(\tau Q_j)} \rho(B_j)^{a/q-b/p+\beta'} && \text{since } \beta' \ge 0 \\
&\le CC_1 \, \rho(\Omega)^{a/q-b/p+\beta'} \|g\|_{L^p_{w_b}(\tau Q_j)} && \text{since } \beta' + \frac{a}{q} - \frac{b}{p} \ge 0 \text{ by (1-15)} \\
&= CC_1 \rho(\Omega)^{a/q-b/p+\beta'} \|g\|_{L^p_{w_b}(B_j \cap \Omega)}
\end{aligned}
$$

since $B_j = \tau Q_j \subset \Omega$ in the present case. Here $\rho(\Omega) = \sup_{z\in\Omega} \rho(z)$ as usual.

Next consider a typical B_j that is not a $\tau\delta$-ball: $B_j = B(x_j, r_j)$, $x_j = x_{B_j} \in \Omega$ and $r_j = r(B_j) > \tau\delta d(x_j)$. We will now use the notion of M-good and M-bad points to extend the notion of an s-John domain by allowing the function ϕ to vary with the starting point x of the curve γ, using ϕ_s and $\mathscr{J}_M$ in Definition 1.2 for M-bad points and M-good points respectively:

Convention 3.1. We adopt a convention for choosing curves that connect points x of the s-John domain Ω to the central point x': If $x \in \Omega^M_g$, we choose the curve γ from x to x' that corresponds to picking $\phi = \mathscr{J}_M$, while if $x \in \Omega^M_\flat$, we choose the curve corresponding to $\phi = \phi_s$.

We abuse earlier terminology by referring to these curves as curves from x to x' *guaranteed by the $\phi_{s,M}$-John condition*. Furthermore, given x and t, we denote by $\phi_{s,M}(t)$ either $\phi_s(t)$ or $\mathscr{J}_M(t)$, depending on whether $x \in \Omega^M_\flat$ or $x \in \Omega^M_g$.

Let γ be the $\phi_{s,M}$-John curve connecting x_j to x', with $\phi_{s,M}$ equal to either $\mathscr{J}_M$ or ϕ_s depending on whether x_j is an M-good or M-bad point. The balls Q_i in $\mathscr{C}(B_j)$ are δ-Whitney balls centered on γ, and they lie in B_j and (by (1-6) and Proposition 2.3(d)) satisfy $\tau r(Q_i) \ge \delta\phi_{s,M}(r_j/(2\alpha\kappa))$. Furthermore, the enlarged balls τQ_i lie in B_j by definition (see (1-6)), and they have bounded intercepts as we now show. In fact, Observation 2.1(2b) applied with δ replaced by $\tau\delta$ to the $\tau\delta$-balls τQ_i shows that if two such $\tau Q_i, \tau Q_k$ intersect, then $d(x_{Q_i}) \sim d(x_{Q_k})$, and therefore $r(Q_i) \sim r(Q_k)$. Then τQ_i have bounded intercepts (for example, this follows from [Chua and Wheeden 2008, Lemma 2.5] applied with $\mathscr{F} = \{Q_i\}$ and $N = \tau$).

Let $I_j = \{Q_i\}$ be a family of *disjoint* balls in $\mathscr{C}_{\phi_{s,M}}(B_j)$. Since balls in $\mathscr{C}_{\phi_{s,M}}(B_j)$ have bounded intercepts uniformly in j, it is a union of a bounded number of families of disjoint balls; see the proof of [Chua and Wheeden 2008, Lemma 2.5]. Because of disjointness, we know by Proposition 2.3(e) that the number of Q_i in I_j with radius between ε and 2ε is at most $(2/\varepsilon)\phi^{-1}_{s,M}(2\varepsilon/\delta)$. Our strategy for verifying (1-10) will be first to estimate the portion of $A(B_j)\mu^*_a(B_j)^{1/q}$ that corresponds to summing over I_j, and then to sum over different I_j and different B_j.

First suppose that x_j is an M-bad point of Ω and $r_j \le 1$. In case $s = 1$, we choose M (depending on c_s) so large that there are no M-bad points, and so we may now assume $s > 1$. Since x_j is M-bad, we have $r(Q_i) \ge Cr_j^s$ for all $Q_i \in I_j$. For the part of $A(B_j)\mu_a^*(B_j)^{1/q}$ that corresponds to summing over I_j, we have in case $p > 1$,

$$\begin{aligned}\sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &= \sum_{Q_i \in I_j} a_*(Q_i)\|g\|_{L^p_w(\tau Q_i)}\mu_a^*(B_j)^{1/q}\\ &\le C\sum_{Q_i \in I_j} a_*(Q_i)\rho(Q_i)^{-b/p}\|g\|_{L^p_{w_b}(\tau Q_i)}\,\mu_a^*(B_j)^{1/q}\\ &\qquad\qquad\text{since } \rho(z_1) \sim \rho(z_2) \text{ for any two } z_1, z_2 \in \tau Q_i \text{ by (3-1)}\\ &\le C\Big(\sum_{Q_i \in I_j}\|g\|^p_{L^p_{w_b}(\tau Q_i)}\Big)^{1/p}\Big(\sum_{Q_i \in I_j}\Big(\frac{a_*(Q_i)}{\rho(Q_i)^{b/p}}\Big)^{p'}\Big)^{1/p'}\mu_a^*(B_j)^{1/q}.\end{aligned}$$

We now show that $\mu_a^*(B_j) \le Cr_j^a\mu(B_j)$ since x_j is an M-bad point and Ω_0 confines the M-bad points. By definition of ρ^*, it suffices to show that $\rho(B_j) \le Cr_j$. We first estimate $\rho(B_j')$, where B_j' is the δ-Whitney ball concentric with B_j:

$$\begin{aligned}\rho(B_j') &= \sup_{B_j'}\rho \le C(\kappa)\inf_{B_j'}\rho && \text{by (3-1)}\\ &= C(\kappa)d(B_j', \Omega_0) \le C(\kappa)\overline{M}r(B_j') && \text{by (1-4)}\\ &= C(\kappa)\overline{M}\delta d(x_j) \le Cr_j\end{aligned}$$

by the assumptions about B_j presently in force. Thus

$$\rho(B_j) \le \kappa(\rho(B_j') + r_j) \le C\,r_j$$

as desired. It follows that the earlier expression is at most

$$\le C\|g\|_{L^p_{w_b}(\bigcup \tau Q_i)}\bigg(\sum_{l=0}^{L_j}\sum_{2^l r_0 \le r(Q_i) < 2^{l+1}r_0}\Big(\frac{a_*(Q_i)}{\rho(Q_i)^{b/p}}\mu(B_j)^{1/q}\Big)^{p'}\bigg)^{1/p'} r_j^{a/q},$$

where $r_0 = \min\{r(Q_i) : Q_i \in I_j\} \ge Cr_j^s$ and $2^{L_j}r_0 \sim \max\{r(Q_i) : Q_i \in I_j\} \le 2\kappa r_j$, and where we used the fact that the τQ_i have bounded overlaps.

