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MIXED INTERIOR AND BOUNDARY NODAL BUBBLING SOLUTIONS FOR A sinh-POISSON EQUATION

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We consider here the semilinear equation $\Delta u + 2\varepsilon^2 \sinh u = 0$ posed on a bounded smooth domain Ω in $\mathbb{R}^2$ with homogeneous Neumann boundary condition, where $\varepsilon > 0$ is a small parameter. We show that for any given nonnegative integers k and l with $k + l \geq 1$, there exists a family of solutions u_ε that develops $2k$ interior and $2l$ boundary singularities for ε sufficiently small, with the property that

$$2\varepsilon^2 \sinh u_\varepsilon \rightharpoonup 8\pi \sum_{i=1}^{2k} (-1)^{i-1} \delta_{\xi_i} + 4\pi \sum_{i=1}^{2l} (-1)^{i-1} \delta_{\xi_i},$$

where $(\xi_1, \ldots, \xi_{2(k+l)})$ are critical points of some functional defined explicitly in terms of the associated Green function.

1. Introduction

The two-dimensional sinh-Poisson equation

$$\Delta u + 2\varepsilon^2 \sinh u = 0 \tag{1-1}$$

arises in various important contexts, notably as a vorticity equation in classical hydrodynamics [Gurarie and Chow 2004; Chow et al. 1998; Kuvshinov and Schep 2000; Mallier and Maslowe 1993], in physico-chemical hydrodynamics [Probstein 1994] and in the geometry of constant mean curvature surfaces [Wente 1986]. In the vorticity connection, it occurs in a remarkable manner out of natural relaxation states in the long-time computation of two-dimensional fluid motion [Mallier and Maslowe 1993] (see also the references therein). In geometry, the sinh-Poisson equation plays a very important role in the study of the construction of constant mean curvature surfaces initiated by Wente [1986]. Wente's seminal work then

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led to work by Steffen [1986], Struwe [1986] and Brezis and Coron [1984], which completed the understanding of the blow-up for constant mean curvature surfaces from a geometric point of view. Spruck [1988] was the first to study the sinh-Poisson equation from an analytic point of view. Recently, the asymptotic behavior of solutions to (1-1) was studied on a closed Riemann surface in [Ohtsuka and Suzuki 2006] and [Jost et al. 2008]. The authors applied the so-called "symmetrization method" and "Pohozaev identity", respectively, to show that there possibly exist two different types of blow-up for a family of solutions to (1-1). Conversely, Bertolucci and Pistoia [2007] tried to construct blow-up solutions to (1-1) with Dirichlet boundary conditions for $n = 2$, and proved that for ε positive and small enough, there exist at least two pairs of solutions that change sign exactly once, that concentrate in the domain and that have their nodal lines intersecting the boundary.

In [Wei et al. 2011] and [Wei 2009] the Neumann problem

$$\begin{cases} \Delta u + 2\varepsilon^2 \sinh u = 0 & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1-2}$$

was considered, where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\varepsilon > 0$ is a parameter. The authors showed a concentration phenomena of solutions to (1-2) in the domain in [Wei et al. 2011], and on the boundary in [Wei 2009].

In this paper, we continue the study of the existence of solutions to (1-2). We prove that there exists a family of solutions u_ε that concentrate positively and negatively in the domain and its boundary.

To state our results, we need to introduce some notation. First, let us define the corresponding Green function for the Neumann problem:

$$\begin{cases} -\Delta G(x, y) = \delta_y(x) - 1/|\Omega| & \text{in } \Omega, \\ \partial G/\partial \nu = 0 & \text{on } \partial\Omega, \\ \int_\Omega G(x, y)dx = 0. \end{cases} \tag{1-3}$$

The regular part of $G(x, y)$ is defined depending on whether y lies in the domain or on its boundary as

$$H(x, y) = \begin{cases} G(x, y) + \dfrac{1}{2\pi} \log|x - y| & \text{for } y \in \Omega, \\ G(x, y) + \dfrac{1}{\pi} \log|x - y| & \text{for } y \in \partial\Omega. \end{cases} \tag{1-4}$$

In this way, $H(\cdot, y)$ is of class $C^{1,\alpha}$ in $\overline{\Omega}$.

For $k + l \geq 1$ and points ξ_j for $j = 1, \dots, 2(k + l)$, with $\xi_j \in \Omega$ for $j \leq 2k$ and $\xi_j \in \partial\Omega$ for $2k + 1 \leq j \leq 2(k + l)$, we define

$$\varphi_{2(k+l)}(\xi_1, \dots, \xi_{2(k+l)}) = \sum_{j=1}^{2(k+l)} c_j^2 H(\xi_j, \xi_j) + \sum_{j \neq i} c_j c_i (-1)^{j+i} G(\xi_j, \xi_i) \tag{1-5}$$

and denote

$$\mathcal{M}_d := \Big\{\xi = (\xi_1, \dots, \xi_{2k}, \xi_{2k+1}, \dots, \xi_{2(k+l)}) \in \Omega^{2k} \times \partial\Omega^{2l} \;\Big|\; \min_{j\neq i}|\xi_j - \xi_i| \geq d,\; \min_{j=1,\dots,2k} \operatorname{dist}(\xi_j, \partial\Omega) \geq d\Big\},$$

where $c_i = 8\pi$ for $i = 1, \dots, 2k$ and $c_i = 4\pi$ for $i = 2k+1, \dots, 2(k+l)$.

Definition 1.1 [Esposito et al. 2006]. We say that ξ is a C^0*-stable critical point* of $\varphi_m : \mathcal{M}_d \to \mathbb{R}$ if for any sequence of functions $\varphi_m^n : \mathcal{M}_d \to \mathbb{R}$ such that $\varphi_m^n \to \varphi_m$ uniformly on compact sets of $\mathcal{M}_d$, the function φ_m^n has a critical point ξ_n such that $\varphi_m^n(\xi_n) \to \varphi_m(\xi)$.

In particular, if ξ is a strict local minimum/maximum point of φ_m, then ξ is a C^0-stable critical point.

Theorem 1.2 (main result). *Let k and l be nonnegative integers with $k + l \geq 1$. Assume $\xi^* \in \mathcal{M}_d$ is a C^0-stable critical point of $\varphi_{2(k+l)}$. Then for any sufficiently small $\varepsilon > 0$, there is a solution u_ε to (1-2) with the property that*

$$2\varepsilon^2 \int_\Omega |\sinh u_\varepsilon|\,dx \to 8\pi(2k+l) \quad \text{as } \varepsilon \to 0. \tag{1-6}$$

More precisely, for any sequence $\{\varepsilon_n\}_{n\geq 1}$ that tends to 0, there is a subsequence and $2(k+l)$ points $\xi_i \in \overline{\Omega}$ for $i = 1, \dots, 2(k+l)$, with $\xi_j \in \Omega$ for $j \leq 2k$ and $\xi_j \in \partial\Omega$ for $2k+1 \leq j \leq 2(k+l)$, and positive constants μ_i for $i = 1, \dots, 2(k+l)$ such that

$$u_\varepsilon(x) = \sum_{i=1}^{2(k+l)} (-1)^{i-1}\Big(\log \frac{1}{(\varepsilon^2\mu_i^2 + |x-\xi_i|^2)^2} + c_i H(x, \xi_i)\Big) + o(1) \tag{1-7}$$

and

$$2\varepsilon^2 \sinh u_\varepsilon \rightharpoonup 8\pi \sum_{i=1}^{2k}(-1)^{i-1}\delta_{\xi_i} + 4\pi \sum_{i=2k+1}^{2(k+l)} (-1)^{i-1}\delta_{\xi_i} \tag{1-8}$$

in the sense of measure. Moreover, the constants μ_i are given by

$$\log(8\mu_i^2) = c_i H(\xi_i, \xi_i) + \sum_{j\neq i}(-1)^{j+i} c_j G(\xi_i, \xi_j).$$

The $l = 0$ (or $k = 0$) case of this theorem was proved in [Wei et al. 2011] (or [Wei 2009]). The conditions that $\xi^* \in \mathcal{M}_d$ be a C^0-stable critical point of $\varphi_{2(k+l)}$ is perhaps not necessary. Here, we need it only because of the technique we will use. In particular, for the case $k = l = 1$ and $\Omega = B = B(0, 1)$, the unit ball in $\mathbb{R}^2$, we don't need the condition and can obtain the existence and the profile of sign-changing solutions that concentrate positively and negatively at different points $\xi_1, \xi_2 \in B$ and $\xi_3, \xi_4 \in \partial B$. More precisely:

Theorem 1.3. *Let $k = l = 1$. Then, there exists a solution u_ε to (1-2) that concentrates at different points $\xi_1, \xi_2 \in B$ and $\xi_3, \xi_4 \in \partial B$, according to (1-6), (1-7) and (1-8) with $k = l = 1$, as ε goes to* 0.

Del Pino and Wei [2006] considered the problem $-\Delta u + u = \lambda e^u$ under Neumann boundary conditions and built a solution with $\lambda \int_\Omega e^u$ uniformly bounded and boundary-interior concentrating, such that $\lambda e^u \rightharpoonup 8\pi \sum_{j=1}^{k} \delta_{\xi_j} + 4\pi \sum_{j=k+1}^{m} \delta_{\xi_j}$. For basic cells, they used explicit solutions of

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u dx < +\infty$$

given by

$$U_{\mu,\xi} = \log \frac{8\mu^2}{(\mu^2 + |x-\xi|^2)^2} \quad \text{for } \mu > 0 \text{ and } \xi \in \mathbb{R}^2.$$

In this paper, we will also construct solutions predicted by the theorems using these ones, but suitably scaled and projected so that it works for the nonlinearity we consider here. A special feature of our problem is presence of *mixed positive-negative boundary-interior* bubbling solutions. This is a new concentration phenomenon. To capture such solutions, we use the so-called localized energy method, which combines Lyapunov–Schmidt reduction and variational techniques. Such a scheme was been used in many works; see for instance [Dávila et al. 2005; del Pino et al. 2005; del Pino and Wei 2006] and references therein. Here we follow [del Pino and Wei 2006; Wei et al. 2011; Wei 2009], but we will overcome some of the difficulties that the mixed concentration phenomenon brings by delicate analysis.

2. Ansatz for the solution

In this section we will provide a first approximation for the solution of the problem (1-2) predicted by Theorems 1.2 and 1.3. Let us fix $k+l \geq 1$. For $i = 1, \dots, 2(k+l)$, let $\xi_i \in \overline{\Omega}$ and let μ_i be positive numbers to be chosen later. We define

$$u_i(x) = \log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + |x-\xi_i|^2)^2}. \tag{2-1}$$

The ansatz is

$$U(x) = \sum_{i=1}^{2(k+l)} (-1)^{i-1} (u_i(x) + H_i^\varepsilon(x)) \tag{2-2}$$

where $H_i^\varepsilon(x)$ is a correction term defined as the solution of

$$\begin{cases} \Delta H_i^\varepsilon = \varepsilon^2 \dfrac{1}{|\Omega|} \displaystyle\int_\Omega e^{u_i} & \text{in } \Omega, \\ \dfrac{\partial H_i^\varepsilon}{\partial \nu} = -\dfrac{\partial u_i}{\partial \nu} & \text{on } \partial\Omega \end{cases} \tag{2-3}$$

with the property that

$$\int_\Omega H_i^\varepsilon(x)dx = -\int_\Omega u_i\,dx. \tag{2-4}$$

This function resembles the shape of the regular part of the Green's function. Indeed, the following estimate for H_i^ε holds true.

Lemma 2.1. *For any* $0<\alpha<1$

$$H_i^\varepsilon(x) = c_i H(x,\xi_i) - \log(8\mu_i^2) + O(\varepsilon) \tag{2-5}$$

holds uniformly in $\overline{\Omega}$, *where* H *is the regular part of the Green function defined by* (1-4).

