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A smooth flat Riemannian manifold is called an *exceptional domain* if it admits positive harmonic functions having vanishing Dirichlet boundary data and constant (nonzero) Neumann boundary data. In analogy with minimal surfaces, a representation formula is derived and applied to the classification of exceptional domains. Some interesting open problems are proposed along the way.

1. Introduction

Given an m-dimensional Riemannian manifold (M, g) and a smooth bounded domain Ω in M, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplace–Beltrami operator under an identically zero Dirichlet boundary condition. The critical points of the functional

$$\Omega \mapsto \lambda_1(\Omega)$$

under the volume constraint $\mathrm{Vol}\,\Omega = \alpha$, where $\alpha \in (0, \mathrm{Vol}\,M)$ is fixed, are called *extremal domains*. Smooth extremal domains are characterized by the property that the eigenfunctions associated with the first eigenvalue of the Laplace–Beltrami operator have constant Neumann boundary data [Soufi 2007]. In other words, a smooth domain is extremal if and only if there exists a positive function u_1 and a constant λ_1 such that

$$\Delta_g u_1 + \lambda_1\, u_1 = 0$$

in Ω with $u_1 = 0$ and $\nabla_n u_1$ constant on $\partial\Omega$, where n denotes the inward unit normal vector to $\partial\Omega$.

The theory of extremal domains is very reminiscent of the theory of constant mean curvature surfaces or hypersurfaces. To give some credit to this assertion, we recall that J. Serrin [1971] proved that the only compact, smooth, extremal domains in Euclidean space are round balls, paralleling the well-known result of Alexandrov asserting that round spheres are the only (embedded) compact constant mean curvature hypersurfaces in Euclidean space. More recently, F. Pacard and

P. Sicbaldi [2009] proved the existence of extremal domains close to small geo-
desic balls centered at critical points of the scalar curvature function, paralleling an
earlier result of R. Ye [1991], which provides constant mean curvature topological
spheres (with high mean curvature) close to small geodesic spheres centered at
nondegenerate critical points of the scalar curvature function.

   We propose the following:

Definition 1.1. A smooth domain $\Omega \subset \mathbb{R}^m$ is said to be an *exceptional domain* if
it supports positive harmonic functions having identically zero Dirichlet boundary
data and constant (nonzero) Neumann boundary data. Any such harmonic function
is called a *roof function*.

   Exceptional domains arise as limits under scaling of sequences of extremal do-
mains, just like minimal surfaces arise as limits under scaling of sequences of
constant mean curvature surfaces. As explained above, there is a formal corre-
spondence between extremal domains and constant mean curvature surfaces. In
this note, we try to explain that there is also a strong analogy between exceptional
domains and minimal surfaces. More generally, we propose:

Definition 1.2. An m-dimensional flat Riemannian manifold M is said to be *excep-
tional* if it supports positive harmonic functions having identically zero Dirichlet
boundary data and constant (nonzero) Neumann boundary data. Any such har-
monic function is called a *roof function*.

   Our results raise the problem of the classification of (unbounded) smooth m-
dimensional exceptional manifolds. In trying to address this classification problem,
we provide a Weierstrass-type representation formula for exceptional flat surfaces.
When the dimension $m = 2$, we give nontrivial examples of exceptional domains
that are embedded in $\mathbb{R}^2$, and we prove a half-space result for exceptional domains
that are conformal to a half-plane.

2. A nontrivial example of an exceptional domain in $\mathbb{R}^2$

The property of being an exceptional domain is preserved under the action of the
group of similarities of $\mathbb{R}^m$ (generated by isometries and dilations). We first give
trivial examples of exceptional domains in $\mathbb{R}^m$:

 (i) The half-space $\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 > 0\}$ is an exceptional domain in
     $\mathbb{R}^m$, since the function $u(x) = x_1$ is a positive harmonic function with identi-
     cally zero Dirichlet boundary data and constant Neumann boundary data.

 (ii) The complement of a ball of radius 1 in $\mathbb{R}^m$ is an exceptional domain since the
     function u defined by $u(x) := \log|x|$ when $m = 2$ and by $u(x) := 1 - |x|^{2-m}$
     when $m \geq 3$ is positive, harmonic, and has 0 Dirichlet and constant Neumann
     data on the unit sphere.

(iii) The product $\Omega \times \mathbb{R}^k$ is an exceptional domain in $\mathbb{R}^m$ provided $\Omega \subset \mathbb{R}^{m-k}$ is an exceptional domain in $\mathbb{R}^{m-k}$.

In dimension $m = 2$, there exists (up to a similarity) at least one other exceptional domain. To describe this domain, we make use of the invariance of the Laplace operator under conformal transformations. The idea is that there exists a (somehow natural) unbounded, positive harmonic function U with identically zero Dirichlet boundary condition on an infinite strip in $\mathbb{R}^2$. This function does not have constant Neumann data, but we can then look for a conformal transformation h which has the property that the pullback of the harmonic function U by h has constant Neumann boundary data on the boundary of the image of the strip by h.

