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A smooth flat Riemannian manifold is called an *exceptional domain* if it admits positive harmonic functions having vanishing Dirichlet boundary data and constant (nonzero) Neumann boundary data. In analogy with minimal surfaces, a representation formula is derived and applied to the classification of exceptional domains. Some interesting open problems are proposed along the way.

1. Introduction

Given an m -dimensional Riemannian manifold (M, g) and a smooth bounded domain Ω in M , we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplace–Beltrami operator under an identically zero Dirichlet boundary condition. The critical points of the functional

$$\Omega \mapsto \lambda_1(\Omega)$$

under the volume constraint $\text{Vol } \Omega = \alpha$, where $\alpha \in (0, \text{Vol } M)$ is fixed, are called *extremal domains*. Smooth extremal domains are characterized by the property that the eigenfunctions associated with the first eigenvalue of the Laplace–Beltrami operator have constant Neumann boundary data [Soufi 2007]. In other words, a smooth domain is extremal if and only if there exists a positive function u_1 and a constant λ_1 such that

$$\Delta_g u_1 + \lambda_1 u_1 = 0$$

in Ω with $u_1 = 0$ and $\nabla_n u_1$ constant on $\partial\Omega$, where n denotes the inward unit normal vector to $\partial\Omega$.

The theory of extremal domains is very reminiscent of the theory of constant mean curvature surfaces or hypersurfaces. To give some credit to this assertion, we recall that J. Serrin [1971] proved that the only compact, smooth, extremal domains in Euclidean space are round balls, paralleling the well-known result of Alexandrov asserting that round spheres are the only (embedded) compact constant mean curvature hypersurfaces in Euclidean space. More recently, F. Pacard and

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P. Sicbaldi [2009] proved the existence of extremal domains close to small geodesic balls centered at critical points of the scalar curvature function, paralleling an earlier result of R. Ye [1991], which provides constant mean curvature topological spheres (with high mean curvature) close to small geodesic spheres centered at nondegenerate critical points of the scalar curvature function.

We propose the following:

Definition 1.1. A smooth domain $\Omega \subset \mathbb{R}^m$ is said to be an *exceptional domain* if it supports positive harmonic functions having identically zero Dirichlet boundary data and constant (nonzero) Neumann boundary data. Any such harmonic function is called a *roof function*.

Exceptional domains arise as limits under scaling of sequences of extremal domains, just like minimal surfaces arise as limits under scaling of sequences of constant mean curvature surfaces. As explained above, there is a formal correspondence between extremal domains and constant mean curvature surfaces. In this note, we try to explain that there is also a strong analogy between exceptional domains and minimal surfaces. More generally, we propose:

Definition 1.2. An m -dimensional flat Riemannian manifold M is said to be *exceptional* if it supports positive harmonic functions having identically zero Dirichlet boundary data and constant (nonzero) Neumann boundary data. Any such harmonic function is called a *roof function*.

Our results raise the problem of the classification of (unbounded) smooth m -dimensional exceptional manifolds. In trying to address this classification problem, we provide a Weierstrass-type representation formula for exceptional flat surfaces. When the dimension $m = 2$, we give nontrivial examples of exceptional domains that are embedded in \mathbb{R}^2 , and we prove a half-space result for exceptional domains that are conformal to a half-plane.

2. A nontrivial example of an exceptional domain in \mathbb{R}^2

The property of being an exceptional domain is preserved under the action of the group of similarities of \mathbb{R}^m (generated by isometries and dilations). We first give trivial examples of exceptional domains in \mathbb{R}^m :

- (i) The half-space $\{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_1 > 0\}$ is an exceptional domain in \mathbb{R}^m , since the function $u(x) = x_1$ is a positive harmonic function with identically zero Dirichlet boundary data and constant Neumann boundary data.
- (ii) The complement of a ball of radius 1 in \mathbb{R}^m is an exceptional domain since the function u defined by $u(x) := \log|x|$ when $m = 2$ and by $u(x) := 1 - |x|^{2-m}$ when $m \geq 3$ is positive, harmonic, and has 0 Dirichlet and constant Neumann data on the unit sphere.

(iii) The product $\Omega \times \mathbb{R}^k$ is an exceptional domain in \mathbb{R}^m provided $\Omega \subset \mathbb{R}^{m-k}$ is an exceptional domain in \mathbb{R}^{m-k} .

In dimension $m = 2$, there exists (up to a similarity) at least one other exceptional domain. To describe this domain, we make use of the invariance of the Laplace operator under conformal transformations. The idea is that there exists a (somehow natural) unbounded, positive harmonic function U with identically zero Dirichlet boundary condition on an infinite strip in \mathbb{R}^2 . This function does not have constant Neumann data, but we can then look for a conformal transformation h which has the property that the pullback of the harmonic function U by h has constant Neumann boundary data on the boundary of the image of the strip by h .

To proceed, it is convenient to identify \mathbb{R}^2 with the complex plane \mathbb{C} .

