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Special groups are an axiomatization of the algebraic theory of quadratic forms over fields. It is known that any finite reduced special group is the special group of some field. We show that any special group that is the projective limit of a projective system of finite reduced special groups is also the special group of some field.

1. Introduction

The theory of special groups is an axiomatization of the algebraic theory of quadratic forms, introduced in [Dickmann and Miraglia 2000]. The class of special groups, together with its morphisms, forms a category. As for other such axiomatizations, the main examples of special groups are provided by fields, in this case by applying the special group functor, which associates to each field F a special group $G(F)$ describing the theory of quadratic forms over F.

The category of special groups is equivalent to that of abstract Witt rings via covariant functors, while the category of reduced special groups is equivalent, via the restriction of the same covariant functors, to the category of reduced abstract Witt rings (see [Dickmann and Miraglia 2000, 1.25 and 1.26]; recall that the special group of a field F is reduced if and only if F is formally real and Pythagorean). The category of reduced special groups is also equivalent, via contravariant functors, to the category of abstract spaces of orderings; see Chapter 3 of the same reference.

The question whether it is possible to realize every (reduced) special group as the special group of some (formally real, Pythagorean) field is still open, but the case of finite reduced special groups (actually of reduced special groups of finite chain length) has been positively answered by the combination of two results: Kula [1979], building on techniques introduced in [Bröcker 1977] for the field case, showed that the product of two finite special groups of (formally real, Pythagorean) fields is still the special group of some (formally real, Pythagorean) field; then Marshall [1980] showed that every finite reduced special group can be constructed from the special group of any real closed field by applying a finite number of times the operations of product and extension. (Marshall's result is actually stated and

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proved for abstract spaces of orderings.) Since the extension of the special group of a (formally real, Pythagorean) field is still the special group of a (formally real, Pythagorean) field, it shows that every finite reduced special group (or reduced special group of finite chain length) is realized as the special group of a field.

After finite reduced special groups, the simplest objects to consider are probably projective limits of finite reduced special groups, that is, profinite reduced special groups. They have already been studied, for example, in [Astier and Tressl 2005; Lira de Lima 1997; Mariano 2003], and notably in [Kula et al. 1984], where the question of the realization of these special groups by fields is considered and where it is shown (as Corollary 4.7) that every profinite reduced special group is *isomorphic to a quotient* of the reduced special group of some field.

In this paper, we improve on this result by showing that every profinite reduced special group is *isomorphic* to the special group of some (necessarily formally real and Pythagorean) field.

2. Preliminaries

Definition 2.1. Let A, B, A', B' be objects in a category $\mathcal{C}$, and let $\lambda : A \to B$, $\lambda' : A' \to B'$ be $\mathcal{C}$-morphisms. Then λ, λ' are said to be *naturally identified* (in symbols, $\lambda \cong \lambda'$) if and only if there are $\mathcal{C}$-isomorphisms $i_A : A \to A'$, $i_B : B \to B'$ such that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A' \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda'} \\ B & \xrightarrow{i_B} & B' \end{array}$$

commutes. In this case, we also say that λ and λ' are *naturally identified via* i_A, i_B.

On special groups. We assume some familiarity with the theory of special groups, as presented in [Dickmann and Miraglia 2000], and only introduce the following notation:

If G is a special group, $\mathrm{Ssat}(G)$ denotes the poset of saturated subgroups of G, ordered by inclusion. We recall that if $\Delta \in \mathrm{Ssat}(G)$, then G/Δ is a reduced special group if and only if $\Delta \subsetneq G$, if and only if $-1 \notin \Delta$.

Definition 2.2. A profinite reduced special group is the projective limit of a projective system of finite reduced special groups.

If $(G'_i, f'_{ij})_{i \le j \in I}$ is a projective system of finite reduced special groups, where $(I, \le)$ is a downward directed poset, and if G is the projective limit of this system, the fact that G is indeed a special group (with the structure induced by its inclusion in the product $\prod_{i \in I} G'_i$) follows immediately from [Dickmann and Miraglia 2003,

Theorem 3.24]. Moreover, as proved in [Lira de Lima 1997, Proposition 1.9.11], it is always possible to describe G as the projective limit of a projective system $(G_i, f_{ij})_{i\leq j\in I}$ having the following properties:

(1) For every $i \in I$, G_i is G/Δ_i with Δ_i saturated subgroup of G of finite index;

(2) For every $i \leq j \in I$, $\Delta_i \subseteq \Delta_j$ and f_{ij} is the canonical projection of special groups induced by this inclusion.

We briefly sketch the argument: Let $\iota : G \hookrightarrow \prod_{j\in I} G'_j$ be the canonical embedding given by the definition of projective limit, and let $\pi_i : \prod_{j\in I} G'_j \to G'_i$ be the canonical projection. We define $\Delta_i := \ker(\pi_i \circ \iota)$, $G_i := G/\Delta_i$ and, for $i \leq j \in I$, f_{ij} to be the canonical projection induced by $\Delta_i \subseteq \Delta_j$. The system $(G_i, f_{ij})_{i\leq j\in I}$ is a projective system, whose projective limit is isomorphic to G, via the map $g \in G \mapsto (g.\Delta_i)_{i\in I} \in \varprojlim(G/\Delta_i, f_{ij})_{i\leq j\in I}$.

Remark 2.3. If $\mathcal{M} = (M_i, f_{ij})_{i\leq j\in I}$ is any projective system, and if $i' \in I$, by restricting this system to the set $I' := \{i \in I \mid i \leq i'\}$ we obtain a new system

$$\mathcal{M}' := (M_i, f_{ij})_{i\leq j\in I'}.$$

Since I' is coinitial in I, $\mathcal{M}$ and $\mathcal{M}'$ have isomorphic projective limits, and $\mathcal{M}'$ possesses the following extra property:

(3) The index set of the projective system has a maximum element.

Definition 2.4. We call *adequate* a projective system of special groups that satisfies conditions (1), (2) and (3) above.

We will adhere to the following convention throughout this paper: Let $(I, \leq)$ be a downward directed poset. If $(I, \leq)$ has a maximum element, we will denote it by $\top$, and if $(I, \leq)$ has a minimum element (which happens for instance if I is finite), we will denote it by $\bot$.

Let G_0, G_1 be abstract groups and denote by $\pi_0 : G_0 \times G_1 \twoheadrightarrow G_0 : (g_0, g_1) \mapsto g_0$, $\pi_1 : G_0 \times G_1 \twoheadrightarrow G_1 : (g_0, g_1) \mapsto g_1$ the canonical projections and by $\iota_0 : G_0 \rightarrowtail G_0 \times G_1 : g_0 \mapsto (g_0, 1)$, $\iota_1 : G_1 \rightarrowtail G_0 \times G_1 : g_1 \mapsto (1, g_1)$ the canonical injections.

The statements in the next paragraph are straightforward.

Fact 2.5. *Let G_0, G_1 be special groups. Then the canonical map*

$$\begin{aligned} \psi : \mathrm{Ssat}(G_0 \times G_1) &\to \mathrm{Ssat}(G_0) \times \mathrm{Ssat}(G_1) \\ \Delta &\mapsto (\iota_0^{-1}[\Delta], \iota_1^{-1}[\Delta]) = (\pi_0[\Delta], \pi_1[\Delta]) \end{aligned}$$

is an order-preserving bijection, whose inverse is $(\Delta_0, \Delta_1) \overset{\psi^{-1}}{\mapsto} \Delta_0 \times \Delta_1$. In particular, if $\Delta \in \mathrm{Ssat}(G_0 \times G_1)$ and $(\Delta_0, \Delta_1) := (\iota_0^{-1}[\Delta], \iota_1^{-1}[\Delta])$, then $\Delta = \Delta_0 \times \Delta_1$ and Δ is proper if and only if Δ_0 or Δ_1 is proper. Moreover:

- *The canonical surjective morphism of special groups*

$$G_0 \times G_1 \twoheadrightarrow G_0/\Delta_0 \times G_1/\Delta_1$$

induces a natural isomorphism of special groups[1]

$$\bar{q}_\Delta : (G_0 \times G_1)/\Delta \xrightarrow{\cong} G_0/\Delta_0 \times G_1/\Delta_1.$$

- *If* $\Delta \subseteq \Delta' \in \mathrm{Ssat}(G_0 \times G_1)$, *then the projection* $(G_0 \times G_1)/\Delta \twoheadrightarrow (G_0 \times G_1)/\Delta'$ *is naturally identified, via the isomorphisms* $\bar{q}_\Delta, \bar{q}_{\Delta'}$, *with the* (*product*) *projection* $G_0/\Delta_0 \times G_1/\Delta_1 \twoheadrightarrow G_0/\Delta_0' \times G_1/\Delta_1'$.

On projective systems of (valued) fields. Let $(I, \leq)$ be a poset. For each $i, j \in I$ such that $i \leq j$ we define $d(i, j) := \max\{\text{length of a chain from } i \text{ to } j\} \in \mathbb{N} \cup \{\infty\}$. If $i \not\leq j$ then we set $d(i, j) := -\infty$. Of course, if i and j are comparable, we have $d(i, j) = d(j, i)$ if and only if $i = j$, if and only if $d(i, j) = 0$.

We will often consider $(I, \leq)$ as a directed graph whose vertices are the elements of I, and where there is an edge from i to j if and only if $i \leq j$ and $d(i, j) = 1$.

We first remark that it is possible to describe some projective systems of fields as projective systems whose morphisms are all inclusions.

Remark 2.6. Let $\mathscr{F} := (F_i, f_{ij})_{i \leq j \in I}$ be a projective system of fields over a downward directed poset $(I, \leq)$ with maximum element $\top \in I$. Then there is an isomorphic projective system of fields $\mathscr{F}' = (F_i', \iota_{ij})_{i \leq j \in I}$ such that, if $i \leq j \in I$, then $F_i' \subseteq F_j'$ and the morphism of fields $\iota_{ij} : F_i' \to F_j'$ is the inclusion. The projective limit of the system $\mathscr{F}$ is thus isomorphic to the intersection of the fields F_i', $i \in I$.

We briefly sketch the argument. For each $i \in I$, we define $F_i' := f_{i\top}[F_i] \subseteq F_\top$. Since for $i \leq j \in I$, $f_{i\top} = f_{j\top} \circ f_{ij}$, we obtain $F_i' \subseteq F_j'$, so we can define ι_{ij} to be this inclusion. It follows that $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic via the morphisms $(f_{i\top})_{i \in I}$. Therefore: $\varprojlim (F_j, f_{ij})_{i \leq j \in I} \cong \varprojlim (F_j', \iota_{ij})_{i \leq j \in I} \cong \bigcap_{i \in I} F_i' \subseteq F_\top'$.

The next results lead to Corollary 2.10, which shows that any finite projective system of fields of characteristic zero, whose index set has a maximum element, is isomorphic to the projective system given by the residues of a finite projective system of valued fields. We first fix some notation:

If (K, v) is a valued field, we denote by Kv or by $\overline{K}$ (if there is no risk of confusion about which valuation we consider) the residue field of v, by vK its value group, by O_K the valuation ring associated to v and by M_K its maximal ideal (if there is no ambiguity about the valuation v under consideration). If $a \in O_K$, we denote by av or $\bar{a}$ (once again if there is no risk of confusion) the class of a in the residue field $\overline{K}$. Finally, if v has rank one, K^v denotes a completion of K with respect to v.

[1] That are reduced if Δ is proper or, otherwise, the trivial special group $\{1\}$.

If $\mathscr{F} = (F_i, \xi_{ij})_{i \le j \in I}$ is a projective system of fields, we denote by $G(\mathscr{F})$ the system $(G(F_i), G(\xi_{ij}))_{i \le j \in I}$ obtained from $\mathscr{F}$ by applying the special group functor G.

If $\mathscr{F} = ((F_i, v_i), \xi_{ij})_{i \le j \in I}$ is a projective system of valued fields, we denote by $\operatorname{res}\mathscr{F}$ or $\mathscr{F}v$ the induced residue projective system $(F_i v_i, (\xi v)_{ij})_{i \le j \in I}$, where the $(\xi v)_{ij}$ are the induced morphisms of fields.

If a projective system of fields or of valued fields is denoted by $(F_i)_{i \in I}$ or $(F_i, v_i)_{i \in I}$, without mention of the morphisms, it means that the morphisms are all inclusions (from a field within all fields with larger index).

