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AN EXAMPLE OF A SINGULAR METRIC ARISING FROM THE BLOW-UP LIMIT IN THE CONTINUITY APPROACH TO KÄHLER–EINSTEIN METRICS

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A family of Kähler metrics with Calabi's symmetry on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ arises from the continuity method for finding Kähler–Einstein metrics. We study the blow-up limit of this family.

1. Introduction

Let M be a compact Kähler manifold with $c_1(M) > 0$. In algebraic geometry, M is called a Fano manifold. It is an important problem to study the existence of Kähler–Einstein metrics on such manifolds. In contrast to the $c_1 < 0$ and $c_1 = 0$ cases, there may be no Kähler–Einstein metrics on a given Fano manifold. Yau, Tian and Donaldson have conjectured that the existence of Kähler–Einstein metrics on M is equivalent to the K-polystability of M; see [Tian 1997; Donaldson 2002].

To find a Kähler–Einstein metric on M, one usually reduces the problem to solving a family of complex Monge–Ampère equations with parameter $\lambda \in [0, 1]$ via the continuity method, as Yau did in [1978]. If M does not admit a Kähler–Einstein metric, then the solutions of this family must blow up as $\lambda \to t_0$ for some $t_0 \in [0, 1]$. Since the solutions of this family give rise to a family of Kähler metrics with strictly positive Ricci curvature and the same volume, the compactness theorem of Gromov implies that this family contains a subfamily converging to a compact metric space with a length metric. The study of this limit space should be helpful in understanding the relationship between Kähler–Einstein metrics and stabilities in geometric invariant theory.

In this paper, we study a simple example, namely the blow-up of $\mathbb{C}P^2$ at one point, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, with a Calabi symmetric metric as the background metric. Note that $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is a ruled surface $\mathbb{P}(\underline{\mathbb{C}} \oplus U)$, where $\underline{\mathbb{C}}$ and U are the trivial line bundle and the universal bundle over $\mathbb{C}P^1$, respectively. It is well known that M is Fano and the automorphism group of M is not reductive [Calabi 1982].

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Therefore by Matsushima's theorem [1957], there are no Kähler–Einstein metrics on M. So if one uses the continuity method to solve the Kähler–Einstein metric equation on M with parameter $\lambda \in [0, 1]$, the parameter λ at which the equation is solvable could not reach 1. Recently, G. Székelyhidi showed that the Monge–Ampère equation is solvable if and only if the parameter λ is less than 6/7, if one chooses a Calabi symmetric metric as a background Kähler metric [2009].[1,2] There are two distinguished divisors E_1 and E_2, respectively defined as the zero section and the infinity section of the ruled surface M. A Calabi symmetric Kähler metric g on M is defined by a convex function u in $t \in (-\infty, \infty)$ with its Kähler form ω_g given by

$$\omega_g = \sqrt{-1}\partial\bar{\partial}u \quad \text{in } \mathbb{C}^2 \setminus \{0\}, \tag{1-1}$$

where $t = \log(|z_1|^2 + |z_2|^2)$ and (z_1, z_2) are the standard coordinates on $\mathbb{C}^2 \setminus \{0\} \cong M \setminus (E_1 \cup E_2)$. Székelyhidi's result implies that the Kähler metrics g_λ arising from the solutions of Monge–Ampère equations will blow up as $\lambda \to 6/7$.

On the other hand, by a general theorem of Cheeger and Colding [1997], there exists a subsequence of metrics g_{λ_i} that converges in the Gromov–Hausdorff sense to a limit metric space g_∞ whose singular set has Hausdorff codimension at least 2. On the regular part, g_∞ is C^α-continuous. It is an interesting problem to study the geometry of the limit space.

Theorem 1.1. (1) *Among the Kähler metrics g_λ arising from the continuity method for finding Kähler–Einstein metrics, there exists a sequence converging smoothly in the Cheeger–Gromov sense to a singular Kähler metric g_∞ on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. The limit g_∞ is smooth on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus E_2$ and has conically symmetric singularities on E_2 with the same conical angle $10\pi/7$ along one direction. Moreover, g_∞ on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus E_1 \cup E_2$ is defined by a strictly increasing convex function $\psi_\infty(t)$ on $(-\infty, \infty)$, which satisfies the equation*

$$\psi'\psi'' = e^{13t/7 - 6\psi/7}. \tag{1-2}$$

(2) *The Ricci curvature of g_∞ is given by*

$$\operatorname{Ric}(g_\infty) = \sqrt{-1}\partial\bar{\partial}\left(\tfrac{1}{7}t + \tfrac{6}{7}\psi_\infty\right) \quad \textit{on } \mathbb{C}^2 \setminus \{0\}. \tag{1-3}$$

In particular, the Ricci curvature of g_∞ is bounded.