Proposition 2.1. *The domain $\Omega := \{w \in \mathbb{C} : |\operatorname{Im} w| < \pi/2 + \cosh(\operatorname{Re} w)\}$ is an exceptional domain.*

To prove this result, we define the infinite strip

$$S := \{z \in \mathbb{C} : \operatorname{Im} z \in (-\pi/2, \pi/2)\}$$

and the mapping

$$F(z) := z + \sinh z.$$

Observe that $\Omega = F(S)$. The proof of [Proposition 2.1](#) follows from the next two lemmas.

Lemma 2.2. *The mapping F is a conformal diffeomorphism from S into Ω .*

Proof. We can write

$$F(z) - F(z') = (z - z') \int_0^1 (1 + \cosh(tz + (1-t)z')) dt.$$

In particular

$$(2-1) \quad \langle z - z', F(z) - F(z') \rangle = |z - z'|^2 \left(1 + \int_0^1 \operatorname{Re} \cosh(tz + (1-t)z') dt \right),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{C} . Now, for all $x + iy \in S$, we have

$$\operatorname{Re} \cosh(x + iy) = \cosh x \cos y \geq 0.$$

This, together with (2-1), implies immediately that F restricted to S is injective. Also,

$$|\partial_z \Lambda(z)|^2 = |1 + \cosh z|^2 = (\cosh x + \cos y)^2.$$

Therefore $\partial_z F$ does not vanish in S . Thus F is a local diffeomorphism, and because the mapping F is holomorphic, it is conformal. \square

We define the real-valued function u on Ω by the identity

$$u(F(z)) = \operatorname{Re} \cosh z \quad \text{for all } z \in S.$$

Lemma 2.3. *The function u is harmonic and positive in Ω , vanishes and has constant Neumann boundary data on $\partial\Omega$.*

Proof. The function W defined in \mathbb{C} by $W(z) := \operatorname{Re} \cosh z$ is harmonic. Indeed, as mentioned in the proof of the previous lemma, $W(x + iy) = \cosh x \cos y$. Hence W is both harmonic and positive in S , and vanishes on ∂S . The mapping F being a conformal diffeomorphism from S to Ω , we conclude the function u is both harmonic and positive in Ω , and vanishes on $\partial\Omega$. We claim that u has constant Neumann data on $\partial\Omega$. Indeed, by definition,

$$u(F(z)) = \frac{1}{2}(\cosh z + \cosh \bar{z}).$$

Since F is holomorphic, differentiation with respect to z yields

$$2 \partial_z u(F(z)) = \frac{\sinh z}{1 + \cosh z}.$$

Therefore

$$|\nabla u|^2(F(z)) = \frac{\cosh x - \cos y}{\cosh x + \cos y},$$

where $z = x + iy$. On $\partial\Omega$, we have $y = \pm\pi/2$ and hence $|\nabla u| \equiv 1$. Since we already know that $u = 0$ on $\partial\Omega$, we conclude that u has constant Neumann boundary data. \square

Lemmas 2.2 and 2.3 complete the proof that $\Omega = F(S)$ is an exceptional domain in \mathbb{R}^2 with roof function u .

Remark 2.4. We suspect that this example generalizes to any dimension $m \geq 3$: specifically, there should exist a rotationally symmetric exceptional domain in \mathbb{R}^m for all $m \geq 3$.

3. Toward a global representation formula

Let M be an *exceptional flat surface* (an exceptional domain of dimension 2) with smooth boundary ∂M . Let \tilde{M} be its universal cover and let $\partial\tilde{M}$ be the preimage of ∂M by the covering map $\tilde{M} \rightarrow M$. In the following, we exclude the uninteresting case where $\partial M = \emptyset$.

By assumption, M is a flat surface. Hence \tilde{M} is naturally endowed with a flat Riemannian metric g and hence with an induced complex structure, which is conformal to the standard one. Also, there exists an orientation-preserving isometric immersion $F : (\tilde{M}, g) \rightarrow (\mathbb{C}, g_{\mathbb{C}})$, where $g_{\mathbb{C}}$ is the canonical Euclidean metric on

\mathbb{C} ; this induces a smooth immersion of $\partial\tilde{M}$. Observe that F is holomorphic and that $\|dF\|_g = 1$ in $\tilde{M} \cup \partial\tilde{M}$. We define the holomorphic $(1, 0)$ -form

$$\Phi := dF = \partial_z F dz.$$

Observe that Φ does not vanish and admits a smooth extension to $\tilde{M} \cup \partial\tilde{M}$.