Proof. The regular part of Green's function $H(x,\xi_i)$ satisfies

$$\begin{cases} \Delta H(x,\xi_i) = \dfrac{1}{|\Omega|} & \text{in } \Omega,\\ \dfrac{\partial H}{\partial \nu}(x,\xi_i) = \dfrac{4}{c_i}\dfrac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} & \text{on } \partial\Omega. \end{cases} \tag{2-6}$$

Now we define $z_\varepsilon(x) = H_i^\varepsilon(x) + \log(8\mu_i^2) - c_i H(x,\xi_i)$. Then

$$\begin{cases} \Delta z_\varepsilon = \varepsilon^2 \dfrac{1}{|\Omega|}\displaystyle\int_\Omega e^{u_i} - \dfrac{c_i}{|\Omega|} & \text{in } \Omega,\\ \dfrac{\partial z_\varepsilon}{\partial \nu} = 4\dfrac{(x-\xi_i)\cdot\nu(x)}{\varepsilon^2\mu_i^2+|x-\xi_i|^2} - 4\dfrac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} & \text{on } \partial\Omega. \end{cases}$$

First, by the definition of u_i, we have

$$\begin{aligned} \varepsilon^2\int_\Omega e^{u_i} &= \varepsilon^2\int_\Omega \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2+|x-\xi_i|^2)^2}\\ &= 8\varepsilon^2\int_{\Omega/\varepsilon\mu_i}\frac{\mu_i^2}{(\varepsilon^2\mu_i^2+\varepsilon^2\mu_i^2y^2)^2}\varepsilon^2\mu_i^2\\ &= 8\int_{\Omega/\varepsilon\mu_i}\frac{dy}{(1+y^2)^2}\\ &= 2c_i\Big(\int_0^\infty \frac{tdt}{(1+t^2)^2} + O\Big(\int_{1/\varepsilon\mu_i}^\infty \frac{tdt}{(1+t^2)^2}\Big)\Big)\\ &= c_i + O(\varepsilon^2\mu_i^2) \end{aligned} \tag{2-7}$$

Next, for $\xi_i\in\Omega$ with $i=1,\dots,2k$, we have

$$\frac{\partial H_i^\varepsilon}{\partial\nu} = 4\frac{(x-\xi_i)\cdot\nu(x)}{\varepsilon^2\mu_i^2+|x-\xi_i|^2} = 4\frac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} + O(\varepsilon^2) \quad \text{for all } \xi_i\in\Omega,\ x\in\partial\Omega.$$

For $\xi_i \in \partial\Omega$ with $i = 2k+1, \dots, 2(k+l)$, we have

$$\lim_{\varepsilon\to 0} \frac{\partial H_i^\varepsilon}{\partial \nu} = 4\frac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} \quad \text{for all } x \neq \xi_i. \tag{2-8}$$

We claim that for any $p > 1$ there exists $C > 0$ such that

$$\Big\| \frac{\partial H_i^\varepsilon}{\partial \nu} - 4\frac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} \Big\|_{L^p(\partial\Omega)} \le C\varepsilon^{1/p}. \tag{2-9}$$

It is not difficult to prove that the inequality

$$|(x-\xi_i)\cdot\nu(x)| \le C|x-\xi_i|^2 \quad \text{for all } x \in \partial\Omega \tag{2-10}$$

holds for $\xi_i \in \partial\Omega$ by assuming that $\xi_i = 0$ and that near the origin $\partial\Omega$ is the graph of a function $P : (-\delta, \delta) \to \mathbb{R}$ with $P(0) = P'(0) = 0$. Now from (2-10) we obtain

$$\begin{aligned} \Big| \frac{\partial H_i^\varepsilon}{\partial \nu} - 4\frac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} \Big| &= 4\varepsilon^2\mu_i^2 \frac{|(x-\xi_i)\cdot\nu(x)|}{|x-\xi_i|^2(\varepsilon^2\mu_i^2+|x-\xi_i|^2)} \\ &\le \frac{C\varepsilon^2}{\varepsilon^2\mu_i^2+|x-\xi_i|^2}. \end{aligned} \tag{2-11}$$

Thus for $\lambda > 0$ small but fixed,

$$\Big| \frac{\partial H_i^\varepsilon}{\partial \nu} - 4\frac{(x-\xi_i)\cdot\nu(x)}{|x-\xi_i|^2} \Big| \le C\varepsilon^2 \quad \text{for all } |x-\xi_i| \ge \lambda,\ x \in \partial\Omega. \tag{2-12}$$

Letting $p > 1$ and changing variables $x - \xi_i = \varepsilon y \mu_i$, we have

$$\begin{aligned} \int_{B_\lambda(\xi_i)\cap\partial\Omega} \Big| \frac{\varepsilon^2}{\varepsilon^2\mu_i^2+|x-\xi_i|^2} \Big|^p &= C\varepsilon \int_{B_{\lambda/\varepsilon\mu_i}(0)\cap\partial\Omega_\varepsilon} \Big| \frac{1}{1+|y|^2} \Big|^p dy \\ &= C\varepsilon \int_0^{\lambda/\varepsilon\mu_i} \frac{1}{(1+t^2)^p} dt \le C\varepsilon. \end{aligned}$$

This, combined with (2-11) and (2-12), shows that (2-9) holds.

By elliptic regularity theory, we obtain $z_\varepsilon \in W^{1+s,p}(\Omega)$ for any $p \ge 1$, with $0 < s < 1/p$. On the other hand, from the Poincaré inequality we get

$$\Big\| z_\varepsilon - \frac{1}{|\Omega|}\int_\Omega z_\varepsilon \Big\|_{W^{1+s,p}(\Omega)} \le C\|\nabla z_\varepsilon\|_{L^p(\Omega)} \le C\varepsilon^{1/p}.$$

This implies the existence of a constant M such that

$$z_\varepsilon(x) = M + O(\varepsilon^\alpha) \quad \text{for any } \alpha \in (0,1),$$

uniformly in $\overline{\Omega}$, where $M = \lim_{\varepsilon\to 0}|\Omega|^{-1}\int_\Omega z_\varepsilon dx$.

To obtain the result, we only need to show $M=0$. First, by the definition of z_ε we have

$$M=\lim_{\varepsilon\to 0}\Big(\frac{1}{|\Omega|}\int_\Omega H_i^\varepsilon(x)dx+\log(8\mu_i^2)-\frac{c_i}{|\Omega|}\int_\Omega H(x,\xi_i)dx\Big). \tag{2-13}$$

The direct computation from (2-4) shows that

$$\begin{aligned}\int_\Omega H_i^\varepsilon(x)&=-\int_\Omega\Big(\log(8\mu_i^2)+\log\frac{1}{(\varepsilon^2\mu_i^2+|x-\xi_i|^2)^2}\Big)\\&=-|\Omega|\log(8\mu_i^2)+2\int_\Omega\log\Big(1+\frac{\varepsilon^2\mu_i^2}{|x-\xi_i|^2}\Big)-4\int_\Omega\log\frac{1}{|x-\xi_i|}\\&=-|\Omega|\log(8\mu_i^2)+c_i\int_\Omega H(x,\xi_i)dx+O(\varepsilon^2\log\varepsilon^{-1}),\end{aligned}$$

where the last equality is consequence of the definition of H and the property of the Green function. Therefore (2-13) implies $M=0$. □

In $\Omega_\varepsilon=\Omega/\varepsilon$, let $v(y)=u(\varepsilon y)$; then solving problem (1-2) is equivalent to solving

$$\begin{cases}\Delta v(y)+2\varepsilon^4\sinh v=0 & \text{in }\Omega_\varepsilon,\\ \partial v/\partial\nu=0 & \text{on }\partial\Omega_\varepsilon.\end{cases} \tag{2-14}$$

We will seek a solution v of (2-14) of the form

$$v(y)=V(y)+\phi(y)\quad\text{for all }y\in\Omega_\varepsilon, \tag{2-15}$$

where

$$V(y)=\sum_{i=1}^{2(k+l)}(-1)^{i-1}(u_i(\varepsilon y)+H_i^\varepsilon(\varepsilon y)). \tag{2-16}$$

Problem (2-14) can be restated: Find a solution ϕ to

$$\begin{cases}\Delta\phi+W\phi+R+N(\phi)=0 & \text{in }\Omega_\varepsilon,\\ \partial\phi/\partial\nu=0 & \text{on }\partial\Omega_\varepsilon,\end{cases} \tag{2-17}$$

where

$$W=2\varepsilon^4\cosh V, \tag{2-18}$$

$$N(\phi)=2\varepsilon^4(\sinh(V+\phi)-\phi\cosh V-\sinh V)\quad\text{(the nonlinear term)}, \tag{2-19}$$

$$R=\Delta V+2\varepsilon^4\sinh V\quad\text{(the error term)}. \tag{2-20}$$

We choose the parameters μ_i as

$$\log(8\mu_i^2)=H(\xi_i,\xi_i)+\sum_{j\neq i}(-1)^{j+i}G(\xi_i,\xi_j). \tag{2-21}$$

From Appendix A, we have for all $y \in \Omega_\varepsilon$ the estimates

$$|R(y)| \le C\varepsilon^{\alpha} \sum_{i=1}^{2(k+l)} \frac{1}{1+|y-\xi_i'|^3}, \tag{2-22}$$

$$W(y) = \sum_{i=1}^{2(k+l)} \frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}(1+\theta_\varepsilon(y)), \tag{2-23}$$

with

$$|\theta_\varepsilon(y)| \le C\varepsilon^{\alpha} + C\varepsilon \sum_{i=1}^{2(k+l)} |y-\xi_i'|, \tag{2-24}$$

where $\xi_i' = \xi_i/\varepsilon$.

3. Analysis of the linearized problem

In this section we study the solvability of the problem

$$\begin{cases} -\Delta\phi = W\phi + h + \displaystyle\sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji}\chi_i Z_{ji} + c_0\chi Z & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{3-1}$$

with

$$\int_{\Omega_\varepsilon} \chi_i Z_{ji}\phi = 0 \quad \text{for } i = 1, \dots, 2(k+l),\ \ j = 1, J_i, \tag{3-2}$$

$$\int_{\Omega_\varepsilon} \chi Z\phi = 0, \tag{3-3}$$

where W is a function that satisfies (2-23) and (2-24), $h \in L^\infty(\Omega_\varepsilon)$, $c_0, c_{ji} \in \mathbb{R}$, the functions χ, χ_i, Z and Z_{ji} will be defined below, $J_i = 2$ for $i = 1, \dots, 2k$, and $J_i = 1$ for $i = 2k+1, \dots, 2(k+l)$.

Define z_{ji} by

$$z_{0i} = \frac{1}{\mu_i} - 2\frac{\mu_i}{\mu_i^2+|y|^2} \quad \text{and} \quad z_{ji} = \frac{y_j}{\mu_i^2+|y|^2}.$$

It is well known that any solution to

$$\Delta\phi + \frac{8\mu_i^2}{(\mu_i^2+|y|^2)^2}\phi = 0\,, \quad |\phi| \le C(1+|y|)^{\sigma} \tag{3-4}$$

is a linear combination of z_{ji} for $j = 0, 1, 2$; see [Chen and Lin 2002, Lemma 2.1].

Next, we fix a large constant R_0 and a nonnegative smooth function $\overline{\chi} : \mathbb{R} \to \mathbb{R}$ such that $\overline{\chi}(r) = 1$ for $r \leq R_0$, $\overline{\chi}(r) = 0$ for $r > R_0 + 1$, and $0 \leq \overline{\chi} \leq 1$.