By (1-2), one sees that the limit metric g_∞ is not a Kähler–Ricci soliton. This situation is quite different from the case of Kähler–Ricci flow studied in [Zhu 2007], where it was shown that the evolved Kähler metrics arising from the Kähler–Ricci

[1] Actually, Székelyhidi proved that the maximal solvable parameter λ is independent of the background metrics we choose.

[2] Chi Li [2009] has calculated the maximal solvable parameter λ for all toric Fano manifolds.

flow on a given toric Fano manifold will converge smoothly to a Kähler–Ricci soliton in the Cheeger–Gromov sense if the initial Kähler metric is toric. (See also [Koiso 1990] for the special case $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ with a Calabi symmetric metric as the initial metric.) The existence of Kähler–Ricci solitons on a toric Fano manifold was proved in [Wang and Zhu 2004]. Note that $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is a toric Fano manifold and that a Calabi symmetric metric is toric.

It is well known that the limit metric space of a sequence of 4-dimensional Riemannian manifolds with Ricci curvature bounded from below and with sectional curvature bounded in the L^2 norm can only have isolated singularities [Anderson 2005; Cheeger et al. 2002]. Theorem 1.1 gives an example of limit metric space with nonisolated singularities. Note that here the sequence of 4-dimensional Riemannian manifolds have only lower bound on their Ricci curvature (without the condition for sectional curvature).

In Section 2, we reduce the Monge–Ampère equations to a family of ordinary differential equations using Calabi's symmetry conditions. In Section 3, we use the Futaki invariant [1983] to give a simple proof to the "only if" part of Székelyhidi's result and to get some crucial estimates. The convergence problem is discussed in Section 4. Theorem 1.1 is finally proved in Section 5 by studying the structure of the singular limit metric. We remark that Theorem 1.1 still holds for the higher dimensional blow-up space $\mathbb{C}P^n \# \overline{\mathbb{C}P^n}$ according to our proof.

2. Reduction of the equation under Calabi's symmetry conditions

Let (M, g) be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$, where the Kähler class $[\omega_g]$ equals $2\pi c_1(M)$. To study the existence of Kähler–Einstein metrics on M, we use the continuity method. Consider the complex Monge–Ampère equations

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}}) e^{h-\lambda\phi} \tag{2-1}$$

with parameter $\lambda \in [0, 1]$, where h is a Ricci potential of g defined by

$$\mathrm{Ric}(g) - \omega_g = \sqrt{-1}\partial\bar{\partial}h.$$

See [Yau 1978; Tian 1987]. If (2-1) is solvable at $\lambda = 1$, then the solution ϕ will define a Kähler–Einstein metric whose Kähler form given by $\omega_g + \sqrt{-1}\partial\bar{\partial}\phi$. In our case $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we choose a background Kähler metric g satisfying Calabi's symmetry conditions, namely, g is defined by a convex function u in $t \in (-\infty, \infty)$, so that

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} u(t) = e^{-t} u'(t) \delta_{\alpha\beta} + e^{-2t} \bar{z}_\alpha z_\beta (u''(t) - u'(t)). \tag{2-2}$$

As Calabi pointed out [1982], g can extend across E_1 and E_2 if and only if the following hold:[3]

(1) The function $u_0(r)$ defined for all $r > 0$ by

$$u_0(r) = u_0(e^t) = u(t) - t \tag{2-3}$$

is extendable by continuity to a smooth function at $r = 0$ satisfying $u_0'(0) > 0$.

(2) The function $u_\infty(r)$ defined for all $r > 0$ by

$$u_\infty(r) = u_\infty(e^{-t}) = u(t) - 3t \tag{2-4}$$

is extendable by continuity to a smooth function at $r = 0$ satisfying $u_\infty'(0) > 0$.

Let $v(t) := -\log\det(g_{\alpha\bar{\beta}}) = 2t - \log u'(t) - \log u''(t)$. Then the Ricci curvature is

$$R_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} v(t) = e^{-t} v'(t)\delta_{\alpha\beta} + e^{-2t}\bar{z}_\alpha z_\beta (v''(t) - v'(t)). \tag{2-5}$$

