Pacific Journal of Mathematics

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Volume 250 No. 2 April 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 250, No. 2, 2011

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We extend the Burghelea–Haller analytic torsion to the twisted de Rham complexes, and compare it with the twisted refined analytic torsion defined by Huang. Finally, we briefly discuss the Cappell–Miller analytic torsion.

1. Introduction

Let E be a unitary flat vector bundle on a closed Riemannian manifold M. Ray and Singer [1971] defined an analytic torsion associated to (M, E) and proved that it does not depend on the Riemannian metric on M. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on M (see [Milnor 1966]). This conjecture was later proved in two celebrated papers [Cheeger 1979; Müller 1978]. Müller [1993] generalized this result to the case when E is a unimodular flat vector bundle on M. Inspired by the considerations of Quillen [1985], Bismut and Zhang [1992] reformulated the above Cheeger–Müller theorem as an equality between the Reidemeister and Ray–Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundle over M. The method used by Bismut and Zhang is different from that of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [1982] on the de Rham complex.

Braverman and Kappeler [2007b; 2007c; 2008] defined the refined analytic torsion for a flat vector bundle over an odd dimensional manifold and showed that it equals the Turaev torsion [1989] (see also [Farber and Turaev 2000]) up to multiplication by a complex number of absolute value one. Burghelea and Haller [2007; 2008], following a suggestion of Müller, defined a generalized analytic torsion associated to a nondegenerate symmetric bilinear form on a flat vector bundle over an arbitrary dimensional manifold and make an explicit conjecture between this generalized analytic torsion and the Turaev torsion. This conjecture was proved up to sign in [Burghelea and Haller 2010] and in full generality in [Su and Zhang 2008]. Cappell and Miller [2010] used non-self-adjoint Laplace operators to define

MSC2000: primary 58J52; secondary 19K56.
Keywords: analytic torsion, symmetric bilinear form, de Rham complex.

another complex-valued analytic torsion and used the method in [Su and Zhang 2008] to prove an extension of the Cheeger–Müller theorem.

Mathai and Wu [2008; 2010b] generalized the classical Ray–Singer analytic torsion to the twisted de Rham complex with an odd degree closed differential form H. In [Mathai and Wu 2010a], they defined and studied analytic torsion of $\mathbb{Z}_2$-graded elliptic complexes. Huang [2010a] generalized Braverman and Kappeler's refined analytic torsion to the twisted de Rham complex, proved a duality theorem and compared it with the twisted Ray–Singer metric.

In this paper, supposing there exists a nondegenerate symmetric bilinear form on the flat vector bundle E, we generalize the Burghelea–Haller analytic torsion to the twisted de Rham complex. For the odd dimensional manifold, we also compare it with the twisted refined analytic torsion and the twisted Ray–Singer metric.

The rest of this paper is organized as follows. In Section 2, supposing there exists a $\mathbb{Z}_2$-graded nondegenerate symmetric bilinear form on a $\mathbb{Z}_2$-graded finite dimensional complex, we define a symmetric bilinear torsion on it. In Section 3, we generalize the Burghelea–Haller analytic torsion to the twisted de Rham complex. In Section 4, when the dimension of the manifold is odd, we show that the twisted Burghelea–Haller analytic torsion is independent of the Riemannian metric g, the symmetric bilinear form b and the representative H in the cohomology class $[H]$. In Section 5, we compare this new torsion with the twisted refined analytic torsion. In Section 6, we briefly discuss the Cappell–Miller analytic torsion on the twisted de Rham complex of an odd dimensional manifold.

2. Symmetric bilinear torsion on a finite dimensional $\mathbb{Z}_2$-graded complex

Consider a cochain complex

$$0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n \longrightarrow 0$$

of finite dimensional complex vector space. Set

$$C^{\bar{k}} = \bigoplus_{i = k \bmod 2} C^i, \quad k = 0, 1.$$

Let

$$(C^\bullet, d) : \cdots \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} C^{\bar{1}} \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} \cdots \tag{2-1}$$

be a $\mathbb{Z}_2$-graded cochain complex. Denote its cohomology by $H^{\bar{k}}$, $k = 0, 1$. Set

$$\det(C^\bullet, d) = \det C^{\bar{0}} \otimes (\det C^{\bar{1}})^{-1}, \quad \det(H^\bullet, d) = \det H^{\bar{0}} \otimes (\det H^{\bar{1}})^{-1}.$$