For $i = 1, \dots, 2k$ (corresponding to the interior bubble case), we define

$$\chi_i(y) = \overline{\chi}(|y - \xi_i'|), \quad Z_{ji}(y) = z_{ji}(y - \xi_i') \quad \text{for } j = 0, 1, 2, \; i = 1, \dots, 2k.$$

For $i = 2k+1, \dots, 2(k+l)$ (corresponding to the boundary bubble case), first we strength the boundary similarly to [del Pino and Wei 2006]. Let us concentrate on $\xi_i \in \partial\Omega$. Without loss of generality, we assume that $\xi_i = 0$ and the unit outward normal at ξ_i is $(0, -1)$. Let $P(x_1)$ be the defining function for the boundary $\partial\Omega$ in a neighborhood $B_\rho(\xi_i)$, that is,

$$\Omega \cap B_\rho(\xi_i) = \{(x_1, x_2) \mid x_2 > P(x_1), (x_1, x_2) \in B_\rho(\xi_i)\},$$

and then define $F_i : B_\rho(\xi_i) \cap \mathcal{N} \to \mathbb{R}^2$ by $F_i = (F_{i1}, F_{i2})$, where

$$F_{i1} = x_1 + \frac{x_2 - P(x_1)}{1 + |P'(x_1)|^2} P'(x_1) \quad \text{and} \quad F_{i2} = x_2 - P(x_1).$$

Then we set

$$F_i^\varepsilon(y) = \varepsilon^{-1} F_i(\varepsilon y)$$

and define

$$\chi_i(y) = \overline{\chi}(F_i^\varepsilon(y)), \quad Z_{ji}(y) = z_{ji}(F_i^\varepsilon(y)) \quad \text{for } j = 0, 1, \; i = 2k+1, \dots, 2(k+l).$$

It is important to observe that F_i preserves the Neumann boundary condition and

$$\Delta Z_{0i} + \frac{8\mu_i}{(\mu_i^2 + |y - \xi_i'|^2)^2} Z_{0i} = O\Big(\frac{\varepsilon^\alpha}{(1+|y-\xi_i'|)^3}\Big).$$

Let $0 < b < 1$ and define for all $i = 1, \dots, 2(k+l)$,

$$Z(y) = \begin{cases} \min\{1/\mu_i - \varepsilon^b, Z_{0i}(y)\} & \text{if } |y - \xi_i'| < \delta/\varepsilon, \\ 1/\mu_i - \varepsilon^b & \text{if } |y - \xi_i'| \geq \delta/\varepsilon \end{cases} \tag{3-5}$$

and $\chi = \sum_{i=1}^{2(k+l)} \chi_i$.

Now let us introduce the norms

$$\|h\|_\infty = \sup_{y \in \Omega_\varepsilon} |h(y)| \quad \text{and} \quad \|h\|_* = \sup_{y \in \Omega_\varepsilon} \frac{|h(y)|}{\varepsilon^2 + \sum_{i=1}^{2(k+l)} (1 + |y - \xi_i'|)^{-2-\sigma}},$$

where we fix $0 < \sigma < 1$, reserving the precise choice for later. Our main result in this section is stated as follows:

Proposition 3.1. *Let $d > 0$ and let k, l be nonnegative integers with $k + l \geq 1$. Then there exists a ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, any $2(k+l)$-points $(\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$ and any $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$,*

$c_0, c_{ji} \in \mathbb{R}$ to (3-1), with $i = 1, \dots, 2(k+l)$ and $j = 1, J_i$. Moreover there is a positive C independent of ε such that

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon|\|h\|_*,$$
$$\max\{|c_0|, |c_{ji}|\} \le C\|h\|_* \quad \textit{for } i = 1, \dots, 2(k+l),\ j = 1, J_i.$$

We begin to prove this result by studying a linear problem

$$\begin{cases} -\Delta\phi = h + W\phi & \text{in } \Omega_\varepsilon, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3-6}$$

together with orthogonality conditions (3-2) and (3-3).

Proposition 3.2. *Let $h \in L^\infty(\Omega_\varepsilon)$. For fixed $d > 0$ there exist $\varepsilon_0 > 0$ and C such that if $0 < \varepsilon < \varepsilon_0$, $\xi = (\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$ and $\phi \in L^\infty(\Omega_\varepsilon)$ is a solution of (3-6) such that (3-2) and (3-3) hold, then*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C\log\varepsilon^{-1}\|h\|_*,$$

where C is independent of ε.

We will prove this estimate by contradiction assuming that there exist a sequence $\varepsilon \to 0$, points $(\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$ (we omit the dependence on ε in the notation) and functions $h, \phi \in L^\infty(\Omega_\varepsilon)$ such that

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} = 1 \quad \text{and} \quad \log\varepsilon^{-1}\|h\|_* = o(1). \tag{3-7}$$

Fix $0 < \gamma < \beta < 1/2$ and consider the function η given by

$$\eta(r) = \begin{cases} 1 & \text{if } r < \varepsilon^{-\gamma}, \\ \dfrac{\log\varepsilon^{-\beta} - \log r}{\log\varepsilon^{-\beta} - \log\varepsilon^{-\gamma}} & \text{if } \varepsilon^{-\gamma} < r < \varepsilon^{-\beta}, \\ 0 & \text{if } r > \varepsilon^{-\beta}. \end{cases} \tag{3-8}$$

Let $\tilde\eta$ be a radial smooth cut-off function on $\mathbb{R}^2$ such that $\tilde\eta(r) \equiv 1$ for $r < \varepsilon^{-\beta}$, $\tilde\eta \equiv 0$ for $r > 2\varepsilon^{-\beta}$, $|\tilde\eta'(r)| \le C\varepsilon^\beta$ and $|\tilde\eta''(r)| \le C\varepsilon^{2\beta}$. Then we set

$$\eta_{1i}(y) = \begin{cases} \eta(|y - \xi_i'|) & \text{for } i = 1, \dots, 2k, \\ \eta(|F_i^\varepsilon(y)|) & \text{for } i = 2k+1, \dots, 2(k+l); \end{cases}$$
$$\eta_{2i}(y) = \begin{cases} \tilde\eta(|y - \xi_i'|) & \text{for } i = 1, \dots, 2k, \\ \tilde\eta(|F_i^\varepsilon(y)|) & \text{for } i = 2k+1, \dots, 2(k+l); \end{cases}$$
$$a_{0i} = \frac{1}{\mu_i((4/c_i)\log\varepsilon^{\gamma-1} + H(\xi_i, \xi_i))}$$

and also

$$\widehat{Z}_{0i}(y) = Z_{0i}(y) - \mu_i^{-1} + a_{0i}G(\varepsilon y, \xi_i).$$

Now define a test function

$$\tilde{Z}_{0i} = \eta_{1i} Z_{0i} + \varepsilon(1-\eta_{1i})\eta_{2i}\widehat{Z}_{0i}.$$

Given ϕ satisfying (3-6) and the orthogonality conditions (3-2) and (3-3), let

$$\tilde{\phi} = \phi - \sum_{i=1}^{2(k+l)} d_i \tilde{Z}_{0i},$$

where the numbers d_i are chosen so that $\int_{\Omega_\varepsilon} \chi_i Z_{0i}\tilde{\phi} = 0$ for any $i = 1, \ldots, 2(k+l)$, namely $d_i = \int_{\Omega_\varepsilon} \chi_i Z_{0i}\phi / \int_{\Omega_\varepsilon} \chi_i Z_{0i}^2$. Observe that

$$d_i = O(1) \quad \text{and} \quad \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} = O(1).$$

Moreover, $\tilde{\phi}$ satisfies

$$\text{(3-9)} \qquad \begin{cases} -\Delta\tilde{\phi} = W\tilde{\phi} + h - \displaystyle\sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i}) & \text{in } \Omega_\varepsilon, \\ \partial\tilde{\phi}/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

and the orthogonality condition

$$\text{(3-10)} \qquad \int_{\Omega_\varepsilon} \chi_i Z_{ji}\tilde{\phi} = 0 \quad \text{for all } i = 1, \ldots, 2(k+l),\ \ j = 0, 1, J_i,$$

where $L := -\Delta - W$.

To reach a contradiction it is sufficient to establish the following:

Lemma 3.3. $\tilde{\phi} \to 0$ *uniformly in* Ω_ε.

Lemma 3.4. $d_i \to 0$ *for all* $i = 1, \ldots, 2(k+l)$.

We postpone proofs of these lemmas and mention first some key steps.

Lemma 3.5. *For all* $i = 1, \ldots, 2(k+l)$ *and* $R > 0$, *we have*

$$\tilde{\phi} \to 0 \quad \textit{uniformly in } \Omega_\varepsilon \cap B_R(\xi_i').$$

Proof. Assume that for some $R > 0$ and $i = 1, \ldots, 2(k+l)$ there is a $c > 0$ such that $\sup_{B_R(\xi_i')} |\tilde{\phi}| \geq c > 0$ for a subsequence $\varepsilon \to 0$. Let us translate and rotate Ω_ε so that $\xi_i' = 0$ and Ω_ε approaches the upper half plane $\mathbb{R}^2_+$. By the elliptic estimate, $\tilde{\phi} \to \tilde{\phi}_0$ uniformly on compact sets and $\tilde{\phi}_0$ is a nontrivial bounded solution of (3-4). Then we conclude that $\tilde{\phi}_0$ is a linear combination of z_{ji} for $j = 0, 1, J_i$. On the other hand, we can take the limit in the orthogonality relations (3-10), observing that the limits of the functions Z_{ji} are just rotations and translations of z_{ji}, and we find that $\int_{\mathbb{R}^2_+} \chi\tilde{\phi}_0 z_{ji} = 0$. This contradicts the fact that $\tilde{\phi}_0 \not\equiv 0$. □

Lemma 3.6. $\overline{\tilde{\phi}} \equiv \dfrac{1}{|\Omega_\varepsilon|} \displaystyle\int_{\Omega_\varepsilon} \tilde{\phi} \to 0.$

Proof. By potential theory we have

$$\tilde{\phi}(y) - \overline{\tilde{\phi}} = \int_{\Omega_\varepsilon} G(\varepsilon y, \varepsilon z)\Big(W\tilde{\phi} + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i})\Big) dz,$$

where G is the Green function defined by (1-3).

Note that since

$$\int_{\Omega_\varepsilon} W\tilde{\phi} + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i}) = 0$$

and

$$G(\varepsilon y, \varepsilon z) = -\frac{4}{c_i} \log \varepsilon - \frac{4}{c_i} \log|y - z| + H(\varepsilon y, \varepsilon z),$$

we have

$$\begin{aligned} \text{(3-11)} \quad & \tilde{\phi}(y) - \overline{\tilde{\phi}} \\ &= \frac{1}{8\pi} \int_{\Omega_\varepsilon} \Big(H(\varepsilon y, \varepsilon z) - \frac{4}{c_i} \log|y - z|\Big)\Big(W\tilde{\phi} + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde{Z}_{0i})\Big) dz. \end{aligned}$$

Since $\tilde{\phi}(y) \to 0$ uniformly on sets of the form $|y - \xi_i'| < R$, we can select a sequence $R_\varepsilon \to \infty$ such that

$$\tilde{\phi}(y) \to 0 \quad \text{uniformly for } |y - \xi_i'| < R_\varepsilon.$$

We can assume $R_\varepsilon \to \infty$ as slowly as we need.

Select a point $y_m \in \Omega_\varepsilon$ for $m = 1, \dots, 2k$ or $y_m \in \partial\Omega_\varepsilon$ for $m = 2k+1, \dots, 2(k+l)$, such that $|y_m - \xi_m'| = R_\varepsilon$. We claim that when we evaluate (3-11) at y_m, all terms in the right side of (3-11) converge to zero except for

$$\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i}) dz = \frac{2\pi}{\mu_i} \delta_{mi} + o(1),$$

where δ_{mi} is Kronecker's delta.

Claim 1. $\displaystyle\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i}) dz = \frac{2\pi}{\mu_i} \delta_{mi} + o(1).$

This is proved in Appendix B.

Claim 2. $\displaystyle\int_{\Omega_\varepsilon} \log|y - z| h(z) dz = o(1)$ *uniformly for* $y \in \Omega_\varepsilon$.

Proof. Observe that $\log|y-z| = O(\log\varepsilon^{-1})$ for $|y-z|>R$, where $R>0$ is fixed, and that $\int_{\Omega_\varepsilon\cap B_R(y)}\big|\log|y-z|\big|dz \le C$. Then

$$\Big|\int_{\Omega_\varepsilon}\log|y-z|h\,dz\Big| \le C\log\varepsilon^{-1}\|h\|_* = o(1). \qquad \square$$

Claim 3. $\displaystyle\int_{\Omega_\varepsilon}\log|y-z|W\tilde\phi\,dz = o(1).$

Proof. It suffices to show that $\log\varepsilon^{-1}\int_{\Omega_\varepsilon}W\tilde\phi\,dz = o(1)$. Integrating equation (3-9), we have

$$\int_{\Omega_\varepsilon}W\tilde\phi + h - \sum_{i=1}^{2(k+l)} d_i L(\tilde Z_{0i}) = 0.$$

The claim then follows from (B-10) and (3-7). $\square$

Claim 4. $\displaystyle A \equiv \int_{\Omega_\varepsilon} H(\varepsilon y,\varepsilon z)(W\tilde\phi + h - L(\tilde Z_{0i})) = o(1)$ *uniformly for* $y\in\Omega_\varepsilon$.

This is proved in Appendix B.