Note that $\rho(Q_i) \sim r(Q_i)$ by applying (3-1) and the argument just used showing that $\rho(B_j')$ is less than a multiple of $r(B_j')$. Also, the number of terms in the inner sum above is at most $C(2^l r_0)^{-1+(1/s)}$. Therefore, by (1-14) and since $\bigcup \tau Q_i \subset B_j$, the last expression is bounded by

$$CC_1\|g\|_{L^p_{w_b}(B_j \cap \Omega)}\bigg(\sum_{l=0}^{L_j}(2^l r_0)^{(\beta - b/p - \eta'/p)p' - (s-1)/s}\bigg)^{1/p'} r_j^{(\eta+a)/q}.$$

Recall that

$$\chi = \frac{s(\beta p - b - \eta') - (s-1)(p-1)}{sp} = \beta - \frac{b}{p} - \frac{\eta'}{p} - \frac{s-1}{sp'}. \tag{3-3}$$

Therefore,

$$\sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} \le CC_1 \|g\|_{L^p_{w_b}(B_j\cap\Omega)} \left(\sum_{l=0}^{L_j} (2^l r_0)^{p'\chi}\right)^{1/p'} r_j^{(\eta+a)/q}.$$

In case $\chi \ge 0$, this is at most

$$\begin{aligned} CC_1 \|g\|_{L^p_{w_b}(B_j\cap\Omega)} (1+L_j)^{1/p'} r_j^{\chi+(\eta+a)/q} & \\ &\le CC_1 r_j^{\chi+(\eta+a)/q} (1+|\log r_j|)^{1/p'} \|g\|_{L^p_{w_b}(B_j\cap\Omega)}, \end{aligned}$$

where the $(1+L_j)^{1/p'}$ term is present only if $\chi = 0$ and where we used that $2^{L_j} \le 2\kappa(r_j/r_0) \le Cr_j^{1-s}$; recall that $r_j \le 1$ in the present case. For any positive real numbers u, v, α with $u < v \le 1$, we have $u^\alpha(1+|\log u|) \le c_\alpha v^\alpha(1+|\log v|)$, and therefore, since by assumption $\chi \ge 0$ and $\eta + a \ge 0$ with $\chi > 0$ if $\eta + a = 0$, we obtain the estimate

$$\le CC_1 \operatorname{diam}(\Omega)^{\chi+(\eta+a)/q} (1+|\log \operatorname{diam}(\Omega)|)^{1/p'} \|g\|_{L^p_{w_b}(B_j\cap\Omega)}.$$

Again, the factor $(1+\log \operatorname{diam}(\Omega))^{1/p'}$ is needed only when $\chi = 0$.

In case $\chi < 0$,

$$\begin{aligned} \sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &\le CC_1 \|g\|_{L^p_{w_b}(B_j\cap\Omega)} r_0^\chi r_j^{(\eta+a)/q} \\ &\le CC_1 r_j^{s\chi+(\eta+a)/q} \|g\|_{L^p_{w_b}(B_j\cap\Omega)} \\ &\le CC_1 \operatorname{diam}(\Omega)^{s\chi+(\eta+a)/q} \|g\|_{L^p_{w_b}(B_j\cap\Omega)} \end{aligned}$$

since $s\chi + (\eta+a)/q \ge 0$ by (1-16).

Similarly, when $p = 1$ (still assuming x_j is an M-bad point and $r_j \le 1$, and recalling that $r(Q_i) \ge Cr_j^s$ if $B_i \in I_j$),

$$\begin{aligned} \sum_{Q_i \in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &\le C \sum \|g\|_{L^1_{w_b}(\tau Q_i)} \left(\sup_{Q_i \in I_j} \frac{a_*(Q_i)}{\rho(Q_i)^b}\right) \mu_a^*(B_j)^{1/q} \\ &\le C\|g\|_{L^1_{w_b}(B_j\cap\Omega)} \left(\sup_{Q_i \in I_j} \frac{a_*(Q_i)\mu(B_j)^{1/q}}{\rho(Q_i)^b}\right) r_j^{a/q} \\ &\le CC_1 \|g\|_{L^1_{w_b}(B_j\cap\Omega)} \left(\sup_{Q_i \in I_j} r(Q_i)^{\beta-b-\eta'}\right) r_j^{(\eta+a)/q}. \end{aligned}$$

Since $p=1$, we have $\chi=\beta-b-\eta'$, and $\sup_{Q_i\in I_j} r(Q_i)^{\beta-b-\eta'}$ is at most a multiple of r_j^{χ} if $\chi\ge 0$ and a multiple of $r_j^{s\chi}$ if $\chi<0$. Thus the last expression is at most

$$CC_1\|g\|_{L^1_{w_b}(B_j\cap\Omega)}\begin{cases}\operatorname{diam}(\Omega)^{\chi+(\eta+a)/q} & \text{if } \chi\ge 0,\\ \operatorname{diam}(\Omega)^{s\chi+(\eta+a)/q} & \text{if } \chi<0,\end{cases}$$

which is equal to

$$CC_1\operatorname{diam}(\Omega)^{\varepsilon}\|g\|_{L^1_{w_b}(B_j\cap\Omega)}.$$

Our estimation of the portion of $A(B_j)\mu_a^*(B_j)^{1/q}$ that corresponds to summing over I_j is now complete in case x_j is an M-bad point and $r_j\le 1$.

Next we will estimate the same portion of $A(B_j)\mu_a^*(B_j)^{1/q}$ in the remaining cases that $x_j\in\Omega_g^M$, or both $x_j\in\Omega_\flat^M$ and $r_j>1$. The case $s=1$ is included in the first of these by choosing M to be the value in the definition of a 1-John domain. In either case, $\Phi_{s,M}(r_j/(2\alpha\kappa))\sim r_j$ (for the second case, recall that $\phi_s(t)=c_st$ when $t\ge 1$). Thus $r(Q_i)\sim r_j$ if $Q_i\in I_j$, and consequently while the argument will be similar to the one above, it will be simpler.

Let us show that $\rho^*(B_j)\sim\rho(Q_i)$ for such Q_i. Since $\rho^*(B_j)=\rho(B_j)+r_j$ and $\rho(B_j)\ge\rho(Q_i)\ge r(Q_i)\sim r_j$, then $\rho^*(B_j)\sim\rho(B_j)$, and it suffices to show that $\rho(B_j)\le C\rho(Q_i)$. But the quasitriangle inequality gives the desired

$$\rho(B_j)\le C(\kappa)(\rho(Q_i)+r_j)\sim\rho(Q_i).$$

Thus, in either of the remaining cases,

$$\begin{aligned}\sum_{Q_i\in I_j} a(Q_i)\mu_a^*(B_j)^{1/q} &= \sum_{Q_i\in I_j}\|g\|_{L^p_w(\tau Q_i)}a_*(Q_i)\mu_a^*(B_j)^{1/q}\\ &\le C\sum_{Q_i\in I_j}\|g\|_{L^p_{w_b}(\tau Q_i)}a_*(Q_i)\mu(B_j)^{1/q}\rho(Q_i)^{a/q-b/p}\\ &\le CC_1\sum_{Q_i\in I_j} r_j^{\beta'}\|g\|_{L^p_{w_b}(\tau Q_i)}\rho(Q_i)^{a/q-b/p} && \text{by (1-13)}\\ &\le CC_1\sum_{Q_i\in I_j}\rho(Q_i)^{\beta'+a/q-b/p}\|g\|_{L^p_{w_b}(\tau Q_i)} && \text{since } \beta'\ge 0\\ &\le CC_1\rho(\Omega)^{\beta'+a/q-b/p}\|g\|_{L^p_{w_b}(B_j\cap\Omega)} && \text{by (1-15).}\end{aligned}\tag{3-4}$$

We have now estimated $\sum_{Q_i\in I_j} a(Q_i)\mu_a^*(B_j)^{1/q}$ in all cases. The corresponding estimates of the full sum

$$A(B_j)\mu_a^*(B_j)^{1/q}=\sum_{Q_i\in\mathscr{C}(B_j)} a(Q_i)\mu_a^*(B_j)^{1/q},\quad \text{with } \mathscr{C}(B_j)=\mathscr{C}_{\phi_{s,M}}(B_j),$$

are comparable. To verify (1-10), it remains to raise these estimates to the power p and add them over those B_j in a disjoint collection $I=\{B_j\}$.