Since all solutions ϕ of (2-1) are symmetric, it becomes

$$(u' + \phi')(u'' + \phi'') = e^{2t - u - \lambda\phi},$$

which we can rewrite as

$$\psi'\psi'' = e^{2t - (\lambda\psi + (1-\lambda)u)}, \tag{2-6}$$

where $\psi = u + \phi$. Note that the volume of g is computed by

$$\begin{aligned} \mathrm{Vol}(M, g) &= \int_{\mathbb{C}^2\setminus\{0\}} u''u'e^{-2t} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ &= \mathrm{Vol}(S^3) \int_{-\infty}^{\infty} u''u'dt = 4\,\mathrm{Vol}(S^3), \end{aligned} \tag{2-7}$$

where $\mathrm{Vol}(S^3)$ denotes the volume of the unit sphere in $\mathbb{R}^4$. So we may normalize u so that

$$\int_{-\infty}^{+\infty} e^{2t - u(t)} dt = 4. \tag{2-8}$$

3. Application of the Futaki invariant

For a convex function $\psi(t)$ on $(-\infty, \infty)$ satisfying the boundary conditions (2-3) and (2-4), we consider the integral

$$I = \int_{-\infty}^{+\infty} (2\psi'\psi'' - \psi'^2\psi'' - \psi''^2 - \psi'\psi''')dt. \tag{3-1}$$

[3] This is also clear from our proof of Proposition 5.2.

One can show that if ψ is a defining function of a Calabi symmetric metric on $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, then I is just the Futaki invariant evaluated at the holomorphic vector field $z_1\partial/\partial z_1 + z_2\partial/\partial z_2$, where z_1 and z_2 are the standard coordinates on $\mathbb{C}^2 \setminus \{0\} \simeq M \setminus (E_1 \cup E_2)$.

Now by the boundary conditions, we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} 2\psi'\psi'' dt = \psi'^2\big|_{-\infty}^{+\infty} = 8, \\ I_2 &= \int_{-\infty}^{+\infty} -\psi'^2\psi'' dt = -\tfrac{1}{3}\psi'^3\big|_{-\infty}^{+\infty} = -\tfrac{26}{3}, \\ I_3 &= \int_{-\infty}^{+\infty} -\psi''^2 - \psi'\psi''' dt = -(\psi'\psi'')\big|_{-\infty}^{+\infty} = 0. \end{aligned}$$

These equalities imply that $I = -2/3 \neq 0$. In particular, we see that there are no Kähler–Einstein metrics on M.

Proposition 3.1. *Equation* (2-6) *is solvable only if* $\lambda < 6/7$.

Proof. According to the boundary conditions, the integral I should equal $-2/3$. But by the equation, we have

$$I = (1-\lambda)\int_{-\infty}^{+\infty} (u' - \psi')\psi'\psi'' dt = \frac{13(1-\lambda)}{3} - \frac{1-\lambda}{2}\int_{-\infty}^{+\infty} \psi'^2 u'' dt.$$

Note that $\psi'^2 < 9$, we have $-\frac{2}{3} = I > -\frac{14}{3}(1-\lambda)$. So $\lambda < 6/7$. □

We can get more information from the integral I.

Lemma 3.2. *For any fixed* t_0, *we have*

$$\lim_{\lambda\to 6/7}\int_{t_0}^{+\infty} \psi_\lambda'\psi_\lambda'' dt = 0. \tag{3-2}$$

In particular, the functions ψ_λ' *converge uniformly to the constant function* 3 *on* $[t_0, +\infty)$ *when* $\lambda \to 6/7$.

Proof. The identity $I \equiv -2/3$ is equivalent to

$$A_\lambda := \int_{-\infty}^{+\infty} u'\psi_\lambda'\psi_\lambda'' dt = \frac{26}{3} - \frac{2}{3(1-\lambda)}.$$

It follows that $\lim_{\lambda\to 6/7} A_\lambda = 4$. On the other hand, we have

$$\begin{aligned} A_\lambda &> \int_{-\infty}^{t_0} \psi_\lambda'\psi_\lambda'' dt + u'(t_0)\int_{t_0}^{+\infty} \psi_\lambda'\psi_\lambda'' dt \\ &= 4 + (u'(t_0) - 1)\int_{t_0}^{+\infty} \psi_\lambda'\psi_\lambda'' dt. \end{aligned} \tag{3-3}$$

This implies that

$$0 < \int_{t_0}^{+\infty} \psi_\lambda' \psi_\lambda'' dt < \frac{1}{u'(t_0)-1}(A_\lambda - 4) \to 0.$$

Thus

$$\tfrac{1}{2}(3^2 - (\psi_\lambda'(t_0))^2) \to 0 \quad \text{as } \lambda \to 6/7,$$

that is, $\psi_\lambda'(t_0) \to 3$ as $\lambda \to 6/7$. By the monotonicity of ψ_λ', the functions ψ_λ' converge uniformly to 3 on $[t_0, +\infty)$. □

4. Convergence

Now we analyze the behavior of ψ_λ as $\lambda \nearrow 6/7$.

