

*Pacific
Journal of
Mathematics*

MINIMAL SETS OF A RECURRENT DISCRETE FLOW

HATTAB HAWETE

MINIMAL SETS OF A RECURRENT DISCRETE FLOW

HATTAB HAWETE

S. G. Dani, giving a counterexample to a result in a paper of Knight, showed that recurrent transitive flows can admit multiple minimal sets. Here we show that such a phenomenon occurs on a wider scale.

Let (X, T) be a discrete flow, where X is a compact metric space and T is a self-homeomorphism of X . For $x \in X$, the set $\{T^n(x) : n \in \mathbb{Z}\}$ is called the *orbit* of x and is denoted by $O(x, T)$. A set W is a *minimal set* of (X, T) if for all $x \in W$ we have $\overline{O(x, T)} = W$. The study of minimal sets of such a system is a central question in topological dynamics. Zorn's lemma ensures the existence of at least a minimal set of (X, T) . If X is a minimal set, (X, T) is called a *minimal flow*.

A point x of X is *recurrent* if $T^{n_k}(x) \rightarrow x$ for some sequence $n_k \rightarrow +\infty$. When each point of X is recurrent we say that (X, T) is a *recurrent flow*. All periodic points are recurrent. The standard example of a nonperiodic recurrent point is any point in the irrational flow on the circle \mathbb{S}^1 . Every point in a minimal set is recurrent, so the existence of minimal sets implies the existence of recurrent points.

Knight [1987] purported to prove that, if X is a compact recurrent orbit closure in (X, T) , then any pair of orbit closures intersect and, in particular, X contains a unique compact minimal set. Dani [1991] pointed out with a counterexample that this statement is false.

In [Theorem 0.1](#) below we enlarge the class of known counterexamples. More specifically, for any weakly mixing, minimal, uniformly rigid system (X, T) the system $(X \times X, T \times T)$, defined by $(T \times T)(x_1, x_2) = (T(x_1), T(x_2))$ for $(x_1, x_2) \in X \times X$, is a recurrent and transitive system with multiple minimal sets.

(Recall that the a discrete flow (X, T) is called

- *transitive* if there exists $x_0 \in X$ with a dense orbit;
- *ergodic* if for all two open subsets U and V there exists n such that $T^n U \cap V$ is nonempty;
- *weakly mixing* if the discrete flow $(X \times X, T \times T)$ is ergodic;

MSC2010: 54H20.

Keywords: minimal set, discrete flow, uniformly rigid, weakly mixing.

- *uniformly rigid* if there exists a sequence $n_k \rightarrow +\infty$ such that

$$\lim_{n_k \rightarrow +\infty} \sup_{x \in X} d(T^{n_k} x; x) = 0,$$

where d is the metric on X .)

Minimal uniformly rigid weakly mixing systems exist; see [Glasner and Maon 1989, Proposition 6.5].

Theorem 0.1. *Let (X, T) be a minimal uniformly rigid weakly mixing system. Then $(X \times X, T \times T)$ is transitive and recurrent, and admits infinitely many minimal sets.*

Proof. Let (X, T) be a minimal uniformly rigid weakly mixing system.

Step 1: $(X \times X, T \times T)$ is transitive. Since (X, T) is weakly mixing, $(X \times X, T \times T)$ is ergodic. But for discrete flows on compact spaces, ergodicity is equivalent to transitivity; see [de Vries 1993], for example. Because $X \times X$ is compact, this means that $(X \times X, T \times T)$ is transitive.

Step 2: $(X \times X, T \times T)$ is recurrent. Since (X, T) is a uniformly rigid flow, there is a sequence $n_k \rightarrow +\infty$ such that

$$\lim_{n_k \rightarrow +\infty} \sup_{x \in X} d(T^{n_k} x, x) = 0.$$

For each point (x, y) point of $X \times X$ we have

$$\lim_{n_k \rightarrow +\infty} (T \times T)^{n_k}(x, y) = \lim_{n_k \rightarrow +\infty} (T^{n_k} x, T^{n_k} y) = (x, y).$$

Thus (x, y) is a recurrent point and so $(X \times X, T \times T)$ is a recurrent discrete flow.