Thus, if $s = 1$,

$$\sum_I A(B_j)^p \mu_a^*(B_j)^{p/q} \le C(C_1)^p \rho(\Omega)^{p(\beta'+a/q-b/p)} \|g\|^p_{L^p_{w_b}(\Omega)}$$
$$= C(C_1)^p \rho(\Omega)^{p\epsilon'} \|g\|^p_{L^p_{w_b}(\Omega)}$$

by (1-15) since the B_j are disjoint. Note that C is independent of M, $\overline{M}$, η, $\tilde{\eta}$, β. In any of the other cases,

$$\sum_I A(B_j)^p \mu_a^*(B_j)^{p/q} \le$$
$$C(C_1)^p \|g\|^p_{L^p_{w_b}(\Omega)} \begin{cases} \max\{\rho(\Omega)^{p\varepsilon'}, \operatorname{diam}(\Omega)^{p\varepsilon}\} & \text{if } \chi \ne 0, \\ \max\{\rho(\Omega)^{p\varepsilon'}, \operatorname{diam}(\Omega)^{p\varepsilon}(1+|\log \operatorname{diam}(\Omega)|)^{p-1}\} & \text{if } \chi = 0. \end{cases}$$

It now follows that (1-10) holds with $C_0^p \mu(\Omega)^{p/q}$ there taken to be the right sides of the estimates above. This verifies condition (3) of Theorem B.

Then (1-11) of Theorem B implies that (1-17) and (1-18) hold provided

$$\{B(x, r_j^x) : x \in \Omega,\ j \in \mathbb{N}\}$$

is a Vitali-type cover with respect to (μ_a, μ_a^*). However, this follows from the analogous assumption in Theorem 1.6 for μ_a and the fact that $\mu_a \le \mu_a^*$. The proof of part (i) of Theorem 1.6 is now complete.

We next prove part (ii), that is, the case $1 \le q \le p$. Recall that we have obtained the following estimates in the proof of part (i) for the balls

$$\{B_\alpha\} = \{B_j^x : x \in \Omega,\ j \in \mathbb{N}\}$$

as in (1-5):

First, if $B_\alpha = B(x, r)$ is a $\tau\delta$-ball, or if $x \in \Omega_g^M$, or if both $x \in \Omega_\flat^M$ and $r \ge 1$, then — see the reasoning before (3-4), note that $r(Q_i) \sim r(B_\alpha)$ and $\rho(Q_i) \sim \rho(B_\alpha)$ for any $Q_i \in \mathscr{C}(B_\alpha)$ now, and recall that $r(Q_i) \le \rho(Q_i)$ by (3-1) —

$$A(B_\alpha)\mu_a^*(B_\alpha)^{1/q} \le CC_1 \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)} r(B_\alpha)^{\beta'} \rho(B_\alpha)^{a/q-b/p}$$
$$\le CC_1 \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)} r(B_\alpha)^{\min\{\varepsilon',\beta'\}} \rho(\Omega)^{\max\{0,a/q-b/p\}},$$

where $\varepsilon' = \beta' + a/q - b/p$.

Second, suppose $B_\alpha = B(x, r)$, $x \in \Omega_\flat^M$, and $1 > r \ge \tau\delta d(x)$. If $\chi \ne 0$,

$$A(B_\alpha)\mu_a^*(B_\alpha)^{1/q} \le CC_1\, r(B_\alpha)^\varepsilon \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)},$$

but if $\chi = 0$,

$$A(B_\alpha)\mu_a^*(B_\alpha)^{1/q} \le CC_1 r(B_\alpha)^\varepsilon (1+|\log r(B_\alpha)|)^{1/p'} \|g\|_{L^p_{w_b}(B_\alpha\cap\Omega)}.$$

Assuming (1-19), there exists $0 < \theta < 1$ such that $(p - q\theta)\tilde{\eta}/(pq\theta) < \varepsilon$ and $(p - q\theta)\tilde{\eta}'/(pq\theta) < \min\{\varepsilon', \beta'\}$. Part (ii) will then also follow from Theorem B. Indeed, for example, when $\chi \neq 0$, if I is a collection of pairwise disjoint balls with center in $\Omega_\flat^M$ and $\tau\delta d(x) \le r < 1$, then it follows from Hölder's inequality that

$$\sum_{B_\alpha \in I} (A(B_\alpha)\mu_a^*(B_\alpha)^{1/q})^{q\theta} \le C C_1^{q\theta} \biggl(\sum_{B_\alpha \in I} \|g\|^p_{L^p_{w_b}(B_\alpha \cap \Omega)}\biggr)^{q\theta/p} \biggl(\sum_{B_\alpha \in I} (r(B_\alpha)^{\varepsilon q\theta})^{p/(p-q\theta)}\biggr)^{(p-q\theta)/p}.$$

However, by (1-19) and the hypothesis of Theorem 1.6(ii),

$$\sum (r(B_\alpha)^{\varepsilon q\theta})^{p/(p-q\theta)} = \sum_{k=-\infty}^{0} \sum_{\lambda^k < r(B_\alpha) \le \lambda^{k+1}} r(B_\alpha)^{\varepsilon q\theta p/(p-q\theta)} \le \sum_{k=-\infty}^{0} M_1 \lambda^{-\tilde{\eta}k} \lambda^{(k+1)\varepsilon q\theta p/(p-q\theta)} \le C.$$

When $s = 1$, the constant C is independent of M, M_1, $\overline{M}$, β, η, $\tilde{\eta}$, η'. □

Remark 3.2. Checking through the proof of Theorem 1.6, we note that instead of requiring $\rho(x) = d(x, \Omega_0)$, it suffices to assume that ρ is any nonnegative function satisfying the following properties (with $\rho(B)$ defined to be $\sup_{x \in B} \rho(x)$):

(i) $\rho(x) \sim \rho(B)$ if $x \in B$ for any δ-ball B in Ω;

(ii) $r(B) \le C\rho(B)$ for any δ-ball B in Ω;

(iii) $\rho(\tilde{B}) \le C(\rho(B) + r(\tilde{B}))$ for all balls $B \subset \tilde{B}$ with both centers in Ω;

(iv) $\rho(Q) \sim r(Q)$ for all δ-Whitney balls Q along s-John curves from M-bad points.

In case Ω is a 1-John domain, (iv) is redundant as there are then no M-bad points. If $\rho(x) = d(x, \Omega_0)$ with $\Omega_0 \subset \Omega^c$, the first three properties of course hold, and (iv) will hold if Ω_0 confines all the M-bad points of Ω.

The same remark applies also to Theorem 1.10. Furthermore, in Theorem 1.8(i), only the first three properties are needed since $s = 1$, while in Theorem 1.8 part (ii), one can substitute (i)–(iii) above for the condition that $\rho(x) = d(x, \Omega_0)$ with $\Omega_0 \subset \Omega^c$, and substitute (iv) for the condition that Ω_0 confines all M-bad points. Finally, Theorem 1.12 remains valid for any nonnegative function ρ that satisfies all four properties (on Euclidean balls instead of quasimetric balls) instead of choosing $\rho(x) = d_E(x, \Omega_0)$ and assuming condition $(*)$ there. In fact, condition $(*)$ is used in Theorem 1.12 to ensure that Ω_0 confines all M-bad points.