Then we have a canonical isomorphism between the determinant lines

$$\phi : \det(C^\bullet, d) \to \det(H^\bullet, d). \tag{2-2}$$

Suppose that there is a nondegenerate symmetric bilinear form on $C^{\bar{k}}$, $k=0,1$. Then it induces a nondegenerate symmetric bilinear form $b_{\det H^\bullet(C^\bullet,d)}$ on the determinant line $\det(H^\bullet,d)$ via the isomorphism (2-2). Let $d_{\bar{k}}^{\#}$ be the adjoint of $d_{\bar{k}}$ with respect to the nondegenerate symmetric bilinear form and define

$$\Delta_{b,\bar{k}} = d_{\bar{k}}^{\#} d_{\bar{k}} + d_{\overline{k+1}} d_{\overline{k+1}}^{\#}.$$

Let λ be the generalized eigenvalue of $\Delta_{b,\bar{k}}$ and let $C_b^{\bar{k}}(\lambda)$ be the generalized λ-eigenspace of $\Delta_{b,\bar{k}}$. Then we have a b-orthogonal decomposition

$$C^{\bar{k}} = \bigoplus_{\lambda} C_b^{\bar{k}}(\lambda) \tag{2-3}$$

and the inclusion $C_b^{\bar{k}}(0) \to C^{\bar{k}}$ induces an isomorphism in cohomology. Particularly, we obtain a canonical isomorphism

$$\det H^\bullet(C_b^\bullet(0)) \cong \det H^\bullet(C^\bullet). \tag{2-4}$$

Proposition 2.1. *The following identity holds*:

$$\begin{aligned} b_{\det H^\bullet(C^\bullet,d)} &\\ &= b_{\det H^\bullet(C_b^\bullet(0),d)} \cdot \det(d_{\bar{0}}^{\#} d_{\bar{0}}|_{C_b^{\bar{0},\perp}(0)\cap \operatorname{im} d_{\bar{0}}^{\#}})^{-1} \cdot \det(d_{\bar{1}}^{\#} d_{\bar{1}}|_{C_b^{\bar{1},\perp}(0)\cap \operatorname{im} d_{\bar{1}}^{\#}}), \end{aligned} \tag{2-5}$$

where $C_b^{\bar{k},\perp}(0) = \bigoplus_{\lambda\neq 0} C_b^{\bar{k}}(\lambda)$, $k=0,1$.

Proof. Same as [Burghelea and Haller 2007, Lemma 3.3]. Suppose $(C_1^\bullet, b_1)$ and $(C_2^\bullet, b_2)$ are finite-dimensional $\mathbb{Z}_2$-graded complexes equipped with $\mathbb{Z}_2$-graded nondegenerate symmetric bilinear forms. Clearly, $H^\bullet(C_1^\bullet \oplus C_2^\bullet) = H^\bullet(C_1^\bullet) \oplus H^\bullet(C_2^\bullet)$ and we obtain a canonical isomorphism of determinant lines

$$\det H^\bullet(C_1^\bullet \oplus C_2^\bullet) = \det H^\bullet(C_1^\bullet) \otimes \det H^\bullet(C_2^\bullet).$$

Then we have

$$b_{\det H^\bullet(C_1^\bullet \oplus C_2^\bullet)} = b_{\det H^\bullet(C_1^\bullet)} \otimes b_{\det H^\bullet(C_2^\bullet)}.$$

In view of the b-orthogonal decomposition (2-3) we may therefore without loss of generality assume $\ker \Delta_{b,\bar{k}} = 0$, $k=0,1$. Then by the lemma just cited we have

$$C^{\bar{k}} = \operatorname{im} d_{\overline{k+1}} \oplus \operatorname{im} d_{\bar{k}}^{\#}.$$

This decomposition is b-orthogonal and invariant under Δ_b. Thus we have the exact complexes

$$0 \longrightarrow C^{\bar{0}} \cap \operatorname{im} d_{\bar{0}}^{\#} \xrightarrow{\;d_{\bar{0}}\;} C^{\bar{1}} \cap \operatorname{im} d_{\bar{0}} \longrightarrow 0,$$
$$0 \longrightarrow C^{\bar{1}} \cap \operatorname{im} d_{\bar{1}}^{\#} \xrightarrow{\;d_{\bar{1}}\;} C^{\bar{0}} \cap \operatorname{im} d_{\bar{1}} \longrightarrow 0.$$

Then from [Burghelea and Haller 2007, Example 3.2], we get the proposition. □

3. Symmetric bilinear torsion on the twisted de Rham complexes

In this section, we suppose that there is a fiberwise nondegenerate symmetric bilinear form on E. Then we define a symmetric bilinear torsion on the determinant line of the twisted de Rham complex.

Twisted de Rham complexes. In this section, we review the twisted de Rham complexes from [Mathai and Wu 2008].