We let $u : M \rightarrow \mathbb{R}^+$ be a roof function on M and, with a slight abuse of notation, we denote its lift also by $u : \tilde{M} \rightarrow \mathbb{R}^+$. The roof function u can be normalized so that

$$(3-1) \quad \|\nabla u\|_g = 1$$

on ∂M . We consider the harmonic conjugate function $v : \tilde{M} \rightarrow \mathbb{R}$ (uniquely defined up to some additive constant) that is the solution of

$$(3-2) \quad \partial_z(u - i v) = 0 \quad (\text{and hence } \partial_{\bar{z}}(u + i v) = 0).$$

We set

$$U := u + i v.$$

Recall that U is a holomorphic function from \tilde{M} into \mathbb{C} . The property that u takes positive values in M and vanishes on ∂M can be translated into the fact that U maps \tilde{M} to $\mathbb{C}^+ := \{w \in \mathbb{C} : \text{Re } w > 0\}$ and $\partial\tilde{M}$ to $i\mathbb{R}$. Since $\Phi \neq 0$ on \tilde{M} , there exists a unique holomorphic function h on \tilde{M} such that $dU = \partial_z U dz = h \Phi$. We deduce from the fact that u vanishes on $\partial\tilde{M}$ and from (3-1) that $\nabla_n U = 1$, where n denotes the inward unit normal vector to $\partial\tilde{M}$. Hence

$$(3-3) \quad \|\partial_z U\|_g = 1 \quad \text{on } \partial\tilde{M}.$$

Now, condition (3-1) translates into the fact that $\|\Phi\|_g = \|dF\|_g = 1 = \|dU\|_g$ on $\partial\tilde{M}$. Clearly, this is equivalent to the fact that $|h| = 1$ on $\partial\tilde{M}$. Therefore, we end up with the following data:

- (i) An oriented, simply connected complex surface \tilde{M} with smooth boundary $\partial\tilde{M}$.
- (ii) A holomorphic function U , defined on \tilde{M} , which takes values in \mathbb{C}^+ and maps $\partial\tilde{M}$ into $i\mathbb{R}$.
- (iii) A holomorphic function h , defined on \tilde{M} , such that $|h| = 1$ on $\partial\tilde{M}$, and for which the 1-form Φ defined by $\Phi := (1/h)dU$ does not vanish on \tilde{M} .

By analogy with the theory of minimal surfaces, we call these data the Weierstrass-type representation formula for exceptional flat surfaces.

Conversely, given a set of such data, we can define the map $F : \tilde{M} \rightarrow \mathbb{C}$ by integrating $dF = \Phi$. Thanks to (iii), this map is an immersion and its image is an immersed exceptional flat surface with roof function given by $u = \text{Re } U$. In

Section 4, we will give some explicit examples of such constructions when $\partial\tilde{M}$ is equal to $\partial D \setminus \{\alpha_1, \dots, \alpha_n\}$, where $\alpha_1, \dots, \alpha_n$ is a finite collection of points on $\partial D = S^1$.

Example 3.1. Here is a (rather pathological) illustration of this Weierstrass-type formula. Consider $M = \mathbb{C}^+$, the function $U(z) = z$ and

$$F(z) = \int_0^z e^{-\sinh \zeta} d\zeta.$$

Note that $\partial_z F$ is $2i\pi$ -periodic, and this implies that $F(z + 2i\pi) = F(z) + C$, where the constant C is given explicitly by

$$C := i \int_0^{2\pi} e^{-i \sin s} ds.$$

Moreover, for $x > 0$,

$$F(x + iy) = F(iy) + \int_0^x e^{-\sinh(s+iy)} ds$$

converges to $+\infty$ as $x \rightarrow +\infty$ if $y = 0$, but this quantity is bounded if $|y - \pi| < \pi/2$, and even admits a finite limit as $x \rightarrow +\infty$.

Hence, in addition to the regular boundary $F(i\mathbb{R})$, which is a smooth periodic curve, the image of F has a singular boundary: the set of limits of $F(x + iy)$ as x tends to $+\infty$, for the values of y for which this limit exists. The roof function tends to infinity along this singular boundary.

4. Examples of exceptional flat surfaces

Thanks to the Weierstrass-type representation in the previous section, we can give many nontrivial examples of exceptional flat surfaces. We keep the notation from that section.

The construction makes use of an integer $n \in \mathbb{N} \setminus \{0\}$ and the Riemann surface $D = \{z \in \mathbb{C} : |z| < 1\}$. On D , we define the holomorphic functions

$$h(z) = z^{n-1} \quad \text{and} \quad U(z) := \frac{1 + z^n}{1 - z^n}.$$

The 1-form Φ is given by

$$\Phi(z) := \frac{2n}{(1 - z^n)^2} dz.$$

Both U and Φ have singularities at the n -th roots of unity. The function F is then obtained by integrating Φ , and the roof function u is defined by $u = \operatorname{Re} U$.

(i) When $n = 1$, we can take

$$F(z) = \frac{1+z}{1-z}.$$

In this case, we simply have $F(D) = \mathbb{C}^+$, and we recover the fact that the half-plane is an exceptional domain. This is the counterpart of the plane in the framework of minimal surfaces.

(ii) When $n = 2$, we can take

$$F(z) = \frac{2z}{1-z^2} + \log \frac{z+1}{z-1}.$$

In this case, the exceptional flat surface found can be isometrically embedded in \mathbb{C} , and hence $F(D)$ is an exceptional domain. In fact, $F(D)$ corresponds (up to some similarity) to the domain Ω , which was defined in [Proposition 2.1](#). This exceptional domain is the counterpart of the catenoid.