To proceed, it is convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Proposition 2.1. *The domain* $\Omega := \{w \in \mathbb{C} : |\text{Im } w| < \pi/2 + \cosh(\text{Re } w)\}$ *is an exceptional domain.*

To prove this result, we define the infinite strip

$$S := \{z \in \mathbb{C} : \text{Im } z \in (-\pi/2, \pi/2)\}$$

and the mapping

$$F(z) := z + \sinh z.$$

Observe that $\Omega = F(S)$. The proof of Proposition 2.1 follows from the next two lemmas.

Lemma 2.2. *The mapping F is a conformal diffeomorphism from S into Ω.*

Proof. We can write

$$F(z) - F(z') = (z - z') \int_0^1 \left(1 + \cosh(tz + (1-t)z')\right) dt.$$

In particular

$$(2\text{-}1) \quad \langle z - z', F(z) - F(z') \rangle = |z - z'|^2 \left(1 + \int_0^1 \text{Re} \cosh\left(tz + (1-t)z'\right) dt\right),$$

where $\langle \, \cdot \, , \, \cdot \, \rangle$ denotes the scalar product in $\mathbb{C}$. Now, for all $x + i\,y \in S$, we have

$$\text{Re} \cosh(x + i\,y) = \cosh x \, \cos y \geq 0.$$

This, together with (2-1), implies immediately that F restricted to S is injective. Also,

$$|\partial_z \Lambda(z)|^2 = |1 + \cosh z|^2 = (\cosh x + \cos y)^2.$$

Therefore $\partial_z F$ does not vanish in S. Thus F is a local diffeomorphism, and because the mapping F is holomorphic, it is conformal. $\qquad\square$

We define the real-valued function u on Ω by the identity

$$u(F(z)) = \operatorname{Re} \cosh z \quad \text{for all } z \in S.$$

Lemma 2.3. *The function u is harmonic and positive in Ω, vanishes and has constant Neumann boundary data on $\partial\Omega$.*

Proof. The function W defined in $\mathbb{C}$ by $W(z) := \operatorname{Re} \cosh z$ is harmonic. Indeed, as mentioned in the proof of the previous lemma, $W(x + i\,y) = \cosh x \, \cos y$. Hence W is both harmonic and positive in S, and vanishes on ∂S. The mapping F being a conformal diffeomorphism from S to Ω, we conclude the function u is both harmonic and positive in Ω, and vanishes on $\partial\Omega$. We claim that u has constant Neumann data on $\partial\Omega$. Indeed, by definition,

$$u(F(z)) = \tfrac{1}{2}(\cosh z + \cosh \bar{z}).$$

Since F is holomorphic, differentiation with respect to z yields

$$2\,\partial_z u(F(z)) = \frac{\sinh z}{1 + \cosh z}.$$

Therefore

$$|\nabla u|^2(F(z)) = \frac{\cosh x - \cos y}{\cosh x + \cos y},$$

where $z = x + i\,y$. On $\partial\Omega$, we have $y = \pm\pi/2$ and hence $|\nabla u| \equiv 1$. Since we already know that $u = 0$ on $\partial\Omega$, we conclude that u has constant Neumann boundary data. $\qquad\square$

Lemmas 2.2 and 2.3 complete the proof that $\Omega = F(S)$ is an exceptional domain in $\mathbb{R}^2$ with roof function u.

Remark 2.4. We suspect that this example generalizes to any dimension $m \geq 3$: specifically, there should exist a rotationally symmetric exceptional domain in $\mathbb{R}^m$ for all $m \geq 3$.

3. Toward a global representation formula

Let M be an *exceptional flat surface* (an exceptional domain of dimension 2) with smooth boundary ∂M. Let $\tilde{M}$ be its universal cover and let $\partial\tilde{M}$ be the preimage of ∂M by the covering map $\tilde{M} \to M$. In the following, we exclude the uninteresting case where $\partial M = \varnothing$.

By assumption, M is a flat surface. Hence $\tilde{M}$ is naturally endowed with a flat Riemannian metric g and hence with an induced complex structure, which is conformal to the standard one. Also, there exists an orientation-preserving isometric immersion $F : (\tilde{M}, g) \to (\mathbb{C}, g_{\mathbb{C}})$, where $g_{\mathbb{C}}$ is the canonical Euclidean metric on

$\mathbb{C}$; this induces a smooth immersion of $\partial\tilde{M}$, Observe that F is holomorphic and that $\|dF\|_g = 1$ in $\tilde{M}\cup\partial\tilde{M}$. We define the holomorphic $(1,0)$-form

$$\Phi := dF = \partial_z F\, dz.$$

Observe that Φ does not vanish and admits a smooth extension to $\tilde{M}\cup\partial\tilde{M}$.