Lemma 2.7. *Let (K, v) be a henselian valued field of residue characteristic zero and let L be a subfield of K. Let N be a subfield of $\overline{K}$ such that $\overline{L} \subseteq N \subseteq \overline{K}$. Then there is a field M such that $L \subseteq M \subseteq K$ and $\overline{M} = N$. Moreover, if $[N : \overline{L}]$ is algebraic, respectively finite, then M can be chosen such that $[M : L]$ is algebraic, respectively finite.*

Proof. Write $N = \overline{L}(X)(\alpha_i, i \in \beta)$, where X is a transcendence basis of N over $\overline{L}$ and $(\alpha_i, i \in \beta)$ is a (possibly infinite) tuple of elements that are algebraic over $\overline{L}(X)$, indexed by an ordinal β. Let Y be a set of transcendental elements over L such that $\overline{Y} = X$. By [Engler and Prestel 2005, Corollary 2.2.2], the restriction of v to $L(Y)$ is the Gauss extension of v from L to $L(Y)$. In particular, $\overline{L(Y)} = \overline{L}(X)$.

We now proceed by induction on $k \in \beta$ to find elements $a_i \in K$, $i < k$, such that $\overline{L(Y)(a_i, i < k)} = \overline{L}(X)(\alpha_i, i < k)$.

If $k = 0$ there is nothing to prove since $\overline{L(Y)} = \overline{L}(X)$.

Assume we have found all a_i for $i < k$. Let $N_k = L(Y)(a_i, i < k)$ and $M_k = \overline{L}(X)(\alpha_i, i < k)$. By hypothesis we have $\overline{N}_k = M_k$. Let $P \in O_K[T]$ be a unitary polynomial such that $\overline{P}$ is the minimal polynomial of α_k over $\overline{L}(X)$. Let a_k be a root of P in K such that $\bar{a}_k = \alpha_k$ (it exists since (K, v) is henselian of residue characteristic zero). We have $\overline{N_k(a_k)} \supseteq M_k(\alpha_k)$ and the fundamental inequality [Engler and Prestel 2005, Theorem 3.3.4] tells us that

$$[\overline{N_k(a_k)} : \overline{N_k}] \le [N_k(a_k) : N_k] (\le \deg P).$$

Since $[M_k(\alpha_k) : M_k] = \deg P$, it follows that $\overline{N_k(a_k)} = M_k(\alpha_k)$, which is the desired result. □

Definition 2.8. Let (K, v) be a valued field and let $(E_i)_{i<n}$ and $(F_i)_{i<n}$ be two sequences of fields of the same length n. We say that $(F_i)_{i<n}$ is a good residue of $(E_i)_{i<n}$ in (K, v) if

(1) $E_i \subseteq K$ and $F_i \subseteq \overline{K}$ for $i < n$;

(2) For every $A \subseteq \{0, \dots, n-1\}$, $\overline{\langle E_i, i \in A \rangle} = \langle F_i, i \in A \rangle$ (where $\langle L_i, i \in A \rangle$ denotes the compositum of the fields L_i).

Lemma 2.9. *Let (K, v) be a henselian valued field of residue characteristic zero, and let $(E_i)_{i<n}$ and $(F_i)_{i<n}$ be two sequences of fields of length n such that $(F_i)_{i<n}$ is a good residue of $(E_i)_{i<n}$ in (K, v). Let $(F'_i)_{i<m}$ be a sequence of subfields of $\bar{K}$, and let, for $i \in \{0, \dots m-1\}$*

$$A_i = \{j \in \{0, \dots, n-1\} \mid F_j \subseteq F'_i\}.$$

Then there is a sequence $(E'_i)_{i<m}$ of subfields of K such that

(1) *for every $i \in \{0, \dots, m-1\}$ and every $j \in A_i$, $E_j \subseteq E'_i$ and* $\operatorname{trdeg} E'_i \mid E_j = \operatorname{trdeg} F'_i \mid F_j$;

(2) *$(F'_i)_{i<m}$ is a good residue of $(E'_i)_{i<m}$.*

Proof. We will use the following reformulation of Remark 4.1.2(3) in [Engler and Prestel 2005]:

Fact 1. Let (N, w) be a valued field and let $P, Q \in O_N[T]$ and $R \in N[T]$ be such that $P = QR$. Assume that Q is primitive (that is, $w'(Q) = 0$, where w' is the Gauss extension of w to $N[T]$, i.e., $\min_{i \le k} w(a_i) = 0$ if one writes $Q = a_0 + \cdots + a_k T^k$). Then $R \in O_N[T]$.

Proof. Write $P = aP_1$ and $R = cR_1$ with $a, c \in N$ and $P_1, R_1 \in N[T]$ such that $w'(P_1) = w'(R_1) = 0$ (so $P_1, R_1 \in O_N[T]$). Then $w(c) = w'(Q) + w'(cR_1) = w'(QcR_1) = w'(QR) = w'(P) \ge 0$ since $P \in O_N[T]$. This yields $R = cR_1$ with $w'(R) = w(c) + w'(R_1) = w(c) \ge 0$, i.e., $R \in O_N[T]$. □

We next fix some notation. For $A \subseteq \{0, \dots, n-1\}$ we denote by F_A the field $\langle F_i, i \in A\rangle$ and similarly by E_A the field $\langle E_i, i \in A\rangle$.

For $i < m$ let $X_i = \{x_{i1}, \dots, x_{ik_i}\}$ be a transcendence basis of F'_i over $F_{A_i} = \bar{E}_{A_i}$, and let $Y_i = \{y_{i1}, \dots, y_{ik_i}\} \subseteq K$ be a set of transcendental elements over E_{A_i} such that $\bar{Y}_i = X_i$. Note that by [Engler and Prestel 2005, corollary 2.2.2], it implies that the restriction of v to $E_{A_i}(Y_i)$ is the Gauss extension of v from E_{A_i} to $E_{A_i}(Y_i)$. In particular we have $\overline{E_{A_i}(Y_i)} = \overline{E_{A_i}}(X_i) = F_{A_i}(X_i)$ (the last equality holds because $(F_i)_{i<n}$ is a good residue of $(E_i)_{i<n}$).

Write $F'_i = F_{A_i}(X_i)(\alpha_i)$, where $\alpha_i = (\alpha_{ij})_{j \in \beta_i}$ is a (possibly infinite) tuple of elements algebraic over $F_{A_i}(X_i)$. For $i < m$ and $j \in \beta_i$ let $P_{ij} \in O_{E_{A_i}(Y_i)}[T]$ be a unitary polynomial such that $\bar{P}_{ij}$ is the minimal polynomial of α_{ij} over $\overline{E_{A_i}(Y_i)} = F_{A_i}(X_i)$, and let $a_{ij} \in O_K$ be a root of P_{ij} with $\bar{a}_{ij} = \alpha_{ij}$ (a_{ij} exists since (K, v) is henselian of residue characteristic zero). We take for E'_i the field $E_{A_i}(Y_i)(a_i)$, where $a_i = (a_{ij})_{j \in \beta_i}$. The first conclusion of the lemma is obviously satisfied. Let $A \subseteq \{0, \dots, m-1\}$.

Claim. Let L be a subfield of K such that (L, v) is henselian, $\langle F'_i, i \in A\rangle \subseteq \bar{L}$, and $\langle E_j, j \in A_i, i \in A\rangle(Y_i, i \in A) \subseteq L$. Then $a_i \in L$ for every $i \in A$, i.e., $L \supseteq \langle E'_i, i \in A\rangle$.

Proof. Let $i \in A$ and $j \in \beta_i$. Since $\alpha_{ij} \in \bar{L}$ and (L, v) is henselian (of residue characteristic zero), there is $b_{ij} \in O_L$ such that $\bar{b}_{ij} = \alpha_{ij}$ and b_{ij} is a root of P_{ij}. Assume $b_{ij} \neq a_{ij}$. Then we can write $P_{ij}(T) = (T - a_{ij})(T - b_{ij})R(T)$ in $E_{A_i}(Y_i, a_{ij}, b_{ij})$. But P_{ij}, $(T - a_{ij})$, $(T - b_{ij})$ each lie in $O_{E_{A_i}(Y_i, a_{ij}, b_{ij})}[T]$ and $(T - a_{ij})(T - b_{ij})$ is primitive, so by Fact 1 we have $R(T) \in O_{E_{A_i}(Y_i, a_{ij}, b_{ij})}[T]$. Going to the residue field $\bar{K}$ we get $\bar{P}_{ij}(T) = (T - \alpha_{ij})^2 \bar{R}(T)$, so α_{ij} is root of order at least 2 of $\bar{P}_{ij}$, which is impossible since $\bar{P}_{ij}$ is the minimal polynomial of α_{ij} and $\operatorname{char} \bar{K} = 0$. So $a_{ij} = b_{ij} \in L$. End of proof of the claim. □

We have $E'_A = \langle E'_i, i \in A\rangle = \langle \langle E_j, j \in A_i\rangle(Y_i)(a_i), i \in A\rangle = \langle E_j, j \in A_i, i \in A\rangle(Y_i, i \in A)(a_i, i \in A)$, and

$$\begin{aligned}\overline{\langle E_j, j \in A_i, i \in A\rangle(Y_i, i \in A)} &= \overline{\langle E_j, j \in A_i, i \in A\rangle}(X_i, i \in A)\\ &= \langle F_j, j \in A_i, i \in A\rangle(X_i, i \in A).\end{aligned}$$

Moreover,

$$\langle F'_i, i \in A\rangle = \langle \langle F_j, j \in A_i\rangle(X_i)(\alpha_i), i \in A\rangle = \langle F_j, j \in A_i, i \in A\rangle(X_i, i \in A)(\alpha_i, i \in A).$$

So $\langle F'_i, i \in A\rangle$ is an algebraic extension of $\overline{\langle E_j, j \in A_i, i \in A\rangle(Y_i, i \in A)}$. In particular (see Lemma 2.7) there is an algebraic extension E'' of

$$\langle E_j, j \in A_i, i \in A\rangle(Y_i, i \in A)$$

(inside K) such that $\bar{E}'' = \langle F'_i, i \in A\rangle$. Let $\tilde{E}$ be the henselian closure of E'' in (K, v). We have $\bar{\tilde{E}} = \langle F'_i, i \in A\rangle$, $E'' \subseteq \tilde{E}$. By the claim, since $\tilde{E}$ is henselian and $\tilde{E} \supseteq \langle F'_i, i \in A\rangle$, we have $a_i \in \tilde{E}$ for every $i \in A$. It implies $E'_A \subseteq \tilde{E}$, which gives, taking residues $\bar{E}'_A \subseteq \bar{\tilde{E}} = \langle F'_i, i \in A\rangle$. But by construction of the E'_i we obviously have $\bar{E}'_A \supseteq \langle F'_i, i \in A\rangle$. It follows that $\bar{E}'_A = \langle F'_i, i \in A\rangle$. □

Corollary 2.10. *Let $\mathcal{F} = (F_i)_{i \in I}$ be a finite projective system of fields of characteristic zero and let $\perp$ be the minimum of I. Assume that $(I, \leq)$ has a maximum $\top$ and let $(E_\perp, v_\perp)$ be a valued field such that $E_\perp v_\perp \cong F_\perp$. Then there is a projective system of valued fields $(E_i, v_i)_{i \in I}$ such that $(F_i)_{i \in I} \cong \operatorname{res}(E_i, v_i)_{i \in I}$ and, for every $i \in I$, $\operatorname{trdeg} E_i \mid E_\perp = \operatorname{trdeg} F_i \mid F_\perp$. Moreover*:

- *We can assume that all (E_i, v_i), $i \in I$, are henselian.*
- *If $v_\perp$ has rank one, then we can choose the valuations v_i, $i \in I$, such that they all have rank one.*

Proof. We first show that there is a projective system of fields $\mathcal{F}' = (F'_i)_{i \in I}$ with $\mathcal{F}' \cong \mathcal{F}$ and there is an extension (K, v) of $(E_\perp, v_\perp)$ such that $\bar{K} = F'_\top$, and such that v has rank one if $v_\perp$ has rank one. In particular $\bar{K} \supseteq F'_i$ for every $i \in I$.

Indeed, write $F_\top = F_\perp(X)(\bar{a})$, where X is a set of elements transcendental over $F_\perp$ and $\bar{a}$ is a sequence of elements algebraic over $F_\perp(X)$. Take Y a set of

indeterminates with the same cardinality as X and consider the Gauss extension w of $v_\perp$ to $E_\perp(Y)$. Then $\overline{E_\perp(Y)} \cong F_\perp(X)$. Note that w has rank one if $v_\perp$ has rank one. Using now for instance [Endler 1963, Satz 1], we find an (algebraic) extension (K, v) of $(E_\perp(Y), w)$ such that $\overline{K}$ and $F_\top$ are isomorphic via a map which we denote by $h : \overline{K} \to F_\top$ (and with v of rank one if w has rank one). Define $F_i' := h^{-1}[F_i]$. This justifies the claim in the first paragraph of the proof.