(iii) Finally when $n \geq 3$, the exceptional flat surfaces we find cannot be isometrically embedded in \mathbb{C} anymore. They are counterparts of the minimal n -noids described in [\[Jorge and Meeks 1983\]](#).

Let us analyze this example further. The function U can be written as

$$U(z) = -\frac{1}{n} \sum_{k=1}^n \frac{z + \alpha^k}{z - \alpha^k},$$

where $\alpha := e^{i2\pi/n}$. In particular, $\text{Re } U$ is nothing but a multiple of the sum of the Poisson kernel on the unit disc with poles at $1, \alpha, \dots, \alpha^{n-1}$. Next,

$$dU = z^{n-1} \frac{2n}{(1-z^n)^2} dz,$$

so the function h is cooked up to counterbalance the zero of dU and ensure that Φ does not vanish in the unit disk, while keeping the condition $|dU|^2 = |\Phi|^2$ on ∂D .

To generalize the example, consider n distinct points $\alpha_1, \dots, \alpha_n \in \mathcal{S}^1 \subset \mathbb{C}$ and $a_1, \dots, a_n > 0$. We define

$$(4-1) \quad U(z) := -\sum_{k=1}^n a_k \frac{z + \alpha_k}{z - \alpha_k}.$$

It is easy to check that $\text{Re } U$ is positive (since each function $z \mapsto -\frac{z + \alpha_k}{z - \alpha_k}$ maps D to \mathbb{C}^+) and vanishes on $\partial D \setminus \{\alpha_1, \dots, \alpha_n\}$. We have

$$\prod_{k=1}^n (z - \alpha_k)^2 dU = P(z) dz,$$

where P is a polynomial that depends on the choice of points $\alpha_1, \dots, \alpha_n$ and weights a_1, \dots, a_n . Assume that P does not vanish on ∂D and denote by z_1, \dots, z_l the roots of P in the unit disc, counted with multiplicity. We simply define

$$h(z) := \prod_{j=1}^l \frac{z - z_j}{z \bar{z}_j - 1}$$

and the 1-form Φ by $\Phi := (1/h) dU$. Integration of Φ yields a $2n$ -dimensional family of exceptional flat surfaces immersed in \mathbb{C} .

5. A global Weierstrass-type representation

In this section, we show that exceptional flat surfaces whose immersion in \mathbb{C} have finitely many *regular ends* and are locally finite coverings of \mathbb{C} are precisely the examples in the previous section. We use the notations introduced in [Section 3](#), and we set

$$\widehat{M} := M \cup \partial M.$$

We further assume that M is *simply connected* and that $\partial M \neq \emptyset$. In particular, M has the conformal type of the unit disk D , and without loss of generality, we can assume that M is indeed equal to D and consider \bar{D} as a natural compactification of M . We denote by F an orientation preserving, holomorphic, isometric immersion $F : (\widehat{M}, g) \rightarrow (\mathbb{C}, g_{\mathbb{C}})$. Recall that $\|dF\|_g = 1$ on ∂M . Some natural hypotheses are needed:

(H-1) M has *finitely many ends*. This means that

$$\partial M = \partial D \setminus \bigcup_{j=1}^n E_j = \bigcup_{j=1}^n I_j,$$

where each $E_j \subset S^1$ is a closed arc and $I_j \subset S^1$ is an open arc.

(H-2) F is *proper*. This means that $F(w)$ tends to infinity as w tends to $\bigcup_{j=1}^n E_j$.

(H-3) *Each end of M is regular*. This means the image of $I_j := (\theta_j^-, \theta_j^+)$ by F is a curve Γ_j asymptotically parallel to fixed directions at infinity. In other words, there exist two unit vectors τ_j^- and $\tau_j^+ \in S^1 \subset \mathbb{C}$ such that

$$\lim_{\theta \in I_j, \theta \rightarrow \theta_j^\pm} \frac{F(e^{i\theta})}{|F(e^{i\theta})|} = \tau_j^\pm.$$

This is the case, for example, if we assume each Γ_j has finite total curvature.

(H-4) *The mapping F is a locally finite covering*. This means there exists $d \in \mathbb{N}^*$ such that, for any $z \in \mathbb{C}$, the cardinal of $\{\zeta \in M : F(\zeta) = z\}$ is at most d .

We now state the main result of this section.

Theorem 5.1. *Assume that M is a simply connected exceptional flat surface and let $F : M \rightarrow \mathbb{C}$ be an isometric immersion. Further assume that (H-1)–(H-4) holds and identify M with D . Then there exist $\mu \in \mathbb{R}$, n distinct points $\alpha_1, \dots, \alpha_n \in S^1$ and n constants $a_1, \dots, a_n > 0$ such that*

$$dF = e^{i\mu} \prod_{k=1}^m \frac{\bar{z}_k z - 1}{z - z_k} dU,$$

where $z_1, \dots, z_m \in \bar{D}$ denote the zeros of dU , counted with multiplicity, and where

$$U(z) := - \sum_{j=1}^n a_j \frac{z + \alpha_j}{z - \alpha_j} \quad \text{in } \bar{D}.$$

The proof is divided into a few lemmas and propositions. We start by analyzing the ends E_j and show that they reduce to isolated points $\alpha_1, \dots, \alpha_n$. Next we analyze the behavior of F near the points α_j and show that F does not have any essential singularity there. Then we proceed with the analysis of the function U and show that it has the expected form. The proof is completed with the study of the function h .