We now return to the proof of Lemma 3.6. From claims above, we get

$$\tilde\phi(y_i) - \bar{\tilde\phi} = \frac{8\pi d_i}{c_i\mu_i} + o(1) \quad \text{for all } i = 1,\dots,2(k+l). \tag{3-12}$$

But the orthogonality condition (3-3) implies that

$$\sum_{i=1}^{2(k+l)} d_i a_i = 0, \quad \text{where } a_i = \int_{\Omega_\varepsilon}\chi_i Z_{0i}^2 > 0. \tag{3-13}$$

Multiplying (3-12) by $c_i a_i\mu_i$, adding and using (3-13), we find

$$\sum_{i=1}^{2(k+l)} c_i\mu_i a_i\tilde\phi(y_i) - a\bar{\tilde\phi} = o(1), \quad \text{where } a = \sum_{i=1}^{2(k+l)} c_i a_i\mu_i.$$

Since $\tilde\phi(y_i)\to 0$ and a is bounded away from zero, we get that $\bar{\tilde\phi} = o(1)$. $\square$

Proof of Lemma 3.3. Let $\check\phi = \tilde\phi(x/\varepsilon)$, with $x\in\Omega$. Then $\check\phi$ satisfies

$$\begin{cases} -\Delta\check\phi(x) = \varepsilon^{-2}\big(\check W\check\phi + h + \sum_{i=1}^{2(k+l)} d_i(\Delta\check Z_{0i} + \check W\check Z_{0i})\big) & \text{in } \Omega,\\ \partial\check\phi/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\check W(x) = W(x/\varepsilon)$, $\check Z_{0i}(x) = \tilde Z_{0i}(x/\varepsilon)$ and $\check h(x) = h(x/\varepsilon)$. For given $\delta>0$, let $E_\delta = \Omega\setminus\bigcup_{i=1}^{2(k+l)} B_\delta(\xi_i)$. Then

$$\frac{1}{\varepsilon^2}\|\check h\|_{L^\infty(E_\delta)} \le C\|h\|_* \to 0 \quad \text{and} \quad \frac{1}{\varepsilon^2}\|\check W\check\phi\|_{L^\infty(E_\delta)} \le C\varepsilon^2.$$

Furthermore, in E_δ we have $\check{Z}_{0i} \equiv 0$. Recalling $\|\check{\phi}\|_{L^\infty(\Omega)} \le 1$ and $|\Omega|^{-1}\int_\Omega \check{\phi} \to 0$, we obtain $\check{\phi} \to 0$ uniformly in E_δ and this implies

$$\tilde{\phi} \to 0 \quad \text{uniformly in } \Omega_\varepsilon \setminus \bigcup_{i=1}^{2(k+l)} B_{\delta/\varepsilon}(\xi_i') \quad \text{for any } \delta > 0.$$

For a given $R_1 > 0$, let $A_i = B_{\delta/\varepsilon}(\xi_i') \setminus B_{R_1}(\xi_i')$. Given $\varepsilon > 0$ small enough, there exist $R_1 > 1$ independent of ε (if necessary we can choose R_1 large enough) and $\psi_i : \Omega_\varepsilon \cap A_i \to \mathbb{R}$ smooth and positive such that

$$\begin{cases} -\Delta\psi_i - W\psi_i \ge C|y-\xi_i'|^{-2-\sigma} + \varepsilon^2 & \text{in } \Omega_\varepsilon \cap A_i\ , \\ \partial\psi_i/\partial\nu \ge 0 & \text{on } \partial\Omega_\varepsilon \cap A_i, \\ \psi_i > 0 & \text{in } \Omega_\varepsilon \cap A_i, \\ \psi_i \ge c > 0 & \text{on } \partial A_i \cap \Omega_\varepsilon, \end{cases}$$

where $C, c > 0$ can be chosen independent of ε and ψ_i is bounded uniformly in $\Omega_\varepsilon \cap A_i$. Let Ψ_0 be the unique solution of

$$\Delta\Psi_0 - \varepsilon^4\Psi_0 + \varepsilon^2 = 0 \quad \text{in } \Omega_\varepsilon, \qquad \partial\Psi_0/\partial\nu = \varepsilon \quad \text{on } \partial\Omega_\varepsilon,$$

and take $\psi_{1i} = 1 - r^{-\sigma}$, where $r = |y - \xi_i'|$. Then we claim that the function

$$\psi_i(y) = \frac{4}{\sigma^2}(C\Psi_0 + \psi_{1i})$$

satisfies the requirements.

In fact, a simple calculation shows that

$$-\Delta\psi_{1i} = \sigma^2 r^{-2-\sigma}.$$

If $\xi_i' \in \Omega_\varepsilon$, we have

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = O(\varepsilon^{1+\sigma}) \quad \text{on } \partial\Omega_\varepsilon.$$

If $\xi_i' \in \partial\Omega_\varepsilon$ and $|y - \xi_i'| > R$, we have

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = \sigma\frac{(y-\xi_i')\cdot\nu_\varepsilon}{r^{2+\sigma}} \quad \text{on } \partial\Omega_\varepsilon.$$

As before, we write $\partial\Omega_\varepsilon$ near ξ_i' as the graph $\{(y_1, y_2) \mid y_2 = \varepsilon^{-1}P(\varepsilon y_1)\}$ with $P(0) = P'(0) = 0$. Then we have

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = \frac{\sigma}{r^{2+\sigma}}\frac{y_1P'(\varepsilon y_1) - P(\varepsilon y_1)}{\sqrt{1+P'(\varepsilon y_1)^2}} = \frac{\sigma}{r^{2+\sigma}}\frac{O(\varepsilon r^2)}{\sqrt{1+O(\delta^2)}} = O\Big(\frac{\varepsilon}{r^\sigma}\Big)$$

for all $R < r < \delta/\varepsilon$. Thus we see that

$$\frac{\partial\psi_{1i}}{\partial\nu_\varepsilon} = o(\varepsilon) \quad \text{on } \partial\Omega_\varepsilon.$$

Therefore, for $|y-\xi_i'|>R$ with $i=1,\dots,2(k+l)$, where R is large, we have by the definition of ψ_i and the fact that $W\le 1/(1+|y-\xi_i'|^4)$ that

$$-\Delta\psi_i - W\psi_i = \frac{C}{\sigma^2}(\varepsilon^2-\varepsilon^4\Psi_0) - \frac{4}{\sigma^2}\frac{C\Psi_0+\psi_{1i}}{1+r^4} + \frac{C}{r^{2+\sigma}} \ge \varepsilon^2 + \frac{C}{r^{2+\sigma}}.$$

And on $\partial\Omega_\varepsilon$,

$$\frac{\partial\psi_i}{\partial\nu_\varepsilon} \ge C\varepsilon.$$

This verifies the claim.

Thanks to the barrier ψ_i, we deduce that the following maximum principle holds in $\Omega_\varepsilon\cap A_i$. If $\phi\in H^1(\Omega_\varepsilon\cap A_i)$ satisfies

$$\begin{cases} -\Delta\phi - W\phi \ge 0 & \text{in } \Omega_\varepsilon\cap A_i, \\ \phi\ge 0 & \text{on } \partial\Omega_\varepsilon\cap A_i, \end{cases}$$

then $\phi\ge 0$ in $\Omega_\varepsilon\cap A_i$.

Let h be bounded and $\tilde\phi$ be a solution of (3-9) satisfying (3-10). We first claim that $\|\tilde\phi\|_{L^\infty(\Omega_\varepsilon\cap A_i)}$ can be controlled in terms of

$$\sum_{i=1}^{2(k+l)} |d_i|\|L(\tilde Z_{0i})\|_*, \qquad \sup_{\Omega_\varepsilon\cap\partial A_i}|\tilde\phi|, \qquad \text{and} \qquad \|h\|_*.$$

Indeed, set

$$\Phi = C\Big(\sup_{\Omega_\varepsilon\cap\partial A_i}|\tilde\phi| + \|h\|_* + \sum_{i=1}^{2(k+l)}|d_i|\|L(\tilde Z_{0i})\|_*\Big)\psi_i.$$

By the maximum principle above, we have $|\tilde\phi|\le\Phi$ in $\Omega_\varepsilon\cap A_i$. Since ψ_i is uniformly bounded, we get

$$|\tilde\phi| \le C\Big(\sup_{\Omega\varepsilon\cap\partial B_{R_1}(\xi_i')}|\tilde\phi| + \sup_{\Omega\varepsilon\cap\partial B_{\delta/\varepsilon}(\xi_i')}|\tilde\phi| + \|h\|_* + \sum_{i=1}^{2(k+l)}|d_i|\|L(\tilde Z_{0i})\|_*\Big)$$

in $\Omega_\varepsilon\cap A_i$. But $\|h\|_*=o(1)$ by the assumption, $\sup_{\Omega\varepsilon\cap\partial B_{R_1}}(\xi_i')|\tilde\phi|\to 0$ by Lemma 3.5, and $\sup_{\Omega\varepsilon\cap\partial B_{\delta/\varepsilon}(\xi_i')}|\tilde\phi|\to 0$ as shown above. At the same time, we also know $|d_i|=O(1)$ and $\|L(\tilde Z_{0i})\|_*=O(\varepsilon^{2\gamma})=o(1)$ from (B-10), this proves the result. □

Proof of Lemma 3.4. We take $\tilde Z_{0i}$ as test function to (3-9), obtaining

$$\sum_{i=1}^{2(k+l)} d_i\int_{\Omega_\varepsilon} L(\tilde Z_{0i})\tilde Z_{0i} = \int_{\Omega_\varepsilon}\tilde\phi(\Delta\tilde Z_{0i}+W\tilde Z_{0i}) + \int_{\Omega_\varepsilon} h\tilde Z_{0i}. \tag{3-14}$$

Observe that

$$\left|\int_{\Omega_\varepsilon} \tilde{Z}_{0i} h\right| \le \|h\|_* \|\tilde{Z}_{0i}\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1} \|h\|_* \frac{1}{\log \varepsilon^{-1}} = o(1) \frac{1}{\log \varepsilon^{-1}}, \tag{3-15}$$

and

$$\left|\int_{\Omega_\varepsilon} \tilde{\phi}(\Delta \tilde{Z}_{0i} + W \tilde{Z}_{0i})\right| \le \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \|L(\tilde{Z}_{0i})\|_* = o(1) \frac{1}{\log \varepsilon^{-1}}. \tag{3-16}$$

It is not difficult to show as above that

$$\left|\int_{\Omega_\varepsilon} L(\tilde{Z}_{0i}) \tilde{Z}_{0i}\right| \ge \frac{C}{\log \varepsilon^{-1}}. \qquad \square$$

Proof of Proposition 3.1. First we prove that for any ϕ, c_{ji}, c_0 and any solution to (3-1), we have the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1} \|h\|_*. \tag{3-17}$$

From Proposition 3.2, we obtain that

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \log \varepsilon^{-1} \Big(\|h\|_* + \sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} |c_{ji}| + |c_0| \Big). \tag{3-18}$$

So it suffices to estimate the values of the constants a_{ji} and c_0.

To this end, we multiple (3-1) by Z_{ji} and integrate to find

$$\int_{\Omega_\varepsilon} L(\phi) Z_{ji} = \int_{\Omega_\varepsilon} h Z_{ji} + c_{ji} \int_{\Omega_\varepsilon} \psi_i Z_{ji}^2. \tag{3-19}$$

Note that $Z_{ji} = O(1/(1+|y-\xi_i|))$ for $j \ne 0$, so

$$\int_{\Omega_\varepsilon} h Z_{ji} = O(\|h\|_*) \tag{3-20}$$

and

$$\int_{\Omega_\varepsilon} L(\phi) Z_{ji} = \int_{\Omega_\varepsilon} L(Z_{ji}) \phi + \int_{\partial\Omega_\varepsilon} \frac{\partial Z_{ji}}{\partial \nu} \phi = O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-21}$$

Substituting (3-20) and (3-21) into (3-19), we obtain

$$|C_{ji}| = O(\|h\|_*) + O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-22}$$

On the other hand, multiplying (3-1) by Z we get

$$c_0 \int_{\Omega_\varepsilon} \chi Z^2 = \int_{\Omega_\varepsilon} L(\phi) Z - \int_{\Omega_\varepsilon} h Z. \tag{3-23}$$

Estimating as before, we have

$$\int_{\Omega_\varepsilon} hZ = O(\|h\|_*) \tag{3-24}$$

and

$$\int_{\Omega_\varepsilon} L(\phi)Z = \int_{\Omega_\varepsilon} L(Z)\phi = O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-25}$$

Thus it follows from (3-23)–(3-25) that

$$|c_0| = O(\|h\|_*) + O(\varepsilon \log \varepsilon^{-1} \|\phi\|_{L^\infty(\Omega_\varepsilon)}). \tag{3-26}$$

From (3-22) and (3-26) we see that the desired bound holds.