Proof of Theorem 1.8. For part (i), let Ω be a 1-John domain, and fix τ and δ with $\tau \geq 1$ and $0 < \tau\delta < 1/(2\kappa^2)$. As noted in the remark following Proposition 2.4, Proposition 2.3 provides a collection $W = \{B\}$ of δ-balls for which the Boman chain conditions listed in the hypothesis of Proposition 2.4 hold, with B_0 in the proposition chosen to be the ball $\mathcal{Q}_L = B(x', \delta d(x'))$ of Proposition 2.3(c), which we denote by B'. Moreover, by Proposition 2.3(a), each ball $B \in W$ contains a concentric δ/λ^2-Whitney ball and lies inside a concentric δ-Whitney ball. By part (b) of the same proposition, the enlarged balls $\{\tau B\}_{B\in W}$ have bounded overlaps. Thus, assuming the hypothesis of Theorem 1.8(i) and applying Proposition 2.4, we have, with C depending on $c_s, q, \kappa, A_\sigma, D_\sigma, a$ and δ,

$$\begin{aligned}
\|f - f_{B'}\|^q_{L^q_{\sigma_a}(\Omega)} &\leq C \sum_{B\in W} \|f - f_B\|^q_{L^q_{\sigma_a}(B)} \\
&\leq C \sum_{B\in W} \rho(B)^a \|f - f_B\|^q_{L^q_\sigma(B)} && \text{by (3-1) and } a \geq 0 \\
&\leq C \sum_{B\in W} \rho(B)^a \sigma(B)\big(a_*(B)\|g\|_{L^p_w(\tau B)}\big)^q && \text{by (1-12) since now } p_0 = q.
\end{aligned}$$

Recall that if $\lambda^{-2}\tilde{B} \subset B \subset \tilde{B}$, where $\tilde{B}$ is the δ-Whitney ball concentric with B, then by (1-25),

$$\sigma(B)a_*(B)^q \leq C_1^q r(B)^{\beta' q}.$$

Combining estimates, we obtain

$$\begin{aligned}
\|f - f_{B'}\|^q_{L^q_{\sigma_a}(\Omega)} &\leq C(C_1)^q \sum_{B\in W} \rho(B)^a r(B)^{\beta' q} \|g\|^q_{L^p_w(\tau B)} \\
&\leq C(C_1)^q \sum_{B\in W} \rho(B)^{a+\beta' q} \rho(B)^{-bq/p} \|g\|^q_{L^p_{w_b}(\tau B)},
\end{aligned}$$

with C depending also on b and τ, since $r(B) \leq \rho(B)$ by (3-1), $\beta' \geq 0$ and ρ is essentially constant on τB by (3-1) applied to $\tau\delta$-balls. Note that the condition $\beta' \geq 0$ need not hold if $\rho(B) \leq cr(B)$ for all δ/λ^2-Whitney balls, and then the constant C also depends on c. Finally, since $\varepsilon' \geq 0$, we obtain the bound

$$C(C_1)^q \rho(\Omega)^{a+\beta' q - bq/p} \sum_{B\in W} \|g\|^q_{L^p_{w_b}(\tau B)} \leq C(C_1)^q \rho(\Omega)^{a+\beta' q-bq/p} \|g\|^q_{L^p_{w_b}(\Omega)}$$

using the bounded overlap property of $\{\tau B\}_{B\in W}$ and the fact that $q \geq p$. Now Theorem 1.8(i) follows.

Next, let us prove part (ii). Thus suppose $s \geq 1$, $p_0 = p = q = 1$ and the hypotheses of Theorem 1.8(ii) hold. Let W be a covering of Ω that satisfies the properties in Proposition 2.3. Fix M and for each $x \in \Omega$, let

$$\mathscr{C}_x = \{R_0, R_1, \ldots, R_L\}, \quad \text{where } L = L_x,$$

be a chain of δ-balls as in the first part of property (c) in Proposition 2.3 with $\phi = \phi_{s,M}$; this chain is denoted there by $\{B_i\}_{i=0}^{L}$. The point x itself lies in the first ball R_0 in $\mathscr{C}_x$, and the last ball R_L satisfies $\lambda^{-2}B' \subset R_L \subset B'$ where $B' = B(x', \delta d(x'))$ is the "central" ball. Moreover, R_L is the same for all $x \in \Omega$, and we denote $R_L = B''$. As in the proof of [Chua and Wheeden 2008, Lemma 3.1],

$$\|f_{R_0} - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \le \sum_{j=1}^{L} \|f_{R_j} - f_{R_{j-1}}\|_{L^1_{\sigma_a}(R_0)}$$

$$= \sum_{j=1}^{L} \frac{\sigma_a(R_0)}{\sigma_a(R_j \cap R_{j-1})} \|f_{R_j} - f_{R_{j-1}}\|_{L^1_{\sigma_a}(R_j \cap R_{j-1})}$$

$$\le \sum_{j=1}^{L} \frac{\sigma_a(R_0)}{\sigma_a(R_j \cap R_{j-1})} \big(\|f - f_{R_{j-1}}\|_{L^1_{\sigma_a}(R_j \cap R_{j-1})} + \|f - f_{R_j}\|_{L^1_{\sigma_a}(R_j \cap R_{j-1})}\big)$$

$$\le C(a, A_\sigma, D_\sigma, \kappa) \sum_{j=0}^{L} \frac{\sigma_a(R_0)}{\sigma_a(R_j)} \|f - f_{R_j}\|_{L^1_{\sigma_a}(R_j)} \tag{3-5}$$

since σ_a is δ-doubling.

Let $W_\flat = \{R : R \in \mathscr{C}_x, x \in \Omega^M_\flat\}$ and $W_g = \{R : R \in \mathscr{C}_x, x \in \Omega^M_g\}$. Also, let $W_{\flat 0}$ and W_{g0} be the subsets of $W_\flat$ and W_g consisting of those R_0 that are the first entry in $\mathscr{C}_x$ as x ranges over $\Omega^M_\flat$ or over Ω^M_g respectively. We will not distinguish between $W_{\flat 0}$ and the subset of Ω that is covered by the balls in $W_{\flat 0}$, and similarly for W_{g0}. Then $\Omega^M_\flat \subset W_{\flat 0}$, $\Omega^M_g \subset W_{g0}$, and $\Omega = W_{\flat 0} \cup W_{g0}$. Hence

$$\|f - f_{B''}\|_{L^1_{\sigma_a}(\Omega)} \le \|f - f_{B''}\|_{L^1_{\sigma_a}(W_{\flat 0})} + \|f - f_{B''}\|_{L^1_{\sigma_a}(W_{g0})}. \tag{3-6}$$

For the first term on the right of (3-6), we have

$$\begin{aligned}\|f - f_{B''}\|_{L^1_{\sigma_a}(W_{\flat 0})} &\le \sum_{R_0 \in W_{\flat 0}} \|f - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \\ &\le \sum_{R_0 \in W_{\flat 0}} \|f - f_{R_0}\|_{L^1_{\sigma_a}(R_0)} + \sum_{R_0 \in W_{\flat 0}} \|f_{R_0} - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