We let $u : M \to \mathbb{R}^+$ be a roof function on M and, with a slight abuse of notation, we denote its lift also by $u : \tilde{M} \to \mathbb{R}^+$. The roof function u can be normalized so that

$$(3\text{-}1) \qquad\qquad \|\nabla u\|_g = 1$$

on ∂M. We consider the harmonic conjugate function $v : \tilde{M} \to \mathbb{R}$ (uniquely defined up to some additive constant) that is the solution of

$$(3\text{-}2) \qquad \partial_z(u - i\,v) = 0 \qquad (\text{and hence } \partial_{\bar{z}}(u + i\,v) = 0\,).$$

We set

$$U := u + i\,v.$$

Recall that U is a holomorphic function from $\tilde{M}$ into $\mathbb{C}$. The property that u takes positive values in M and vanishes on ∂M can be translated into the fact that U maps $\tilde{M}$ to $\mathbb{C}^+ := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ and $\partial\tilde{M}$ to $i\,\mathbb{R}$. Since $\Phi \neq 0$ on $\tilde{M}$, there exists a unique holomorphic function h on $\tilde{M}$ such that $dU = \partial_z U\, dz = h\,\Phi$. We deduce from the fact that u vanishes on $\partial\tilde{M}$ and from (3-1) that $\nabla_n U = 1$, where n denotes the inward unit normal vector to $\partial\tilde{M}$. Hence

$$(3\text{-}3) \qquad\qquad \|\partial_z U\|_g = 1 \qquad \text{on } \partial\tilde{M}.$$

Now, condition (3-1) translates into the fact that $\|\Phi\|_g = \|dF\|_g = 1 = \|dU\|_g$ on $\partial\tilde{M}$. Clearly, this is equivalent to the fact that $|h| = 1$ on $\partial\tilde{M}$. Therefore, we end up with the following data:

  (i) An oriented, simply connected complex surface $\tilde{M}$ with smooth boundary $\partial\tilde{M}$.

 (ii) A holomorphic function U, defined on $\tilde{M}$, which takes values in $\mathbb{C}^+$ and maps $\partial\tilde{M}$ into $i\,\mathbb{R}$.

(iii) A holomorphic function h, defined on $\tilde{M}$, such that $|h| = 1$ on $\partial\tilde{M}$, and for which the 1-form Φ defined by $\Phi := (1/h)dU$ does not vanish on $\tilde{M}$.

By analogy with the theory of minimal surfaces, we call these data the Weierstrass-type representation formula for exceptional flat surfaces.

Conversely, given a set of such data, we can define the map $F : \tilde{M} \to \mathbb{C}$ by integrating $dF = \Phi$. Thanks to (iii), this map is an immersion and its image is an immersed exceptional flat surface with roof function given by $u = \operatorname{Re} U$. In

Section 4, we will give some explicit examples of such constructions when $\partial \tilde{M}$ is equal to $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1, \ldots, \alpha_n$ is a finite collection of points on $\partial D = S^1$.

Example 3.1. Here is a (rather pathological) illustration of this Weierstrass-type formula. Consider $M = \mathbb{C}^+$, the function $U(z) = z$ and

$$F(z) = \int_0^z e^{-\sinh \zeta} \, d\zeta.$$

Note that $\partial_z F$ is $2i\pi$-periodic, and this implies that $F(z + 2i\pi) = F(z) + C$, where the constant C is given explicitly by

$$C := i \int_0^{2\pi} e^{-i \sin s} \, ds.$$

Moreover, for $x > 0$,

$$F(x + iy) = F(iy) + \int_0^x e^{-\sinh(s+iy)} \, ds$$

converges to $+\infty$ as $x \to +\infty$ if $y = 0$, but this quantity is bounded if $|y - \pi| < \pi/2$, and even admits a finite limit as $x \to +\infty$.

Hence, in addition to the regular boundary $F(i\mathbb{R})$, which is a smooth periodic curve, the image of F has a singular boundary: the set of limits of $F(x + iy)$ as u tends to $+\infty$, for the values of y for which this limit exists. The roof function tends to infinity along this singular boundary.

4. Examples of exceptional flat surfaces

Thanks to the Weierstrass-type representation in the previous section, we can give many nontrivial examples of exceptional flat surfaces. We keep the notation from that section.

The construction makes use of an integer $n \in \mathbb{N} \setminus \{0\}$ and the Riemann surface $D = \{z \in \mathbb{C} : |z| < 1\}$. On D, we define the holomorphic functions

$$h(z) = z^{n-1} \quad \text{and} \quad U(z) := \frac{1 + z^n}{1 - z^n}.$$

The 1-form Φ is given by

$$\Phi(z) := \frac{2n}{(1 - z^n)^2} \, dz.$$

Both U and Φ have singularities at the n-th roots of unity. The function F is then obtained by integrating Φ, and the roof function u is defined by $u = \operatorname{Re} U$.

(i) When $n = 1$, we can take

$$F(z) = \frac{1+z}{1-z}.$$

In this case, we simply have $F(D) = \mathbb{C}^+$, and we recover the fact that the half-plane is an exceptional domain. This is the counterpart of the plane in the framework of minimal surfaces.