To keep notation simple, we assume $\mathscr{F} = \mathscr{F}'$ as above. We construct the valued fields (E_i, v_i) (for $i \neq \perp$) as subfields of K endowed with the restriction of the valuation v. Since the valuation will always be v, we only look for the subfields E_i. Let $\perp$ be the minimum of I. We find the fields E_i by induction on $d(\perp, i)$ (note that $d(\perp, \top) = \max_{j\in I} d(\perp, j)$).

For $l \in \{0, \dots, d(\perp, \top)\}$, let $D_l = \{i \in I \mid d(\perp, i) = l\}$.

If $d(\perp, i) = 0$, then, by hypothesis and by the claim above, we already have the subvalued field $(E_\perp, v_\perp) \subseteq (K, v)$. Note that since $D_0 = \{\perp\}$ the sequence of fields $(F_i)_{i\in D_0}$ is a good residue of $(E_i)_{i\in D_0}$ in (K, v).

Assume we have found a system of fields $(E_i)_{i\in I, d(\perp,i)\leq l}$ such that $\mathrm{res}(E_i, v \restriction E_i) = F_i$ for $i \in I$ so that $d(\perp, i) \leq l$ and $(F_i)_{i\in D_l}$ is a good residue of $(E_i)_{i\in D_l}$ in (K, v). We write $D_{l+1} = \{i_k \mid k < m\}$, then we apply Lemma 2.9 with $(F_k')_{k<m} = (F_{i_k})_{k<m}$, and obtain in this way a sequence $(E_k')_{k<m}$. We define the fields E_i for $i \in D_{l+1}$ by $(E_{i_k})_{k<m} = (E_k')_{k<m}$.

Finally, we can replace $(E_\top, v_\top)$ by one of it henselian closures, and each (E_i, v_i) by its henselian closure inside $(E_\top, v_\top)$. The new residue system is isomorphic to the previously defined residue system, which shows that we can assume that all (E_i, v_i) are henselian. □

3. Main results

Our main result, Corollary 3.3, is a direct consequence of the next two theorems, whose proofs are given in Sections 4 and 5 respectively.

Theorem 3.1. *Let $\mathscr{K} := (K_i, f_{ij})_{i\leq j\in I}$ be a projective system of fields (respectively formally real Pythagorean fields) such that $G(K_i)$ is finite for every $i \in I$. Let $(G_i, \lambda_{ij})_{i\leq j\in I} = G(\mathscr{K})$ and let G be the projective limit of this projective system of finite special groups. Then G is isomorphic to the special group of some field (respectively formally real Pythagorean field).*

Theorem 3.2. *Let $\mathscr{G} := (G_i, \lambda_{ij})_{i\leq j\in I}$ be an adequate projective system of finite reduced special groups* (see Definition 2.4). *Then there is a projective system $\mathscr{K}$ of formally real Pythagorean fields whose morphisms are inclusions, such that $\mathscr{G} \cong G(\mathscr{K})$.*

Now consider a profinite reduced special group G. Say it is the projective limit of the system $\mathscr{G} = (G_i, f_{ij})_{i\leq j\in I}$ of finite reduced special groups. Let i' be any

element in I and consider the system $\mathscr{G}'$ equal to $\mathscr{G}$ restricted to indices in $I' := \{i \in I \mid i \leq i'\}$. The special group G is the projective limit of the system $\mathscr{G}'$, whose index set I' has a maximum element $\top = i'$. We can now use the strategy outlined after Definition 2.2 to express G as an adequate projective system whose index set is I'. Applying Theorem 3.2 then Theorem 3.1 now yields:

Corollary 3.3. *Every profinite reduced special group is isomorphic to the special group of some formally real Pythagorean field.*

4. Proof of Theorem 3.1

If $(I, \leq)$ is a downward directed poset and $i \in I$, then $i^{\leftarrow}$ denotes $\{j \in I \mid j \leq i\}$ and $i^{\rightarrow}$ denotes $\{j \in I \mid j \geq i\}$.

We first assume the following reductions:

(1) I has a maximum $\top$ ($I = \top^{\leftarrow}$).

(2) All the K_i, $i \in I$, are subfields of the field $M := K_\top$, and the morphisms $f_{ij} : K_i \to K_j$ are inclusions. In particular, the projective limit of the system $\mathscr{K}$ is isomorphic to the intersection of the fields K_i, $i \in I$.

These assumptions can safely be made because for the original projective system of fields $\mathscr{K} := (K_j, f_{jk})_{j \leq k \in I}$ and for each $i' \in I$ fixed,

(i) the set $i'^{\leftarrow}$ is a coinitial subset of I, and

(ii) if $j \leq i' \in I$, we can identify K_j with the subfield $K'_j := f_{ji'}[K_j]$ of $K_{i'}$, and the morphisms $f_{jk} : K_j \to K_k$ are naturally identified with inclusions $\iota_{jk} : K'_j \hookrightarrow K'_k$.

The reductions above give us

$$\varprojlim (K_j, f_{jk})_{j \leq k \in I} \cong \varprojlim (K_j, f_{jk})_{j \leq k \in i'^{\leftarrow}} \cong \varprojlim (K'_j, \iota_{jk})_{j \leq k \in i'^{\leftarrow}} \cong \bigcap_{j \in i'^{\leftarrow}} f_{ji'}[K_j] \subseteq K_{i'}$$

and

$$G := \varprojlim (G(K_j), G(f_{jk}))_{j \leq k \in I} \cong \varprojlim (G(K_j), G(f_{jk}))_{j \leq k \in i'^{\leftarrow}} \cong \varprojlim (G(K'_j), G(\iota_{jk}))_{j \leq k \in i'^{\leftarrow}}.$$

Now consider the language $L = L_R \cup \{R_i\}_{i \in I}$, where L_R is the language of rings and the R_i are unary relation symbols. We turn M into an L-structure by interpreting each R_i in M by the subfield K_i.

Let N be an $|I|^+$-saturated elementary extension of M in the language L. (See [Chang and Keisler 1990, Chapter 5 and Lemma 5.1.2] or [Hodges 1993, p. 480 and Corollary 10.2.2] for the definition of saturated models and the existence result we just used. Note that this notion of saturation is not linked to the existing one

for subgroups of special groups.) Each $F_i := R_i^N$ is a subfield of N, and the fields F_i form a projective system of fields $\mathscr{F}$ (since for $k \leq i$, $j \in I$ the sentence "$R_k \subseteq R_i \cap R_j$" is in the theory of M). Moreover, for every $i \in I$, $K_i \hookrightarrow F_i$ is an L_R-elementary embedding and therefore induces an isomorphism of special groups

$$G(K_i) \xrightarrow{\cong} G(F_i)$$

(since the special groups $G(K_i)$, being finite, are described in the theory of M). More generally $G(\mathscr{K}) \cong G(\mathscr{F})$, so $G \cong \varprojlim G(\mathscr{K}) \cong \varprojlim G(\mathscr{F})$.

Let $F := \bigcap_{i\in I} F_i$ and define

$$\begin{aligned} \xi : \ G(F) &\to \varprojlim G(\mathscr{F}) \\ a \cdot \dot{F}^2 &\mapsto (a \cdot \dot{F}_i^{\,2})_{i\in I}. \end{aligned}$$

We show that ξ is an isomorphism of special groups, which yields $G \cong G(F)$ as needed (in particular, if the fields K_i, $i \in I$, are formally real Pythagorean, then $F = \bigcap_{i\in I} F_i$ is formally real Pythagorean, since $G(F)$ is a reduced special group).

Step 1. It is clear that ξ is well-defined and is a morphism of groups.

Step 2. ξ is a morphism of special groups. Indeed, it is clear that ξ sends -1 to -1. Let $a \cdot \dot{F}^2, b \cdot \dot{F}^2 \in G(F)$ be such that $a \cdot \dot{F}^2 \in D_{G(F)}\langle 1, b \cdot \dot{F}^2\rangle$. There are then $c, d \in F$ such that, for all $i \in I$, $a = c^2 + bd^2$ in F_i. Then $a \cdot \dot{F}_i^{\,2} \in D_{G(F_i)}\langle 1, b \cdot \dot{F}_i^{\,2}\rangle$ for every $i \in I$, and therefore $\xi(a \cdot \dot{F}^2) \in D_{G'}\langle 1, \xi(b \cdot \dot{F}^2)\rangle$.

Step 3. ξ is surjective: Let $a = (a_i \cdot \dot{F}_i^2)_{i\in I} \in \varprojlim G(F_i)$. So for all $i \leq j \in I$, $a_i \cdot \dot{F}_j^2 = a_j \cdot \dot{F}_j^2$. We want $x \in N$ satisfying the set of formulas

$$\Delta := \{x \in F_i\}_{i\in I} \cup \{x = a_i \bmod \dot{F}_i^2\}_{i\in I}.$$

Every finite part of Δ is satisfied in N since $a = (a_i \cdot \dot{F}_i^2)_{i\in I} \in \varprojlim G(F_i)$ (it suffices to take $x = a_k$, where k is less than every one of the indices $i \in I$ occurring in this finite part). By $|I|^+$-saturation, Δ has a solution x in N. Then $\xi(x) = (a_i \cdot \dot{F}_i^2)$.

The rest of the proof relies on the following lemma.

Lemma 4.1. *Let $n \in \mathbb{N}$ and let $P(X_1, \ldots, X_n) \in F[X_1, \ldots, X_n]$. Assume the equation $P(X_1, \ldots, X_n) = 0$ has a solution in every F_i, $i \in I$. Then the same equation has a solution in F.*

Proof. We are looking for $\bar{x} \in N$ such that the set of formulas

$$\Sigma := \{P(\bar{x}) = 0\} \cup \{\bar{x} \in F_i\}_{i\in I}$$

is satisfied in N. Since the F_i, together with the inclusions between them, form a projective system, every finite part of Σ has a solution, and by the $|I|^+$-saturation of N, Σ has a solution in N. □

We go back to proving that ξ is an isomorphism:

Step 4. ξ is injective: Let $a = a \cdot \dot{F}^2 \in G(F)$ be such that $\xi(a) = 1$, i.e., $a \in \dot{F}_i^{\,2}$ for every $i \in I$, i.e., the polynomial $X^2 - a$ has a root in each F_i, $i \in I$. By Lemma 4.1, $X^2 - a$ has a root in F, hence $a \in \dot{F}^2$.

Step 5. ξ is a monomorphism of special groups: Let $a, b \in F$ be such that, for every $i \in I$, $a \cdot \dot{F}_i^2 \in D\langle 1, b \cdot \dot{F}_i^2\rangle$. Let $P(X, Y) = a - (X^2 + bY^2) \in F[X, Y]$. By hypothesis, $P(X, Y) = 0$ has a solution in each F_i, hence a solution in F by Lemma 4.1, which means $a \in D_{G(F)}\langle 1, b\rangle$.

5. Proof of Theorem 3.2

Reducing to a finite projective system. Since $\mathcal{G}$ is adequate, the set $(I, \leq)$ has a maximum element $\top$.

In this subsection we show that it is enough to prove Theorem 3.2 when $\mathcal{G} := (G_i, f_{ij})_{i \leq j \in I}$ is a finite projective system of special groups such that I has a maximum (which we will also denote by $\top$).

Let L be the language $\{0, 1, -, +, \cdot\} \cup \{F_i \mid i \in I\} \cup \{Q_i^g \mid i \in I, g \in G_i\}$, where $0, 1$ are constant symbols, $-$ is a unary function symbol, $+, \cdot$ are binary function symbols and F_i, Q_i^g are unary predicate symbols, for each $i \in I$ and $g \in G_i$. Denote by λ_i the *inverse* of the bijection $g \in G_i \mapsto Q_i^g$, $i \in I$. The projective system of fields we are looking for is a model of the theory Ω consisting of (first-order) L-sentences that are informally described in the four items below:

(1) the interpretation of the unary predicate $F_\top$ is the universe of the L-structure (i.e., $\forall x(F_\top(x))$) and "$(F_\top, 0, 1, +, \cdot)$ is a field";

(2) for every $i \leq j \in I$:
"$F_i \subseteq F_j$" and "$(F_i, 0, 1, +, \cdot)$ is a subfield of the field $(F_\top, 0, 1, +, \cdot)$" (technically speaking, $+$ and $\cdot$ are functional symbols globally defined whose restrictions to F_i give internal operations on F_i);

(3) for every $i \in I$:
"λ_i is an isomorphism of special groups $G(F_i) \to G_i$";

(4) for every $i \leq j \in I$:
"the morphism of special groups induced by the inclusion $F_i \subseteq F_j$ is naturally identified with f_{ij}, via the isomorphisms λ_i, λ_j".

It is clear how to describe the expressions in items (1) and (2) by first-order L-sentences. For the reader's convenience, we add a more explicit description of the L-sentences involved in the two remaining items: the hypothesis that the special groups G_i are all finite ensures that the prescription in item (3) can be encoded by a set of first-order L-sentences.