To estimate II, note by (3-5) that if $R_0 \in W_{\flat 0}$, then

$$\|f_{R_0} - f_{B''}\|_{L^1_{\sigma_a}(R_0)} \le C(a, A_\sigma, D_\sigma, \kappa) \sum_{R \in W_\flat;\, R_0 \subset R^*} \frac{\sigma_a(R_0)}{\sigma_a(R)} \|f - f_R\|_{L^1_{\sigma_a}(R)},$$

where $R^* \subset C[\phi_s^{-1}(Cr(R))/r(R)]R \cap B(x_R, \operatorname{diam}(\Omega))$. In fact, by (2-1), R^* is chosen (depending at most on δ, κ, ϕ_s and M) so that for each ball R_j in (3-5), we have $R_0 \subset R_j^*$ assuming that $R_0 \in W_{\flat 0}$. Adding over R_0 gives

$$\begin{aligned}\mathrm{II} &\le C \sum_{R\in W_\flat}\Big(\sum_{R_0\in W_{\flat 0};\,R_0\subset R^*}\sigma_a(R_0)\Big)\frac{1}{\sigma_a(R)}\|f-f_R\|_{L^1_{\sigma_a}(R)}\\ &\le C\sum_{R\in W_\flat}\frac{\sigma_a(R^*)}{\sigma_a(R)}\|f-f_R\|_{L^1_{\sigma_a}(R)}\end{aligned}$$

since the balls in W have bounded overlaps. Clearly term I has the same bound, and consequently the first term on the right of (3-6) satisfies

$$\begin{aligned}\|f-f_{B''}\|_{L^1_{\sigma_a}(W_{\flat 0})} &\le C\sum_{R\in W_\flat}\frac{\sigma_a(R^*)}{\sigma_a(R)}\|f-f_R\|_{L^1_{\sigma_a}(R)}\\ &\le C\sum_{R\in W_\flat}\frac{\sigma_a(R^*)a_*(R)}{\rho(R)^b}\|g\|_{L^1_{w_b}(\tau R)} \qquad \text{by (1-12) and (3-1)}\\ &\le C\Big(\sup_{R\in W_\flat}\frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b}\Big)\sum_{R\in W}\|g\|_{L^1_{w_b}(\tau R)}\\ &\qquad\qquad \text{since } \sigma_a(R^*)\le\rho(R^*)^a\sigma(R^*)\\ &\le C\Big(\sup_{R\in W_\flat}\frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b}\Big)\|g\|_{L^1_{w_b}(\Omega)}\end{aligned}\tag{3-7}$$

since $\{\tau R : R\in W\}$ has bounded overlaps.

The same argument with R^* replaced by CR can be used to estimate the second term on the right of (3-6) since (2-1) guarantees that in (3-5) we have $R_0 \subset CR_j$ when $R_0 \in W_{g0}$. This gives

$$\|f-f_{B''}\|_{L^1_{\sigma_a}(W_{g0})} \le C\Big(\sup_{R\in W_g}\frac{\sigma(CR)\rho(CR)^a a_*(R)}{\rho(R)^b}\Big)\|g\|_{L^1_{w_b}(\Omega)}.\tag{3-8}$$

To estimate the supremum in (3-8), note that every $R\in W$ is a δ-ball and so satisfies $r(R)\le\rho(R)$ by (3-1). Also, by Proposition 2.3, $\lambda^{-2}Q\subset R\subset Q$ for the δ-Whitney ball Q concentric with R. Recall that we now assume a version of (1-14) for such balls with μ replaced by σ and $p=q=1$. Also $\rho(CR)\le C\rho(R)$ from the definition of $\rho(R)$, and $\sigma(CR)\le C\sigma(R)$ since σ is δ-doubling. Thus

$$\frac{\sigma(CR)\rho(CR)^a a_*(R)}{\rho(R)^b}\le CC_1\,\rho(R)^{a-b}r(R)^{\beta+\eta-\eta'}\le CC_1\,\rho(R)^{a-b+\beta+\eta-\eta'}$$

since $\beta+\eta-\eta'\ge 0$ by hypothesis. Using $a-b+\beta+\eta-\eta'\ge 0$ due to (1-16) with $p=q=1$ (see Remark 1.7(6)), we obtain

$$\sup_{R\in W_g}\frac{\sigma(CR)\rho(CR)^a a_*(R)}{\rho(R)^b}\le CC_1\,\rho(\Omega)^{a-b+\beta+\eta-\eta'}.\tag{3-9}$$

The same estimate holds for the part of the supremum in (3-7) that is extended over those $R \in W_\flat$ with $r(R) \ge 1$, since $R^* \subset CR \subset B(x_R, \operatorname{diam}(\Omega))$ for such R. To estimate the remaining part, namely the part corresponding to $r(R) \le 1$, we first apply our version of (1-14) to the pair (R^*, R) and note that $r(R^*) = Cr(R)^{1/s}$ when $r(R) \le 1$, obtaining

$$\sup_{R\in W_\flat;\, r(R)\le 1} \frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b} \le C \sup_{R\in W_\flat;\, r(R)\le 1} \frac{\rho(R^*)^a}{\rho(R)^b}\, r(R)^{\beta-\eta'+\eta/s}. \tag{3-10}$$

To further estimate (3-10), let us show that $r(R) \sim \rho(R)$ for any $R \in W_\flat$. In fact, $\rho(R) \ge r(R)$ by (3-1). Also

$$\rho(R) = \sup_{z\in R} \rho(z) \le C \inf_{z\in R} \rho(z) = Cd(R, \Omega_0) \quad \text{by (3-1)},$$

and it is enough to show that $d(R, \Omega_0) \le Cr(R)$ if $R \in W_\flat$. This follows directly from (1-4) if $R \in W_\flat$ is centered on an s-John curve leading from an M-bad point, and it then follows for general $R \in W_\flat$ by using Proposition 2.3(c) to find a subball of R of comparable radius that is centered on such a curve. Then if $R \in W_\flat$ and $r(R) \le 1$,

$$\rho(R^*) \le \kappa\,(\rho(R) + r(R^*)) \le C(\rho(R) + r(R)^{1/s}) \le C\rho(R)^{1/s},$$

and consequently, by (3-10),

$$\begin{aligned}\sup_{R\in W_\flat; r(R)\le 1} \frac{\sigma(R^*)\rho(R^*)^a a_*(R)}{\rho(R)^b} &\le C \sup_{R\in W} \rho(R)^{a/s-b+\beta-\eta'+\eta/s} \\ &\le C\rho(\Omega)^{a/s-b+\beta-\eta'+\eta/s}\end{aligned} \tag{3-11}$$

since

$$a/s - b + \beta - \eta' + \eta/s \ge 0$$

by (1-16) (with $p = q = 1$). The estimate (3-11) holds even if Ω_0 does not confine the M-bad points provided we assume in addition that $\beta - \eta' + \eta/s \ge 0$; this follows by simply majorizing the factor $r(R)^{\beta-\eta'+\eta/s}$ in (3-10) by $\rho(R)^{\beta-\eta'+\eta/s}$ and using the inequality $r(R) \le \rho(R)$ when estimating $\rho(R^*)$ above.