(ii) When $n = 2$, we can take

$$F(z) = \frac{2z}{1-z^2} + \log\frac{z+1}{z-1}.$$

In this case, the exceptional flat surface found can be isometrically embedded in $\mathbb{C}$, and hence $F(D)$ is an exceptional domain. In fact, $F(D)$ corresponds (up to some similarity) to the domain Ω, which was defined in Proposition 2.1. This exceptional domain is the counterpart of the catenoid.

(iii) Finally when $n \geq 3$, the exceptional flat surfaces we find cannot be isometrically embedded in $\mathbb{C}$ anymore. They are counterparts of the minimal n-noids described in [Jorge and Meeks 1983].

Let us analyze this example further. The function U can be written as

$$U(z) = -\frac{1}{n}\sum_{k=1}^{n}\frac{z+\alpha^k}{z-\alpha^k},$$

where $\alpha := e^{i2\pi/n}$. In particular, $\operatorname{Re} U$ is nothing but a multiple of the sum of the Poisson kernel on the unit disc with poles at $1, \alpha, \ldots, \alpha^{n-1}$. Next,

$$dU = z^{n-1}\frac{2n}{(1-z^n)^2}\,dz,$$

so the function h is cooked up to counterbalance the zero of dU and ensure that Φ does not vanish in the unit disk, while keeping the condition $|dU|^2 = |\Phi|^2$ on ∂D.

To generalize the example, consider n distinct points $\alpha_1, \ldots, \alpha_n \in S^1 \subset \mathbb{C}$ and $a_1, \ldots, a_n > 0$. We define

$$(4\text{-}1)\qquad U(z) := -\sum_{k=1}^{n} a_k\frac{z+\alpha_k}{z-\alpha_k}.$$

It is easy to check that $\operatorname{Re} U$ is positive (since each function $z \mapsto -\dfrac{z+\alpha_k}{z-\alpha_k}$ maps D to $\mathbb{C}^+$) and vanishes on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$. We have

$$\prod_{k=1}^{n}(z-\alpha_k)^2\,dU = P(z)\,dz,$$

where P is a polynomial that depends on the choice of points $\alpha_1, \ldots, \alpha_n$ and weights $a_1, \ldots, a_n$. Assume that P does not vanish on ∂D and denote by $z_1, \ldots, z_l$ the roots of P in the unit disc, counted with multiplicity. We simply define

$$h(z) := \prod_{j=1}^{l} \frac{z - z_j}{z \bar{z}_j - 1}$$

and the 1-form Φ by $\Phi := (1/h)\, dU$. Integration of Φ yields a $2n$-dimensional family of exceptional flat surfaces immersed in $\mathbb{C}$.

5. A global Weierstrass-type representation

In this section, we show that exceptional flat surfaces whose immersion in $\mathbb{C}$ have finitely many *regular ends* and are locally finite coverings of $\mathbb{C}$ are precisely the examples in the previous section. We use the notations introduced in Section 3, and we set

$$\widehat{M} := M \cup \partial M.$$

We further assume that M is *simply connected* and that $\partial M \neq \varnothing$. In particular, M has the conformal type of the unit disk D, and without loss of generality, we can assume that M is indeed equal to D and consider $\bar{D}$ as a natural compactification of M. We denote by F an orientation preserving, holomorphic, isometric immersion $F : (\widehat{M}, g) \to (\mathbb{C}, g_{\mathbb{C}})$. Recall that $\|dF\|_g = 1$ on ∂M. Some natural hypotheses are needed:

(H-1) *M has finitely many ends.* This means that

$$\partial M = \partial D \setminus \bigcup_{j=1}^{n} E_j = \bigcup_{j=1}^{n} I_j,$$

where each $E_j \subset S^1$ is a closed arc and $I_j \subset S^1$ is an open arc.

(H-2) *F is proper.* This means that $F(w)$ tends to infinity as w tends to $\bigcup_{j=1}^{n} E_j$.

(H-3) *Each end of M is regular.* This means the image of $I_j := (\theta_j^-, \theta_j^+)$ by F is a curve Γ_j asymptotically parallel to fixed directions at infinity. In other words, there exist two unit vectors τ_j^- and $\tau_j^+ \in S^1 \subset \mathbb{C}$ such that

$$\lim_{\theta \in I_j,\, \theta \to \theta_j^\pm} \frac{F(e^{i\theta})}{|F(e^{i\theta})|} = \tau_j^\pm.$$

This is the case, for example, if we assume each Γ_j has finite total curvature.

(H-4) *The mapping F is a locally finite covering.* This means there exists $d \in \mathbb{N}^*$ such that, for any $z \in \mathbb{C}$, the cardinal of $\{\zeta \in M : F(\zeta) = z\}$ is at most d.

We now state the main result of this section.