As promised, we first analyze the sets E_j :

Lemma 5.2. *Under the assumptions of Theorem 5.1, there exists a finite number of points $\alpha_1, \dots, \alpha_n \in \partial D = S^1$ such that $\widehat{M} = \bar{D} \setminus \{\alpha_1, \dots, \alpha_n\}$.*

Proof. We need to show that each interval E_j is reduced to a point. This essentially follows from the fact that the capacity of E_j vanishes.

Suppose, for a contradiction, that E_j is an arc of positive arc length for some j , and take some $l \in (0, \pi/2)$ and an arc $E \subset E_j$ of length l . Our problem being invariant under the action of fractional linear transformations of the unit disk, we can assume without loss of generality that E is the image of $[-l/2, l/2]$ under $s \mapsto e^{is}$. Reducing l if necessary, we can also assume that the opposite arc $-E$, the image of $[-l/2, l/2]$ under $s \mapsto -e^{is}$, is contained in $S^1 \setminus \bigcup_{j=1}^n E_j$.

Recall that for any smooth function defined on (a, b) which satisfies $f(b) = 1$ and $f(a) = 0$, we have

$$1 = f(b) - f(a) = \int_a^b f'(s) ds \leq \left(\int_a^b (f')^2(s) ds \right)^{1/2} \sqrt{b - a}.$$

If in addition, $b - a \leq 2$, we conclude that

$$\int_a^b (f')^2(s) ds \geq \frac{1}{2}.$$

Now assume that we are given a smooth function $f : \bar{D} \rightarrow \mathbb{R}$ such that $f = 1$ on E and $f = 0$ on $-E$. Using the previous inequality, we can write

$$(5-1) \quad \int_D \|\nabla f\|_{g_{\mathbb{C}}}^2 dx dy \geq \int_{D \cap \{|x| < \sin(l/2)\}} |\partial_y f|^2 dx dy \geq \int_{|x| \leq \sin(l/2)} \frac{1}{2} dx = \sin \frac{l}{2}.$$

Given $R > r > 0$, define $\chi : \mathbb{C} \rightarrow \mathbb{R}$ by

$$\chi(z) = \begin{cases} 0 & \text{if } |z| \leq r, \\ \frac{\log(|z|/r)}{\log(|z|/R)} & \text{if } r \leq |z| \leq R, \\ 1 & \text{if } R \leq |z|, \end{cases}$$

and we define $f : D \rightarrow \mathbb{R}$ by $f := \chi \circ F$. Since F is conformal, we can write

$$\int_D \|\nabla f\|_{g_{\mathbb{C}}}^2 dx dy = \int_D \|\nabla \chi\|_g^2 d \text{vol}_g.$$

Now, using (H-4), we conclude that

$$(5-2) \quad \int_D \|\nabla f\|_g^2 d \text{vol}_g \leq d \int_{\mathbb{C}} \|\nabla \chi\|_{g_{\mathbb{C}}}^2 dx dy = d \frac{2\pi}{\log(R/r)}.$$

Fixing $r > 0$ large enough, we can ensure that f is identically equal to 0 on $-E$. Using (H-2), we see that f is identically equal to 1 on each E_j , and in particular on E . Therefore f can be used in (5-1), which together with (5-2) yields

$$2\pi d \geq \sin \frac{l}{2} \log \frac{R}{r}$$

independently of $R > r$. Letting R tend to infinity, we get a contradiction, and the proof is complete. □

Therefore, we now know that $E_j := \{\alpha_j\}$. Without loss of generality, we can assume that $\alpha_1, \dots, \alpha_n$ are arranged counterclockwise along S^1 . We agree that $\alpha_0 := \alpha_n$ and $\alpha_{n+1} := \alpha_1$, and that for each $j = 1, \dots, n$, the arc I_j is positively oriented and joins α_j to α_{j+1} . We now analyze the singularities of F close to α_j .

Given $j = 1, \dots, n$, we denote by $S(\alpha_j, r)$ the circle of radius $r > 0$ centered at α_j . We define

$$\gamma_j := \bar{D} \cap S(\alpha_j, r),$$

which we assume to be oriented clockwise. The angle $\theta_j \in \mathbb{R}$ at α_j is defined by

$$\theta_j := - \lim_{r \rightarrow 0} \int_{\gamma_k} F^* d\theta,$$

where $d\theta := \text{Im } dz/z$. Thanks to (H-3), θ_j is well defined, and we have

$$\tau_j^- = e^{i\theta_j} \tau_{j-1}^+.$$

Lemma 5.3. *Under the assumption of Theorem 5.1, the function*

$$H_j(z) := (z - \alpha_j)^{\theta_j/\pi} F(z)$$

is holomorphic in a neighborhood of α_j in $\bar{D} \setminus \{\alpha_j\}$, and $H_j(\alpha_j) \neq 0$.