Let $w_\lambda = -(2t-(1-\lambda)u-\lambda\psi_\lambda)$. Then w_λ is strictly convex. Let $p_\lambda \in M$, so that $w_\lambda(p_\lambda) = \inf_{x\in M} w_\lambda(x) = C_\lambda$. Clearly, $p_\lambda \in M\backslash(E_1 \bigcup E_2) \cong \mathbb{C}^2\backslash\{0\}$, so we may abuse the notation to identify p_λ with its coordinate in $\mathbb{C}^2\backslash\{0\}$. Let $t_\lambda = \log|p_\lambda|^2$.

Lemma 4.1. *When $\lambda \to 6/7$, we have $t_\lambda \to -\infty$.*

Proof. Suppose that there is a subsequence $\lambda_i \to 6/7$ but $t_\lambda \geq -C > -\infty$. Since $w_\lambda'(t_\lambda) = 0$, we have

$$\psi_\lambda'(-C) \leq \psi_\lambda'(t_\lambda) = \frac{2}{\lambda} - \frac{1-\lambda}{\lambda} u'(t_\lambda) \leq \frac{2}{\lambda}.$$

Then we can easily get a contradiction from this and Lemma 3.2. □

We now introduce a family of modified functions of ψ_λ by

$$\tilde{\psi}_\lambda(t) = \psi_\lambda(t+t_\lambda) - \lambda^{-1}(2t_\lambda - (1-\lambda)u(t_\lambda)).$$

Then $\tilde{\psi}_\lambda$ satisfies the equation

$$\tilde{\psi}''\tilde{\psi}' = e^{(2-(1-\lambda)u'(t_\lambda))t - \lambda\tilde{\psi} + (1-\lambda) f_\lambda(t)}, \tag{4-1}$$

where

$$f_\lambda(t) = -(u(t+t_\lambda) - u(t_\lambda) - u'(t_\lambda)t) = u_0(e^{t_\lambda}) - u_0(e^{t+t_\lambda}) + (u'(t_\lambda) - 1)t.$$

It is clear that $\lim_{\lambda\to 6/7} f_\lambda(t) = 0$ for any t.

Proposition 4.2. *There exist a sequence of convex functions $\tilde{\psi}_{\lambda_i}$, where $\lambda_i \to 6/7$, and a smooth convex function ψ_∞ defined on $(-\infty, \infty)$, such that the $\tilde{\psi}_{\lambda_i}$ converge locally uniformly and smoothly to ψ_∞, which satisfies the equation*

$$\psi''\psi' = e^{(13/7)t - (6/7)\psi} \quad \textit{for } t \in (-\infty, \infty). \tag{4-2}$$

Proof. It suffices to prove that

$$|C_\lambda| \le C.$$

In fact, if this is true, we see that all the $\tilde{\psi}_\lambda$ are uniformly bounded on any bounded intervals. As a consequence, by (4-1), the $\tilde{\psi}''_\lambda$ are also uniformly bounded on any bounded intervals. Then again by (4-1), it is easy to see that the C^k norms of the $\tilde{\psi}_\lambda$ are locally uniformly bounded. Thus there exist a sequence of convex functions $\tilde{\psi}_\lambda$ that converges locally uniformly in C^k norm to a convex function ψ_∞ defined on $(-\infty, \infty)$. On the other hand, by Lemma 4.1, the t_λ go to $-\infty$ as $\lambda \to 6/7$. Hence, by (4-1) and the fact that $f_\lambda(t) \to 0$ as $\lambda \to 6/7$, we conclude that ψ_∞ is in fact smooth and satisfies (4-2).

Now we prove the the boundedness of C_λ. By the boundary conditions, we have

$$\int_{-\infty}^{\infty} (\psi''_\lambda \psi'_\lambda) dt = \tfrac{1}{2}(\psi'^2_\lambda(\infty) - \psi'^2_\lambda(-\infty)) = 4. \tag{4-3}$$

Then by the convexity of w_λ and the fact $|w'_\lambda| \le 1$, it is easy to get a lower bound of C_λ. So we only need to obtain an upper bound. For simplicity, we write $w = w_\lambda$ and $\psi = \psi_\lambda$.