Item (3): for each $i \in I$:

- for each $g \in G_i$, "$Q_i^g \subseteq \dot{F}_i$" and "$Q_i^g = a\dot{F}_i^2$, for some $a \in \dot{F}_i$";
- for each $g, g' \in G_i$ such that $g \neq g'$, "$Q_i^g \cap Q_i^{g'} = \varnothing$";
- "$\dot{F}_i = \bigcup\{Q_i^g \mid g \in G_i\}$" (as G_i is a finite special group, this can be described by a first-order L-sentence);
- "$1 \in Q_i^1$ and $-1 \in Q_i^{-1}$";
- for each $g, g' \in G_i$, "for each a, a', if $a \in Q_i^g$ and $a' \in Q_i^{g'}$ then $a \cdot a' \in Q_i^{gg'}$";
- for each $g, g' \in G_i$ such that $g' \in D_{G_i}(1, g)$, "for each a, a', if $a \in Q_i^g$ and $a' \in Q_i^{g'}$ then there are $x, y \in F_i$ such that $a' = x^2 + ay^2$";
- for each $g, g' \in G_i$ such that $g' \notin D_{G_i}(1, g)$, "for each a, a', if $a \in Q_i^g$ and $a' \in Q_i^{g'}$ then for all $x, y \in F_i$, $a' \neq x^2 + ay^2$".

Item (4): for each $i \leq j \in I$:

By the axioms above: since $F_i \subseteq F_j$ and we have the partitions

$$\dot{F}_i/\dot{F}_i^2 = \{Q_i^g \mid g \in G_i\} \quad \text{and} \quad \dot{F}_j/\dot{F}_j^2 = \{Q_j^{g'} \mid g' \in G_j\},$$

then for each $g \in G_i$ there is a unique $g' \in G_j$ such that $Q_i^g \subseteq Q_j^{g'}$. In this way we obtain a function $q_{ij} : \dot{F}_i/\dot{F}_i^2 \to \dot{F}_j/\dot{F}_j^2$. Clearly $q_{ij}(a.\dot{F}_i^2) = a.\dot{F}_j^2$, for every $a \in \dot{F}_i$, i.e., q_{ij} is the special group morphism induced by the inclusion $F_i \subseteq F_j$. We add a new list of axioms expressing that $\lambda_j \circ q_{ij} = f_{ij} \circ \lambda_i$. A direct examination of the equivalent condition $q_{ij} = \lambda_j^{-1} \circ f_{ij} \circ \lambda_i$ shows that these axioms must be

$$\text{for each } g \in G_i, \text{ "} Q_i^g \subseteq Q_j^{f_{ij}(g)} \text{"}.$$

Using now the compactness theorem (see [Chang and Keisler 1990, Theorem 1.3.22] or [Hodges 1993, Theorem 6.1.1]), to find a model of this theory we only need to find a model of every finite part $\Omega_0 \subseteq \Omega$. Let J be the set of elements of I occurring in this finite part Ω_0, together with $\top$. Since I is downward directed, we can assume that J is also downward directed (taking a larger set J if necessary), that is J has a first element $\bot$. In particular J determines a finite projective system of special groups whose index set that has a maximum and a minimum.

Description of the proof by induction. We therefore assume from now on that the index poset $(I, \leq)$ is finite and that it has a minimum $\bot$ and a maximum $\top$. We find a finite projective system $\mathcal{H}$ of Pythagorean fields of characteristic 0 such that $\mathcal{G} \cong G(\mathcal{H})$ by induction on the construction of $G_\bot$ by products and extensions. For the purpose of the proof, we allow the (nonreduced) special group $\{1\}$ to appear in $\mathcal{G}$.

Recall that since $\mathcal{G}$ is an adequate projective system, the morphisms $f_{ij}, i \leq j \in I$, are quotients by saturated subgroups (see the paragraph after Definition 2.2).

If $G_\perp \cong \{1\}$, then all special groups in the system are trivial and all morphisms are isomorphisms. We can obviously realize such a system by taking $F_i = A$, $i \in I$, where A is any fixed algebraically closed field of characteristic 0.

If $G_\perp \cong \mathbb{Z}_2$, then all special groups in the system are isomorphic to $\mathbb{Z}_2$ or to $\{1\}$ and all morphisms are isomorphisms or naturally identified with $\mathbb{Z}_2 \twoheadrightarrow \{1\}$. We can obviously realize such a system by simply selecting a real closed field R and an algebraically closed field A such that $R \subseteq A$.

If $G_\perp \cong G'_\perp \times G''_\perp$, since all morphisms and special groups in the systems are quotients of $G_\perp$ by (larger and larger) saturated subgroups, and using Fact 2.5, the whole projective system $(G_i, f_{ij})_{i\le j\in I}$ splits according to the product $G_\perp \cong G'_\perp \times G''_\perp$ into two adequate projective systems of finite special groups:

$$(G'_i, f'_{ij})_{i\le j\in I} \quad \text{and} \quad (G''_i, f''_{ij})_{i\le j\in I}. \tag{5-1}$$

(Note that, for each $i \in I$, if G_i is reduced, then either both G'_i and G''_i are reduced or one of them is the trivial special group $\{1\}$ and the other is reduced.) By induction the systems in (5-1) are realized by two projective systems of Pythagorean fields of characteristic 0: $\mathscr{F}' = (F'_i)_{i\in I}$ and $\mathscr{F}'' = (F''_i)_{i\in I}$ (where the morphisms are inclusions), so we just need to "glue" them together. For this we use results from [Kula 1979], which describe how to realize a finite product of finite special groups when each one is already realized. This is achieved in the next subsection.

If $G_\perp \cong G'[H]$, as above, the morphisms of special groups in the projective system are quotients of $G_\perp$ by (larger and larger) saturated subgroups Δ_i. This case is dealt with starting on page 279, using results from [Becher 2002].

Gluing, the product case. The next several pages are taken by the proof of the following result.

Theorem 5.1. *Let $(I, \le)$ be a finite downward directed index set with first element $\perp$ and last element $\top$. Let $\mathscr{F}' = (F'_i)_{i\in I}$, $\mathscr{F}'' = (F''_i)_{i\in I}$ be finite projective systems of fields of characteristic* 0, *where the morphisms are inclusions and such that for every $i \in I$ $G(F'_i)$ and $G(F''_i)$ are finite special groups. Then there is a finite projective system $\mathscr{F} = (F_i)_{i\in I}$ of fields of characteristic* 0 (*where the morphisms are inclusions*) *such that*

$$G(\mathscr{F}) \cong G(\mathscr{F}') \times G(\mathscr{F}'').$$

Remark 5.2. In this theorem, for each $i \in I$ we have:

(a) F_i is Pythagorean if and only if F'_i *and* F''_i are Pythagorean.

(b) If F_i is Pythagorean, then F_i is formally real if and only if F'_i *or* F''_i is formally real.

We begin with a reformulation of some results from [Kula 1979].

Definition 5.3. Let F be a field equipped with n mutually independent valuations of rank one $v_1, \dots, v_n$, and let f_i be an embedding of F into F^{v_i}, a completion of F with respect to v_i. We say that $(F, f_1, \dots, f_n)$ fulfills the global squares property if, for every $a \in F$,

$$a \in \dot{F}^2 \iff \forall i \in \{1, \dots, n\}\ f_i(a) \in (\dot{F^{v_i}})^2.$$

(Note that the left to right implication always holds.)

Theorem 5.4 [Kula 1979, Corollary 2.5]. *With notation as in Definition 5.3, assume that $(F, f_1, \dots, f_n)$ fulfills the global squares property. Then the map*

$$\begin{aligned} \xi_F : G(F) &\to \textstyle\prod_{i=1}^n G(F^{v_i}) \\ a\dot{F}^2 &\mapsto (f_i(a) \cdot (\dot{F^{v_i}})^2)_{i=1,\dots,n} \end{aligned}$$

is an isomorphism of special groups.

Theorem 5.5 [Kula 1979, Theorem 2.6]. *Let $(L_i, v_{i1}, \dots, v_{in})_{i \in I}$ be a finite projective system of fields equipped with n mutually independent valuations of rank one, and such that I has a maximum element $\top$. Then for every $i \in I$ there is an algebraic extension $\mathscr{C}(L_i)$ of L_i and a morphism of special groups*

$$\eta_i : G(\mathscr{C}(L_i)) \to \prod_{k=1}^n G((L_i)^{v_{ik}})$$

such that

(1) *$\mathscr{C}(L_i) \subseteq \mathscr{C}(L_j)$ for every $j \in I$, $j \geq i$;*

(2) *$G(\mathscr{C}(L_i)) \underset{\eta_i}{\cong} \prod_{k=1}^n G((L_i)^{v_{ik}})$; and*

(3) *the morphism of special groups $\prod_{k=1}^n G((L_i)^{v_{ik}}) \to \prod_{k=1}^n G((L_j)^{v_{jk}})$, given by the product of the morphisms of special groups induced by $(L_i)^{v_{ik}} \subseteq (L_j)^{v_{jk}}$ is naturally identified, via the isomorphisms η_i and η_j, with the morphism of special groups $G(\mathscr{C}(L_i)) \to G(\mathscr{C}(L_j))$ induced by $\mathscr{C}(L_i) \subseteq \mathscr{C}(L_j)$.*

Proof. Since a valuation v_{ik} is the restriction on L_i of the valuation $v_{\top k}$, we drop the first index and simply denote it by v_k. For $k \in \{1, \dots, n\}$ we fix a completion $L_\top^k$ of $L_\top$ with respect to v_k and define, for $i \in I$, L_i^k to be the completion of L_i in $L_\top^k$ with respect to v_k. The systems $(L_i^k)_{i \in I}$, for $k \in \{1, \dots, n\}$, are all projective systems of fields, where the morphisms are the inclusions (since L_i^k is simply the set of limits in $L_\top^k$ of v_k-Cauchy sequences of elements of L_i).

Let K_+ be an algebraic closure of $L_\top$. We define the set

$$\begin{aligned} \mathscr{L} := \big\{&\text{projective systems of fields } (E_i, \iota_{i1}, \dots, \iota_{in})_{i \in I} \\ &\text{such that } L_i \subseteq E_i \subseteq K_+ \text{ with } E_i \mid L_i \text{ algebraic,} \\ &\text{equipped with the } L_i\text{-embeddings of fields } \iota_{ik} : E_i \to L_i^k \text{ for } k = 1, \dots, n\big\}. \end{aligned}$$

(Note that the condition that $(E_i, \iota_{i1}, \dots, \iota_{in})_{i\in I}$ is a projective system implies $\iota_{ik} \subseteq \iota_{jk}$ for $i \leq j \in I$ and $k \in \{1, \dots, n\}$, which is possible since $L_i^k \subseteq L_j^k$.) We equip $\mathcal{L}$ with the partial ordering

$$(E_i, \iota_{i1}, \dots, \iota_{in})_{i\in I} \leq (F_i, \kappa_{i1}, \dots, \kappa_{in})_{i\in I}$$
$$\text{if and only if}$$
$$\text{for every } i \in I \text{ and } k \in \{1, \dots, n\},\ E_i \subseteq F_i \text{ and } \iota_{ik} \subseteq \kappa_{ik}.$$

By Zorn's lemma, $\mathcal{L}$ has a maximal element $(M_i, f_{i1}, \dots, f_{in})_{i\in I}$. We show that, for $j \in I$, $(M_j, f_{j1}, \dots, f_{jn})$ fulfills the global squares property. Let $j \in I$ and let $a \in M_j \setminus \{0\}$ be such that $f_{jk}(a) \in (L_j^k)^{\times 2}$, for $k = 1, \dots, n$. Assume $\sqrt{a} \notin M_j$. Fix a square root $\sqrt{a}$ of a and $\alpha_k \in L_j^k$ such that $\alpha_k^2 = f_{jk}(a)$. Then each morphism f_{jk} can be (properly) extended to $M_j' := M_j(\sqrt{a})$ by sending $\sqrt{a}$ to α_k. Moreover, with $A_j := \{r \in I \mid r \geq j\}$, and since for $r \in A_j$ we have $L_j^k \subseteq L_r^k$, the same reasoning tells us that, for each $r \in A_j$ and $k \in \{1, \dots, n\}$, each morphism f_{rk} can be extended to $M_r' := M_r(\sqrt{a})$ by sending $\sqrt{a}$ to α_k (since $\alpha_k \in L_r^k$). If $r \in I \setminus A_j$, we take $M_r' := M_r$. We obtain in this way $(M_i', f_i', \dots, f_n')_{i\in I}$, a projective system of fields equipped with n morphisms of fields that is (strictly) larger than $(M_i, f_i, \dots, f_n)_{i\in I}$, a contradiction. It follows that $\sqrt{a} \in M_j$ and thus that $(M_j, f_{j1}, \dots, f_{jn})$, for $j \in I$, fulfills the global squares property. If we take $\mathscr{C}(L_i) = M_i$ for $i \in I$, the first conclusion of the theorem then holds, and the second follows by Theorem 5.4, with

$$\begin{aligned} \eta_i : G(M_i) &\xrightarrow{\cong} \prod_{k=1}^{n} G((L_i)^k) \\ a \cdot \dot{M}_i^2 &\mapsto (f_i(a) \cdot (L_i^k)^{\times 2})_{i=1,\dots,n}, \end{aligned}$$

for $i \in I$. The third conclusion is proved in the next lemma. □

Lemma 5.6. *Let the notation be as in Theorem 5.5 and its proof.*

Let $(L, v_1, \dots, v_n) \supseteq (K, v_1 \restriction K, \dots, v_n \restriction K)$ be two fields equipped with n mutually independent valuations of rank one. For $m = 1, \dots, n$ let

- *L^m be a completion of L with respect to v_m and K^m be a completion of K with respect to $v_m \restriction K$ such that $K^m \subseteq L^m$,*
- *f_m be an embedding of K into K^m and g_m be an embedding of L into L^m extending f_m.*

Assume $(K, f_1, \dots, f_n)$ and $(L, g_1, \dots, g_n)$ satisfy the global squares property.