Theorem 5.1. *Assume that* M *is a simply connected exceptional flat surface and let* $F : M \to \mathbb{C}$ *be an isometric immersion. Further assume that* (H-1)–(H-4) *holds and identify* M *with* D. *Then there exist* $\mu \in \mathbb{R}$, n *distinct points* $\alpha_1, \ldots, \alpha_n \in S^1$ *and* n *constants* $a_1, \ldots, a_n > 0$ *such that*

$$dF = e^{i\mu} \prod_{k=1}^{m} \frac{\bar{z}_k \, z - 1}{z - z_k} \, dU,$$

where $z_1, \ldots, z_m \in \bar{D}$ *denote the zeros of* dU, *counted with multiplicity, and where*

$$U(z) := -\sum_{j=1}^{n} a_j \, \frac{z + \alpha_j}{z - \alpha_j} \quad \text{in } \bar{D}.$$

The proof is divided into a few lemmas and propositions. We start by analyzing the ends E_j and show that they reduce to isolated points $\alpha_1, \ldots, \alpha_n$. Next we analyze the behavior of F near the points α_j and show that F does not have any essential singularity there. Then we proceed with the analysis of the function U and show that it has the expected form. The proof is completed with the study of the function h.

As promised, we first analyze the sets E_j:

Lemma 5.2. *Under the assumptions of Theorem 5.1, there exists a finite number of points* $\alpha_1, \ldots, \alpha_n \in \partial D = S^1$ *such that* $\widehat{M} = \bar{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$.

Proof. We need to show that each interval E_j is reduced to a point. This essentially follows from the fact that the capacity of E_j vanishes.

Suppose, for a contradiction, that E_j is an arc of positive arc length for some j, and take some $l \in (0, \pi/2)$ and an arc $E \subset E_j$ of length l. Our problem being invariant under the action of fractional linear transformations of the unit disk, we can assume without loss of generality that E is the image of $[-l/2, l/2]$ under $s \mapsto e^{is}$. Reducing l if necessary, we can also assume that the opposite arc $-E$, the image of $[-l/2, l/2]$ under $s \mapsto -e^{is}$, is contained in $S^1 \setminus \bigcup_{j=1}^{n} E_j$.

Recall that for any smooth function defined on (a, b) which satisfies $f(b) = 1$ and $f(a) = 0$, we have

$$1 = f(b) - f(0) = \int_a^b f'(s) \, ds \le \left(\int_a^b (f')^2(s) \, ds \right)^{1/2} \sqrt{b - a}.$$

If in addition, $b - a \le 2$, we conclude that

$$\int_a^b (f')^2(s) \, ds \ge \tfrac{1}{2}.$$

Now assume that we are given a smooth function $f : \bar{D} \to \mathbb{R}$ such that $f = 1$ on E and $f = 0$ on $-E$. Using the previous inequality, we can write

$$(5\text{-}1) \qquad \int_D \|\nabla f\|^2_{g_\mathbb{C}} \, dx \, dy \geq \int_{D \cap \{|x| < \sin(l/2)\}} |\partial_y f|^2 \, dx \, dy \geq \int_{|x| \leq \sin(l/2)} \tfrac{1}{2} \, dx = \sin \frac{l}{2}.$$

Given $R > r > 0$, define $\chi : \mathbb{C} \to \mathbb{R}$ by

$$\chi(z) = \begin{cases} 0 & \text{if } |z| \leq r, \\ \dfrac{\log(|z|/r)}{\log(|z|/R)} & \text{if } r \leq |z| \leq R, \\ 1 & \text{if } R \leq |z|, \end{cases}$$

and we define $f : D \to \mathbb{R}$ by $f := \chi \circ F$. Since F is conformal, we can write

$$\int_D \|\nabla f\|^2_{g_\mathbb{C}} \, dx \, dy = \int_D \|\nabla f\|^2_g \, d\,\mathrm{vol}_g \,.$$

Now, using (H-4), we conclude that

$$(5\text{-}2) \qquad \int_D \|\nabla f\|^2_g \, d\,\mathrm{vol}_g \leq d \int_\mathbb{C} \|\nabla \chi\|^2_{g_\mathbb{C}} \, dx \, dy = d \, \frac{2\pi}{\log(R/r)}.$$

Fixing $r > 0$ large enough, we can ensure that f is identically equal to 0 on $-E$. Using (H-2), we see that f is identically equal to 1 on each E_j, and in particular on E. Therefore f can be used in (5-1), which together with (5-2) yields

$$2\pi d \geq \sin \frac{l}{2} \, \log \frac{R}{r}$$

independently of $R > r$. Letting R tend to infinity, we get a contradiction, and the proof is complete. $\qquad\square$

Therefore, we now know that $E_j := \{\alpha_j\}$. Without loss of generality, we can assume that $\alpha_1, \ldots, \alpha_n$ are arranged counterclockwise along S^1. We agree that $\alpha_0 := \alpha_n$ and $\alpha_{n+1} := \alpha_1$, and that for each $j = 1, \ldots, n$, the arc I_j is positively oriented and joins α_j to α_{j+1}. We now analyze the singularities of F close to α_j.