Let M be a closed Riemannian manifold and $E \to M$ be a complex flat vector bundle with flat connection ∇. Let H be an odd-degree closed differential form on M. We set $\Omega^{\bar{0}} = \Omega^{\mathrm{even}}(M, E)$, $\Omega^{\bar{1}} = \Omega^{\mathrm{odd}}(M, E)$ and $\nabla^H = \nabla + H\wedge$. We define the twisted de Rham cohomology groups as

$$H^{\bar{k}}(M, E, H) = \frac{\ker\bigl(\nabla^H : \Omega^{\bar{k}}(M, E) \to \Omega^{\overline{k+1}}(M, E)\bigr)}{\operatorname{im}\bigl(\nabla^H : \Omega^{\overline{k+1}}(M, E) \to \Omega^{\bar{k}}(M, E)\bigr)}, \quad k = 0, 1.$$

Suppose H is replaced by $H' = H - dB$ for some $B \in \Omega^{\bar{0}}(M)$, then there is an isomorphism $\varepsilon_B = e^B \wedge \cdot : \Omega^{\bullet}(M, E) \to \Omega^{\bullet}(M, E)$ satisfying

$$\varepsilon_B \circ \nabla^H = \nabla^{H'} \circ \varepsilon_B.$$

Therefore ε_B induces an isomorphism

$$\varepsilon_B : H^{\bullet}(M, E, H) \to H^{\bullet}(M, E, H')$$

on the twisted de Rham cohomology.

The construction of the symmetric bilinear torsion. Suppose that there exists a nondegenerate symmetric bilinear form on E. To simplify notation, let $C^{\bar{k}} = \Omega^{\bar{k}}(X, E)$ and let $d_{\bar{k}} = d_{\bar{k}}^{E,H}$ be the operator ∇^H acting on $C^{\bar{k}}$ $(k = 0, 1)$. Then $d_{\bar{1}} d_{\bar{0}} = d_{\bar{0}} d_{\bar{1}} = 0$ and we have a complex

$$\cdots \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} C^{\bar{1}} \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} \cdots. \tag{3-1}$$

The metric g^M and the symmetric bilinear form b determine together a symmetric bilinear form on $\Omega^{\bullet}(M, E)$ such that if $u = \alpha f$, $v = \beta g \in \Omega^{\bullet}(M, E)$ such that $\alpha, \beta \in \Omega^{\bullet}(M)$, $f, g \in \Gamma(E)$, then

$$\beta_{g,b}(u, v) = \int_M (\alpha \wedge *\beta) b(f, g), \tag{3-2}$$

where $*$ is the Hodge star operator. Denote by $d_{\bar{k}}^{\#}$ the adjoint of $d_{\bar{k}}$ with respect to the nondegenerate symmetric bilinear form (3-2). Then we define the Laplacians

$$\Delta_{b,\bar{k}} = d_{\bar{k}}^{\#} d_{\bar{k}} + d_{\overline{k+1}} d_{\overline{k+1}}^{\#}, \quad k = 0, 1.$$

If λ is in the spectrum of $\Delta_{b,\bar{k}}$, then the image of the associated spectral projection is finite dimensional and contains smooth forms only. Referring to this image as the (generalized) λ-eigenspace of $\Delta_{b,\bar{k}}$ and denoting it by $\Omega^{\bar{k}}_{\{\lambda\}}(M, E)$, there exists $N_\lambda \in \mathbb{N}$ such that

$$(\Delta_{b,\bar{k}} - \lambda)^{N_\lambda}|_{\Omega^{\bar{k}}_{\{\lambda\}}(M,E)} = 0.$$

Therefore for different generalized eigenvalues λ, μ, the spaces $\Omega^{\bar{k}}_{\{\lambda\}}(M, E)$ and $\Omega^{\bar{k}}_{\{\mu\}}(M, E)$ are $\beta_{g,b}$-orthogonal.

For any $a \geq 0$, set

$$\Omega^{\bar{k}}_{[0,a]}(M, E) = \bigoplus_{0\leq|\lambda|\leq a} \Omega^{\bar{k}}_{\{\lambda\}}(M, E).$$

Then $\Omega^{\bar{k}}_{[0,a]}(M, E)$ is finite dimensional and one gets a nondegenerate symmetric bilinear form

$$b_{\det H^\bullet(\Omega^\bullet_{[0,a]},d)} \quad \text{on} \quad \det H^\bullet(\Omega^\bullet_{[0,a]}, d).$$

Let $\Omega^{\bar{k}}_{(a,+\infty)}(M, E)$ denote the $\beta_{g,b}$-orthogonal complement to $\Omega^{\bar{k}}_{[0,a]}(M, E)$.