Let $\lambda : \prod_{m=1}^n G(K^m) \to \prod_{m=1}^n G(L^m)$ be the product of the morphisms of special groups induced by the inclusions $K^m \subseteq L^m$ for $m = 1, \dots, n$, and let $\mu : G(K) \to G(L)$ be the morphism of special groups induced by $K \subseteq L$.

Then λ and μ are naturally identified via the isomorphisms ξ_K and ξ_L given by Theorem 5.4.

Proof. By Theorem 5.4 the isomorphism $G(K) \cong G(K^1) \times \cdots \times G(K^n)$ is

$$\begin{aligned} \xi_K : G(K) &\to G(K^1) \times \cdots \times G(K^n) \\ x \cdot \dot{K}^2 &\mapsto (f_1(x) \cdot (\dot{K^1})^2, \ldots, f_n(x) \cdot (\dot{K^n})^2). \end{aligned}$$

Similarly, the isomorphism between $G(L)$ and $G(L^1) \times \cdots \times G(L^n)$ is

$$\begin{aligned} \xi_L : G(L) &\to G(L^1) \times \cdots \times G(L^n) \\ x \cdot \dot{L}^2 &\mapsto (g_1(x) \cdot (\dot{L^1})^2, \ldots, g_n(x) \cdot (\dot{L^n})^2). \end{aligned}$$

Thus $\lambda = \xi_L \circ \mu \circ \xi_K^{-1}$ since $g_m \restriction K = f_m$ for $m = 1, \ldots, n$. □

We now turn our attention to the two finite projective systems of fields $\mathscr{F}' = (F_i')_{i \in I}$ and $\mathscr{F}'' = (F_i'')_{i \in I}$ of characteristic zero introduced in the statement of Theorem 5.1. We first show that we can assume that the fields in $\mathscr{F}'$ and $\mathscr{F}''$ are at most countable and of finite transcendence degree over $\mathbb{Q}$. This is achieved by the following proposition.

Proposition 5.7 [Kula 1979, Proposition 3.1]. *Let $\mathscr{L} := (L_i)_{i \in I}$ be a finite projective system of fields of characteristic 0 such that $G(L_i)$ is a finite special group for all $i \in I$. There is a map $\mathscr{F}$, defined on $\{L_i\}_{i \in I}$, satisfying the following properties whenever $i \leq j \in I$:*

(1) *$\mathscr{F}(L_i)$ is a countable subfield of L_i with finite transcendence degree over $\mathbb{Q}$.*

(2) *If $\varphi_i : \mathscr{F}(L_i) \hookrightarrow L_i$ is the inclusion map, then $G(\varphi_i) : G(\mathscr{F}(L_i)) \to G(L_i)$ is an isomorphism of special groups.*

(3) *$\mathscr{F}(L_i) \subseteq \mathscr{F}(L_j)$.*

(4) *If $\lambda_{ij} : G(L_i) \to G(L_j)$ is the morphism of special groups induced by $L_i \subseteq L_j$, then the morphism of special groups $G(\mathscr{F}(L_i)) \to G(\mathscr{F}(L_j))$ induced by $\mathscr{F}(L_i) \subseteq \mathscr{F}(L_j)$ is naturally identified with λ_{ij}, via the isomorphisms $G(\varphi_i)$ and $G(\varphi_j)$.*

Proof. The proof is a trivial extension of Kula's. If L is a field with a finite number of square classes, a representative system of $G(L)$ is a finite subset $R(L) = A \cup B$ of L such that

- $A \subseteq \dot{L}$ and $\dot{L}/\dot{L}^2 = A/\dot{L}^2$;
- For every $a_1, a_2 \in A$ with $a_1 \in D_L\langle 1, a_2\rangle$, there are $b_1, b_2 \in B$ such that $a_1 = b_1^2 + a_2 b_2^2$.

Claim: For every $i \in I$ there is a representative system $R(L_i)$ of L_i such that $R(L_i) \subseteq R(L_j)$ whenever $i \leq j$.

Proof of the claim: Direct by induction on $d(\bot, i)$ (just take a system of representatives of L_i and add to it all the $R(L_j)$ for $\bot \leq j < i$).

Then, just as in [Kula 1979], take for $\mathcal{F}(L_i)$ the algebraic closure of $\mathbb{Q}(R(L_i))$ in L_i. □

The following two propositions show that we can assume that $\operatorname{atd}(F_i') = \operatorname{atd}(F_i'')$ for every $i \in I$, where atd denotes the absolute transcendence degree, i.e., the transcendence degree over $\mathbb{Q}$.

Proposition 5.8 ([Kula 1979], Lemma 3.2). *Let $(L_i)_{i\in I}$ be a finite projective system of countable fields of finite absolute transcendence degree. There is a map $\mathcal{T}$ defined on $\{L_i\}_{i\in I}$ satisfying the following properties whenever $i \le j \in I$:*

(1) *$\mathcal{T}(L_i)$ is a countable field extension of L_i.*

(2) $\operatorname{atd}(\mathcal{T}(L_i)) = \operatorname{atd}(L_i) + 1$.

(3) *If $\tau_i : L_i \hookrightarrow \mathcal{T}(L_i)$ is the inclusion map, then $G(\tau_i) : G(L_i) \to G(\mathcal{T}(L_i))$ is an isomorphism of special groups.*

(4) $\mathcal{T}(L_i) \subseteq \mathcal{T}(L_j)$.

(5) *If $\lambda_{ij} : G(L_i) \to G(L_j)$ is the morphism of special groups induced by $L_i \subseteq L_j$, then the morphism of special groups $G(\mathcal{T}(L_i)) \to G(\mathcal{T}(L_j))$ induced by $\mathcal{T}(L_i) \subseteq \mathcal{T}(L_j)$ is naturally identified with λ_{ij} (via $G(\tau_i)$ and $G(\tau_j)$).*

Proof. For $i \in I$ let $K_i := L_i(x)(\sqrt[2^n]{x})_{n\in\mathbb{N}}$ (x is an indeterminate), and consider on K_i the unique extension v_i of the valuation on $L_i(x)$ determined by the irreducible polynomial x. The K_i, together with their inclusions, form a projective system, and the sets $\Phi_i := \{v_i\}$ satisfy the hypothesis of Theorem 5.5. We now apply the map $\mathcal{E}$ defined in Theorem 5.5 to the projective system of the K_i and get the projective system of the $\mathcal{T}(L_i)$. Since L_i is countable, K_i and $\mathcal{T}(L_i) = \mathcal{E}(K_i)$ are countable. Kula's proof of [Kula 1979, lemma 3.2] shows that the second and third claims of the proposition hold, and the last two hold by Theorem 5.5. □

Proposition 5.9. *There exist finite projective systems $\mathcal{K}' = (K_i')_{i\in I}$ and $\mathcal{K}'' = (K_i'')_{i\in I}$ of fields of characteristic 0 such that*

(1) *$G(\mathcal{K}') \cong G(\mathcal{F}')$ and $G(\mathcal{K}'') \cong G(\mathcal{F}'')$, and*

(2) *for every $i \in I$, $\operatorname{atd}(K_i') = \operatorname{atd}(K_i'') < \infty$.*

Proof. We assume there is some $i \in I$ such that $\operatorname{atd}(F_i') \neq \operatorname{atd}(F_i'')$ and we proceed by induction on $d(\bot, i)$, the maximal length of a chain from $\bot$ to i.

- $d(\bot, i) = 0$, i.e., $i = \bot$. Let $t := \max\{\operatorname{atd}(F_\bot'), \operatorname{atd}(F_\bot'')\}$. We then apply Proposition 5.8 as many times as necessary to the system $\mathcal{F}'$ or $\mathcal{F}''$ (the one that does no realize the maximum), and we obtain two new systems $\mathcal{F}'_{(0)}$ and $\mathcal{F}''_{(0)}$ indexed by I, whose fields of index $\bot$ have same (finite) absolute transcendence degree t.

- $d(\perp, i) = n > 0$. We now proceed by induction on the number of i's with $d(\perp, i) = n$ and $\operatorname{atd}(F'_i) \neq \operatorname{atd}(F''_i)$. We fix one of them: i_1. By induction we can assume that the projective systems $\mathscr{F}'$ and $\mathscr{F}''$ satisfy $\operatorname{atd}(F'_j) = \operatorname{atd}(F''_j)$ for every $j \in I$, $d(\perp, j) < n$. We consider the systems $\mathscr{F}' \restriction i_1^{\rightarrow}$ and $\mathscr{F}'' \restriction i_1^{\rightarrow}$. By applying Proposition 5.8, we get two new systems $\mathscr{T}'$ and $\mathscr{T}''$, indexed by $i_1^{\rightarrow}$ whose fields indexed by i_1 have same absolute transcendence degree. We replace, in $\mathscr{F}'$, respectively $\mathscr{F}''$, the subsystem $\mathscr{F}' \restriction i_1^{\rightarrow}$ by $\mathscr{T}'$, respectively $\mathscr{F}'' \restriction i_1^{\rightarrow}$ by $\mathscr{T}''$ and we write $\mathscr{F}'_{(1)}$, $\mathscr{F}''_{(1)}$ for the new sets of fields. Since every field has been replaced by a field extension, we still get projective systems of fields and, moreover, $G(\mathscr{F}'_{(1)}) \cong G(\mathscr{F}')$ and $G(\mathscr{F}''_{(1)}) \cong G(\mathscr{F}'')$. Now $\operatorname{atd}(F'_{(1)i_1}) = \operatorname{atd}(F''_{(1)i_1}) < \infty$, and we proceed by induction. □

So, from now on, we assume that our two finite projective systems of fields $\mathscr{F}'$ and $\mathscr{F}''$ consist of countable fields having the same finite transcendence degree over $\mathbb{Q}$ at each index.

Remark 5.10. Let K be a field equipped with two independent valuations v_1 and v_2 and let (L, w_1, w_2) be an extension of (K, v_1, v_2). Then w_1 and w_2 are independent. Indeed, if it were not the case, then w_1 and w_2 would define the same topology on L (see [Engler and Prestel 2005, Theorem 2.3.4]), and therefore the same induced topologies on K, which coincide with the topologies defined by v_1 and v_2. It shows that v_1 and v_2 define the same topology on K, a contradiction since they are independent (again by the theorem just cited).

Lemma 5.11. *There are two henselian valued fields $(E'_\perp, v')$ and $(E''_\perp, v'')$ both containing $\mathbb{Q}(X)$, such that*

(1) *v' and v'' are of rank one,*

(2) *$E'_\perp v' \cong F'_\perp$ and $E''_\perp v'' \cong F''_\perp$,*

(3) *$\operatorname{atd} E'_\perp = \operatorname{atd} F'_\perp + 1 = \operatorname{atd} F''_\perp + 1 = \operatorname{atd} E''_\perp$,*

(4) *$v'E'_\perp$ and $v''E''_\perp$ are divisible, and*

(5) *the restrictions of v' and v'' to $\mathbb{Q}(X)$ are independent.*

(In (4), two-divisible is actually enough for our purposes.)