Now consider the Hilbert space

$$H = \left\{ \phi \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \chi Z\phi = 0, \int_{\Omega_\varepsilon} \chi_i Z_{ji}\phi = 0 \text{ for } i = 1, \dots, 2(k+l),\ j = 1, J_i \right\}$$

with the norm $\|\phi\|_H^2 = \int_{\Omega_\varepsilon} |\nabla \phi|^2$. Problem (3-1) is equivalent to finding $\phi \in H$ such that

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla\psi - \int_{\Omega_\varepsilon} W\phi\psi = \int_{\Omega_\varepsilon} h\psi \quad \text{for all } \psi \in H.$$

By Fredholm's alternative, this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by the a priori estimate (3-17). □

Remark. The result of Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3-1) defines a continuous linear map from $L^\infty(\Omega_\varepsilon)$, with norm $\|\cdot\|_*$, into $L^\infty(\Omega_\varepsilon)$. Moreover, the operator T is differential with respect to the variables ξ'_m. In fact, computations similar to those used in [Wei et al. 2011] yield the estimate

$$\|\partial_{\xi'_m} \phi\|_{L^\infty(\Omega_\varepsilon)} \le C(\log \varepsilon^{-1})^2 \|h\|_*. \tag{3-27}$$

4. The nonlinear problem with constraints

Let us introduce a small parameter τ and consider

$$V_1(y) = V(y) + \tau Z(y) \quad \text{for } y \in \Omega_\varepsilon, \tag{4-1}$$

where V and Z are given by (2-16) and (3-5). Then we set

$$W_1 = 2\varepsilon^4 \cosh V_1, \qquad R_1 = \Delta V_1 + 2\varepsilon^4 \sinh V_1$$

and

$$N_1(\phi_1) = 2\varepsilon^4(\sinh(V_1 + \phi_1) - \phi_1 \cosh V_1 - \sinh V_1).$$

Now we consider the following auxiliary nonlinear problem:

$$
\begin{cases}
\Delta\phi_1+W_1\phi_1+R_1+N_1(\phi_1)+\sum_{i=1}^{2(k+l)}\sum_{j=1}^{J_i} c_{ji}\chi_i Z_{ji}+c_0\chi Z=0 & \text{in } \Omega_\varepsilon,\\
\partial\phi_1/\partial\nu=0 & \text{on } \partial\Omega_\varepsilon,\\
\int_{\Omega_\varepsilon}\chi Z\phi_1=0,\quad \int_{\Omega_\varepsilon}\chi_i Z_{ji}\phi_1=0 & \text{for all } i=1,\dots,2(k+l),\ j=1,J_i.
\end{cases}
\tag{4-2}
$$

Then we can follow the proofs [Wei et al. 2011, Lemma 4.1 and Theorem 4.2] to obtain the following results; we omit the details.

Lemma 4.1. *Let $k+l\ge 1$, $d>0$, $\alpha\in(0,1)$ and $\tau=O(\varepsilon^\theta)$ with $\theta>\alpha/2$. Then there exist $\varepsilon_0>0$ and $C>0$ such that for $0<\varepsilon<\varepsilon_0$ and for any $\xi_1,\dots,\xi_{2(k+l)}\in\mathcal{M}_d$, problem (4-2) admits a unique solution ϕ_1, c_0, c_{ji} for $i=1,\dots,2(k+l)$, $j=1,J_i$, such that*

$$\|\phi_1\|_{L^\infty(\Omega_\varepsilon)}\le C\varepsilon^\alpha. \tag{4-3}$$

Furthermore, the function $(\tau,\xi')\to\phi_1(\tau,\xi')\in C(\overline{\Omega}_\varepsilon)$ is C^1 and

$$
\begin{aligned}
&\|D_{\xi'}\phi_1\|_{L^\infty(\Omega_\varepsilon)}\le C|\log\varepsilon|^2(\varepsilon+\varepsilon^{2\theta}+\varepsilon^{2\alpha}),\\
&\|D_\tau\phi_1\|_{L^\infty(\Omega_\varepsilon)}\le C(\varepsilon^\alpha+\varepsilon^\theta)|\log\varepsilon|.
\end{aligned}
\tag{4-4}
$$

Lemma 4.2. *Let $k+l\ge 1$ and $d>0$. For any $0<\alpha<1$ there exist $\varepsilon_0>0$ and $C>0$ such that for $0<\varepsilon<\varepsilon_0$ and any $(\xi_1,\dots,\xi_{2(k+l)})\in\mathcal{M}_d$, there exists a unique τ with $|\tau|=O(\varepsilon^\alpha)$ such that problem (4-2) admits a unique solution ϕ, c_0, c_{ji} for $i=1,\dots,2(k+l)$, $j=1,J_i$ with $c_0=0$ and such that*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)}\le C\varepsilon^\alpha. \tag{4-5}$$

Furthermore, the function $\xi'\mapsto\phi(\xi')$ is C^1 and

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)}\le C\varepsilon^\alpha|\log\varepsilon|^2.$$

5. Variational reduction and expansion of the energy

In view of Lemmas 4.1 and 4.2, given $\xi=(\xi_1,\dots,\xi_{2(k+l)})\in\mathcal{M}_d$, we set $\phi(\xi)$ and $c_{ji}(\xi)$ to be the unique solution to (4-2) with $c_0=0$ satisfying the bounds (4-3) and (4-4). Let

$$J_\varepsilon(v)=\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla v|^2dx-2\varepsilon^4\int_{\Omega_\varepsilon}\cosh v\,dx$$

and define

$$F_\varepsilon(\xi)=J_\varepsilon(V_1(\xi')+\phi(\xi')), \tag{5-1}$$

where $\xi' = \xi/\varepsilon$ and $V_1(\xi') = V(\xi') + \tau(\xi')Z(\xi')$ with $\tau(\xi)$ given by Lemma 4.2.

Lemma 5.1. *If $\xi = (\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$ is a critical point of F_ε, then*

$$v = V_1(\xi') + \phi(\xi')$$

is a critical point of J_ε, that is, a solution to (2-14).

Proof. A direct computation gives

$$\frac{\partial F_\varepsilon}{\partial \xi_m} = \varepsilon^{-1} \frac{\partial J_\varepsilon(V_1(\xi') + \phi(\xi'))}{\partial \xi'_m} = \varepsilon^{-1} DJ_\varepsilon(V_1(\xi') + \phi(\xi'))\Big(\frac{\partial V_1(\xi')}{\partial \xi'_m} + \frac{\partial \phi(\xi')}{\partial \xi'_m}\Big).$$

Since $V_1(\xi') + \phi(\xi')$ solves (4-2) with $c_0 = 0$, we have

$$\frac{\partial F_\varepsilon}{\partial \xi_m} = \varepsilon^{-1} \sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji} \int_{\Omega_\varepsilon} \chi_i Z_{ji} \Big(\frac{\partial V_1(\xi')}{\partial \xi'_m} + \frac{\partial \phi(\xi')}{\partial \xi'_m}\Big).$$

From the assumption $DF_\varepsilon(\xi) = 0$, we obtain

$$\sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji} \int_{\Omega_\varepsilon} \chi_i Z_{ji} \Big(\frac{\partial V_1(\xi')}{\partial \xi'_m} + \frac{\partial \phi(\xi')}{\partial \xi'_m}\Big) = 0 \quad \text{for all } m = 1, \dots, 2(k+l).$$

Since

$$\|\partial_{\xi'_m} \phi(\xi')\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^\alpha |\log \varepsilon|^2 \quad \text{and} \quad \partial_{\xi'_m} V(\xi') = (-1)^m Z_{jm} + o(1)$$

for $j = 1, J_i$, where $o(1)$ is in the L^∞-norm as a direct consequence of (4-1), it follows that

$$\sum_{i=1}^{2(k+l)} \sum_{j=1}^{J_i} c_{ji} \int_{\Omega_\varepsilon} \chi_i Z_{ji}((-1)^m Z_{jm} + o(1)) = 0 \quad \text{for all } m = 1, \dots, 2(k+l),$$

which is a strictly diagonal dominant system. This implies that $c_{ji} = 0$ for all $i = 1, \dots, 2(k+l)$, $j = 1, J_i$. □

A key step in seeking the critical points of the functional F_ε is finding its expected closeness to the functional $J_\varepsilon(V_1(\xi))$. The procedure is completely similar to that of [Wei et al. 2011, Theorem 5.2], so we omit it here.

Lemma 5.2. *The expansion*

$$F_\varepsilon(\xi) = J_\varepsilon(V) + \theta_\varepsilon(\xi)$$

holds with $|\theta_\varepsilon(\xi)| + |\nabla \theta_\varepsilon(\xi)| = o(1)$ uniformly on points in $\mathcal{M}_d$.

Now we will give an asymptotic estimate of $J_\varepsilon(V)$, where V is defined by (2-16) and J_ε is given as above.

Lemma 5.3. *Let $k+l \geq 1$, let $d > 0$, let μ_i be given by (2-21) and let V be the function defined in (2-16). Then the expansion*

$$J_\varepsilon(V) = -\frac{1}{2}\sum_{i=1}^{2(k+l)} c_i\Big(c_i H(\xi_i,\xi_i) + \sum_{j,j\neq i}(-1)^{j+i}c_j G(\xi_j,\xi_i)\Big) + 2\sum_{i=1}^{2(k+l)} c_i \log\varepsilon^{-1} + \sum_{i=1}^{2(k+l)} c_i(\log 8 - 2) + O(\varepsilon^\alpha). \tag{5-2}$$

holds uniformly on points $\xi = (\xi_1, \dots, \xi_{2(k+l)}) \in \mathcal{M}_d$.

Proof. Recall the definition of $V(y) = \sum_{i=1}^{2(k+l)}(-1)^{i-1}(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y))$. We find that it satisfies

$$\begin{cases} -\Delta V = \varepsilon^4 \displaystyle\sum_{i=1}^{2(k+l)}(-1)^{i-1}\Big(e^{u_i(\varepsilon y)} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} e^{u_i(\varepsilon y)}\Big) & \text{in } \Omega_\varepsilon,\\ \partial V/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{5-3}$$

We will compute the two terms in $J_\varepsilon(V)$.

First, by (5-3) we have

$$\begin{aligned} \int_{\Omega_\varepsilon}|\nabla V|^2 &= \int_{\Omega_\varepsilon}(-\Delta V)V \\ &= \int_{\Omega_\varepsilon}\Big(\varepsilon^4\sum_{j=1}^{2(k+l)}(-1)^{j-1}\Big(e^{u_j(\varepsilon y)} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} e^{u_j(\varepsilon y)}\Big)\Big) \\ &\qquad\qquad \times\Big(\sum_{i=1}^{2(k+l)}(-1)^{i-1}\Big(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)\Big)\Big) \\ &= \varepsilon^4\sum_{j,i}(-1)^{j+i}\int_{\Omega_\varepsilon}\Big(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)\Big)e^{u_j(\varepsilon y)} \\ &\quad - \frac{\varepsilon^4}{|\Omega_\varepsilon|}\Big(\sum_{j=1}^{2(k+l)}(-1)^{j-1}\int_{\Omega_\varepsilon} e^{u_j(\varepsilon y)}\Big)\Big(\int_{\Omega_\varepsilon}\sum_{i=1}^{2(k+l)}(-1)^{i-1}\Big(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)\Big)\Big) \\ &= \varepsilon^4\sum_{j,i}(-1)^{j+i}\int_{\Omega_\varepsilon}(u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y))e^{u_j(\varepsilon y)} + O(\varepsilon), \end{aligned}$$

where the last equality is due to the fact $\varepsilon^4\sum_{j=1}^{2(k+l)}(-1)^{j-1}\int_{\Omega_\varepsilon} e^{u_j(\varepsilon y)} = O(\varepsilon^4)$, which can be easily deduced from (2-7).