For the subcomplexes $(\Omega^{\overline{k+1}}_{(a,+\infty)}(M, E), d)$, since the operators $d_{\bar{k}} d^{\#}_{\bar{k}}$ and $\Delta_{b,\overline{k+1}}$ are equal and invertible on $\operatorname{im}(d_{\bar{k}}) \cap \Omega^{\overline{k+1}}_{(a,+\infty)}(M, E)$, we have

$$P_{\bar{k}} := d^{\#}_{\bar{k}}(d_{\bar{k}} d^{\#}_{\bar{k}})^{-1} d_{\bar{k}} = d^{\#}_{\bar{k}}(\Delta_{b,\overline{k+1}})^{-1} d_{\bar{k}} \tag{3-3}$$

is a pseudodifferential operator of order 0 and satisfies

$$P^2_{\bar{k}} = P_{\bar{k}}.$$

By definition we have

$$\begin{aligned}\zeta(s, d^{\#}_{\bar{k}} d_{\bar{k}}|_{\operatorname{im} d^{\#}_{\bar{k}} \cap \Omega^{\bar{k}}_{(a,+\infty)}(M,E)}) &= \operatorname{Tr}(\Delta^{-s}_{b,\bar{k}} P_{\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}) \\ &= \operatorname{Tr}(P_{\bar{k}} \Delta^{-s}_{b,\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}).\end{aligned} \tag{3-4}$$

Then $\zeta(s, d^{\#}_{\bar{k}} d_{\bar{k}}|_{\operatorname{im} d^{\#}_{\bar{k}} \cap \Omega^{\bar{k}}_{(a,+\infty)}(M,E)})$ has a meromorphic extension to the whole complex plane and, by [Wodzicki 1984, Section 7], it is regular at 0. So by [Wodzicki 1984; Grubb and Seeley 1995], we have the following analogue of [Mathai and Wu 2008, Theorem 2.1].

Theorem 3.1. *For $k = 0, 1$, $\zeta(s, d^{\#}_{\bar{k}} d_{\bar{k}}|_{\operatorname{im} d^{\#}_{\bar{k}} \cap \Omega^\bullet_{(a,+\infty)}(M,E)})$ is holomorphic in the half plane for $\operatorname{Re}(s) > n/2$ and extends meromorphically to $\mathbb{C}$ with possible poles at $\{(n-l)/2,\ l = 0, 1, 2, \dots\}$ only, and is holomorphic at $s = 0$.*

Then for $k = 0, 1$ and any $a \geq 0$, the regularized zeta determinant

$$\det{}'(d^{\#}_{\bar{k}} d_{\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}) := \exp\big(-\zeta'(0, d^{\#}_{\bar{k}} d_{\bar{k}}|_{\operatorname{im} d^{\#}_{\bar{k}} \cap \Omega^{\bar{k}}_{(a,+\infty)}(M,E)})\big). \tag{3-5}$$

is well defined.

Proposition 3.2. *The symmetric bilinear form on* $\det H^\bullet(\Omega^\bullet(M,E,H),d)$ *given by*

$$b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E),d)} \cdot \det{}'(d^\#_{\bar 0} d_{\bar 0}|_{\Omega^{\bar 0}_{(a,+\infty)}(M,E)})^{-1} \cdot \big(\det{}'(d^\#_{\bar 1} d_{\bar 1}|_{\Omega^{\bar 1}_{(a,+\infty)}(M,E)})\big) \tag{3-6}$$

is independent of the choice of $a \geq 0$.

Proof. Let $0 \leq a < c < \infty$. We have

$$(\Omega^{\bar k}_{[0,c]}(M,E), d_{\bar k}) = (\Omega^{\bar k}_{[0,a]}(M,E), d_{\bar k}) \oplus (\Omega^{\bar k}_{(a,c]}(M,E), d_{\bar k}), \tag{3-7}$$

$$(\Omega^{\bar k}_{(a,+\infty)}(M,E), d_{\bar k}) = (\Omega^{\bar k}_{(a,c]}(M,E), d_{\bar k}) \oplus (\Omega^{\bar k}_{(c,+\infty)}(M,E), d_{\bar k}). \tag{3-8}$$

By the definition of the determinant,

$$\det{}'(d^\#_{\bar k} d_{\bar k}|_{\Omega^{\bar k}_{(a,+\infty)}(M,E)}) = \det{}'(d^\#_{\bar k} d_{\bar k}|_{\Omega^{\bar k}_{(a,c]}(M,E)}) \cdot \det{}'(d^\#_{\bar k} d_{\bar k}|_{\Omega^{\bar k}_{(c,+\infty)}(M,E)}). \tag{3-9}$$

Applying Proposition 2.1 to (3-7),

$$b_{\det H^\bullet(\Omega^\bullet_{[0,c]})} = b_{\det H^\bullet(\Omega^\bullet_{[0,a]})} \cdot \det{}'(d^\#_{\bar 0} d_{\bar 0}|_{\Omega^{\bar 0}_{(a,c]}(M,E)})^{-1} \cdot \big(\det{}'(d^\#_{\bar 1} d_{\bar 1}|_{\Omega^{\bar 1}_{(a,c]}(M,E)})\big).$$

Then we get the proposition. □

Definition 3.3. The symmetric bilinear form defined by (3-6) is called the Ray–Singer symmetric bilinear torsion on $\det H^\bullet(\Omega^\bullet(M,E,H),d)$ and is denoted by $\tau_{b,\nabla,H}$.