Let B_0 be the interval defined by

$$B_0 := \{t \in (-\infty, \infty) \mid C_\lambda \le w(t) \le C_\lambda + 1\}.$$

Then there exist exact two numbers s_0 and t_0 with $s_0 < t_0$ such that $w(s_0) = w(t_0) = C_\lambda + 1$. Clearly $t_\lambda \in B_0$, and it holds that

$$\psi'' \ge c_0 e^{-C_\lambda} \quad \text{on } B_0.$$

So

$$w'' \ge \lambda c_0 e^{-C_\lambda} \ge \tfrac{1}{2} c_0 e^{-C_\lambda}. \tag{4-4}$$

We want to show that

$$R := \tfrac{1}{2}(t_0 - s_0) \le \sqrt{\frac{4}{c_0}} e^{C_\lambda/2}. \tag{4-5}$$

In fact we consider the function on $\mathbb{R}$ defined by

$$v(t) = \tfrac{1}{4} c_0 e^{-C_\lambda} (|t - \tfrac{1}{2}(s_0 + t_0)|^2 - R^2) + C_\lambda + 1.$$

Then it is clear that $v(t)$ satisfies

$$v'' = \tfrac{1}{2} c_0 e^{-C_\lambda} \text{ on } B_0 \quad \text{and} \quad v(s_0) = v(t_0) = C_\lambda + 1. \tag{4-6}$$

Thus by (4-4) and (4-6), we get

$$(w - v)'' \ge 0 \text{ on } B_0 \quad \text{and} \quad w(t) = v(t) \text{ for } t = s_0 \text{ and } t = t_0.$$

It follows from the convexity that

$$w \le v \quad \text{on } B_0.$$

In particular,

$$C_\lambda \le w(\tfrac{1}{2}(s_0+t_0)) \le v(\tfrac{1}{2}(s_0+t_0)) = -\tfrac{1}{4}c_0 e^{-C_\lambda} R^2 + C_\lambda + 1.$$

This implies (4-5).

For $k \ge 1$, we choose a family of closed sets

$$B_k := \{t \in (-\infty, \infty) \mid k + C_\lambda \le w(t) \le C_\lambda + k + 1\}.$$

Then there are s_k and t_k with $s_k < t_{k-1}$, for $k \ge 1$, such that

$$B_k = [s_{k-1}, s_k] \cup [t_{k-1}, t_k].$$

By the convexity of w, it is easy to see $w'(t_0), -w'(s_0) \ge 1/(2R)$, and so

$$-w'(s), w'(t) \ge 1/(2R) \quad \text{for all } s \le s_0 \text{ and } t \ge t_0.$$

Thus

$$t_k - t_{k-1} \le 2R \quad \text{and} \quad s_k - s_{k-1} \le 2R.$$

Hence by (4-5), we get

$$s_k - s_{k-1},\, t_k - t_{k-1} \le 2R \le 2\sqrt{\frac{4}{c_0}} e^{C_\lambda/2}.$$

It follows that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-w} dt &= \sum_k \int_{B_k} e^{-w} dt \\ &\le \sum_k 4\sqrt{\frac{4}{c_0}} e^{C_\lambda/2} e^{-C_\lambda - k} \\ &= 4\sqrt{\frac{4}{c_0}} e^{-C_\lambda/2} \sum_k e^{-k} \le Ce^{-C_\lambda/2}. \end{aligned} \tag{4-7}$$

This inequality and (4-3) imply that $4 \le Ce^{-C_\lambda/2}$. □

According to Proposition 4.2, we can define a Kähler metric ω_∞ on $\mathbb{C}^2 \setminus \{0\}$ by $\sqrt{-1}\partial\bar\partial\psi_\infty$. Then we have the following convergence of g_λ.

Proposition 4.3. *There exists a sequence of biholomorphic maps σ_{λ_i} on M, with $\lambda_i \to 6/7$, such that the $\sigma_{\lambda_i}^* \omega_{g_{\lambda_i}}$ converge to ω_∞ on $\mathbb{C}^2 \setminus \{0\}$ smoothly as $\lambda_i \to 6/7$. In particular, the $(M \setminus (E_1 \cup E_2), \omega_{g_{\lambda_i}})$ converge to $(\mathbb{C}^2 \setminus \{0\}, \omega_\infty)$ in the Cheeger–Gromov sense.*

Proof. Let σ_λ be the biholomorphic map on $\mathbb{C}^2 \setminus \{0\}$ defined by

$$\sigma_\lambda(z_1, z_2) = (e^{t_\lambda} z_1, e^{t_\lambda} z_2).$$

Clearly this action fixes the points $\{0\}$ and ∞. Thus the action can extend to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Furthermore,

$$\sigma_\lambda^* \omega_{g_\lambda} = \sqrt{-1}\partial\bar{\partial}\sigma_\lambda^* \psi_\lambda = \sqrt{-1}\partial\bar{\partial}\tilde{\psi}_\lambda \quad \text{on } \mathbb{C}^2 \setminus \{0\}.$$

By Proposition 4.2, we see that there exist a sequence of parameters λ_i such that $\sigma_\lambda^* \omega_{g_{\lambda_i}}$ converge locally uniformly and smoothly to ω_∞. □

5. Properties of the limit metric

Now we discuss the structure of ω_∞ near E_1 and E_2.