Proof. Without loss of generality, we can assume that $\alpha_j = 1$. By right composing F with the conformal transformation $z \mapsto (1 - z)/(1 + z)$, we can replace D by \mathbb{C}^+ . Now we define

$$G(z) := F(z)^{-\pi/\theta_j}.$$

Observe that $G(0) = 0$ by (H-2). Moreover, (H-3) and the definition of θ_j imply that the image by G of a neighborhood of 0 in $i\mathbb{R}$ is a \mathcal{C}^1 -curve, and hence analytic. In particular, there exists some conformal transformation T such that, for some $r > 0$, the image by $T \circ G$ of $i(-r, r)$ is a straight line segment in $i\mathbb{R}$. Then it is possible to extend $T \circ G$ to a function \tilde{G} defined on a neighborhood of 0 in \mathbb{C} by setting

$$\tilde{G}(z) = \begin{cases} T(G(z)) & \text{if } \text{Im } z \geq 0, \\ -\overline{T(G(-\bar{z}))} & \text{if } \text{Im } z \leq 0. \end{cases}$$

The resulting \tilde{G} is bounded in a neighborhood of 0 in \mathbb{C} and holomorphic away from 0. It is well known that the singularity is then removable and hence it is holomorphic. Therefore \tilde{G} is actually holomorphic in a neighborhood of 0. In particular, we can write

$$G(z) = z^k H(z)$$

near 0, where H is a holomorphic function that does not vanish at 0. Going back to the definition of G , this implies that

$$F(z) = (z - \alpha_j)^{-k\theta_j/\pi} H_j(z),$$

where H_j is holomorphic in a neighborhood of α_j and does not vanish at α_j . But the definition of θ_j readily implies that $k = 1$. This completes the proof. \square

As a corollary, we conclude that

$$(5-3) \quad H(z) := F(z) \prod_{j=1}^n (z - \alpha_j)^{\theta_j/\pi}$$

is a bounded holomorphic function in \bar{D} . Moreover, since F tends to infinity as z approaches α_j , this implies that $\theta_j > 0$.

We now make use of the fact that M is an exceptional domain, and hence there is a roof function $u : \hat{M} \rightarrow [0, +\infty)$. We can define the holomorphic function $U := u + iv$, where $v : \hat{M} \rightarrow \mathbb{R}$ is the (real-valued) harmonic conjugate of u . The purpose of the next result is to show that U is precisely given by (4-1).

Lemma 5.4. *Under the assumptions of Theorem 5.1, there exist n constants $a_1, \dots, a_n > 0$ such that*

$$U(z) = - \sum_{j=1}^n a_j \frac{z + \alpha_j}{z - \alpha_j}.$$

Proof. First, it is possible to extend the function U to all $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_n\}$ by defining V to be equal to U in $\bar{D} \setminus \{\alpha_1, \dots, \alpha_n\}$, and

$$V(z) := -\overline{U(1/\bar{z})}$$

when $z \in \mathbb{C} \setminus \bar{D}$. The key observation is that, since $\operatorname{Re} U = 0$ on $\partial D \setminus \{\alpha_1, \dots, \alpha_n\}$, the function V is continuous and in fact holomorphic on $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_n\}$. Moreover, V converges to $V(\infty) := -\overline{U(0)}$ at infinity.

We proceed with the proof that the function V has no essential singularity at any α_j ; it will follow from Picard's theorem. By definition, $\operatorname{Re} V$ vanishes on I_j and is positive in D . Therefore the outward normal derivative of $\operatorname{Re} V$ on I_j is negative. This implies that the tangential derivative of $\operatorname{Im} V$ on I_j does not vanish and hence that $\operatorname{Im} V$ is strictly monotone on each I_j . This shows that there exists some neighborhood \mathcal{V} of α_j in \mathbb{C} such that any element of $i\mathbb{R}$ is achieved by V at most twice in \mathcal{V} (that is, at most once on I_j and at most once on I_{j-1} and certainly not in $\mathcal{V} \setminus \partial D$, since V takes values in $\mathbb{C} \setminus i\mathbb{R}$ away from ∂D). Picard's big theorem [Conway 1978] then implies that α_j is not an essential singularity of V . Hence α_j is either a removable singularity of V or a pole.

Since $\|\nabla u\|_g \equiv 1$ on ∂M , this forces $|\partial_z U| = |\partial_z F|$ on ∂M , and since $|\partial_z F|$ tends to $+\infty$ at α_j , so does $|\partial_z U|$. Hence all α_j are poles of V .

We are now interested in the zeros of V . Since $\operatorname{Re} V$ takes positive values in D and negative values in $(\mathbb{C} \cup \{\infty\}) \setminus \bar{D}$, we already know that the only possible zeros of V are on ∂D . We have already seen that, along I_j , the function V equals iv , where v is strictly monotone. Further, since α_{j-1} and α_j are poles of this function, $|V|$ must converge to $+\infty$ as we approach either α_{j-1} or α_j . Because of the continuity of v along each I_j it follows that v vanishes exactly at one point β_j on each I_j . Moreover, this zero is simple: if it had order $k > 1$, the zero set of $\operatorname{Re} V$ near β_j would contain k curves intersecting at β_j , and this would force $\operatorname{Re} V = \operatorname{Re} U$ to vanish in D , in contradiction with our hypothesis.