4. Symmetric bilinear torsion under metric and flux deformations

In this section, we will use the methods in [Mathai and Wu 2008] to study the dependence of the torsion on the metric g, the symmetric bilinear form b and the flux H.

Variation of the torsion with respect to the metric and symmetric bilinear form. We assume that M is a closed compact oriented manifold of odd dimension. Suppose the pair (g_u, b_u) is deformed smoothly along a one-parameter family with parameter $u \in \mathbb{R}$. Let $Q_{\bar k}$ be the spectral projection onto $\Omega^{\bar k}_{[0,a]}(M,E)$ and $\Pi_{\bar k} = 1 - Q_{\bar k}$ be the spectral projection onto $\Omega^{\bar k}_{(a,+\infty)}(M,E)$. Let

$$\alpha = *_u^{-1}\frac{\partial *_u}{\partial u} + b_u^{-1}\frac{\partial b_u}{\partial u}.$$

Lemma 4.1. *Under the assumptions above,*

$$\frac{\partial}{\partial u}\log\Big(\det{}'(d^\#_{\bar 0} d_{\bar 0}|_{\Omega^{\bar 0}_{(a,+\infty)}(M,E)})^{-1} \cdot \big(\det{}'(d^\#_{\bar 1} d_{\bar 1}|_{\Omega^{\bar 1}_{(a,+\infty)}(M,E)})\big)\Big) = -\sum_{k=0,1}(-1)^k \operatorname{Tr}(\alpha Q_{\bar k}). \tag{4-1}$$

Proof. While $d_{\bar{k}}$ is independent of u, we have

$$\frac{\partial d_{\bar{k}}^{\#}}{\partial u} = -[\alpha, d_{\bar{k}}^{\#}].$$

Using $P_{\bar{k}} d_{\bar{k}}^{\#} = d_{\bar{k}}^{\#}$, $d_{\bar{k}} P_{\bar{k}} = d_{\bar{k}}$ and $P_{\bar{k}}^2 = P_{\bar{k}}$, we get $d_{\bar{k}}^{\#} d_{\bar{k}} P_{\bar{k}} = P_{\bar{k}} d_{\bar{k}}^{\#} d_{\bar{k}} = d_{\bar{k}}^{\#} d_{\bar{k}}$ and

$$P_{\bar{k}} \frac{\partial P_{\bar{k}}}{\partial u} P_{\bar{k}} = 0.$$

Following the $\mathbb{Z}$-graded case, we set

$$\begin{aligned}(4\text{-}2)\qquad f(s,u) &= \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s-1} \operatorname{Tr}(e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}} P_{\bar{k}} |_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)})\, dt \\ &= \Gamma(s) \sum_{k=0,1} (-1)^k \zeta(s, d_{\bar{k}}^{\#} d_{\bar{k}} |_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}).\end{aligned}$$

Using the above identities on $P_{\bar{k}}$, the trace property and by an application of Duhamel's principal, we get

$$\begin{aligned}(4\text{-}3)\quad \frac{\partial f}{\partial u} &= \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s-1} \operatorname{Tr}\Big(t[\alpha, d_{\bar{k}}^{\#}] d_{\bar{k}} e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}} \Pi_{\bar{k}} + e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}} \frac{\partial P_{\bar{k}}}{\partial u} P_{\bar{k}} \Pi_{\bar{k}} \Big)\, dt \\ &= \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s-1} \operatorname{Tr}\Big(t\alpha [d_{\bar{k}}^{\#}, d_{\bar{k}} e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}}] \Pi_{\bar{k}} + P_{\bar{k}} e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}} \frac{\partial P_{\bar{k}}}{\partial u} \Pi_{\bar{k}} \Big)\, dt \\ &= \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s-1} \operatorname{Tr}\Big(t\alpha (e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}} d_{\bar{k}}^{\#} d_{\bar{k}} - e^{-t d_{\bar{k}} d_{\bar{k}}^{\#}} d_{\bar{k}} d_{\bar{k}}^{\#}) \Pi_{\bar{k}} \\ &\qquad\qquad + e^{-t d_{\bar{k}}^{\#} d_{\bar{k}}} P_{\bar{k}} \frac{\partial P_{\bar{k}}}{\partial u} \Pi_{\bar{k}} \Big)\, dt \\ &= \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s} \operatorname{Tr}\big(\alpha e^{-t \Delta_{b,\bar{k}}} \Delta_{b,\bar{k}} \Pi_{\bar{k}} \big)\, dt \\ &= -\sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s} \frac{\partial}{\partial t} \operatorname{Tr}\big(\alpha (e^{-t\Delta_{b,\bar{k}}} \Pi_{\bar{k}}) \big)\, dt.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}(4\text{-}4)\quad \frac{\partial f}{\partial u} &= s \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s-1} \operatorname{Tr}\big(\alpha (e^{-t\Delta_{b,\bar{k}}} \Pi_{\bar{k}}) \big)\, dt \\ &= s \sum_{k=0,1} (-1)^k \left(\int_0^1 + \int_1^{+\infty} \right) t^{s-1} \operatorname{Tr}\left(\alpha e^{-t\Delta_{b,\bar{k}}} (1 - Q_{\bar{k}}) \right) dt.\end{aligned}$$