Lemma 5.1. *Let* $a := \lim_{t\to-\infty} \psi'_\infty(t)$ *and* $b := \lim_{t\to\infty} \psi'_\infty(t)$. *Then we have* $a = 1$ *and* $b = 3$.

Proof. Since $\mathrm{Ric}(\omega_\lambda) \geq \lambda\omega_\lambda$, by the Bonnet–Myers theorem, the diameters are uniformly bounded. Then by the Bishop–Gromov volume comparison theorem, we have

$$\mathrm{Vol}(B_r(x), \omega_\lambda) \geq Cr^n \quad \text{for all } x \in M \text{ and } r \leq 1.$$

This means the family of metrics ω_λ are noncollapsing. Then by a result of Cheeger and Colding [1997, Theorem 5.4], the convergent sequence ω_{λ_i} of metrics satisfy

$$\lim_{\lambda_i \to 6/7} \mathrm{Vol}(M, \omega_{\lambda_i}) = \mathrm{Vol}(M, \omega_\infty).$$

On the other hand,

$$\begin{aligned} \mathrm{Vol}(M, \omega_\lambda) &= \int_{\mathbb{C}^2\setminus\{0\}} \psi''\psi' e^{-2t} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ &= \mathrm{Vol}(S^3) \int_{-\infty}^{\infty} \psi''\psi' dt = 4\,\mathrm{Vol}(S^3) \end{aligned}$$

and

$$\mathrm{Vol}(M, \omega_\infty) = \tfrac{1}{2}\,\mathrm{Vol}(S^3)(b^2 - a^2).$$

It is obvious that $a \geq 1$ and $b \leq 3$. The claim follows. □

Proposition 5.2. *The metric* ω_∞ *can extend to a smooth metric on* $M \setminus E_2$.

Proof. In the standard coordinates on $\mathbb{C}^2$, we can express ω_∞ as

$$\begin{aligned} \omega_\infty &= \sqrt{-1}\partial\bar{\partial}\psi_\infty \\ &= \sqrt{-1}\sum_{\alpha,\beta}(e^{-t}\psi'_\infty\delta_{\alpha\beta} + e^{-2t}(\psi''_\infty - \psi'_\infty)\bar{z}_\alpha z_\beta) dz_\alpha \wedge d\bar{z}_\beta, \end{aligned} \tag{5-1}$$

where $z = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$ and $t = \log|z|^2$. We will use the coordinate transformation

$$w_1 = z_1/z_2 \quad \text{and} \quad w_2 = z_2$$

near $z = (z_1, z_2) = 0$. In fact, this transformation blows up a neighborhood of 0 to a neighborhood of E_1 in M. Since ω_∞ is symmetric, we may consider the behavior of ω_∞ along $z = (0, z_2)$ with $|z_2| \ll 1$ under this coordinate transformation. By (5-1), it is easy to see the components of the metric at $(0, z_2)$ are given by

$$g_{1\bar{1}} = e^{-t}\psi'_\infty(t), \quad g_{2\bar{2}} = e^{-t}\psi''_\infty(t), \quad g_{1\bar{2}} = 0.$$

Then, in the new coordinate system w, we have

$$\tilde{g}_{1\bar{1}} = \psi'_\infty(t), \quad \tilde{g}_{2\bar{2}} = e^{-t}\psi''_\infty(t), \quad \tilde{g}_{1\bar{2}} = w_2\bar{w}_1e^{-t}\psi'_\infty = 0. \tag{5-2}$$

On the other hand, by (4-2) and Lemma 3.2, we see that for any $\alpha < 1$ there is a uniform constant C_1 such that

$$\psi''_\infty(t) \le C_1e^{\alpha t} \quad \text{for all } t \le 0.$$

This implies

$$1 \le \psi'_\infty(t) \le 1 + C_2e^{\alpha t},$$

and so we get $|\psi_\infty - t| \le C_2$. Thus again by (4-2), we obtain

$$C_3^{-1} \le e^{-t}\psi''_\infty(t) \le C_3 \quad \text{for all } t \le 0. \tag{5-3}$$

This means that

$$C^{-1} \le \tilde{g}_{2\bar{2}} \le C \quad \text{for all } t \le 0$$

and for some uniform constant C. Moreover from the argument above, one can show that $g_1(s) := \tilde{g}_{2\bar{2}}$ can extend to a continuous function on the interval $[0, 1)$, where $s = e^t$. In fact, we will prove that $g_1(s)$ is C^∞ at $s = 0$ in the following.