Proof. Let $\{y_1, \ldots, y_k\}$ be a finite transcendence basis of $F'_\perp$ over $\mathbb{Q}$, and let E be $\mathbb{Q}(y_1, \ldots, y_k)(X)$, equipped with the valuation v determined by the irreducible polynomial $X \in \mathbb{Q}(y_1, \ldots, y_k)[X]$. Then $\bar{E} \cong \mathbb{Q}(y_1, \ldots, y_k)$, $vE = \mathbb{Z}$ and $F'_\perp$ is isomorphic to an algebraic extension of $\bar{E}$. By [Endler 1963, Satz 1], there is an algebraic extension $E'_\perp$ of E and an extension v' of v to $E'_\perp$ such that $E'_\perp = F'_\perp$ and $v'E'_\perp$ is divisible of rank one.

To construct $(E''_\perp, v'')$, we proceed as above but start with the valuation on $\mathbb{Q}(y_1, \dots, y_k)(X)$ associated to the irreducible polynomial $X - 1$. Obviously, $v'' \restriction \mathbb{Q}(X)$ and $v' \restriction \mathbb{Q}(X)$ are independent over $\mathbb{Q}(X)$. □

We now apply Corollary 2.10 twice (with the valued fields $(E'_\perp, v')$ and $(E''_\perp, v'')$ given by Lemma 5.11), and get two projective systems of henselian valued fields $\mathscr{E}' = (E'_i, v'_i)_{i\in I}$ and $\mathscr{E}'' = (E''_i, v''_i)_{i\in I}$ equipped with valuations of rank one, such that $\mathrm{res}(\mathscr{E}') \cong \mathscr{F}'$ and $\mathrm{res}(\mathscr{E}'') \cong \mathscr{F}''$. Up to renaming the transcendental elements, we can assume that for every $i \in I$ there is a finite set X_i of transcendental elements over $\mathbb{Q}$ and an algebraic closure Q_i of $\mathbb{Q}(X_i)$ such that $E'_i, E''_i \subseteq Q_i$, and such that, for every $i \leq j \in I$ $X_i \subseteq X_j$ and $Q_i \subseteq Q_j$.

Since, for $i \in I$, E'_i and E''_i are both subfields of Q_i, we can consider the projective system of valued fields $(E'_i \cap E''_i, v'_i, v''_i)_{i\in I}$. Note that v'_i and v''_i are independent by Remark 5.10 and Lemma 5.11(5). We recall now the following special case of a result from [Heinemann 1985]:

Theorem 5.12. *Let K be a field equipped with two independent valuations v_1 and v_2. Fix an algebraic closure $\tilde{K}$ of K. Let (H_i, v_i), for $i = 1, 2$, be henselian extensions of (K, v_i) such that $H_1, H_2 \subseteq \tilde{K}$ and $K = H_1 \cap H_2$.*

Then (H_i, v_i) is a henselization of (K, v_i), for $i = 1, 2$.

Applying this result, we obtain that, for every $i \in I$, (E'_i, v'_i) is a henselization of $(E'_i \cap E''_i, v'_i)$ and (E''_i, v''_i) is a henselization of $(E'_i \cap E''_i, v''_i)$. In particular:

(1) $v'(E'_i \cap E''_i)$ and $v'(E'_i \cap E''_i)$ are two-divisible;

(2) $\mathrm{res}(E'_i \cap E''_i, v'_i)_{i\in I} \cong \mathscr{F}'$ and $\mathrm{res}(E'_i \cap E''_i, v''_i)_{i\in I} \cong \mathscr{F}''$;

(3) v' and v'' are independent on $E'_i \cap E''_i$ (by Lemma 5.11.(5) and Remark 5.10).

We now apply Theorem 5.5 to the system $(E'_i \cap E''_i, v'_i, v''_i)_{i\in I}$ and get the system $(\mathscr{C}(E'_i \cap E''_i))_{i\in I}$, which satisfies

$$G((\mathscr{C}(E'_i \cap E''_i))_{i\in I}) \cong (G((E'_i \cap E''_i)^{v'_i}) \times G((E'_i \cap E''_i)^{v''_i}), g'_{ij} \times g''_{ij})_{i\leq j\in I},$$

where g'_{ij}, respectively g''_{ij}, is the map induced by $(E'_i \cap E''_i)^{v'_i} \subseteq (E'_j \cap E''_j)^{v'_j}$, respectively by $(E'_i \cap E''_i)^{v''_i} \subseteq (E'_j \cap E''_j)^{v''_j}$. We claim that this last projective system of (Pythagorean) fields is isomorphic to $(G(F'_i) \times G(F''_i), f'_{ij} \times f''_{ij})_{i\leq j\in I}$. It suffices to check that, for instance, the projective system $(G((E'_i \cap E''_i)^{v'_i}), g'_{ij})_{i\leq j\in I}$ is isomorphic to $(G(F'_i), f'_{ij})_{i\leq j\in I}$. This is the content of the remainder of this section.

Since $((E'_i \cap E''_i)^{v'_i}, v'_i)$ is an immediate extension of $(E'_i \cap E''_i, v'_i)$, we have $\mathrm{res}((E'_i \cap E''_i)^{v'_i}, v'_i)_{i\in I} \cong \mathscr{F}'$, so

$$G(\mathrm{res}((E'_i \cap E''_i)^{v'_i}, v'_i)_{i\in I}) \cong G(\mathscr{F}') = (G(F'_i), f'_{ij})_{i\leq j\in I}$$

and it suffices to show that $(G((E_i'\cap E_i'')^{v_i'}, g_{ij}')_{i\le j\in I}$ and $G(\mathrm{res}((E_i'\cap E_i'')^{v_i'}, v_i')_{i\in I}))$ are isomorphic.

We are now in position to conclude by using the following adaptation of the Baer-Krull theorem [Dickmann and Miraglia 2000, Theorem 1.33]. Recall that the functor G is well defined, in general, from the category of *unitary commutative rings* into the category of L_{SG}*-structures*.

Lemma 5.13. *Let (K, v) be a valued field, $i : O_K \to K$ be the inclusion and $q : O_K \to \overline{K}$ be the projection on the quotient $(\overline{K} = O_K/M_K)$. Suppose that $v(2) = 0$.*

(1) *The L_{SG}-structures $G(K)$ and $G(\overline{K})$ are special groups. The induced L_{SG}-morphism $G(i) : G(O_K) \to G(K)$ is injective and the induced L_{SG}-morphism $G(q) : G(O_K) \to G(\overline{K})$ is surjective.*

(2) *If (K, v) is 2-henselian and $vK = 2vK$, then $G(i) : G(O_K) \to G(K)$ and $G(q) : G(O_K) \to G(\overline{K})$ are L_{SG}-isomorphisms. In particular, the L_{SG}-structure $G(O_K)$ is a special group.*

(3) *If $(K', v') \supseteq (K, v)$ is a valued field extension, then $O_K \subseteq O_{K'}$, $M_K \subseteq M_{K'}$ and the diagram of special groups below is commutative (where the vertical arrows are induced by the field extension).*

$$\begin{array}{ccccc} G(\overline{K}) & \xleftarrow{G(q)} & G(O_K) & \xrightarrow{G(i)} & G(K) \\ \downarrow & & \downarrow & & \downarrow \\ G(\overline{K}') & \xleftarrow{G(q')} & G(O_{K'}) & \xrightarrow{G(i')} & G(K') \end{array}$$

Proof. (1) Since $v(2) = 0$, 2 is invertible in the rings K, O_K and $\overline{K}$, and therefore, as K and $\overline{K}$ are fields, the L_{SG}-structures $G(K)$ and $G(\overline{K})$ are special groups ([Dickmann and Miraglia 2000, Theorem 1.32 p.23]). As $q : O_K \to \overline{K}$ is a surjective ring homomorphism, it induces a surjective group homomorphism $\dot{O}_K/\dot{O}_K^2 \to \dot{\overline{K}}/\dot{\overline{K}}^2$ and therefore $G(q) : G(O_K) \to G(\overline{K})$ is a surjective L_{SG}-morphism. Now let $a \in \dot{O}_K$ such that $a.\dot{K}^2 = 1.\dot{K}^2$; i.e.,there is $b \in \dot{K}$ such that $a = b^2$, then $2v(b) = v(a) = 0$ and $b \in \dot{O}_K$; therefore $\ker(G(i)) = \{1.\dot{O}_K^2\}$ and $G(i) : G(O_K) \to G(K)$ is an injective L_{SG}-morphism.

(2) We first prove that $G(q)$ is an L_{SG}-isomorphism.

Let $a \in \dot{O}_K$ such that $q(a).\dot{\overline{K}}^2 = 1.\dot{\overline{K}}^2$ then, as $q : O_K \to \overline{K}$ is a surjective ring homomorphism, there is $b \in \dot{O}_K$ such that $q(a) = q(b^2)$. Consider now the polynomial $P(t) = t^2 - a$ in $O_K[t]$: it is a quadratic monic polynomial such that $q(b) \in \overline{K}$ is a root of $P^q(t) = t^2 - q(a)$ in $\overline{K}$ and this root is simple (since $q(a) \neq 0$ and $\mathrm{char}(\overline{K}) \neq 2$). The hypothesis (K, v) 2-henselian then entails that there is $b' \in O_K$ such that $q(b') = q(b)$ and $P(b') = 0$; i.e., $a = b'^2$, for some $b' \in \dot{O}_K$

(because $a \in \dot{O}_K$). Therefore $\ker(G(q)) = \{1.\dot{O}_K^2\}$ and $G(q)$ is an injective L_{SG}-morphism.

We show that whenever $a \in \dot{O}_K$, we have

$$D_{G(\overline{K})}\langle 1, G(q)(a\dot{O}_K^2)\rangle \subseteq G(q)[D_{G(O_K)}\langle 1, a\dot{O}_K^2\rangle].$$

Since q and $G(q)$ are surjective, it is enough to prove that for any $z \in \dot{O}_K$ such that there are $x, y \in O_K$ with $q(z) = 1.q(x)^2 + q(a).q(y)^2$, there are $z' \in \dot{O}_K$ and $x', y' \in O_K$ with $z' = 1.x'^2 + a.y'^2$ and

$$q(z').\dot{\overline{K}}^2 = q(z).\dot{\overline{K}}^2.$$

We recall that $\dot{O}_K = O_K \setminus M_K$ and we split the proof into four cases:

- $x \in M_K$, $y \in M_K$: it is not possible because $q(z) \neq 0$.
- $x \in \dot{O}_K$, $y \in M_K$: then $q(z) = q(x^2)$ and the quadratic monic polynomial $P(t) = t^2 - z$ over O_K has a root in $\overline{K}$, and this root is simple (because $q(z) \neq 0$ and $\mathrm{char}(\overline{K}) \neq 2$). By the hypothesis that (K, v) is 2-henselian, P has then a root x' in O_K. Taking this x' as well as $z' := z$ and $y' := 0$ proves the result.
- $x \in M_K$, $y \in \dot{O}_K$: then $q(z) = q(ay^2)$ and the polynomial $P(t) = t^2 - a^{-1}z$ has a root in $\overline{K}$ and this root is again simple (because $q(z), q(a) \neq 0$ and $\mathrm{char}(\overline{K}) \neq 2$). Therefore P has a root $y' \in O_K$. Taking this y' together with $z' := z$ and $x' = 0$ proves the result.
- $x \in \dot{O}_K$, $y \in \dot{O}_K$: then $q((x/y)^2 + a - zy^{-2}) = 0$ and the polynomial $P(t) = t^2 + (a - z')$, with $z' := zy^{-2}$, is a quadratic monic polynomial in $O_K[t]$ such that $q(x/y) \in \overline{K}$ is a root of $P^q(t) = t^2 + q(a - z')$ in $\overline{K}$ and we may suppose this root is simple (because, if not, as $\mathrm{char}(\overline{K}) \neq 2$, then $q(a - zy^{-2}) = 0$ and we can proceed as in the case just above). Then the hypothesis (K, v) 2-henselian entails that there is $x' \in O_K$ such that $0 = P(x') = x'^2 + a - z'$ and $q(x') = q(x/y)$, i.e., such that $z' = 1.x'^2 + a.y'^2$, with $y' := 1$. Therefore $G(q)(z.\dot{O}_K^2) = G(q)(z'.\dot{O}_K^2) \in G(q)[D_{G(O_K)}\langle 1, a\dot{O}_K^2\rangle]$.

We now prove that $G(i)$ is an L_{SG}-isomorphism.

As $vK = 2vK$, for any $a \in \dot{K}$ there is $c \in \dot{K}$ such that $v(ac^2) = 0$, i.e., $ac^2 \in \dot{O}_K$. Therefore $G(i)(ac^2.\dot{O}_K^2) = a.\dot{K}^2$ and $G(i)$ is surjective.