Combining (3-6)–(3-11) gives

$$\|f - f_{B''}\|_{L^1_{\sigma_a}(\Omega)} \le CC_1 \max\bigl\{\rho(\Omega)^{a/s-b+\beta-\eta'+\eta/s},\, \rho(\Omega)^{a-b+\beta+\eta-\eta'}\bigr\}\|g\|_{L^1_{w_b}(\Omega)}.$$

Finally, using a similar approach as in [Chua and Wheeden 2008, Lemma 3.1], we have (recall that $\lambda^{-2}B' \subset B'' \subset B'$)

$$\begin{aligned}
\|f_{B'} - f_{B''}\|_{L^1_{\sigma_a}(\Omega)} &= \sigma_a(\Omega)|f_{B'} - f_{B''}| \\
&\le \frac{\sigma_a(\Omega)}{\sigma(B'')}\Big(\int_{B''}(|f - f_{B'}| + |f - f_{B''}|)d\sigma\Big) \\
&\le \frac{\sigma_a(\Omega)}{\sigma(B'')}\big(\|f - f_{B'}\|_{L^1_\sigma(B')} + \|f - f_{B''}\|_{L^1_\sigma(B'')}\big) \\
&\le C\sigma_a(\Omega)(a_*(B') + a_*(B''))\|g\|_{L^1_w(\tau B')} \\
&\qquad\qquad \text{by (1-12) and the fact that } \sigma \text{ is } \delta\text{-doubling} \\
&\le C\sigma(\Omega)\rho(\Omega)^a \frac{1}{\rho(B')^b}\,\big(a_*(B') + a_*(B'')\big)\,\|g\|_{L^1_{w_b}(\tau B')} \\
&\le CC_1\rho(\Omega)^{a-b+\beta+\eta-\eta'}\|g\|_{L^1_{w_b}(\Omega)}
\end{aligned}$$

using (1-14) for B' and B'' (with μ replaced by σ and $p = q = 1$). This completes the proof of (1-27) by the triangle inequality. □

Proof of Theorem 1.10. For each $x \in \Omega$, choose $\{Q_i^x\}_{i=1}^\infty$ and $\{B_j^x\}_{j=1}^\infty$ as in the proof of Theorem 1.6. For any $\omega > 0$, set

$$b(Q, f^\omega) = a_*(Q)\|Yf^\omega\|_{L^p_w(\tau Q)}.$$

Note that (1-29) holds with this $b(\cdot\,,\cdot)$ by the hypothesis of Theorem 1.10. Also, by the proof of Theorem 1.6(i) (with g there replaced by $|Yf^\omega|$),

$$\sum_{B\in I} A(B, f^\omega)^p \mu_a^*(B)^{p/q} \le (C^*)^p\|Yf^\omega\|^p_{L^p_{w_b}(\Omega)} \tag{3-12}$$

for any collection I of disjoint balls B_j^x. Here C^* is the constant in either (1-17) or (1-18), respectively. This shows that (1-30) holds with (μ_a^*, μ_a) in place of (μ^*, μ), with $\theta = p/q$, and with $h(\Omega, f^\omega)$ defined by

$$h(\Omega, f^\omega) = C^*\|Yf^\omega\|_{L^p_{w_b}(\Omega)}\mu_a(\Omega)^{-1/q}.$$

Then (1-31) requires that

$$h^*(\Omega, f)^q = \sup_{\omega>0}\sum_{k=1}^\infty \|Yf^{2^k\omega}\|^q_{L^p_{w_b}(\Omega)} < \infty.$$

Theorem 1.9 now gives (noting that $\wp = \mu_a(\Omega)/\mu_a(B')$ as in the proof of Theorem 1.6)

$$\frac{1}{\mu_a(\Omega)}\|f - f_{B',\sigma}\|^q_{L^q_{\mu_a}(\Omega)} \le C\frac{\mu_a(\Omega)}{\mu_a(B')}C^{*q}h^*(\Omega, f)\frac{1}{\mu_a(\Omega)} + \Big(\frac{8}{\sigma(B')}\|f - f_{B',\sigma}\|_{L^1_\sigma(B')}\Big)^q,$$

which proves Theorem 1.10. □

Proof of Theorem 1.12. Suppose $\varepsilon > 0$, $M > 1$ and Ω_0 is a subset of Ω^c with $\partial\Omega \cap \big(\bigcup_{x\in\Omega_\flat^M} B(x, \varepsilon)\big) \subset \Omega_0$. We will show that Ω_0 confines the M'-bad points of Ω for suitable M'. Let us first show that if $d(x) \ge \varepsilon/3$ then x is an M'-good point for some $M' > 1$ depending on ε. Indeed, since Ω is an s-John domain, there is a curve $\gamma : [0, l] \to \Omega$ with $\gamma(0) = x$ and $\gamma(l) = x'$ such that

$$|\gamma(t_1) - \gamma(t_2)| \le |t_1 - t_2| \quad \text{and} \quad d(\gamma(t)) \ge c_s \min\{t^s, t\}.$$

If $t \le \varepsilon/(6\kappa)$, then

$$\tfrac{1}{3}\varepsilon \le d(x) \le \kappa(d(\gamma(0), \gamma(t)) + d(\gamma(t))) \le \kappa(t + d(\gamma(t))) \le \kappa(\varepsilon/(6\kappa) + d(\gamma(t))),$$

and consequently, $d(\gamma(t)) \ge \varepsilon/(6\kappa) \ge t$. On the other hand, if $\kappa t \ge \varepsilon/6$, then

$$d(\gamma(t)) > c_s t \min\{1, t^{s-1}\} \ge c_s \min\{1, (\varepsilon/(6\kappa))^{s-1}\}t.$$

Combining estimates shows that $x \in \Omega_g^{M'}$ for suitably large M' depending only on ε, κ, s and c_s. We may assume $M' \ge M$, so that $\Omega_\flat^{M'} \subset \Omega_\flat^M$.

We now show that there is a constant $C > 0$ (independent of x) such that if $x \in \Omega_\flat^{M'}$ and $\gamma : [0, l] \to \Omega$ is the s-John curve connecting x to x', then $d(\gamma(t)) \ge Cd(\gamma(t), \Omega_0)$. We will use the fact that $\Omega_0 \supset \partial\Omega \cap B(x, \varepsilon)$. First, recall that we must have $d(x) < \varepsilon/3$ since $x \in \Omega_\flat^{M'}$. Let us consider two cases.

Case (i): $t < \varepsilon/3$. Then $|\gamma(t) - x| \le t < \varepsilon/3$ and hence

$$d(\gamma(t)) \le |\gamma(t) - x| + d(x) < 2\varepsilon/3.$$

Pick $z \in \partial\Omega$ such that $d(\gamma(t)) = |\gamma(t) - z|$. Then $z \in B(x, \varepsilon)$ since

$$|z - x| \le |z - \gamma(t)| + |\gamma(t) - x| < \varepsilon.$$

Thus $z \in \Omega_0$ and $d(\gamma(t)) \ge d(\gamma(t), \Omega_0)$.

Case (ii): $t \ge \varepsilon/3$. We combine the facts that $d(\gamma(t)) > c_s \min\{t^s, t\} \ge c_{s,\varepsilon}t$ and

$$d(\gamma(t), \Omega_0) \le |\gamma(t) - x| + d(x, \Omega_0) \le t + \varepsilon \le 4t.$$

It follows that Ω_0 confines the M'-bad points of Ω, as desired.