We rewrite (4-2) as

$$[\psi'_\infty{}^2]'_s = 2e^{-(6/7)(\psi_\infty - t)}, \tag{5-4}$$

where f' and $[f]'_s$ are derivatives of f with respect to t and s, respectively. Then by (5-3), it is easy to see that $[(\psi'_\infty)^2]'_s$ is Lipschitz at $s = 0$. It follows that $g_1(s)$ is also Lipschitz at $s = 0$. This implies that $(\psi_\infty - t)'_s$ is Lipschitz at $s = 0$. Thus by (5-4), we can repeat the arguments above to show that $(g_1)'_s(0)$ exists and $(g_1)'_s(s)$ is Lipschitz at $s = 0$. Using the "bootstrap" argument, we see that $g_1(s)$ is C^∞ at $s = 0$.

The argument above also proves that $g_2(s) = \psi'_\infty(t) = \tilde{g}_{1\bar{1}}$ is C^∞ at $s = e^t = 0$. Note that $s = |w_2|^2$. Since the derivative of ω_∞ at $(0, 0)$ along the direction of the other variable w_1 is a function in the variables w_1 and w_2, we see that ω_∞ can extend to a smooth metric on $M \setminus E_2$. □

To analyze the behavior of ω_∞ near $z = \infty$, we introduce the following concept.

Definition 5.3. Let $g = \sum_{i,j} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be a Kähler metric defined on $M^* = M \setminus D$, where D is a smooth subvariety of codimension 1. We say that the metric g has conically symmetric singularities on D along one direction with a conical angle $\alpha\pi$ if for every point $p \in D$, there exists a coordinate system $(U; w_1, \dots, w_n)$ near p such that $w(p) = (0, \dots, 0)$ and in which the components $g_{i\bar{j}}$ of g on $U \setminus D$ are such that the components $(|w_1|^{2-\alpha}) g_{1\bar{1}}$, $g_{1\bar{j}}$ for $j = 1, \dots, n$ and $g_{l\bar{m}}$ for $l, m = 2, \dots, n$ can be extended to a positive definite matrix-valued smooth function on U in the variables $|w_1|^{\alpha/2}, w_2, \bar{w}_2, \dots, w_n, \bar{w}_n$.

Remark 5.4. If $\alpha = 2/k$ for some integer $k \geq 2$ in Definition 5.3, then the metric g has an orbifold structure. In fact, if $\tilde{V}$ is a branched covering of a neighborhood V of p by the map $\pi : (z_1, z_2, \dots, z_n) \mapsto (w_1 = (z_1)^k, w_2 = z_2, \dots, w_n = z_n)$, then $\pi^* g$ can be extended to a smooth Kähler metric on $\tilde{V}$.

Theorem 5.5. (1) *The singular Kähler metric ω_∞ on $\mathbb{C}P^n \# \overline{\mathbb{C}P^n}$ defined by ψ_∞ has conically symmetric singularities lying on the infinity divisor E_2, with the same conical angle $10\pi/7$ along one direction.*

(2) *The Ricci curvature of ω_∞ satisfies the equation*

$$\operatorname{Ric}(\omega_\infty) = \sqrt{-1}\,\partial\bar{\partial}\left(\tfrac{1}{7} t + \tfrac{6}{7} \psi_\infty\right). \tag{5-5}$$

In particular, the Ricci curvature is bounded.

Proof. By Proposition 5.2, it suffices to analyze the behavior of ω_∞ near E_2. We write the homogeneous coordinates on $M \setminus E_1$ (as a subset of $\mathbb{C}P^2$) as $[Z_0, Z_1, Z_2]$, where E_2 is defined by the equation $Z_0 = 0$. Then we have on $M \setminus (E_1 \cup E_2)$

$$z_1 = \frac{Z_1}{Z_0} \quad \text{and} \quad z_2 = \frac{Z_2}{Z_0}.$$

By the symmetry conditions we imposed, we may consider only the behavior of ω_∞ on the open set $U := (M \setminus E_1) \cap \{Z_2 \neq 0\}$. The affine coordinates on U are

$$w_1 = \frac{Z_1}{Z_2} = \frac{z_1}{z_2} \quad \text{and} \quad w_2 = \frac{Z_0}{Z_2} = \frac{1}{z_2}.$$

