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MINIMAL SETS OF A RECURRENT DISCRETE FLOW

HATTAB HAWETE

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S. G. Dani, giving a counterexample to a result in a paper of Knight, showed that recurrent transitive flows can admit multiple minimal sets. Here we show that such a phenomenon occurs on a wider scale.

Let (X, T) be a discrete flow, where X is a compact metric space and T is a self-homeomorphism of X . For $x \in X$, the set $\{T^n(x) : n \in \mathbb{Z}\}$ is called the *orbit* of x and is denoted by $O(x, T)$. A set W is a *minimal set* of (X, T) if for all $x \in W$ we have $\overline{O(x, T)} = W$. The study of minimal sets of such a system is a central question in topological dynamics. Zorn's lemma ensures the existence of at least a minimal set of (X, T) . If X is a minimal set, (X, T) is called a *minimal flow*.

A point x of X is *recurrent* if $T^{n_k}(x) \rightarrow x$ for some sequence $n_k \rightarrow +\infty$. When each point of X is recurrent we say that (X, T) is a *recurrent flow*. All periodic points are recurrent. The standard example of a nonperiodic recurrent point is any point in the irrational flow on the circle \mathbb{S}^1 . Every point in a minimal set is recurrent, so the existence of minimal sets implies the existence of recurrent points.

Knight [1987] purported to prove that, if X is a compact recurrent orbit closure in (X, T) , then any pair of orbit closures intersect and, in particular, X contains a unique compact minimal set. Dani [1991] pointed out with a counterexample that this statement is false.

In Theorem 0.1 below we enlarge the class of known counterexamples. More specifically, for any weakly mixing, minimal, uniformly rigid system (X, T) the system $(X \times X, T \times T)$, defined by $(T \times T)(x_1, x_2) = (T(x_1), T(x_2))$ for $(x_1, x_2) \in X \times X$, is a recurrent and transitive system with multiple minimal sets.

(Recall that the a discrete flow (X, T) is called

- *transitive* if there exists $x_0 \in X$ with a dense orbit;
- *ergodic* if for all two open subsets U and V there exists n such that $T^n U \cap V$ is nonempty;
- *weakly mixing* if the discrete flow $(X \times X, T \times T)$ is ergodic;

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- *uniformly rigid* if there exists a sequence $n_k \rightarrow +\infty$ such that

$$\lim_{n_k \rightarrow +\infty} \sup_{x \in X} d(T^{n_k} x; x) = 0,$$

where d is the metric on X .)

Minimal uniformly rigid weakly mixing systems exist; see [Glasner and Maon 1989, Proposition 6.5].

Theorem 0.1. *Let (X, T) be a minimal uniformly rigid weakly mixing system. Then $(X \times X, T \times T)$ is transitive and recurrent, and admits infinitely many minimal sets.*

Proof. Let (X, T) be a minimal uniformly rigid weakly mixing system.

Step 1: $(X \times X, T \times T)$ is transitive. Since (X, T) is weakly mixing, $(X \times X, T \times T)$ is ergodic. But for discrete flows on compact spaces, ergodicity is equivalent to transitivity; see [de Vries 1993], for example. Because $X \times X$ is compact, this means that $(X \times X, T \times T)$ is transitive.

Step 2: $(X \times X, T \times T)$ is recurrent. Since (X, T) is a uniformly rigid flow, there is a sequence $n_k \rightarrow +\infty$ such that

$$\lim_{n_k \rightarrow +\infty} \sup_{x \in X} d(T^{n_k} x, x) = 0.$$

For each point (x, y) point of $X \times X$ we have

$$\lim_{n_k \rightarrow +\infty} (T \times T)^{n_k}(x, y) = \lim_{n_k \rightarrow +\infty} (T^{n_k} x, T^{n_k} y) = (x, y).$$

Thus (x, y) is a recurrent point and so $(X \times X, T \times T)$ is a recurrent discrete flow.

Step 3: *There are infinitely many minimal sets of $(X \times X, T \times T)$.* Define $D_n = \{(x, T^n(x)) : x \in X\}$. Then D_n is an invariant closed set of $(X \times X, T \times T)$. If F is a nonempty closed $(T \times T)$ -invariant subset of D_n , then so is its projection, say $p_1(F)$, on the first factor. By the minimality of T we get $p_1(F) = X$, and hence $F = D_n$. Thus D_n is minimal for every n . Since (X, T) is minimal it follows that the D_n are pairwise distinct. \square

Remark 0.2. The discrete flow $(X \times X, T \times T)$ does not have fixed points because we chose (X, T) as a minimal discrete flow.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 250 No. 2 April 2011

Realizing profinite reduced special groups	257
VINCENT ASTIER and HUGO MARIANO	
On fibered commensurability	287
DANNY CALEGARI, HONGBIN SUN and SHICHENG WANG	
On an overdetermined elliptic problem	319
LAURENT HAUSWIRTH, FRÉDÉRIC HÉLEIN and FRANK PACARD	
Minimal sets of a recurrent discrete flow	335
HATTAB HAWETE	
Trace-positive polynomials	339
IGOR KLEP	
Remarks on the product of harmonic forms	353
LIVIU ORNEA and MIHAELA PILCA	
Steinberg representation of $\mathrm{GSp}(4)$: Bessel models and integral representation of L -functions	365
AMEYA PITALE	
An integral expression of the first nontrivial one-cocycle of the space of long knots in \mathbb{R}^3	407
KEIICHI SAKAI	
Burghelea–Haller analytic torsion for twisted de Rham complexes	421
GUANGXIANG SU	
$K(n)$ -localization of the $K(n+1)$ -local E_{n+1} -Adams spectral sequences	439
TAKESHI TORII	
Thompson’s group is distorted in the Thompson–Stein groups	473
CLAIRE WLADIS	
Parabolic meromorphic functions	487
ZHENG JIAN-HUA	



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