To finish the proof, we must check that for each $a \in \dot{O}_K$, we have

$$D_{G(K)}\langle 1, G(i)(a\dot{O}_K^2)\rangle \subseteq G(i)[D_{G(O_K)}\langle 1, a\dot{O}_K^2\rangle].$$

Note that if $a = -b^2$ for some $b \in \dot{O}_K$ then, as $2 \in \dot{O}_K$, we have $D_{G(O_K)}\langle 1, a\dot{O}_K^2\rangle = \dot{O}_K/\dot{O}_K^2$. Since $G(i)$ is a surjective group homomorphism, we have

$$G(i)[D_{G(O_K)}\langle 1, a\dot{O}_K^2\rangle] = \dot{\overline{K}}/\dot{\overline{K}}^2$$

and therefore $D_{G(K)}\langle 1, G(i)(a\dot{O}_K^2)\rangle \subseteq G(i)[D_{G(O_K)}\langle 1, a\dot{O}_K^2\rangle]$. Thus we only have to deal with the case $a \notin -\dot{O}_K^2$ and, again as $G(i)$ is a surjective group homomorphism, it is enough to prove that for any $z \in \dot{O}_K$ such that there are $x, y \in K$ with $z = 1.x^2 + a.y^2$, there are $z' \in \dot{O}_K$ and $x', y' \in O_K$ such that $z' = 1.x'^2 + a.y'^2$ and $z'.\dot{K}^2 = z.\dot{K}^2$.

We split the proof into four cases:

- $x, y \in O_K$. Then we simply take $z' := z$, $x' := x$ and $y' := y$.
- $x \in O_K$ and $y \notin O_K$. Then $y^{-1} \in M_K$ and $x/y \in M_K$. Thus $(x/y)^2 \in M_K$ and $1(x/y)^2 + a = zy^{-2} \in M_K$. This implies $a \in M_K$, a contradiction because $a \in \dot{O}_K = O_K \setminus M_K$.
- $x \notin O_K$ and $y = 0$. Then $z = x^2 \notin O_K$, a contradiction.
- $x \notin O_K$ and $y \neq 0$. Then $z = x^2(1 + a(y/x)^2) \in \dot{O}_K$ and $x^{-1} \in M_K$. As $z \in \dot{O}_K$, this implies $(1 + a(y/x)^2) = zx^{-2} \in M_K$, and thus $-a(y/x)^2 \in 1 + M_K$. As (K, v) is 2-henselian and $\mathrm{char}(\overline{K}) \neq 2$, $1 + M_K \subseteq O_K^2$ and as $y \neq 0$, then $-a \in K^2$. But $-a \in \dot{O}_K$, so $-a \in \dot{O}_K \cap K^2 = \dot{O}_K^2$, contradicting the hypothesis $a \in \dot{O}_K \setminus -\dot{O}_K^2$.

(3) It follows directly from the definition of extension of valued fields that the following diagram of (local) rings and (local) homomorphisms is commutative:

$$\begin{array}{ccccc} \overline{K} & \xleftarrow{i} & O_K & \xrightarrow{q} & K \\ \downarrow & & \downarrow & & \downarrow \\ \overline{K}' & \xleftarrow{i'} & O_{K'} & \xrightarrow{q'} & K' \end{array}$$

The result follows by applying the functor G to it. □

Under the hypotheses of Lemma 5.13, the last item gives us in particular the commutative diagram

$$\begin{array}{ccc} G(\overline{K}) & \xrightarrow{\tau_K} & G(K) \\ \downarrow & & \downarrow \\ G(\overline{K}') & \xrightarrow{\tau_{K'}} & G(K') \end{array} \tag{5-2}$$

where the maps $\tau_K := G(q) \circ G(i)^{-1}$ and $\tau_{K'} := G(q') \circ G(i')^{-1}$ are isomorphisms of special groups whenever (K, v) and (K', v') are 2-henselian with 2-divisible value groups, and the vertical maps are induced by the field inclusions.

Since, for $i \leq j \in I$, we have an extension of valued fields $((E_i' \cap E_i'')^{v_i'}, v_i') \subseteq ((E_j' \cap E_j'')^{v_j'}, v_j')$ and these two fields are 2-henselian with divisible value groups,

we conclude that the diagram

$$\begin{array}{ccc} G(\overline{(E_i' \cap E_i'')^{v_i'}}) & \xrightarrow{\tau_i} & G((E_i' \cap E_i'')^{v_i'}) \\ \downarrow & & \downarrow \\ G(\overline{(E_j' \cap E_j'')^{v_j'}}) & \xrightarrow{\tau_j} & G((E_j' \cap E_j'')^{v_j'}) \end{array}$$

is commutative, where the maps τ_i and τ_j are the isomorphisms corresponding to τ_K and $\tau_{K'}$ in (5-2). This concludes the proof of Theorem 5.1.

Gluing, the extension case. Now assume that $G_\perp \cong G'[H]$, the last case discussed on page 269. Here we have $G_i \cong G'[H]/\Delta_i$ for every $i \in I$, where H is a fixed finite group of exponent 2, Δ_i is a saturated subgroup of $G'[H]$ and $\Delta_\perp = \{1\}$. Furthermore, if $i \leq j \in I$ we have $\Delta_i \subseteq \Delta_j$ and f_{ij} is naturally identified with the canonical projection from $G'[H]/\Delta_i$ onto $G'[H]/\Delta_j$.

In view of this, the following theorem is a reformulation of the last case in the induction step (page 269), and this section is devoted to its proof.

Theorem 5.14. *Let $(I, \leq)$ be a finite downward directed index set with first element $\perp$ and last element $\top$. Let G' be a reduced special group and assume that whenever $\mathcal{G} = (G_i, \eta_{ij})_{i \leq j \in I}$ is a projective system of reduced special groups with $G_\perp = G'$, then $\mathcal{G}$ is realized by a projective system of Pythagorean fields of characteristic zero (where the morphisms are inclusions).*

Let H be a finite group of exponent 2 and let $(\Delta_i)_{i \in I}$ be a projective system of saturated subgroups of $G'[H]$, where the morphisms are inclusions. Let $\mathcal{G}'$ be the projective system indexed by I of the special groups $G'[H]/\Delta_i$, where the morphisms are the canonical projections.

Then $\mathcal{G}'$ is realized by a projective system of Pythagorean fields of characteristic zero (where the morphisms are inclusions).

Notation: If G is a special group and H is a group of exponent 2, we will identify G (respectively H) with the subgroup $G \times \{1\}$ (respectively $\{1\} \times H$) in $G[H] = \{(g, h) \mid g \in G, h \in H\}$ and write $g \cdot h$ for the pair (g, h).

As $H \cong H_1 \times H_2$ entails $G'[H] \cong (G'[H_1])[H_2]$, we may assume $\dim_{\mathbb{F}_2} H = 1$, i.e., $H = \{1, h\}$ with $h^2 = 1$ and $h \neq 1$.

We define, for $i \in I$ and $i \leq j \in I$:

$$\begin{aligned} &\Omega_i := \Delta_i \cap G', \quad G_i'' := G'/\Omega_i \ \text{ (note that } \Omega_i \subseteq \Omega_j) \\ &q_{ij} : G_i'' \to G_j'' \text{ the canonical projection,} \\ &\Theta_i := \{(g \cdot \Omega_i).w \in G_i''[H] \mid g.w \in \Delta_i\}. \end{aligned}$$

The following fact is then easily checked:

Fact 5.15. (1) *Ω_i is a saturated subgroup of G'.*

(2) *Θ_i is a saturated subgroup of $G_i''[H]$ with $G_i'' \cap \Theta_i = \{1\}$.*

(3) *The morphism of special groups $q_{ij} \times \mathrm{Id} : G_i''[H] \to G_j''[H]$ is such that $(q_{ij} \times \mathrm{Id})(\Theta_i) \subseteq \Theta_j$ and $(q_{ij} \times \mathrm{Id}) \restriction \Theta_i : \Theta_i \to \Theta_j$ is injective.*

(4) *The map*

$$\begin{aligned} \omega_i : \ G'[H]/\Delta_i &\to (G'/\Omega_i)[H]/\Theta_i \\ (g \cdot h)/\Delta_i &\mapsto ((g/\Omega_i) \cdot h)/\Theta_i \end{aligned}$$

is an isomorphism of special groups.

(5) *The diagram*

$$\begin{array}{ccc} G'[H]/\Delta_i & \xrightarrow{f_{ij}} & G'[H]/\Delta_j \\ \downarrow{\scriptstyle \omega_i} & & \downarrow{\scriptstyle \omega_j} \\ G_i''[H]/\Theta_i & \xrightarrow{\widetilde{q_{ij} \times \mathrm{Id}}} & G_j''[H]/\Theta_j \end{array}$$

commutes, where $\widetilde{q_{ij} \times \mathrm{Id}}$ is the canonical map induced on the quotients.

Note that by hypothesis, since $G_\perp'' = G'$, the projective system $(G_i'', q_{ij})_{i \le j \in I}$ is realized by a system of Pythagorean fields $(K_i)_{i \in I}$ of characteristic zero.

To complete the proof, it is then enough to represent the projective system of special groups $(G_i''[H]/\Theta_i, \widetilde{q_{ij} \times \mathrm{Id}})_{i \le j \in I}$ by some projective system of Pythagorean fields of characteristic zero; this is the content of the following proposition.

Proposition 5.16. *There is a projective system of Pythagorean fields of characteristic zero $(L_i)_{i \in I}$, where the morphisms are inclusions, such that*

$$(G_i''[H]/\Theta_i, \widetilde{q_{ij} \times \mathrm{Id}})_{i \le j \in I} \cong G((L_i)_{i \in I}).$$

The rest of this section now consists in the proof of Proposition 5.16.

Let us denote by γ_{ij} the morphism of special groups induced by $K_i \subseteq K_j$:

$$(G(K_i), \gamma_{ij})_{i \le j \in I} \cong (G_i'', q_{ij})_{i \le j \in I}.$$

We define $M_i = K_i((t))$ for every $i \in I$ and record a well known result:

Lemma 5.17. *$\dot{M}_i/\dot{M}_i^{\,2} = \{at^k \cdot \dot{M}_i^{\,2} \mid a \in \dot{K}_i, k \in \{0, 1\}\}$, and the isomorphism of special groups from $G(M_i)$ to $G(K_i)[H]$ is*

$$\begin{aligned} \lambda_i : \ G(M_i) &\to G(K_i)[H] \\ at^k \cdot \dot{M}_i^{\,2} &\mapsto (a \cdot \dot{K}_i^{\,2})h^k. \end{aligned}$$

Proof. This is exactly [Dickmann and Miraglia 2000, Theorem 1.33], where the explicit definition of the isomorphism is given at the beginning of the proof on page 28. □

It immediately follows that

$$(G(K_i((t))), \gamma_{ij} \times \mathrm{Id})_{i \le j \in I} \cong (G_i''[H], q_{ij} \times \mathrm{Id})_{i \le j \in I}. \tag{5-3}$$

For each $i \in I$, let Γ_i be the saturated subgroup of $G(K_i((t)))$ that corresponds, by the isomorphisms above, to the saturated subgroup Θ_i of $G_i''[H]$. This then yields

$$(G(K_i((t)))/\Gamma_i, \widetilde{\gamma_{ij} \times \mathrm{Id}})_{i \le j \in I} \cong (G_i''[H]/\Theta_i, \widetilde{q_{ij} \times \mathrm{Id}})_{i \le j \in I}$$

(where $\widetilde{\gamma_{ij} \times \mathrm{Id}}$ denotes the induced map on the quotients), which in turn shows that we only have to find a projective system of fields realizing the system

$$(G(K_i((t)))/\Gamma_i, \widetilde{\gamma_{ij} \times \mathrm{Id}})_{i \le j \in I}.$$

Therefore, to keep notation simple, we may assume that $G(K_i)[H] = G_i''[H]$, $\Gamma_i = \Theta_i$, $\gamma_{ij} \times \mathrm{Id} = q_{ij} \times \mathrm{Id}$, and that $\widetilde{q_{ij} \times \mathrm{Id}}$ is the map from $G(K_i)[H]/\Theta_i$ to $G(K_j)[H]/\Theta_j$ induced by $\gamma_{ij} \times \mathrm{Id} = q_{ij} \times \mathrm{Id}$.

In this vein, for every $i \le j \in I$, we will write $G(M_i) = G_i''[H]$, $q_{ij} \times \mathrm{Id}$ will stand for the morphism of special groups induced by the inclusion $M_i \subseteq M_j$, and the diagram

$$\begin{array}{ccc} \dot{M}_i & \longrightarrow & \dot{M}_j \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_j} \\ G(M_i) & \xrightarrow{q_{ij} \times \mathrm{Id}} & G(M_j) \end{array} \tag{5-4}$$

is commutative, where p_i and p_j denote the canonical maps.