Finally, we prove that V has only simple poles. We know that V extends meromorphically to a map on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with neither a pole nor a zero at infinity. Furthermore, V has exactly n simple zeros and n poles; hence these poles must be simple. To summarize, V can be written as a linear combination of the constant function and functions of the form $z \mapsto 1/(z - \alpha_j)$. Without loss of generality, this

amounts to saying that V can be written as

$$V(z) = a - \sum_{j=1}^n a_j \frac{z + \alpha_j}{z - \alpha_j},$$

where a and the a_j are complex numbers. Using the fact that, by construction, $V(1/\bar{z}) = -\overline{V(z)}$, we conclude that $a \in i\mathbb{R}$ and also that $a_j \in \mathbb{R}$. Moreover, since $\text{Re } U$ is positive, this implies that the a_j are positive real numbers. This completes the proof, since U is defined up to the addition of some element of $i\mathbb{R}$. \square

We are now in a position to complete our analysis of the function F . Since F is an immersion, $dF \neq 0$ on \widehat{M} . Hence there exists a unique holomorphic function h on \widehat{M} such that

$$(5-4) \quad \partial_z U = h \partial_z F$$

on \widehat{M} . Moreover, since $\|\nabla u\|_g \equiv 1$ on ∂M , this implies that $|h| \equiv 1$ on ∂M . We now analyze the function h , which will complete the proof of [Theorem 5.1](#).

Lemma 5.5. *Under the assumptions of [Theorem 5.1](#), there exists a constant $e^{i\mu} \in \mathbb{R}$ such that the function h defined by (5-4) has the form*

$$h(z) = e^{-i\mu} \prod_{k=1}^m \frac{z - z_k}{\bar{z}_k z - 1},$$

where z_1, \dots, z_m are the zeros of $\partial_z U$ in D counted with multiplicity.

Proof. The function h is holomorphic in D and satisfies $|h|=1$ on $\partial D \setminus \{\alpha_1, \dots, \alpha_n\}$. We can extend h to a holomorphic function H , defined on $(\mathbb{C} \cup \{\infty\}) \setminus \{\alpha_1, \dots, \alpha_n\}$ by setting $H(z) := h(z)$ for all $z \in \bar{D} \setminus \{\alpha_1, \dots, \alpha_n\}$ and

$$(5-5) \quad \overline{H(z)} := \frac{1}{h(1/\bar{z})}$$

for all $z \in \mathbb{C} \setminus \bar{D}$. Clearly H is locally bounded in $\bar{D} \setminus \{\alpha_1, \dots, \alpha_n\}$, and its only singularities in $(\mathbb{C} \cup \{\infty\}) \setminus \bar{D}$ are poles that are the images by $z \mapsto 1/\bar{z}$ of the zeros of h ; hence H is meromorphic outside $\{\alpha_1, \dots, \alpha_n\}$. But [Lemma 5.3](#) and (5-3) imply that, near α_j , $|H|$ is bounded by a constant times $|z - \alpha_j|^{-k_j}$ for some $k_j > 0$. Therefore α_j is not an essential singularity of H , and hence H is meromorphic in $\mathbb{C} \cup \{\infty\}$.

Observe that $|H(z)| = 1$ on $\partial D \setminus \{\alpha_1, \dots, \alpha_n\}$, and this implies that the points α_j are not poles of H . Therefore, the singularities α_j of H are removable. Also, we have

$$\Delta |H|^2 = 4\partial_z \partial_{\bar{z}} |H|^2 = 4|\partial_z H|^2 \geq 0,$$

and since $|H| = 1$ on ∂D , the maximum principle implies that $|H| \leq 1$ in D .

Since H is bounded in \bar{D} , it does not have poles in this set. This also implies that H has no zeroes in $(\mathbb{C} \cup \{\infty\}) \setminus D$, because otherwise H would have poles in \bar{D} by (5-5). Therefore, if $z_1, \dots, z_m \in D$ denote the zeros of H (counted with multiplicity), the poles of H are given by $1/\bar{z}_1, \dots, 1/\bar{z}_m$ (also counted with multiplicity). It is then a simple exercise to check that H is of the form

$$H(z) = C \prod_{k=1}^m \frac{z - z_k}{\bar{z}_k z - 1}$$

for some constant $C \in \mathbb{C}$. Finally, the condition that $|H(z)| = 1$ on ∂D forces $|C| = 1$. This completes the proof. \square

6. A Bernstein type result for two-dimensional exceptional domains

We prove the following Bernstein type result for two-dimensional exceptional domains.