For all $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and all Euclidean balls $B \subset \Omega$, the L^1, L^1 version of Poincaré's inequality together with Hölder's inequality yield the L^1, L^p version

$$\frac{1}{|B|}\|f - f_B\|_{L^1(B)} \le C\frac{r(B)}{|B|^{1/p}}\|\nabla f\|_{L^p(B)}.$$

We will apply various earlier results with $\sigma = \mu = 1,\ Y = \nabla,\ \beta = 1,\ s \ge 1,$ $a \ge 0,\ b \in \mathbb{R},\ 1/q \ge 1/p - 1/n,\ \tilde{\eta} = \tilde{\eta}' = \eta = \eta' = n,\ \beta' = 1 - n/p + n/q$, and $a_*(B) = Cr(B)^{1-n/p}$. Let us first consider the case $C(\Omega, f) = |B'|^{-1}\int_{B'} f\,dx$. In case $q > p$, we apply Theorem 1.10 to obtain (1-39); note that (1-37) now agrees with (1-16).

For the case $p = q > 1$, note that (1-38) implies (1-37) with strict inequality, that is,

$$\frac{s(n+b-1) - p + 1}{(n+a)p} < \frac{1}{p}.$$

It follows that there exists $q_0 > p$ such that

$$\frac{s(n+b-1) - p + 1}{(n+a)p} \le \frac{1}{q_0},$$

and we can then apply the result from the first part and then use Hölder's inequality to conclude this case. For the case $p = q \ge 1$ and $s = 1$, where we assume (1-37), that is, $b - a \le p$, just apply Theorem 1.8(i), noting that $\beta' + a/q - b/p \ge 0$ follows from (1-37).

For the case $p > q \ge 1$, we apply Theorem 1.6(ii). Conditions (1-38) now agree with (1-19) by arguments like those in Remark 1.7(6). Of course, we will only get a weak-type estimate instead of a strong-type one in this way. However, as the conditions (1-38) are strict inequality, the weak-type estimate will be valid for some $q_0 > q$. Then (1-39) follows from interpolation; see Remark 1.7(10). Finally, in case $q = p = 1$ and $s \ge 1$, recall that we assume $n + a \ge s(n+b-1)$. In fact, (1-39) with $q = p = 1$ and $C(\Omega, f) = |B'|^{-1}\int_{B'} f\,dx$ is true by Theorem 1.8(ii); now $\beta' = 1$ and $a_*(B) = Cr(B)^{1-n}$. Note that (1-16) follows from $n + a \ge s(n+b-1)$ since then $n + a \ge n + b - 1$ if $n + b - 1 \ge 0$, while $n + a \ge n + b - 1$ holds trivially if $n + b - 1 < 0$.

Now (1-39) is clear with $C(\Omega, f) = |B'|^{-1}\int_{B'} f$ in all cases. By the same argument used in [Chua and Wheeden 2008, Remark 1.3], we see that (1-39) also holds with $C(\Omega, f) = (|\mathcal{D}|_{\rho^a dx})^{-1}\int_{\mathcal{D}} f\rho^a dx$ for any $\mathcal{D} \subset \Omega$ such that $|\mathcal{D}| > 0$, provided the constant C in (1-39) also depends on $|\Omega|_{\rho^a dx}/|\mathcal{D}|_{\rho^a dx}$.

Finally, the last sentence in the statement of Theorem 1.12 follows directly from the result in the last sentence of Theorem 1.8(ii) applied with the standard Euclidean structure, that is, with $\beta = 1,\ \eta = \eta' = n$ and $w = \sigma = 1$, since then the requirement in that sentence that $\beta + \eta/s - \eta' \ge 0$ is guaranteed by assuming that $s \le n/(n-1)$. □

Proof of Theorem 1.14.. We will prove the result by applying Theorem 1.10 with $a_*(B) = Cr(B)^{1-n/p}$. For any $(z, z') \in \mathcal{D}$, we first connect (z, z') to the axis $z' = 0$ (say to $(z_1, 0)$) along the line through (z, z') that is orthogonal to the boundary, and then connect $(z_1, 0)$ to $(2, 0)$ by a segment of the axis. Clearly, there exists $1 \geq \tilde{c} = \tilde{c}(s) > 0$ small enough such that $z_1 < 2$ whenever $z < \tilde{c}$, and then the path from (z, z') to $(2, 0)$ lies in $\mathcal{D}$ and is an s-John curve. Moreover, if $z \geq \tilde{c}$ then (z, z') can be connected to $(2, 0)$ by a 1-John curve with constant M depending on $\tilde{c}$. Hence the set of M-bad points $\mathcal{D}_\flat^M \subset \{(z, z') \in \mathcal{D} : z < \tilde{c}\}$, and it is clear that $\mathcal{D}_0$ confines the M-bad points when $B((0, 0), \varepsilon) \cap \partial\mathcal{D} \subset \mathcal{D}_0$ for some $\varepsilon > 0$. Moreover, recall that $\partial\mathcal{D}$ always confines the M-bad points.

Let $\delta = 1/4$. First note that the measure $\mu(E) = |E \cap \mathcal{D}|_{\rho^a dx}$ is δ-doubling for any $a \geq 0$. Let us show that it is also doubling on $\mathcal{D}$ when either $a = 0$ or when $\mathcal{D}_0 = \partial\mathcal{D}$ and $a \geq 0$. By Proposition 2.2(3) and the fact that μ is δ-doubling, we only need to show that there exists $c_1 \geq 1$ such that $\mu(B(x, 2r)) \leq c_1 \mu(B(x, r))$ for all $d(x)/4 \leq r \leq 2$ and $x = (z, z')$ with $z < \tilde{c}$ as $\{(z, z') \in \mathcal{D} : z \geq \tilde{c}\} \subset \mathcal{D}_g^M$.

Consider first the case $x = (z_0, 0)$. Using the fact that $y = t^s$ is convex and $z_0 < 1$, we see by elementary calculus that $d(x)$ is at least the distance between $x = (z_0, 0)$ and the straight line passing through the point (z_0, z_0^s) with slope s; note it suffices to consider only $n = 2$ here. Hence $d(x) \geq z_0^s/\sqrt{1 + s^2} \geq z_0^s/(2s)$. Again by calculus, if $z_0^s/(2s) \leq r \leq 2$ then

$$r^2 \geq \left(\frac{z_0/(2s)^{1/s} + r/2}{2}\right)^{2s} + (r/2)^2$$

since the analogous inequality with $r^{1/s}$ in place of $z_0/(2s)^{1/s}$ is true if $0 \leq r \leq 2$. Hence when $z_0 < 1$ and $x = (z_0, 0)$, the cylinder

$$\left\{(z, z') \in \mathbb{R} \times \mathbb{R}^{n-1} : z_0 + r/4 \leq z \leq z_0 + r/2, |z'| < \left(\frac{z_0/(2s)^{1/s} + r/2}{2}\right)^s\right\}$$

lies inside $B(x, r) \cap \mathcal{D}$ when $d(x) \leq r \leq 2$. It follows that if $x = (z_0, 0)$ and $d(x) \leq r \leq 2$, then

- when $a = 0$, $\mu(B(x, r)) \geq Cr(z_0 + r)^{(n-1)s}$;
- when $a \geq 0$ and $\mathcal{D}_0 = \partial\mathcal{D}$ (so $\rho((z, z')) = d_E((z, z'))$),
$$\mu(B(x, r)) \geq Cr(z_0 + r)^{(n-1+a)s}$$
since $d_E((z, z')) \geq C(z_0 + r)^s$ on a proportional part of the cylinder.

It is easy to see that both of these remain true (for a larger constant C) even if $d(x)/4 \leq r \leq 4$ when $x = (z_0, 0)$, $z_0 < 1$.