Given $j = 1, \ldots, n$, we denote by $S(\alpha_j, r)$ the circle of radius $r > 0$ centered at α_j. We define

$$\gamma_j := \bar{D} \cap S(\alpha_j, r),$$

which we assume to be oriented clockwise. The *angle* $\theta_j \in \mathbb{R}$ at α_j is defined by

$$\theta_j := - \lim_{r \to 0} \int_{\gamma_k} F^* d\theta,$$

where $d\theta := \mathrm{Im} \, dz/z$. Thanks to (H-3), θ_j is well defined, and we have

$$\tau_j^- = e^{i\theta_j} \tau_{j-1}^+.$$

Lemma 5.3. *Under the assumption of Theorem 5.1, the function*

$$H_j(z) := (z - \alpha_j)^{\theta_j/\pi} F(z)$$

is holomorphic in a neighborhood of α_j in $\bar{D} \setminus \{\alpha_j\}$, and $H_j(\alpha_j) \neq 0$.

Proof. Without loss of generality, we can assume that $\alpha_j = 1$. By right composing F with the conformal transformation $z \mapsto (1 - z)/(1 + z)$, we can replace D by $\mathbb{C}^+$. Now we define

$$G(z) := F(z)^{-\pi/\theta_j}.$$

Observe that $G(0) = 0$ by (H-2). Moreover, (H-3) and the definition of θ_j imply that the image by G of a neighborhood of 0 in $i\mathbb{R}$ is a $\mathscr{C}^1$-curve, and hence analytic. In particular, there exists some conformal transformation T such that, for some $r > 0$, the image by $T \circ G$ of $i(-r, r)$ is a straight line segment in $i\mathbb{R}$. Then it is possible to extend $T \circ G$ to a function $\tilde{G}$ defined on a neighborhood of 0 in $\mathbb{C}$ by setting

$$\tilde{G}(z) = \begin{cases} T(G(z)) & \text{if } \operatorname{Im} z \geq 0, \\ -\overline{T(G(-\bar{z}))} & \text{if } \operatorname{Im} z \leq 0. \end{cases}$$

The resulting $\tilde{G}$ is bounded in a neighborhood of 0 in $\mathbb{C}$ and holomorphic away from 0. It is well known that the singularity is then removable and hence it is holomorphic. Therefore $\tilde{G}$ is actually holomorphic in a neighborhood of 0. In particular, we can write

$$G(z) = z^k H(z)$$

near 0, where H is a holomorphic function that does not vanish at 0. Going back to the definition of G, this implies that

$$F(z) = (z - \alpha_j)^{-k\,\theta_j/\pi} H_j(z),$$

where H_j is holomorphic in a neighborhood of α_j and does not vanish at α_j. But the definition of θ_j readily implies that $k = 1$. This completes the proof. $\square$

As a corollary, we conclude that

(5-3)
$$H(z) := F(z) \prod_{j=1}^{n} (z - \alpha_j)^{\theta_j/\pi}$$

is a bounded holomorphic function in $\bar{D}$. Moreover, since F tends to infinity as z approaches α_j, this implies that $\theta_j > 0$.

We now make use of the fact that M is an exceptional domain, and hence there is a roof function $u : \widehat{M} \to [0, +\infty)$. We can define the holomorphic function $U := u + iv$, where $v : \widehat{M} \to \mathbb{R}$ is the (real-valued) harmonic conjugate of u. The purpose of the next result is to show that U is precisely given by (4-1).

Lemma 5.4. *Under the assumptions of Theorem 5.1, there exist n constants a_1, $\ldots, a_n > 0$ such that*

$$U(z) = -\sum_{j=1}^{n} a_j \frac{z + \alpha_j}{z - \alpha_j}.$$

Proof. First, it is possible to extend the function U to all $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$ by defining V to be equal to U in $\bar{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$, and

$$V(z) := -\overline{U(1/\bar{z})}$$

when $z \in \mathbb{C} \setminus \bar{D}$. The key observation is that, since $\operatorname{Re} U = 0$ on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$, the function V is continuous and in fact holomorphic on $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$. Moreover, V converges to $V(\infty) := -\overline{U(0)}$ at infinity.

We proceed with the proof that the function V has no essential singularity at any α_j; it will follow from Picard's theorem. By definition, $\operatorname{Re} V$ vanishes on I_j and is positive in D. Therefore the outward normal derivative of $\operatorname{Re} V$ on I_j is negative. This implies that the tangential derivative of $\operatorname{Im} V$ on I_j does not vanish and hence that $\operatorname{Im} V$ is strictly monotone on each I_j. This shows that there exists some neighborhood $\mathcal{V}$ of α_j in $\mathbb{C}$ such that any element of $i\mathbb{R}$ is achieved by V at most twice in $\mathcal{V}$ (that is, at most once on I_j and at most once on I_{j-1} and certainly not in $\mathcal{V} \setminus \partial D$, since V takes values in $\mathbb{C} \setminus i\mathbb{R}$ away from ∂D). Picard's big theorem [Conway 1978] then implies that α_j is not an essential singularity of V. Hence α_j is either a removable singularity of V or a pole.