For $j \neq i$, we have by a calculation similar to (2-23)

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \varepsilon^4 (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_j(\varepsilon y)} \\
&\quad = \Big(\int_{\Omega_\varepsilon^1} + \int_{\Omega_\varepsilon^2} \Big) (\varepsilon^4 (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_j(\varepsilon y)}) \\
&\quad = \int_{\Omega_\varepsilon^1|_{\xi_j'=0}} \frac{8}{(1+y^2)^2} \big(\log|\xi_i - \xi_j|^{-4} + c_i H(\xi_j, \xi_i) \big) + O(\varepsilon^\alpha) \\
&\quad = c_j c_i G(\xi_j, \xi_i) + O(\varepsilon^\alpha).
\end{aligned}
\tag{5-4}
$$

where $\Omega_\varepsilon^1 := B_{\delta/(\varepsilon\mu_j)}(\xi_j') \cap (\Omega_\varepsilon/\mu_i)$ and $\Omega_\varepsilon^2 := (\Omega_\varepsilon/\mu_i) \setminus \Omega_\varepsilon^1$. For $j = i$, we have

$$
\begin{aligned}
&\varepsilon^4 \int_{\Omega_\varepsilon} (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_i(\varepsilon y)} \\
&\quad = \int_{\Omega_\varepsilon} \frac{8\mu_i^2}{(\mu_i^2 + |y - \xi_i'|^2)^2} \Big(\log \frac{8\mu_i^2}{(\varepsilon^2 \mu_i^2 + |\varepsilon y - \xi_i|^2)^2} + c_i H(\xi_i, \xi_i) \\
&\qquad\qquad - \log(8\mu_i^2) + O(\varepsilon^\alpha) + O(\varepsilon |y - \xi_i'|) \Big) \\
&\quad = 4c_i \log \varepsilon^{-1} + c_i (c_i H(\xi_i, \xi_i) - 2 \log 8\mu_i^2) + 2c_i (\log 8 - 1) + O(\varepsilon^\alpha).
\end{aligned}
$$

So from the choice of μ_i (see (2-21)), we get

$$
\begin{aligned}
\varepsilon^4 \int_{\Omega_\varepsilon} (u_i(\varepsilon y) + H_i^\varepsilon(\varepsilon y)) e^{u_i(\varepsilon y)} &= 4c_i \log \varepsilon^{-1} + 2c_i (\log 8 - 1) \\
&\quad - c_i \Big(c_i H(\xi_i, \xi_i) + 2 \sum_{m, m \neq i} (-1)^{m+i} c_m G(\xi_m, \xi_i) \Big) + O(\varepsilon^\alpha).
\end{aligned}
\tag{5-5}
$$

Combining (5-4) and (5-5), we have

$$
\begin{aligned}
\frac{1}{2} \int_{\Omega_\varepsilon} |\nabla V|^2 &= -\frac{1}{2} \sum_{i=1}^{2(k+l)} c_i \Big(c_i H(\xi_i, \xi_i) + \sum_{j, j \neq i} (-1)^{j+i} c_j G(\xi_j, \xi_i) \Big) \\
&\quad + 2 \sum_{i=1}^{2(k+l)} c_i \log \varepsilon^{-1} + (\log 8 - 1) \sum_{i=1}^{2(k+l)} c_i + O(\varepsilon^\alpha).
\end{aligned}
\tag{5-6}
$$

Next, let us compute the second term in $J_\varepsilon(V)$. Let $\Omega_i^1 = B_{\delta/\varepsilon}(\xi_i') \cap (\Omega_\varepsilon/\mu_i)$. Then

$$
2\varepsilon^4 \int_{\Omega_\varepsilon} \cosh V = 2\varepsilon^4 \sum_{i=1}^{2(k+l)} \int_{\Omega_i^1} \cosh V + O(\varepsilon^2).
$$

Suppose first i is odd. Then

$$\begin{aligned}2\varepsilon^4\int_{\Omega_i^1}\cosh V &= \varepsilon^4\int_{\Omega_i^1} e^V + O(\varepsilon)\\ &= \int_{\Omega_i^1}\varepsilon^4 e^{u_i(\varepsilon y)}\exp\Big(H_i^\varepsilon+\sum_{m\neq i}(-1)^{m-1}(u_m+H_m^\varepsilon)\Big)+O(\varepsilon)\\ &= c_i+O(\varepsilon).\end{aligned}$$

Therefore

$$2\varepsilon^4\int_{\Omega_i^1}\cosh V = c_i+O(\varepsilon). \tag{5-7}$$

Similarly for i even, we also have (5-7). So we obtain

$$2\varepsilon^4\int_{\Omega_\varepsilon}\cosh V = \sum_{i=1}^{2(k+l)} c_i+O(\varepsilon). \tag{5-8}$$

Finally, from (5-6) and (5-8) we conclude that (5-2) holds. □

6. Proof of main theorems

Proof of Theorem 1.2. Let

$$v(y)=V_1(\xi')(y)+\phi(\xi')(y) \quad \text{for } y\in\Omega_\varepsilon,$$

where V_1 is given by (4-1) and ϕ is the unique solution to problem (4-2) with $c_0=0$, whose existence and properties are established in Lemma 4.2. According to Lemma 4.1, v is a solution to problem (2-14) if we adjust ξ so that it is a critical point of the function $F_\varepsilon(\xi)$ defined in (5-1), or equivalently, so that it is a critical point of

$$\tilde F_\varepsilon(\xi)=2\Big(2\sum_{i=1}^{2(k+l)} c_i\log\varepsilon^{-1}+\sum_{i=1}^{2(k+l)} c_i(\log 8-2)-F_\varepsilon(\xi)\Big). \tag{6-1}$$

From Lemmas 5.2 and 5.3 it follows that for $\xi\in\mathcal{M}_d$,

$$\tilde F_\varepsilon(\xi)=\varphi_{2(k+l)}(\xi)+\varepsilon\Theta_\varepsilon(\xi), \tag{6-2}$$

where Θ_ε and $\nabla_\xi\Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon\to 0$. On the other hand, $\tilde F_\varepsilon\to\varphi_{2(k+l)}$ uniformly on compact sets of $\mathcal{M}_d$ as ε goes to 0. Now by Definition 1.1, we deduce that if ε is small enough, there exists a critical point $\xi_\varepsilon\in\mathcal{M}_d$ of $\tilde F_\varepsilon$ such that $\tilde F_\varepsilon\to\varphi_{2(k+l)}(\xi^*)$. Moreover, up to subsequence, $\xi_\varepsilon\to\xi$ as ε tends to 0, with $\varphi_{2(k+l)}(\xi)=\varphi_{2(k+l)}(\xi^*)$. The function $u_\varepsilon(x)=v(y)$ is therefore a solution to (1-2) with the qualitative properties predicted by the theorem, as can be easily shown. □

Proof of Theorem 1.3. First, we recall here some facts about the regular part of the Green function $H(x, y)$ defined by (1-4). If $y \in \Omega$ is a point close to $\partial\Omega$, we let y^* be its uniquely determined reflection with respect to $\partial\Omega$. Now, we consider the auxiliary function

$$H^*(x, y) = -\frac{1}{2\pi} \log \frac{1}{|x - y^*|},$$

and set

$$\psi(x, y) = H(x, y) - H^*(x, y)$$

Then from the equation corresponding to $H(x, y)$ and the elliptic regularity theory, it is not difficult to verify $\psi(x, y)$ is bounded in $\overline{\Omega} \times \overline{\Omega}$ and hence one can derive the estimates

$$H(x, y) = -\frac{1}{2\pi} \log \frac{1}{|x - y^*|} + O(1) \quad \text{for all } x \in \overline{\Omega} \text{ uniformly.} \tag{6-3}$$

If $y \in \partial\Omega$, note that $H(x, y)$ satisfies

$$\begin{cases} \Delta H(x, y) = \dfrac{1}{|\Omega|} & \text{in } \Omega, \\ \dfrac{\partial H}{\partial \nu}(x, y) = \dfrac{1}{\pi} \dfrac{(x - y) \cdot \nu(x)}{|x - y|^2} & \text{on } \partial\Omega. \end{cases}$$

With this and (2-10), we obtain that $x \mapsto H(x, y) \in C^{1,\alpha}(\overline{\Omega})$. On the other hand, by the continuity of the boundary term with respect to y in $L^\infty(\partial\Omega)$, we can get $H(x, y) \in C(\overline{\Omega}, \partial\Omega)$. In particular, $H(x, x)$ is in $C(\partial\Omega)$.

Now, we prove the result. It suffices to show the existence of critical points of the function $\varphi_{2+2}(\xi_1, \dots, \xi_4)$ in $\mathcal{M}_d$. In this case,

$$\begin{aligned} \varphi_{2+2}(\xi_1, \dots, \xi_4) = 16\pi^2\big(&4H(\xi_1, \xi_1) + 4H(\xi_2, \xi_2) + H(\xi_3, \xi_3) + H(\xi_4, \xi_4) \\ &- 4G(\xi_1, \xi_2) + 2G(\xi_1, \xi_3) - 2G(\xi_1, \xi_4) \\ &- 2G(\xi_2, \xi_3) + 2G(\xi_2, \xi_4) - G(\xi_3, \xi_4)\big). \end{aligned} \tag{6-4}$$

We will look for a solution to problem (1-2) with the concentration points ξ given by

$$\xi_1 = (-\lambda, 0), \quad \xi_2 = (\lambda, 0), \quad \xi_3 = (1, 0), \quad \text{and} \quad \xi_4 = (-1, 0) \quad \text{for } \lambda \in (0, 1).$$

Using results obtained in the previous sections (or from the proof of Theorem 1.2), we reduce the problem of finding solution to (1-2) to that finding critical points of the function $\varphi_{2+2}(\lambda) : (0,1) \to \mathbb{R}$ defined by

$$\begin{aligned}
\varphi_{2+2}(\lambda) &:= \varphi_{2+2}(\xi(\lambda)) \\
&= 16\pi^2\Big(H(\xi_3,\xi_3)+H(\xi_4,\xi_4)-\frac{4}{\pi}\log\frac{1}{2-\lambda}+O(1) \\
&\qquad\qquad -\frac{2}{\pi}\log\frac{1}{2\lambda}-\frac{4}{\pi}\log\frac{1}{1-\lambda}+\frac{4}{\pi}\log\frac{1}{1+\lambda}-\frac{1}{\pi}\log\frac{1}{2} \\
&\qquad -H(\xi_1,\xi_2)+H(\xi_1,\xi_3)-H(\xi_1,\xi_4)-H(\xi_2,\xi_3)+H(\xi_2,\xi_4)-H(\xi_3,\xi_4)\Big) \\
&= 32\pi(2\log(2-\lambda)+\log\lambda+2\log(1-\lambda)-2\log(1+\lambda))+O(1).
\end{aligned}$$

Here, we have used the fact that $H(x,y)\in C(\bar{B},\partial B)$ and (6-3). Now there exists a $\lambda_0\in(0,1)$ such that $\varphi_{2+2}(\lambda_0)=\max_{\lambda\in(0,1)}\varphi_{2+2}(\lambda)$, since $\lim_{\lambda\to 0^+}\varphi_{2+2}(\lambda)=\lim_{\lambda\to 1^-}\varphi_{2+2}(\lambda)=-\infty$. Then λ_0 is a C^0-stable critical point of φ_{2+2}, and so the function $\tilde{F}_\varepsilon(\xi)$ defined by (6-1) has a critical point. This proves our result. □

Appendix A.