Define $n_i := \dim_{\mathbb{F}_2} \Theta_i$ for $i \in I$. Note that $\dim_{\mathbb{F}_2} \Theta_i \le \dim_{\mathbb{F}_2} H = 1$, so $n_i \in \{0, 1\}$. Since $(q_{ij} \times \mathrm{Id}) \restriction \Theta_i : \Theta_i \to \Theta_j$ is injective, we have $n_i \le n_j$ whenever $i \le j \in I$. If $n_i = 1$, write $\Theta_i = \{1, a_i h\}$, with $a_i \in G_i''$. In this case, and if $i \le j \in I$, we have $(q_{ij} \times \mathrm{Id})(\Theta_i) = \Theta_j$, so $q_{ij}(a_i) = a_j$.

Lemma 5.18. *There is $b \in \dot{M}_\perp$ such that, for every $i \in I$, $\Theta_i \subseteq \{1, p_i(b)\}$.*

Proof. For every $i \le j \in I$, the map q_{ij} is surjective. In particular the map $q_{\perp\top}$ is surjective and, by diagram (5-4) above, $p_\top(\dot{M}_\perp) = \mathrm{Im}(q_{\perp\top} \times \mathrm{Id}) = G(M_\top)$. Let $b \in \dot{M}_\perp$ be such that $\{1, p_\top(b)\} = \Theta_\top$. Let now $i \in I$ and let $x \in \Theta_i$. Then

$$(q_{i\top} \times \mathrm{Id})(x) \in \Theta_\top = \{1, p_\top(b)\}.$$

If $(q_{i\top} \times \mathrm{Id})(x) = 1$, we get $x = 1 \in \{1, p_i(b)\}$, because $(q_{ij} \times \mathrm{Id}) \restriction : \Theta_i \to \Theta_j$ is an injective group homomorphism. If $(q_{i\top} \times \mathrm{Id})(x) = p_\top(b)$, since diagram (5-4) is commutative, we get $p_\top(b) = (q_{i\top} \times \mathrm{Id})(p_i(b))$, so $(q_{i\top} \times \mathrm{Id})(x) = (q_{i\top} \times \mathrm{Id})(p_i(b))$ and we conclude that $x = p_i(b)$. □

We assume from now on that there is $i \in I$ such that $n_i = 1$ (equivalently, $n_\top = 1$). Otherwise $\Theta_j = \{1\}$ for every $j \in I$, and the projective system of fields $(M_i)_{i\in I}$ realizes the projective system of special groups $(G_i''[H]/\Theta_i, \widetilde{q_{ij} \times \mathrm{Id}})_{i\le j\in I}$.

Recall that an element a of a special group T is called rigid when $a \neq 1$ and $D_T\langle 1, a\rangle = \{1, a\}$ and an element b of T is birigid when b and $-b$ are rigid. If $T = G[H]$, then every element in $G[H] \setminus G$ is birigid (this is essentially the only way to obtain birigid elements in a special group; see [Dickmann and Miraglia 2000, p. 12, Berman's Theorem]).

Since we assume that $n_i = 1$ for some $i \in I$ (in other words $n_\top = 1$), it follows that the element b produced in Lemma 5.18 is birigid in M_i for every $i \in I$.

The next proposition uses the following notation: If K is a field and $a \in K$ then $K(\sqrt[\infty]{a})$ stands for $K(\sqrt[2^n]{a}, n \in \mathbb{N})$.

Proposition 5.19 [Becher 2002, Proposition 8.2]. *Let F be a field, let a be a birigid element in F (i.e., $a \in \dot{F}$ and $a.\dot{F}^2$ is birigid in $G(F)$) and let φ be a quadratic form over F. Let $L := F(\sqrt[\infty]{a})$. Then*

(1) *$\dot{L} = \dot{F}\dot{L}^2$ and $\dot{F} \cap \dot{L}^2 = \dot{F}^2 \cup a\dot{F}^2$;*

(2) *φ is isotropic over L if and only if $\varphi \oplus a\varphi$ is isotropic over F.*

We define, for $i \in I$,

$$L_i = \begin{cases} M_i & \text{if } n_i = 0, \\ M_i(\sqrt[\infty]{b}) & \text{if } n_i = 1. \end{cases}$$

Since $n_i = 1$ implies $n_j = 1$ whenever $i \le j \in I$, the system $(L_i)_{i\in I}$ is a projective system of fields. Note that the following diagram of fields is obviously commutative (with the natural inclusions as morphisms):

$$\begin{array}{ccc} M_i & \longrightarrow & M_j \\ \downarrow & & \downarrow \\ L_i & \longrightarrow & L_j \end{array}$$

which implies that the induced diagram of special groups is also commutative:

$$\begin{array}{ccc} G(M_i) & \xrightarrow{q_{ij}\times \mathrm{Id}} & G(M_j) \\ {\scriptstyle \mu_i}\downarrow & & \downarrow{\scriptstyle \mu_j} \\ G(L_i) & \xrightarrow{\tau_{ij}} & G(L_j) \end{array} \tag{5-5}$$

where $\mu_i : G(M_i) \to G(L_i)$ is the map induced by $M_i \subseteq L_i$ and τ_{ij} is the map induced by $L_i \subseteq L_j$.

Lemma 5.20. *For $i \in I$, let $\pi_i : G(K_i)[H] \to G(K_i)[H]/\Theta_i$ be the canonical projection. Then μ_i is surjective and there is a unique isomorphism of special groups $\xi_i : G(L_i) \to G(K_i)[H]/\Theta_i$ such that the diagram*

$$\begin{array}{ccc} G(M_i) & \xrightarrow{\mu_i} & G(L_i) \\ {\scriptstyle\lambda_i}\downarrow & & \downarrow{\scriptstyle\xi_i} \\ G(K_i)[H] & \xrightarrow{\pi_i} & G(K_i)[H]/\Theta_i \end{array}$$

is commutative. In particular, L_i is Pythagorean.

Proof. The case $n_i = 0$ is trivial, so we assume $n_i = 1$. To avoid unnecessary notational complications, if K is a field and $x \in \dot{K}$, we simply write $\bar{x}$ for the class of x in $\dot{K}/\dot{K}^2$. By Proposition 5.19(1) we know that μ_i is surjective and that $\ker(\mu_i \circ \lambda_i^{-1}) = \{1, \lambda_i(\bar{b})\} = D_{G(K_i)[H]}\langle 1, \lambda_i(\bar{b})\rangle$. In particular, there is a unique isomorphism of groups $\xi_i : G(L_i) \to G(K_i)[H]/\{1, \lambda_i(\bar{b})\}$ such that the following diagram commutes:

$$\begin{array}{ccccc} G(K_i)[H] & \xrightarrow{\lambda_i^{-1}} & G(M_i) & \xrightarrow{\mu_i} & G(L_i) \\ {\scriptstyle\pi_i}\downarrow & & & \swarrow{\scriptstyle\xi_i} & \\ G(K_i)[H]/\{1, \lambda_i(\bar{b})\} & & & & \end{array}$$

We show that ξ_i is an isomorphism of special groups. The image of -1 is clearly -1. Take $\mu_i(\bar{c}), \mu_i(\bar{d}) \in G(L_i)$, where $c, d \in \dot{M}_i$. We have

$$\begin{aligned} \mu_i(\bar{c}) \in D_{G(L_i)}\langle 1, \mu_i(\bar{d})\rangle &\Leftrightarrow c \in D_{L_i}\langle 1, d\rangle \\ &\Leftrightarrow \langle\langle -c, d\rangle\rangle \text{ isotropic over } L_i \\ &\Leftrightarrow \langle\langle -c, d\rangle\rangle \oplus b\langle\langle -c, d\rangle\rangle \text{ isotropic over } M_i, \end{aligned}$$

the last equivalence following from Proposition 5.19(2). Recalling that Pfister forms are isotropic if and only if they are hyperbolic, we continue the chain of equivalences with

$$\begin{aligned} &\Leftrightarrow \langle 1, b\rangle \otimes \langle\langle -c, d\rangle\rangle \text{ isotropic over } M_i \\ &\Leftrightarrow \langle 1, b\rangle \otimes \langle\langle -c, d\rangle\rangle \text{ hyperbolic over } M_i \\ &\Leftrightarrow \langle 1, \lambda_i(\bar{b})\rangle \otimes \langle\langle -\lambda_i(\bar{c}), \lambda_i(\bar{d})\rangle\rangle \text{ hyperbolic in } G(K_i)[H] \\ &\Leftrightarrow \langle 1, \lambda_i(\bar{b})\rangle \otimes \langle\langle -\lambda_i(\bar{c}), \lambda_i(\bar{d})\rangle\rangle \equiv \langle 1, \lambda_i(\bar{b})\rangle \otimes \langle -1, 1, -1, 1\rangle \text{ in } G(K_i)[H] \\ &\Leftrightarrow \langle\langle -\pi_i \circ \lambda_i(\bar{c}), \pi_i \circ \lambda_i(\bar{d})\rangle\rangle \equiv \langle -1, 1, -1, 1\rangle \\ &\qquad\qquad \text{in } G(K_i)[H]/D\langle 1, \lambda_i(\bar{b})\rangle = G(K_i)[H]/\{1, \lambda_i(\bar{b})\}, \end{aligned}$$

the last step following from [Dickmann and Miraglia 2000, Proposition 2.21]. But this last condition is equivalent to $\langle\langle -\pi_i \circ \lambda_i(\bar{c}), \pi_i \circ \lambda_i(\bar{d})\rangle\rangle$ being hyperbolic in

$G(K_i)[H]/\{1, \lambda_i(\bar{b})\}$, and so to $\pi_i \circ \lambda_i(\bar{c})$ lying in $D_{G(K_i)[H]/\{1,\lambda_i(\bar{b})\}}\langle 1, \pi_i \circ \lambda_i(\bar{d})\rangle$. This shows that ξ_i is an isomorphism of special groups. Since $\lambda_i(b)$ is birigid and $|H| = 2$, we obtain that $G(L_i)$, being isomorphic to $G(K_i)[H]/\{1, \lambda_i(\bar{b})\} \cong G(K_i)$, is a reduced special group or $\{1\}$, which entails that L_i is a Pythagorean field. □

Recall that, using the identifications made after (5-3), we have

$$\widetilde{(q_{ij} \times \mathrm{Id})}((a\dot{K_i}^2)h^k \cdot \Theta_i) = (a\dot{K_j}^2)h^k \cdot \Theta_j \quad \text{for } a \in \dot{K}_i \text{ and } k \in \{0, 1\}.$$

Proposition 5.21. *The diagram*

$$\begin{array}{ccc} G(L_i) & \xrightarrow{\tau_{ij}} & G(L_j) \\ {\scriptstyle \xi_i}\downarrow & & \downarrow{\scriptstyle \xi_j} \\ G(K_i)[H]/\Theta_i & \xrightarrow{\widetilde{q_{ij} \times \mathrm{Id}}} & G(K_j)[H]/\Theta_j \end{array}$$

commutes. In particular, $G((L_i)_{i\in I}) \cong (G_i''[H]/\Theta_i, \widetilde{q_{ij} \times \mathrm{Id}})_{i\le j\in I}$ *and*

$$\varprojlim G((L_i)_{i\in I}) \cong \varprojlim (G_i, f_{ij})_{i\le j\in I}.$$

Proof. Since μ_i and μ_j are surjective by Lemma 5.20, the commutative diagram in that same lemma completely determines ξ_i and ξ_j. Let $z = \mu_i(at^k\dot{M_i}^2) \in G(L_i)$ (with $a \in \dot{K}_i$ and $k \in \{0, 1\}$). Then

$$\xi_i(z) = \xi_i \circ \mu_i(at^k\dot{M_i}^2) = \pi_i \circ \lambda_i(at^k\dot{M_i}^2) = \pi_i((a\dot{K_i}^2)h^k) = ((a\dot{K_i}^2)h^k) \cdot \Theta_i. \tag{5-6}$$

where the second equality comes from Lemma 5.20. Applying this, we obtain $\widetilde{(q_{ij} \times \mathrm{Id})} \circ \xi_i(z) = \widetilde{(q_{ij} \times \mathrm{Id})}(((a\dot{K_i}^2)h^k) \cdot \Theta_i) = (a\dot{K_j}^2)h^k \cdot \Theta_j$ and

$$\begin{aligned} \xi_j \circ \tau_{ij}(z) &= \xi_j \circ \tau_{ij} \circ \mu_i(at^k\dot{M_i}^2) \\ &= \xi_j \circ \mu_j \circ (q_{ij} \times \mathrm{Id})(at^k\dot{M_i}^2) && \text{by diagram (5-5)} \\ &= \xi_j \circ \mu_j(at^k\dot{M_j}^2) && \text{since } q_{ij} \times \mathrm{Id} \text{ is induced by } M_i \subseteq M_j \\ &= (a\dot{K_j}^2)h^k \cdot \Theta_j && \text{by (5-6)}, \end{aligned}$$

which finishes the proof. □

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