Since $\|\nabla u\|_g \equiv 1$ on ∂M, this forces $|\partial_z U| = |\partial_z F|$ on ∂M, and since $|\partial_z F|$ tends to $+\infty$ at α_j, so does $|\partial_z U|$. Hence all α_j are poles of V.

We are now interested in the zeros of V. Since $\operatorname{Re} V$ takes positive values in D and negative values in $(\mathbb{C} \cup \{\infty\}) \setminus \bar{D}$, we already know that the only possible zeros of V are on ∂D. We have already seen that, along I_j, the function V equals iv, where v is strictly monotone. Further, since α_{j-1} and α_j are poles of this function, $|V|$ must converge to $+\infty$ as we approach either α_{j-1} or α_j. Because of the continuity of v along each I_j it follows that v vanishes exactly at one point β_j on each I_j. Moreover, this zero is simple: if it had order $k > 1$, the zero set of $\operatorname{Re} V$ near β_j would contain k curves intersecting at β_j, and this would force $\operatorname{Re} V = \operatorname{Re} U$ to vanish in D, in contradiction with our hypothesis.

Finally, we prove that V has only simple poles. We know that V extends meromorphically to a map on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with neither a pole nor a zero at infinity. Furthermore, V has exactly n simple zeros and n poles; hence these poles must be simple. To summarize, V can be written as a linear combination of the constant function and functions of the form $z \mapsto 1/(z - \alpha_j)$. Without loss of generality, this

amounts to saying that V can be written as

$$V(z) = a - \sum_{j=1}^{n} a_j \frac{z + \alpha_j}{z - \alpha_j},$$

where a and the a_j are complex numbers. Using the fact that, by construction, $V(1/\bar{z}) = -\overline{V(z)}$, we conclude that $a \in i\mathbb{R}$ and also that $a_j \in \mathbb{R}$. Moreover, since $\operatorname{Re} U$ is positive, this implies that the a_j are positive real numbers. This completes the proof, since U is defined up to the addition of some element of $i\mathbb{R}$. □

We are now in a position to complete our analysis of the function F. Since F is an immersion, $dF \neq 0$ on $\widehat{M}$. Hence there exists a unique holomorphic function h on $\widehat{M}$ such that

$$(5\text{-}4) \qquad\qquad \partial_z U = h\, \partial_z F$$

on $\widehat{M}$. Moreover, since $\|\nabla u\|_g \equiv 1$ on ∂M, this implies that $|h| \equiv 1$ on ∂M. We now analyze the function h, which will complete the proof of Theorem 5.1.

Lemma 5.5. *Under the assumptions of Theorem 5.1, there exists a constant $e^{i\mu} \in \mathbb{R}$ such that the function h defined by (5-4) has the form*

$$h(z) = e^{-i\mu} \prod_{k=1}^{m} \frac{z - z_k}{\bar{z}_k z - 1},$$

where $z_1, \ldots z_m$ are the zeros of $\partial_z U$ in D counted with multiplicity.

Proof. The function h is holomorphic in D and satisfies $|h| = 1$ on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$. We can extend h to a holomorphic function H, defined on $(\mathbb{C} \cup \{\infty\}) \setminus \{\alpha_1, \ldots, \alpha_n\}$ by setting $H(z) := h(z)$ for all $z \in \bar{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$ and

$$(5\text{-}5) \qquad\qquad \overline{H(z)} := \frac{1}{h(1/\bar{z})}$$

for all $z \in \mathbb{C} \setminus \bar{D}$. Clearly H is locally bounded in $\bar{D} \setminus \{\alpha_1, \ldots, \alpha_n\}$, and its only singularities in $(\mathbb{C} \cup \{\infty\}) \setminus \bar{D}$ are poles that are the images by $z \mapsto 1/\bar{z}$ of the zeros of h; hence H is meromorphic outside $\{\alpha_1, \ldots, \alpha_n\}$. But Lemma 5.3 and (5-3) imply that, near α_j, $|H|$ is bounded by a constant times $|z - \alpha_j|^{-k_j}$ for some $k_j > 0$. Therefore α_j is not an essential singularity of H, and hence H is meromorphic in $\mathbb{C} \cup \{\infty\}$.

Observe that $|H(z)| = 1$ on $\partial D \setminus \{\alpha_1, \ldots, \alpha_n\}$, and this implies that the points α_j are not poles of H. Therefore, the singularities α_j of H are removable. Also, we have

$$\Delta |H|^2 = 4\partial_z \partial_{\bar{z}} |H|^2 = 4 |\partial_z H|^2 \geq 0,$$

and since $|H| = 1$ on ∂D, the maximum principle implies that $|H| \leq 1$ in D.