Proof of (2-22) *and* (2-23). By Lemma 2.1 and the fact that H is C^1 in $\overline{\Omega}$, we have

$$\begin{aligned}
H_j^\varepsilon(\varepsilon y) &= c_j H(\varepsilon y,\xi_j)-\log(8\mu_j^2)+O(\varepsilon^\alpha) \\
&= c_j H(\xi_i,\xi_j)-\log(8\mu_j^2)+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|).
\end{aligned}$$

Let us fix a small constant $\delta>0$. For $|y-\xi_i'|\le\delta/\varepsilon$,

$$\begin{aligned}
&(-1)^{i-1}H_i^\varepsilon(\varepsilon y)+\sum_{j\ne i}(-1)^{j-1}\Big(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_i^2+|\varepsilon y-\varepsilon\xi_j'|^2)^2}+H_j^\varepsilon(\varepsilon y)\Big) \\
&\quad = (-1)^{i-1}\big(c_i H(\xi_i,\xi_i)-\log(8\mu_i^2)\big) \\
&\qquad\quad +\sum_{j\ne i}(-1)^{j-1}\Big(\log\frac{8\mu_j^2}{|\xi_i-\xi_j|^4}+c_j H(\xi_i,\xi_j)-\log(8\mu_j^2)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|) \\
&\quad = (-1)^{i-1}(c_i H(\xi_i,\xi_i)-\log(8\mu_i^2)) \\
&\qquad\qquad\qquad +\sum_{j\ne i}(-1)^{j-1}c_j G(\xi_i,\xi_j)+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)
\end{aligned}$$

which is equal to $O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)$; here first equality follows because

$$\begin{aligned}
\varepsilon^2\mu_j^2+\varepsilon^2|y-\xi_j'|^2 &= \big(|\xi_j-\xi_i|+O(|\varepsilon y-\xi_i|)\big)^2+\varepsilon^2\mu_j^2 \\
&= |\xi_j-\xi_i|^2\Big(1+O\Big(\frac{|\varepsilon y-\xi_i|^2}{|\xi_j-\xi_i|^2}\Big)+\frac{\varepsilon^2\mu_j^2}{|\xi_j-\xi_i|^2}\Big) \\
&= |\xi_j-\xi_i|^2\big(1+O(\varepsilon^2|y-\xi_i'|^2)+O(\varepsilon^2)\big).
\end{aligned}$$

First, we estimate W. For $|y-\xi_i'|\le\delta/\varepsilon$, a direct computation shows

$$
\begin{aligned}
W &= 2\varepsilon^4\cosh V\\
&= \varepsilon^4\exp\Big(\sum_{i=1}^{2(k+l)}(-1)^{i-1}(u_i+H_i^\varepsilon)\Big)+\varepsilon^4\exp\Big(\sum_{i=1}^{2(k+l)}(-1)^{i}(u_i+H_i^\varepsilon)\Big)\\
&= \varepsilon^4\Big(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Big)^{(-1)^{i-1}}\\
&\quad\times\exp\Big((-1)^{i-1}H_i^\varepsilon(\varepsilon y)+\sum_{j\ne i}(-1)^{j-1}\Big(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+\varepsilon^2|y-\xi_j'|^2)^2}+H_j^\varepsilon(\varepsilon y)\Big)\Big)\\
&\quad+\varepsilon^4\Big(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Big)^{(-1)^{i}}\\
&\quad\times\exp\Big((-1)^{i}H_i^\varepsilon(\varepsilon y)+\sum_{j\ne i}(-1)^{j}\Big(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+\varepsilon^2|y-\xi_j'|^2)^2}+H_j^\varepsilon(\varepsilon y)\Big)\Big)\\
&= \varepsilon^4\Big(\Big(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Big)^{(-1)^{i-1}}+\Big(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Big)^{(-1)^{i}}\Big)\\
&\qquad\times\exp\Big[O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\Big]\\
&= \frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}\big(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\big)+O(\varepsilon^4).
\end{aligned}
$$

Therefore

$$
W(y)=\frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|))\quad\text{for all } |y-\xi_i'|<\delta/\varepsilon. \tag{A-1}
$$

Similarly, for $|y-\xi_i'|<\delta/\varepsilon$ we have

$$
\begin{aligned}
&2\varepsilon^4\sinh V\\
&\quad= \varepsilon^4\Big(\Big(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Big)^{(-1)^{i-1}}-\Big(\frac{8\mu_i^2}{\varepsilon^4(\mu_i^2+|y-\xi_i'|^2)^2}\Big)^{(-1)^{i}}\Big)\\
&\qquad\times\exp\big(O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\big)\\
&\quad= (-1)^{i-1}\frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}\big(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_i'|)\big)+O(\varepsilon^4).
\end{aligned} \tag{A-2}
$$

On the other hand, for $|y-\xi_i'|\ge\delta/\varepsilon$, it is easy to see that $W(y)=O(\varepsilon^4)$ and $2\varepsilon^4\sinh V=O(\varepsilon^4)$. This, together with (A-1), implies (2-23) and (2-24).

Next, by our definitions,

$$\begin{aligned}\Delta V &= \sum_{i=1}^{2(k+l)} (-1)^{i-1}\big(\varepsilon^2\Delta u_i(\varepsilon y)+\varepsilon^2\Delta H_i^\varepsilon(\varepsilon y)\big)\\ &= \sum_{i=1}^{2(k+l)} (-1)^{i-1}\Big(-\varepsilon^4 e^{u_i(\varepsilon y)}+\frac{\varepsilon^4}{|\Omega|}\int_\Omega e^{u_i(x)}dx\Big)\\ &= \sum_{i=1}^{2(k+l)} (-1)^{i-1}\Big(-\frac{8\mu_i^2}{(\mu_i^2+|y-\xi_i'|^2)^2}\Big)+\sum_{i=1}^{2(k+l)} (-1)^{i-1}\frac{\varepsilon^4}{|\Omega|}\int_\Omega e^{u_i(x)}dx.\end{aligned}$$

The last term in the above equality can be controlled by $O(\varepsilon^4)$ since from (2-7), we have

$$\varepsilon^2\sum_{i=1}^{2(k+l)} (-1)^{i-1}\int_\Omega e^{u_i} = O(\varepsilon^2|\mu_i-\mu_j|),$$

Combining this with (A-2), we get (2-22). □

Appendix B.

Proof of Claim 1. Since $\eta'(r)$ has a jump at $r=\varepsilon^{-\gamma}$ and $r=\varepsilon^{-\beta}$ and is otherwise smooth, we see that $L(\tilde Z_{0i})$ is a measure.

$$\begin{aligned}L(\tilde Z_{0i}) &= (-\Delta-W)\big(\eta_{1i}Z_{0i}+\varepsilon(1-\eta_{1i})\eta_{2i}\hat Z_{0i}\big)\\ &= -(Z_{0i}-\varepsilon\eta_{2i}\hat Z_{0i})\big([\eta_{1i}'(\varepsilon^{-\gamma})]\mu_{\varepsilon^{-\gamma}}+[\eta_{1i}'(\varepsilon^{-\beta})]\mu_{\varepsilon^{-\beta}}\big)\\ &\quad -2\nabla\eta_{1i}(\nabla Z_{0i}-\varepsilon\hat Z_{0i}\nabla\eta_{2i}-\varepsilon\eta_{2i}\nabla\hat Z_{0i})-\eta_{1i}(\Delta Z_{0i}+WZ_{0i})\\ &\quad -\varepsilon(1-\eta_{1i})(\hat Z_{0i}\Delta\eta_{2i}+\eta_{2i}\Delta\hat Z_{0i}+2\nabla\eta_{2i}\nabla\hat Z_{0i}+W\eta_{2i}\hat Z_{0i})\end{aligned}$$

where $[\eta_{1i}'(r)]=\eta_{1i}'(r^+)-\eta_{1i}'(r^-)$ denotes the jump of η_{1i}' at r, and μ_r is the 1-dimensional measure on the circle of radius r.

Let us consider first the case $m=i$:

$$\int_{\Omega_\varepsilon}\log|y_i-z|L(\tilde Z_{0i}) = \int_{\Omega_\varepsilon}(\log|y_i-z|-\log|\xi_i'-z|)L(\tilde Z_{0i})dz + \int_{\Omega_\varepsilon}\log|\xi_i'-z|L(\tilde Z_{0i})dz. \tag{B-1}$$

Let $r=|z-\xi_i'|$, and note that $\Delta\eta_{2i}=O(\varepsilon^{2\beta})$ and $\nabla\eta_{2i}=O(\varepsilon^\beta)$. For $r<\varepsilon^{-\beta}$, we have

$$\begin{aligned}\eta_{1i}(\Delta Z_{0i}+WZ_{0i}) &= \eta_{1i}(\Delta Z_{0i}+e^{v_i}(1+\theta_\varepsilon)Z_{0i})\\ &\le \frac{8\mu_i^2}{(\mu_i^2+|z-\xi_i'|^2)^2}O(\varepsilon^\alpha+\varepsilon|z-\xi_i'|)+O\Big(\frac{\varepsilon^\alpha}{(1+|y-\xi_i'|)^3}\Big).\end{aligned} \tag{B-2}$$

Thus

$$
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} \eta_{1i}(\Delta Z_{0i}+WZ_{0i})\log|z-\xi_i'|\Big| \\
&\qquad \le \int_{\Omega_\varepsilon} \eta_{1i}\bigg(\frac{8\mu_i^2 O(\varepsilon^\alpha+\varepsilon|z-\xi_i'|)}{(\mu_i^2+|z-\xi_i'|^2)^2}+O\Big(\frac{\varepsilon^\alpha}{(1+|y-\xi_i'|)^3}\Big)\bigg)\log|z-\xi_i'| \\
&\qquad \le C\int_0^{\varepsilon^{-\beta}}\Big(\frac{\varepsilon^\alpha}{(1+r)^3}+\frac{\varepsilon^\alpha+\varepsilon r}{(1+r^2)^2}\Big) r\log r\,dr \\
&\qquad = O\Big((\varepsilon^\alpha+\varepsilon^{1-\beta})\log\varepsilon^{-1}\Big) \\
&\qquad = o(1).
\end{aligned}
\tag{B-3}
$$

For $\varepsilon^{-\gamma}<r<\varepsilon^{-\beta}$,

$$
\begin{aligned}
\frac{1}{\mu_i}-a_{0i}G(\varepsilon z,\xi_i) &= \frac{1}{\mu_i}-\frac{4\log\varepsilon^{-1}-4\log|z-\xi_i'|+c_iH(\varepsilon z,\xi_i)}{\mu_i[4(1-\gamma)\log\varepsilon^{-1}+c_iH(\xi_i,\xi_j)]} \\
&= \frac{\log r-\gamma\log\varepsilon^{-1}+\varepsilon r}{(1-\gamma)\mu_i\log\varepsilon^{-1}}(1+O(\varepsilon)).
\end{aligned}
\tag{B-4}
$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}(1-\eta_{1i})W(\mu_i^{-1}-a_{0i}G)\log|z-\xi_i'|dz \\
&\qquad = \int_{r>\varepsilon^{-\gamma}} O\Big(\frac{\log r-\gamma\log\varepsilon^{-1}+\varepsilon r}{(1-\gamma)\mu_i\log\varepsilon^{-1}}\Big)O(r^{-4r})\log r\,dr \\
&\qquad = O(\varepsilon^{2\gamma}\log\varepsilon^{-1})
\end{aligned}
\tag{B-5}
$$

and

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\nabla\eta_{1i}(\nabla Z_{0i}-\varepsilon\hat Z_{0i}\nabla\eta_{2i}-\varepsilon\eta_{2i}\nabla\hat Z_{0i})\log|z-\xi_i'|dz \\
&\quad = 2\pi\int_{\varepsilon^{-\gamma}}^{\varepsilon^{-\beta}}\frac{-r^{-1}}{(\beta-\gamma)\log\varepsilon^{-1}} \\
&\qquad\qquad \times\Big(O(r^{-3})+O(\varepsilon^{1+\beta})+O\Big(\frac{\varepsilon}{\log\varepsilon^{-1}}(r^{-1}+C)\Big)\Big)r\log r\,dr \\
&\quad = O(\varepsilon^{2\gamma})+O(\frac{\varepsilon^{1-\beta}}{\log\varepsilon^{-1}}).
\end{aligned}
\tag{B-6}
$$

For $r>\varepsilon^{-\gamma}$,

$$
\begin{aligned}
&\hat Z_{0i}\Delta\eta_{2i}+\eta_{2i}\Delta\hat Z_{0i}+2\nabla\eta_{2i}\nabla\hat Z_{0i}+W\eta_{2i}\hat Z_{0i} \\
&\qquad = \hat Z_{0i}\Delta\eta_{2i}+2\nabla\eta_{2i}\nabla\hat Z_{0i}+\eta_{2i}(\Delta Z_{0i}+WZ_{0i}+a_{0i}\Delta G-W\mu_i^{-1}+Wa_{2i}G).
\end{aligned}
$$