Proposition 6.1. *Assume that Ω is a two-dimensional exceptional domain conformal to \mathbb{C}^+ , and let u be a roof function on Ω . We further assume that $\partial_x u > 0$ in Ω . Then Ω is a half-plane.*

Proof. Since we have assumed that Ω is conformal to \mathbb{C}^+ , there exists a holomorphic map $\Psi : \mathbb{C}^+ \mapsto \Omega$. We then define

$$H := (\partial_z u) \circ \Psi.$$

The function H is holomorphic in \mathbb{C}^+ and does not vanish, since we have assumed that $\partial_x u \neq 0$. Moreover, $|H| \equiv 1$ on $\partial\mathbb{C}^+$. We can write $H = e^{i\Theta}$, where Θ is a holomorphic function defined in \mathbb{C}^+ that is real valued on the imaginary axis. This means that $\text{Im } \Theta = 0$ when $\text{Re } z = 0$. Since we have assumed that $\partial_x u > 0$, we also conclude that $\text{Re } \Theta \in (-\pi/2, \pi/2)$.

We can extend Θ as a holomorphic function $\tilde{\Theta}$ in \mathbb{C} as follows:

$$\tilde{\Theta}(z) := \begin{cases} \Theta(z) & \text{if } \text{Re } z \geq 0, \\ \overline{\Theta(-\bar{z})} & \text{if } \text{Re } z < 0. \end{cases}$$

It is easy to check that $\tilde{\Theta}$ is a holomorphic function: in fact, the real part of Θ is extended as an even function of $\text{Re } z$, while the imaginary part of Θ is extended as an odd function of $\text{Re } z$. That $\tilde{\Theta}$ is \mathcal{C}^1 is then a consequence of the fact that $\text{Im } \Theta = 0$ on the imaginary axis, while the holomorphicity of Θ follows from the fact that $\partial_x \text{Re } \Theta = 0$ on the imaginary axis of \mathbb{C} .

The real part of $\tilde{\Theta}$, being a bounded harmonic function, must be constant. Then $\tilde{\Theta}$, being holomorphic, must itself be constant. But this implies that the gradient of u is constant, and hence the level sets of u are straight lines. This implies that

u only depends on one variable, and hence it is an affine function. This completes the proof. \square

Corollary 6.2. *There is no exceptional domain contained in a wedge*

$$\Omega \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq \kappa |\operatorname{Im} z|\}$$

for any $\kappa > 0$.

Proof. The proof is by contradiction. If Ω were such an exceptional domain, there would exist on Ω a roof function u . One can apply the moving plane method [Serrin 1971; Gidas et al. 1979] to prove that $\partial_x u > 0$ and hence that $\partial\Omega$ is a graph over the y -axis. Since Ω is contained in a half-plane, there is no bounded, positive, harmonic function on Ω having 0 boundary data on $\partial\Omega$; otherwise one could use an affine function as a barrier to obtain a contradiction. Certainly, $\Omega \cup \partial\Omega$ is conformal to $\bar{D} \setminus E$, where D is the unit disc and E is a closed arc included in S^1 . Necessarily, E is reduced to a point, since otherwise we can construct bounded, positive, harmonic functions on E that have 0 boundary data on $S^1 \setminus E$, and these would lift to bounded, positive, harmonic function on Ω , with 0 boundary data, a contradiction. Therefore, we conclude that Ω is conformal to \mathbb{C}^+ . The assumptions of Lemma 5.5 are fulfilled, and hence we conclude that Ω is a half-plane, which is a contradiction. \square

7. Open problems

We have no nontrivial example of an exceptional domain in higher dimensions \mathbb{R}^m for $m \geq 3$, besides the ones described in Section 2. In dimension $m = 2$, it is tempting to conjecture that (up to similarity) the only exceptional domains that can be embedded in \mathbb{R}^2 are half-spaces, the complement of a ball and the example discussed in Section 2.

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Volume 250 No. 2 April 2011

Realizing profinite reduced special groups	257
VINCENT ASTIER and HUGO MARIANO	
On fibered commensurability	287
DANNY CALEGARI, HONGBIN SUN and SHICHENG WANG	
On an overdetermined elliptic problem	319
LAURENT HAUSWIRTH, FRÉDÉRIC HÉLEIN and FRANK PACARD	
Minimal sets of a recurrent discrete flow	335
HATTAB HAWETE	
Trace-positive polynomials	339
IGOR KLEP	
Remarks on the product of harmonic forms	353
LIVIU ORNEA and MIHAELA PILCA	
Steinberg representation of $\mathrm{GSp}(4)$: Bessel models and integral representation of L -functions	365
AMEYA PITALE	
An integral expression of the first nontrivial one-cocycle of the space of long knots in \mathbb{R}^3	407
KEIICHI SAKAI	
Burghelea–Haller analytic torsion for twisted de Rham complexes	421
GUANGXIANG SU	
$K(n)$ -localization of the $K(n+1)$ -local E_{n+1} -Adams spectral sequences	439
TAKESHI TORII	
Thompson’s group is distorted in the Thompson–Stein groups	473
CLAIRE WLADIS	
Parabolic meromorphic functions	487
ZHENG JIAN-HUA	