Since H is bounded in $\bar{D}$, it does not have poles in this set. This also implies that H has no zeroes in $(\mathbb{C} \cup \{\infty\}) \setminus D$, because otherwise H would have poles in $\bar{D}$ by (5-5). Therefore, if $z_1, \ldots, z_m \in D$ denote the zeros of H (counted with multiplicity), the poles of H are given by $1/\bar{z}_1, \ldots, 1/\bar{z}_m$ (also counted with multiplicity). It is then a simple exercise to check that H is of the form

$$H(z) = C \prod_{k=1}^{m} \frac{z - z_k}{\bar{z}_k \, z - 1}$$

for some constant $C \in \mathbb{C}$. Finally, the condition that $|H(z)| = 1$ on ∂D forces $|C| = 1$. This completes the proof. $\square$

6. A Bernstein type result for two-dimensional exceptional domains

We prove the following Bernstein type result for two-dimensional exceptional domains.

Proposition 6.1. *Assume that Ω is a two-dimensional exceptional domain conformal to $\mathbb{C}^+$, and let u be a roof function on Ω. We further assume that $\partial_x u > 0$ in Ω. Then Ω is a half-plane.*

Proof. Since we have assumed that Ω is conformal to $\mathbb{C}^+$, there exists a holomorphic map $\Psi : \mathbb{C}^+ \mapsto \Omega$. We then define

$$H := (\partial_z u) \circ \Psi.$$

The function H is holomorphic in $\mathbb{C}^+$ and does not vanish, since we have assumed that $\partial_x u \neq 0$. Moreover, $|H| \equiv 1$ on $\partial \mathbb{C}^+$. We can write $H = e^{i\Theta}$, where Θ is a holomorphic function defined in $\mathbb{C}^+$ that is real valued on the imaginary axis. This means that $\operatorname{Im} \Theta = 0$ when $\operatorname{Re} z = 0$. Since we have assumed that $\partial_x u > 0$, we also conclude that $\operatorname{Re} \Theta \in (-\pi/2, \pi/2)$.

We can extend Θ as a holomorphic function $\tilde{\Theta}$ in $\mathbb{C}$ as follows:

$$\tilde{\Theta}(z) := \begin{cases} \Theta(z) & \text{if } \operatorname{Re} z \geq 0, \\ \overline{\Theta(-\bar{z})} & \text{if } \operatorname{Re} z < 0. \end{cases}$$

It is easy to check that $\tilde{\Theta}$ is a holomorphic function: in fact, the real part of Θ is extended as an even function of $\operatorname{Re} z$, while the imaginary part of Θ is extended as an odd function of $\operatorname{Re} z$. That $\tilde{\Theta}$ is $\mathscr{C}^1$ is then a consequence of the fact that $\operatorname{Im} \Theta = 0$ on the imaginary axis, while the holomorphicity of Θ follows from the fact that $\partial_x \operatorname{Re} \Theta = 0$ on the imaginary axis of $\mathbb{C}$.

The real part of $\tilde{\Theta}$, being a bounded harmonic function, must be constant. Then $\tilde{\Theta}$, being holomorphic, must itself be constant. But this implies that the gradient of u is constant, and hence the level sets of u are straight lines. This implies that

u only depends on one variable, and hence it is an affine function. This completes the proof. $\square$

Corollary 6.2. *There is no exceptional domain contained in a wedge*

$$\Omega \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq \kappa \, |\operatorname{Im} z|\}$$

for any $\kappa > 0$.

Proof. The proof is by contradiction. If Ω were such an exceptional domain, there would exist on Ω a roof function u. One can apply the moving plane method [Serrin 1971; Gidas et al. 1979] to prove that $\partial_x u > 0$ and hence that $\partial\Omega$ is a graph over the y-axis. Since Ω is contained in a half-plane, there is no bounded, positive, harmonic function on Ω having 0 boundary data on $\partial\Omega$; otherwise one could use an affine function as a barrier to obtain a contradiction. Certainly, $\Omega \cup \partial\Omega$ is conformal to $\bar{D} \setminus E$, where D is the unit disc and E is a closed arc included in S^1. Necessarily, E is reduced to a point, since otherwise we can construct bounded, positive, harmonic functions on E that have 0 boundary data on $S^1 \setminus E$, and these would lift to bounded, positive, harmonic function on Ω, with 0 boundary data, a contradiction. Therefore, we conclude that Ω is conformal to $\mathbb{C}^+$. The assumptions of Lemma 5.5 are fulfilled, and hence we conclude that Ω is a half-plane, which is a contradiction. $\square$

7. Open problems

We have no nontrivial example of an exceptional domain in higher dimensions $\mathbb{R}^m$ for $m \geq 3$, besides the ones described in Section 2. In dimension $m = 2$, it is tempting to conjecture that (up to similarity) the only exceptional domains that can be embedded in $\mathbb{R}^2$ are half-spaces, the complement of a ball and the example discussed in Section 2.

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