Step 3: *There are infinitely many minimal sets of $(X \times X, T \times T)$.* Define $D_n = \{(x, T^n(x)) : x \in X\}$. Then D_n is an invariant closed set of $(X \times X, T \times T)$. If F is a nonempty closed $(T \times T)$ -invariant subset of D_n , then so is its projection, say $p_1(F)$, on the first factor. By the minimality of T we get $p_1(F) = X$, and hence $F = D_n$. Thus D_n is minimal for every n . Since (X, T) is minimal it follows that the D_n are pairwise distinct. \square

Remark 0.2. The discrete flow $(X \times X, T \times T)$ does not have fixed points because we chose (X, T) as a minimal discrete flow.

Acknowledgement

The author gratefully acknowledges helpful corrections, comments and suggestions from the referee.

References

- [Dani 1991] S. G. Dani, “A remark on recurrent dynamical systems”, *J. Indian Math. Soc. (N.S.)* **56**:1-4 (1991), 1–6. [MR 93g:58113](#) [Zbl 0864.54033](#)
- [Glasner and Maon 1989] S. Glasner and D. Maon, “Rigidity in topological dynamics”, *Ergodic Theory Dynam. Systems* **9**:2 (1989), 309–320. [MR 90h:54050](#) [Zbl 0661.58027](#)
- [Knight 1987] R. A. Knight, “Minimal sets in recurrent discrete flows”, *Proc. Amer. Math. Soc.* **100**:1 (1987), 195–198. [MR 88f:54076](#) [Zbl 0612.54051](#)
- [de Vries 1993] J. de Vries, *Elements of topological dynamics*, Math. and its Appl. **257**, Kluwer, Dordrecht, 1993. [MR 94m:54098](#) [Zbl 0783.54035](#)

Received February 19, 2010. Revised June 14, 2010.

HATTAB HAWETE
INSTITUT SUPÉRIEUR D’INFORMATIQUE ET DU MULTIMEDIA
ROUTE DE TUNIS KM 10
B.P. 242
SFAX 3021
TUNISIA
hattab.hawete@yahoo.fr

PACIFIC JOURNAL OF MATHEMATICS

<http://www.pjmath.org>

Founded in 1951 by

E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jonathan Rogawski
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
jonr@math.ucla.edu

PRODUCTION

pacific@math.berkeley.edu

Silvio Levy, Scientific Editor

Mathew Cargo, Senior Production Editor

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US \$420/year for the electronic version, and \$485/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

A NON-PROFIT CORPORATION

Typeset in L^AT_EX

Copyright ©2011 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 250 No. 2 April 2011

| | |
|-------------------------------------------------------------------------------------------------------------|-----|
| Realizing profinite reduced special groups | 257 |
| VINCENT ASTIER and HUGO MARIANO | |
| On fibered commensurability | 287 |
| DANNY CALEGARI, HONGBIN SUN and SHICHENG WANG | |
| On an overdetermined elliptic problem | 319 |
| LAURENT HAUSWIRTH, FRÉDÉRIC HÉLEIN and FRANK PACARD | |
| Minimal sets of a recurrent discrete flow | 335 |
| HATTAB HAWETE | |
| Trace-positive polynomials | 339 |
| IGOR KLEP | |
| Remarks on the product of harmonic forms | 353 |
| LIVIU ORNEA and MIHAELA PILCA | |
| Steinberg representation of $\mathrm{GSp}(4)$: Bessel models and integral representation of L -functions | 365 |
| AMEYA PITALE | |
| An integral expression of the first nontrivial one-cocycle of the space of long knots in \mathbb{R}^3 | 407 |
| KEIICHI SAKAI | |
| Burghelea–Haller analytic torsion for twisted de Rham complexes | 421 |
| GUANGXIANG SU | |
| $K(n)$ -localization of the $K(n+1)$ -local E_{n+1} -Adams spectral sequences | 439 |
| TAKESHI TORII | |
| Thompson’s group is distorted in the Thompson–Stein groups | 473 |
| CLAIRE WLADIS | |
| Parabolic meromorphic functions | 487 |
| ZHENG JIAN-HUA | |