So, recalling (B-5), we have

$$
\begin{aligned}
&\varepsilon \int_{\Omega_\varepsilon} (1-\eta_{1i})(\hat{Z}_{0i}\Delta\eta_{2i} + \eta_{2i}\Delta\hat{Z}_{0i} + 2\nabla\eta_{2i}\nabla\hat{Z}_{0i} + W\eta_{2i}\hat{Z}_{0i}) \log|z-\xi_i'|dz \\
&= \varepsilon \int_{\varepsilon^{-\beta}}^{2\varepsilon^{-\beta}} O(\varepsilon^{2\beta}) r \log r dr + \varepsilon \int_{\varepsilon^{-\beta}}^{2\varepsilon^{-\beta}} O(\varepsilon^{\beta}) O\Big(r^{-3} + \frac{\varepsilon}{\log\varepsilon^{-1}}(C+r^{-1})\Big) r \log r dr \\
&\quad + \varepsilon \int_{\varepsilon^{-\gamma}}^{2\varepsilon^{-\beta}} \Big(O\Big(\frac{\varepsilon^{\alpha}+\varepsilon r}{r^4}\Big) + O\Big(\frac{\varepsilon^{\alpha}}{(1+r)^3}\Big) + O\Big(\frac{\varepsilon^2}{\log\varepsilon^{-1}}\Big)\Big) r \log r dr \\
&\quad - \varepsilon \int_{\Omega_\varepsilon} (1-\eta_{1i}) W(\mu_i^{-1} - a_{0i}G) \log|z-\xi_i'|dz,
\end{aligned}
$$

which is equal to $O(\varepsilon\log\varepsilon^{-1})$. A direct computation shows

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} [\eta_{1i}'(\varepsilon^{-\gamma})]\mu_{\varepsilon^{-\gamma}}(Z_{0i} - \varepsilon\eta_{2i}\hat{Z}_{0i}) \log|z-\xi_i'|dz \\
&\qquad = \frac{-\varepsilon^{\gamma}}{(\beta-\gamma)\log\varepsilon^{-1}} \int_{r=\varepsilon^{-\gamma}} (Z_{0i} - \varepsilon\hat{Z}_{0i}) \log|z-\xi_i'| \\
&\qquad = \frac{-\varepsilon^{\gamma}}{(\beta-\gamma)\log\varepsilon^{-1}} \times \frac{1+O(\varepsilon^{2\gamma})}{\mu_i} \times 2\pi\varepsilon^{-\gamma}\log\varepsilon^{-\gamma} \\
&\qquad = \frac{-2\pi\gamma}{\mu_i(\beta-\gamma)} + O(\varepsilon^{2\gamma}).
\end{aligned}
$$

Similarly,

$$
\int_{\Omega_\varepsilon} [\eta_{1i}'(\varepsilon^{-\beta})]\mu_{\varepsilon^{-\beta}}(Z_{0i} - \varepsilon\eta_{2i}\hat{Z}_{0i}) \log|z-\xi_i'|dz = \frac{2\pi\beta}{\mu_i(\beta-\gamma)} + O(\varepsilon^{2\beta}).
$$

Hence

$$
\int_{\Omega_\varepsilon} L(\tilde{Z}_{0i}) \log|z-\xi_i'|dz = \frac{2\pi}{\mu_i} + o(1).
$$

For the first integral in the right side of (B-1), we can assume $R_\varepsilon \to +\infty$ slowly enough so that $\varepsilon^{\gamma}R_\varepsilon \to 0$. Then

$$
\big|\log|y_i - z| - \log|\xi_i' - z|\big| = \Big|\log\frac{|y_i-z|}{r}\Big| \le \Big|\log\frac{|y_i-\xi_i'|+r}{r}\Big| \tag{B-7}
$$

for $r = |\xi_i' - z|$; therefore we have from (B-2)

$$
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} (\log|y_i-z| - \log|\xi_i'-z|)\eta_{1i}(\Delta Z_{0i} + WZ_{0i})dz\Big| \\
&\qquad \le C\int_0^{\varepsilon^{-\beta}} \log(R_\varepsilon r^{-1}+1)\Big(O\Big(\frac{\varepsilon^{\alpha}+\varepsilon r}{(1+r^2)^2}\Big) + O\Big(\frac{\varepsilon^{\alpha}}{(1+r)^3}\Big)\Big) r dr \qquad \text{(B-8)}\\
&\qquad = O(\varepsilon^{\alpha}(R_\varepsilon + \log\varepsilon^{-1})).
\end{aligned}
$$

On the other hand, from (B-7), for $\varepsilon^{-\gamma} \le r = |z - \xi_i'| \le \varepsilon^{-\beta}$ we have

$$\big|\log|y_i - z| - \log|\xi_i' - z|\big| \le C|y_i - \xi_i'|/\varepsilon^{-\gamma}$$

and it follows that

$$\Big|\int_{\Omega_\varepsilon} (\log|y_i - z| - \log|\xi_i' - z|)\big(L(\tilde{Z}_{0i}) + \eta_{1i}(\Delta Z_{0i} + W Z_{0i})\big)dz\Big| = O(\varepsilon^\gamma R_\varepsilon).$$

Thus, from this and (B-8), we obtain

$$\Big|\int_{\Omega_\varepsilon} \Big(\log|y_i - z| - \log|\xi_i' - z|\Big) L(\tilde{Z}_{0i})\Big| = o(1). \tag{B-9}$$

Next, we show that if $m \ne i$, then

$$\int_{\Omega_\varepsilon} \log|y_m - z| L(\tilde{Z}_{0i}) dz = o(1).$$

In fact,

$$\begin{aligned}\int_{\Omega_\varepsilon} &\log|y_m - z| L(\tilde{Z}_{0i}) dz \\ &= \int_{\Omega_\varepsilon} (\log|y_m - z| - \log|y_m - \xi_i'|) L(\tilde{Z}_{0i}) dz + \int_{\Omega_\varepsilon} \log|y_m - \xi_i'| L(\tilde{Z}_{0i}) dz.\end{aligned}$$

We assume that $R_\varepsilon < \varepsilon^{-\gamma}/2$, so that

$$\big|\log|y_m - z| - \log|y_m - \xi_i'|\big| \le \log\Big(1 + \frac{|z - \xi_i'|}{|y_m - \xi_i'|}\Big) = O(\varepsilon|z - \xi_i'|).$$

Thus

$$\Big|\int_{\Omega_\varepsilon} (\log|y_m - z| - \log|y_m - \xi_i'|) L(\tilde{Z}_{0i}) dz\Big| = O\Big(\frac{\varepsilon^{1-\beta}}{\log \varepsilon^{-1}}\Big).$$

Finally,

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0i}) dz = O(\varepsilon^{2\gamma}). \tag{B-10}$$

This implies

$$\int_{\Omega_\varepsilon} \log|y_m - \xi_i'| L(\tilde{Z}_{0i}) dz = o(1).$$

Therefore Claim 1 holds. □

Proof of Claim 4. Let

$$\zeta(r) = \begin{cases} 1 & \text{if } r < \varepsilon^{-1/2}, \\ (\log(\delta/\varepsilon) - \log r)/(\log(\delta/\varepsilon) - \log \varepsilon^{-1/2}) & \text{if } \varepsilon^{-1/2} < r < \delta/\varepsilon, \\ 0 & \text{if } r > \delta/\varepsilon, \end{cases}$$

and set

$$\psi(z)=\sum_{i=1}^{2(k+l)} H(\varepsilon y,\xi_i)\zeta(|z-\xi_i'|).$$

Testing (3-9) by ψ and integrating by parts, we obtain

$$\int_{\Omega_\varepsilon}\Big(W\tilde\phi+h-\sum_{i=1}^{2(k+l)} d_i L(\tilde Z_{0i})\Big)\psi+\int_{\Omega_\varepsilon}\tilde\phi\Delta\psi-\int_{\partial\Omega_\varepsilon}\tilde\phi\frac{\partial\psi}{\partial\nu}=0.$$

Thus

$$A=\int_{\Omega_\varepsilon}(H(\varepsilon y,\varepsilon z)-\psi)\Big(W\tilde\phi+h-\sum_{i=1}^{2(k+l)} d_i L(\tilde Z_{0i})\Big)-\int_{\Omega_\varepsilon}\tilde\phi\Delta\psi+\int_{\partial\Omega_\varepsilon}\tilde\phi\frac{\partial\psi}{\partial\nu}.$$

Since H, ψ and $\tilde\phi$ are bounded,

$$\Big|\int_{\Omega_\varepsilon}(H(\varepsilon y,\varepsilon z)-\psi)h\,dz\Big|\le C\|h\|_*=o(1) \tag{B-11}$$

and

$$\Big|\int_{\Omega_\varepsilon}(H(\varepsilon y,\varepsilon z)-\psi)L(\tilde Z_{0i})\Big|\le C\Big|\int_{\Omega_\varepsilon}L(\tilde Z_{0i})dz\Big|=o(1). \tag{B-12}$$

Also, it is not difficult to show that

$$\int_{\Omega_\varepsilon}\tilde\phi\Delta\psi=O\Big(\frac{1}{\log(\delta/\varepsilon)}\Big)=o(1),\quad \int_{\partial\Omega_\varepsilon}\tilde\phi\frac{\partial\psi}{\partial\nu}=O\Big(\frac{1}{\log(\delta/\varepsilon)}\Big)=o(1). \tag{B-13}$$

For instance, the first integer in (B-13) can be estimated as

$$\Big|\int_{\Omega_\varepsilon}\tilde\phi\Delta\psi\Big|\le\|\tilde\phi\|_{L^\infty(\Omega_\varepsilon)}\int_{\Omega_\varepsilon}|\Delta\psi|.$$

But $\Delta\psi$ is a measure with support on the arcs $r=\varepsilon^{-1/2}$ and $r=\delta/\varepsilon$, where $r=|z-\xi_i'|$, and

$$\int_{\Omega_\varepsilon}|\Delta\psi|=O\Big(\varepsilon^{-1/2}\frac{1}{\varepsilon^{-1/2}\log\varepsilon^{-1}}+\frac{\delta}{\varepsilon}\frac{1}{(\delta/\varepsilon)\log\varepsilon^{-1}}\Big)=O\Big(\frac{1}{\log(\delta/\varepsilon)}\Big)=o(1).$$

Note that for $|z-\xi_i'|>\delta/\varepsilon$, we have $W=O(r^{-4})$, and H and $\tilde\phi$ are bounded; thus

$$\int_{\Omega_\varepsilon\setminus(\bigcup_i B_{\delta/\varepsilon}(\xi_i'))}(H(\varepsilon y,\varepsilon z)-\psi)W\tilde\phi=o(1). \tag{B-14}$$

On the other hand, for $|z-\xi_i'| \le \delta/\varepsilon$, we have $H(\varepsilon y, \varepsilon z) - H(\varepsilon y, \xi_i) = O(\varepsilon|z-\xi_i'|)$ and $W = O((r^2+1)^{-2})$. So

$$\begin{aligned}\Bigl|\int_{\Omega_\varepsilon \cap B_{\varepsilon^{-1/2}}(\xi_i')} (H(\varepsilon y, \varepsilon z) - \psi(z)) W\tilde{\phi}\, dz\Bigr| \\ = \Bigl|\int_{\Omega_\varepsilon \cap B_{\varepsilon^{-1/2}}(\xi_i')} (H(\varepsilon y, \varepsilon z) - H(\varepsilon y, \xi_i)) W\tilde{\phi}\, dz\Bigr| \\ \le C\varepsilon \int_0^{\varepsilon^{-1/2}} \frac{r^2}{(r^2+1)^2} dr = O(\varepsilon^{1/2}) = o(1).\end{aligned} \tag{B-15}$$

In the region $\varepsilon^{-1/2} < r = |z-\xi_i'| < \delta/\varepsilon$, noting the fact that H, ζ and $\tilde{\phi}$ are bounded and that $W = O(r^{-4})$, we find

$$\Bigl|\int_{\Omega_\varepsilon \cap B_{\delta/\varepsilon}(\xi_i') \setminus B_{1/\sqrt{\varepsilon}}(\xi_i')} (H(\varepsilon y, \varepsilon z) - \psi(z)) W\tilde{\phi}\, dz\Bigr| \le C \int_{1/\sqrt{\varepsilon}}^{\delta/\varepsilon} r^{-3} dr = o(1). \tag{B-16}$$

Therefore, Claim 4 follows from (B-10)–(B-16). □

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