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S. G. Dani, giving a counterexample to a result in a paper of Knight, showed that recurrent transitive flows can admit multiple minimal sets. Here we show that such a phenomenon occurs on a wider scale.

Let (X, T) be a discrete flow, where X is a compact metric space and T is a self-homeomorphism of X. For $x \in X$, the set $\{T^n(x) : n \in \mathbb{Z}\}$ is called the *orbit* of x and is denoted by $O(x, T)$. A set W is a *minimal set* of (X, T) if for all $x \in W$ we have $\overline{O(x, T)} = W$. The study of minimal sets of such a system is a central question in topological dynamics. Zorn's lemma ensures the existence of at least a minimal set of (X, T). If X is a minimal set, (X, T) is called a *minimal flow*.

A point x of X is *recurrent* if $T^{n_k}(x) \to x$ for some sequence $n_k \to +\infty$. When each point of X is recurrent we say that (X, T) is a *recurrent flow*. All periodic points are recurrent. The standard example of a nonperiodic recurrent point is any point in the irrational flow on the circle $\mathbb{S}^1$. Every point in a minimal set is recurrent, so the existence of minimal sets implies the existence of recurrent points.

Knight [1987] purported to prove that, if X is a compact recurrent orbit closure in (X, T), then any pair of orbit closures intersect and, in particular, X contains a unique compact minimal set. Dani [1991] pointed out with a counterexample that this statement is false.

In Theorem 0.1 below we enlarge the class of known counterexamples. More specifically, for any weakly mixing, minimal, uniformly rigid system (X, T) the system $(X \times X, T \times T)$, defined by $(T \times T)(x_1, x_2) = (T(x_1), T(x_2))$ for $(x_1, x_2) \in X \times X$, is a recurrent and transitive system with multiple minimal sets.

(Recall that the a discrete flow (X, T) is called

- *transitive* if there exists $x_0 \in X$ with a dense orbit;
- *ergodic* if for all two open subsets U and V there exits n such that $T^n U \cap V$ is nonempty;
- *weakly mixing* if the discrete flow $(X \times X, T \times T)$ is ergodic;

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- *uniformly rigid* if there exists a sequence $n_k \to +\infty$ such that

$$\lim_{n_k \to +\infty} \sup_{x \in X} d(T^{n_k} x; x) = 0,$$

where d is the metric on X.)

Minimal uniformly rigid weakly mixing systems exist; see [Glasner and Maon 1989, Proposition 6.5].

Theorem 0.1. *Let (X, T) be a minimal uniformly rigid weakly mixing system. Then $(X \times X, T \times T)$ is transitive and recurrent, and admits infinitely many minimal sets.*

Proof. Let (X, T) be a minimal uniformly rigid weakly mixing system.

Step 1: *$(X \times X, T \times T)$ is transitive.* Since (X, T) is weakly mixing, $(X \times X, T \times T)$ is ergodic. But for discrete flows on compact spaces, ergodicity is equivalent to transitiveness; see [de Vries 1993], for example. Because $X \times X$ is compact, this means that $(X \times X, T \times T)$ is transitive.

Step 2: *$(X \times X, T \times T)$ is recurrent.* Since (X, T) is a uniformly rigid flow, there is a sequence $n_k \to +\infty$ such that

$$\lim_{n_k \to +\infty} \sup_{x \in X} d(T^{n_k} x, x) = 0.$$

For each point (x, y) point of $X \times X$ we have

$$\lim_{n_k \to +\infty} (T \times T)^{n_k}(x, y) = \lim_{n_k \to +\infty} (T^{n_k} x, T^{n_k} y) = (x, y).$$

Thus (x, y) is a recurrent point and so $(X \times X, T \times T)$ is a recurrent discrete flow.

Step 3: *There are infinitely many minimal sets of* $(X \times X, T \times T)$. Define $D_n = \{(x, T^n(x)) : x \in X\}$. Then D_n is an invariant closed set of $(X \times X, T \times T)$. If F is a nonempty closed $(T \times T)$-invariant subset of D_n, then so is its projection, say $p_1(F)$, on the first factor. By the minimality of T we get $p_1(F) = X$, and hence $F = D_n$. Thus D_n is minimal for every n. Since (X, T) is minimal it follows that the D_n are pairwise distinct. □

Remark 0.2. The discrete flow $(X \times X, T \times T)$ does not have fixed points because we chose (X, T) as a minimal discrete flow.

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