Since α is a smooth tensor and n is odd, the asymptotic expansion as $t \downarrow 0$ for $\operatorname{Tr}(\alpha e^{-t\Delta_{b,\bar{k}}})$ does not contain a constant term. Therefore $\int_0^1 t^{s-1} \operatorname{Tr}(\alpha e^{-t\Delta_{b,\bar{k}}})\, dt$

does not have a pole at $s=0$. On the other hand, because of the exponential decay of $\operatorname{Tr}(\alpha e^{-t\Delta_{b,\bar{k}}}\Pi_{\bar{k}})$ for large t,

$$\int_1^{+\infty} t^{s-1}\operatorname{Tr}(\alpha e^{-t\Delta_{b,\bar{k}}}\Pi_{\bar{k}})$$

is an entire function in s. So

$$\frac{\partial f}{\partial u}\Big|_{s=0} = -s\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}\operatorname{Tr}(\alpha Q_{\bar{k}})\,dt\big|_{s=0} = -\sum_{k=0,1}(-1)^k\operatorname{Tr}(\alpha Q_{\bar{k}}) \tag{4-5}$$

and hence

$$\frac{\partial}{\partial u}\sum_{k=0,1}(-1)^k\zeta(0, d_{\bar{k}}^{\#}d_{\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}) = 0. \tag{4-6}$$

Finally, from (4-5), (4-6), we have

$$\begin{aligned}\det{}'(d_{\bar{0}}^{\#}d_{\bar{0}}|_{\Omega^{\bar{0}}_{(a,+\infty)}(M,E)})^{-1}\cdot\big(\det{}'(d_{\bar{1}}^{\#}d_{\bar{1}}|_{\Omega^{\bar{1}}_{(a,+\infty)}(M,E)})\big)\\ = \exp\biggl(\lim_{s\to 0}\biggl(f(s,u)-\frac{1}{s}\sum_{k=0,1}(-1)^k\zeta\big(0, d_{\bar{k}}^{\#}d_{\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}\big)\biggr)\biggr),\end{aligned} \tag{4-7}$$

and the result follows. □

Lemma 4.2. *Under the same assumptions, along any one-parameter deformation of (g_u, b_u), we have*

$$\frac{\partial}{\partial w}\Big|_u\left(\frac{b_{w,\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E),d)}}{b_{u,\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E),d)}}\right) = \sum_{k=0,1}(-1)^k\operatorname{Tr}(\alpha Q_{\bar{k}}). \tag{4-8}$$

Proof. For sufficiently small $w-u$, the restriction of the spectral projection

$$Q_{\bar{k}}|_{\Omega^{\bar{k}}_{u,[0,a]}(M,E)} : \Omega^{\bar{k}}_{u,[0,a]}(M,E)\to\Omega^{\bar{k}}_{w,[0,a]}(M,E)$$

is an isomorphism of complexes. Then for sufficiently small $w-u$, we have

$$\begin{aligned}&\frac{b_{w,\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E),d)}}{b_{u,\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E),d)}}\\ &= \det\big((\beta_{g_u,b_u}|_{\Omega^{\bar{0}}_{u,[0,a]}(M,E)})^{-1}(Q_{\bar{0}}|_{\Omega^{\bar{0}}_{u,[0,a]}(M,E)})^*\\ &\quad\cdot(\beta_{g_w,b_w}|_{\Omega^{\bar{0}}_{w,[0,a]}(M,E)})\big)\cdot\det\big((\beta_{g_u,b_u}|_{\Omega^{\bar{1}}_{u,[0,a]}(M,E)})^{-1}(Q_{\bar{1}}|_{\Omega^{\bar{1}}_{u,[0,a]}(M,E)})^*\\ &\quad\cdot(\beta_{g_w,b_w}|_{\Omega^{\bar{1}}_{w,[0,a]}(M,E)})\big)^{-1}.\end{aligned} \tag{4-9}$$

Then similarly to [Burghelea and Haller 2007], we get (4-8). □

Combining Lemma 4.1 and Lemma 4.2, we have:

Theorem 4.3. *Let M be a closed, compact manifold of odd dimension, E be a flat vector bundle over M, and H be a closed differential form on M of odd degree. Then the symmetric bilinear torsion $\tau_{b,\nabla,H}$ on the twisted de Rham complex does not depend on the choices of the Riemannian metric on M and the symmetric bilinear form b in a same homotopy class of nondegenerate symmetric bilinear forms on E.*

Variation of analytic torsion with respect to the flux in a cohomology class. We continue to assume that $\dim M$ is odd and use the same notation as above. Suppose the (real) flux form H is deformed smoothly along a one-parameter family with parameter $v \in \mathbb{R}$ in such a way that the cohomology class $[H] \in H^{\bar{1}}(M, \mathbb{R})$ is fixed. Then $\partial H/\partial v = -dB$ for some form $B \in \Omega^{\bar{0}}(M)$ that depends smoothly on v; let

$$\beta = B \wedge \cdot\,.$$

Lemma 4.4. *Under the above assumptions,*

$$\frac{\partial}{\partial v}\log\Big(\det{}'(d_{\bar{0}}^{\#}d_{\bar{0}}|_{\Omega^{\bar{0}}_{(a,+\infty)}(M,E)})^{-1}\cdot\big(\det{}'(d_{\bar{1}}^{\#}d_{\bar{1}}|_{\Omega^{\bar{1}}_{(a,+\infty)}(M,E)})\big)\Big) = -2\sum_{k=0,1}(-1)^k\operatorname{Tr}(\beta Q_{\bar{k}}). \tag{4-10}$$

Proof. As in the proof of Lemma 4.1, we set

$$f(s,v) = \sum_{k=0,1}(-1)^k\int_0^{+\infty} t^{s-1}\operatorname{Tr}(e^{-td_{\bar{k}}^{\#}d_{\bar{k}}}P_{\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)})\,dt.$$

We note that B, hence β is real. Using

$$\frac{\partial d_{\bar{k}}}{\partial v} = [\beta, d_{\bar{k}}],\quad \frac{\partial d_{\bar{k}}^{\#}}{\partial v} = -[\beta^{\#}, d_{\bar{k}}^{\#}],\qquad P_{\bar{k}}^2 = P_{\bar{k}} = P_{\bar{k}}^{\#},\quad P_{\bar{k}}\frac{\partial P_{\bar{k}}}{\partial v}P_{\bar{k}} = 0$$

and Duhamel's principle, similarly to [Mathai and Wu 2008, Lemma 3.5], we get

$$\frac{\partial f}{\partial v} = -2\sum_{k=0,1}(-1)^k\int_0^{+\infty} t^s\frac{\partial}{\partial t}\operatorname{Tr}(\beta e^{-t\Delta_{b,\bar{k}}}\Pi_{\bar{k}})\,dt. \tag{4-11}$$

The rest is similar to the proof of Lemma 4.1. □

Lemma 4.5. *Under the same assumptions, along any one-parameter deformation of H that fixes the cohomology class $[H]$, we have*

$$\frac{\partial}{\partial w}\bigg|_v\left(\frac{b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E,H^w),d)}}{b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E,H^v),d)}}\right) = 2\sum_{k=0,1}(-1)^k\operatorname{Tr}(\beta Q_{\bar{k}}), \tag{4-12}$$

where we identify $\det H^\bullet(M, E, H)$ *along the deformation.*

Proof. For sufficiently small $w - v$, we have

$$Q_{\bar{k}}\varepsilon_B : \Omega^{\bar{k}}_{[0,a]}(M, E, H^v) \to \Omega^{\bar{k}}_{[0,a]}(M, E, H^w)$$

is an isomorphism of complexes and the induced symmetric bilinear form on the determinant line $\det H^\bullet(\Omega^\bullet_{[0,a]}(M, E, H^v), d)$ is

$$\Big(\big(\det(Q_{\bar{k}}\varepsilon_B)^* b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E,H^w),d)}\big)\Big)(\,\cdot\,,\cdot\,) = b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E,H^w),d)}\big(\det(Q_{\bar{k}}\varepsilon_B)\,\cdot\,, \det(Q_{\bar{k}}\varepsilon_B)\,\cdot\,\big), \tag{4-13}$$

where

$$\det(Q_{\bar{k}}\varepsilon_B) : \det H^\bullet\left(\Omega^*_{[0,a]}(M, E, H^v)\right) \to \det H^\bullet\left(\Omega^*_{[0,a]}(M, E, H^w)\right)$$

is the induced isomorphism on the determinant lines. Then we can compare it with $b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E,H^u),d)}$, and similarly to [Mathai and Wu 2008, Lemma 3.7], we get (4-12). □

Combining Lemma 4.4 and Lemma 4.5, we have:

Theorem 4.6. *Let M be a closed, compact manifold of odd dimension, E be a flat vector bundle over M. Suppose H and H' are closed differential forms on M of odd degrees representing the same de Rham cohomology class, and let B be an even form so that $H' = H - dB$. Then the symmetric bilinear torsion satisfies* $(\det \varepsilon_B)^* \tau_{b,\nabla,H'} = \tau_{b,\nabla,H}$.