Next note that if $a \geq 0$, $\mathcal{D}_0 = \mathcal{D}^c$ and $0 < R < 4$, then

$$\mu(B(x, R)) \leq R(z_0 + R)^{(n-1+a)s}. \tag{3-13}$$

Moreover, if $a = 0$ but $\mathcal{D}_0$ may not be $\mathcal{D}^c$, then for $0 < R < 4$,

$$\mu(B(x, R)) \le CR(z_0 + R)^{(n-1)s}.$$

Clearly (3-13) and the last estimate remain valid for $x = (z_0, 0) \in \mathcal{D}$ without the restriction $z_0 < 1$. It is now easy to see that if either $\mathcal{D}_0 = \mathcal{D}^c$ or $a = 0$, then $\mu(B(x, 2r)) \le c_1\mu(B(x, r))$ for any $x = (z_0, 0)$ with $z_0 < \tilde{c}$ and $d(x)/4 \le r \le 2$.

Finally, it remains to consider the case $x = (z_0, z'),\ z' \neq 0$ and $z_0 < \tilde{c}$. Recall that there is $z_1 > z_0$ such that the line connecting $x = (z_0, z')$ to $x_1 = (z_1, 0)$ is orthogonal to the boundary. If $r < 2|(z_0, z') - (z_1, 0)|$, it is easy to see that $B(x, r)$ contains a δ-ball Q with $r(Q) \ge c_2 r$, and hence $\mu(B(x, 2r)) \le c_1\mu(B(x, r))$ since μ is δ-doubling for $\delta = 1/4$. On the other hand, when $r \ge 2|(z_0, z') - (z_1, 0)|$, we have $B(x_1, r/2) \subset B(x, r) \subset B(x_1, 2r)$, and consequently $\mu(B(x, 2r)) \le c_1\mu(B(x, r))$ with a larger c_1 if necessary. We conclude that μ is doubling on $\mathcal{D}$ if either $a = 0$ or $\mathcal{D}_0 = \mathcal{D}^c$ and $a \ge 0$. In particular, by Remark 1.7(1), (1-5) holds for μ in these cases.

We are now ready to show part (2) of Theorem 1.14. Let a, b, p, q satisfy (1-40) and (1-43). It will be convenient to rename a and b by $\tilde{a}$ and $\tilde{b}$ respectively. We will apply Theorem 1.10 with $a = b = 0$ there to the measures

$$\mu(E) = |E \cap \mathcal{D}|_{\rho^{\tilde{a}}dx} \quad \text{and} \quad w(E) = |E \cap \mathcal{D}|_{\rho^{\tilde{b}}dx},$$

where $\rho(x) = d(x, \Omega_0)$. First, let B be any δ-ball in $\mathcal{D}$ and f be any locally Lipschitz function on $\mathcal{D}$. By the unweighted L^1, L^1 Poincaré inequality and the fact that $\rho(x) \sim \rho(B)$ on the δ-ball B, we have for $p \ge 1$ that

$$\frac{1}{\mu(B)}\|f - f_B\|_{L^1_\mu(B)} \le C\frac{r(B)}{w(B)^{1/p}}\|\nabla f\|_{L^p_w(B)}. \tag{3-14}$$

Thus, (1-12) holds with $f_B = |B|^{-1}\int_B f\,dx,\ \tau = 1,\ \sigma = \mu$ and $p_0 = 1$.

To verify (1-14), suppose as in (1-14) that $B = B(x, r)$ and Q satisfy $x \in \mathcal{D}^M_\flat$, $d(x)/4 \le r < 2$ and $Q \in \mathscr{C}(B)$. We first consider the case $x = (z_0, 0)$, with $z_0 < \tilde{c} < 1$. Observe that

$$\mu(B)^{1/q} \le Cr(B)^{1/q}r(Q)^{(n-1+\tilde{a})/q}$$

by (3-13) and the fact that $r(Q) \ge C(z_0 + r(B))^s$ (instead of the usual $cr(B)^s$) since Q has center on the axis between $(z_0, 0)$ and $(2, 0)$. Now, since $r(Q) \sim d(Q, \partial\mathcal{D})$, we have $w(Q) \sim r(Q)^{n+\tilde{b}}$ and hence

$$\mu(B)^{1/q} \le Cr(B)^{1/q}w(Q)^{1/p}r(Q)^{(n-1+\tilde{a})/q-(n+\tilde{b})/p}. \tag{3-15}$$

Since μ is δ-doubling, we can as before extend (3-15) to include the case when $x = (z_0, z') \in \mathcal{D}_b^M$, with $z' \neq 0$. Thus

$$(\mu(B)/r(B))^{1/q} \leq C(w(Q)/r(Q)^{n+\tilde{b}-p(n-1+\tilde{a})/q})^{1/p},$$

which verifies (1-14) with $\eta = 1$ and $\eta' = n + \tilde{b} - p(n-1+\tilde{a})/q$.

Now suppose B is a δ-ball. Then $\mu(B) \sim r(B)^n \rho(B)^{\tilde{a}}$ and $w(B) \sim r(B)^n \rho(B)^{\tilde{b}}$. Hence,

$$\mu(B)^{1/q} w(B)^{-1/p} \leq C\rho(B)^{\tilde{a}/q-\tilde{b}/p} r(B)^{n/q-n/p} \leq C_\Omega r(B)^{\beta'-1},$$

where

$$\beta' = 1 + \frac{n}{q} - \frac{n}{p} + \min\left\{0, \frac{\tilde{a}}{q} - \frac{\tilde{b}}{p}\right\} \quad \text{and} \quad C_\Omega = \begin{cases} \rho(\Omega)^{\tilde{a}/q-\tilde{b}/p} & \text{if } \tilde{a}/q - \tilde{b}/p > 0, \\ 1 & \text{if } \tilde{a}/q - \tilde{b}/p \leq 0 \end{cases}$$

and in case $\tilde{a}/q - \tilde{b}/p > 0$ we have used $r(B) \leq \rho(B) \leq \rho(\Omega)$. Since

$$1 + n(1/q - 1/p) \geq 0 \quad \text{and} \quad 1 + \frac{n+\tilde{a}}{q} - \frac{n+\tilde{b}}{p} \geq 0$$

by (1-43), we have $\beta' \geq 0$.

We now check that by (1-43), with η and η' as above,

$$\frac{\eta + 0}{q} \geq \frac{s(\eta' + 0 - p) + (s-1)(p-1)}{p}.$$

Hence (1-16) holds with a, b, β there chosen as $a = b = 0$ and $\beta = 1$. Part (2) of Theorem 1.14 then follows from Theorem 1.10; see Remark 1.11.

To prove part (1), we will use $\mu(E) = w(E) = |E \cap \mathcal{D}|$. It is clear that (3-14) remains true for all δ-balls B. Next note that instead of (3-15), we have, for all balls B and Q such that $Q \in \mathscr{C}(B)$,

$$(\mu(B)/r(B))^{1/q} \leq C(w(Q)/r(Q)^{n-p(n-1)/q})^{1/p}.$$

Part (1) now follows from Theorem 1.10 (see Remark 1.11) with $a = a$, $b = b$, $\beta = 1$, $\eta = 1$ and $\eta' = n - p(n-1)/q$. □

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SENG-KEE CHUA
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
10, LOWER KENT RIDGE ROAD
SINGAPORE 119076
SINGAPORE
matcsk@nus.edu.sg

RICHARD L. WHEEDEN
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
PISCATAWAY, NJ 08854
UNITED STATES
wheeden@math.rutgers.edu

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