A direct computation shows that the components of the metric ω_∞ at $w = (0, w_2)$ are given by

$$\tilde{g}_{1\bar{1}} = \psi'_\infty(t), \quad \tilde{g}_{2\bar{2}} = e^t \psi''_\infty(t), \quad \tilde{g}_{1\bar{2}} = 0, \tag{5-6}$$

where $t = \log(|z_1|^2 + |z_2|^2) = \log(1/|w_2|^2)$. On the other hand, by (4-2) and the arguments in the proof of Proposition 5.2, one can show that

$$\begin{aligned} &|\psi_\infty - 3t| \leq C, \\ &e^t (\psi_\infty)''(t) = O(e^{(2/7)t}) \quad \text{as } t \to \infty. \end{aligned} \tag{5-7}$$

Moreover, if we set $s = e^{-(5/7)t}$ and rewrite (4-2) as

$$[\psi_\infty'^2]_s' = 2e^{-(6/7)(\psi_\infty - 3t)},$$

then we can prove that $\tilde{g}_1(s) = e^{(5/7)t}\psi_\infty''(t)$ and $\tilde{g}_2(s) = \psi_\infty - 3t$ are both C^∞ at $s = 0$. Hence we have proved that ω_∞ has a conical structure at each point in E_2 with the same conical angle $(10/7)\pi$.

By (4-2), we see that the Ricci curvature of ω_∞ satisfies (5-5). By the local formula (5-6) of ω_∞ near E_2, the Ricci curvature is bounded. □

Theorem 1.1 follows from Theorem 5.5 and Proposition 5.2.

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References

[Anderson 2005] M. T. Anderson, "Orbifold compactness for spaces of Riemannian metrics and applications", *Math. Ann.* **331**:4 (2005), 739–778. MR 2006c:53029 Zbl 1071.53025

[Calabi 1982] E. Calabi, "Extremal Kähler metrics", pp. 259–290 in *Seminar on Differential Geometry*, edited by S. T. Yau, Ann. of Math. Stud. **102**, Princeton Univ. Press, 1982. MR 83i:53088 Zbl 0487.53057

[Cheeger and Colding 1997] J. Cheeger and T. H. Colding, "On the structure of spaces with Ricci curvature bounded below, I", *J. Differential Geom.* **46**:3 (1997), 406–480. MR 98k:53044 Zbl 0902.53034

[Cheeger et al. 2002] J. Cheeger, T. H. Colding, and G. Tian, "On the singularities of spaces with bounded Ricci curvature", *Geom. Funct. Anal.* **12**:5 (2002), 873–914. MR 2003m:53053 Zbl 1030.53046

[Donaldson 2002] S. K. Donaldson, "Scalar curvature and stability of toric varieties", *J. Differential Geom.* **62**:2 (2002), 289–349. MR 2005c:32028 Zbl 1074.53059

[Futaki 1983] A. Futaki, "An obstruction to the existence of Einstein Kähler metrics", *Invent. Math.* **73**:3 (1983), 437–443. MR 84j:53072 Zbl 0506.53030

[Koiso 1990] N. Koiso, "On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics", pp. 327–337 in *Recent topics in differential and analytic geometry*, edited by T. Ochiai, Adv. Stud. Pure Math. **18**, Academic Press, Boston, MA, 1990. MR 93d:53057 Zbl 0739.53052

[Li 2009] C. Li, "Greatest lower bounds on Ricci curvature for toric Fano manifolds", preprint, 2009. arXiv 0909.3443

[Matsushima 1957] Y. Matsushima, "Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne", *Nagoya Math. J.* **11** (1957), 145–150. MR 20 #995 Zbl 0091.34803

[Székelyhidi 2009] G. Székelyhidi, "Greatest lower bounds on the Ricci curvature of Fano manifolds", preprint, 2009. arXiv 0903.5504

[Tian 1987] G. Tian, "On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$", *Invent. Math.* **89**:2 (1987), 225–246. MR 88e:53069 Zbl 0599.53046

[Tian 1997] G. Tian, "Kähler–Einstein metrics with positive scalar curvature", *Invent. Math.* **130**:1 (1997), 1–37. MR 99e:53065 Zbl 0892.53027

[Wang and Zhu 2004] X.-J. Wang and X. Zhu, "Kähler–Ricci solitons on toric manifolds with positive first Chern class", *Adv. Math.* **188**:1 (2004), 87–103. MR 2005d:53074 Zbl 1086.53067

[Yau 1978] S. T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I", *Comm. Pure Appl. Math.* **31**:3 (1978), 339–411. MR 81d:53045 Zbl 0362.53049

[Zhu 2007] X. Zhu, "Kähler–Ricci flow on a toric manifold with positive first Chern class", preprint, 2007. arXiv math/0703486

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YALONG SHI
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
CHINA
ylshi@math.pku.edu.cn

XIAOHUA ZHU
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
CHINA
xhzhu@math.pku.edu.cn

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