5. Compare with the refined analytic torsion

In this section, we will compare the symmetric bilinear torsion $\tau_{b,\nabla,H}$ with the refined analytic torsion $\rho_{\mathrm{an}}(\nabla^H)$ defined in [Huang 2010a]. The main theorem of this section is the following.

Theorem 5.1. *Let M be a closed odd dimensional manifold, E be a complex vector bundle over M with connection ∇, H be a closed odd-degree differential form on M. Suppose there exists a nondegenerate symmetric bilinear form on E. Then*

$$\tau_{b,\nabla,H}\left(\rho_{\mathrm{an}}(\nabla^H)\right) = \pm e^{-2\pi i\left(\eta(\nabla^H) - \operatorname{rank} E\cdot\eta_{\mathrm{trivial}}\right)}. \tag{5-1}$$

(Here $\eta(\nabla^H)$ and η_{trivial} are defined in [Huang 2010a].)

We will use the method in [Braverman and Kappeler 2007a] to prove the theorem and the proof will be given later.

Let h be a Hermitian metric on E. One can construct the Ray–Singer analytic torsion as an inner product on $\det H^\bullet(M, E, H)$, or equivalently as a metric on the determinant line; see [Huang 2010a, (6.13)]. We denote the resulting inner product by $\tau_{h,\nabla,H}$. Then by our Theorem 5.1 and Theorem 6.2 of the same reference, we get:

Corollary 5.2. *If* $\dim M$ *is odd*, $\left|\dfrac{\tau_{b,\nabla,H}}{\tau_{h,\nabla,H}}\right| = 1.$

The dual connection. Let M be an odd dimensional closed manifold and E be a flat vector bundle over M, with flat connection ∇. Assume that there exists a nondegenerate symmetric bilinear form b on E. The dual connection ∇' to ∇ on E with respect to the form b is defined by the formula

$$db(u,v) = b(\nabla u, v) + b(u, \nabla' v), \quad u, v \in \Gamma(M, E).$$

We denote by E' the flat vector bundle (E, ∇').

Choices of the metric and the spectral cut. Until the end of this section we fix a Riemannian metric g on M and set $\mathcal{B}^H = \mathcal{B}(\nabla^H, g) = \Gamma\nabla^H + \nabla^H\Gamma$ and $\mathcal{B}'^H = \mathcal{B}'(\nabla'^H, g) = \Gamma\nabla'^H + \nabla'^H\Gamma$, where $\Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ is the chirality operator defined by

$$\Gamma\omega = i^{(n+1)/2} * (-1)^{q(q+1)/2}\omega, \quad \omega \in \Omega^q(M, E).$$

We also fix $\theta \in (-\pi/2, 0)$ such that both θ and $\theta+\pi$ are Agmon angles for the odd signature operator $\mathcal{B}^H$. One easily checks that

$$(\nabla^H)^\# = \Gamma\nabla'^H\Gamma, \quad (\nabla'^H)^\# = \Gamma\nabla^H\Gamma, \quad \text{and} \quad (\mathcal{B}^H)^\# = \mathcal{B}'^H. \tag{5-2}$$

As $\mathcal{B}^H$ and $(\mathcal{B}^H)^\#$ have the same spectrum it then follows that

$$\eta(\mathcal{B}'^H) = \eta(\mathcal{B}^H) \quad \text{and} \quad \mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}'^H) = \mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}^H). \tag{5-3}$$

Proof of Theorem 5.1. The symmetric bilinear form $\beta_{g,b}$ induces a nondegenerate symmetric bilinear form

$$H^j(M, E') \otimes H^{n-j}(M, E) \to \mathbb{C}, \quad j = 0, \dots, n,$$

and, hence, identifies $H^j(M, E')$ with the dual space of $H^{n-j}(M, E)$. Using the construction of [Huang 2010a, Section 5.1] (with $\tau : \mathbb{C} \to \mathbb{C}$ be the identity map) we obtain a linear isomorphism

$$\alpha : \det H^\bullet(M, E, H) \to \det H^\bullet(M, E', H). \tag{5-4}$$

Lemma 5.3. *Let* $E \to M$ *be a complex vector bundle over a closed oriented odd dimensional manifold* M *endowed with a nondegenerate bilinear form* b *and let* ∇ *be a flat connection on* E*. Let* ∇' *denote the connection dual to* ∇ *with respect to* b*. Let* H *be a closed odd-degree differential form on* M*. Then*

$$\alpha\left(\rho_{\mathrm{an}}(\nabla^H)\right) = \rho_{\mathrm{an}}(\nabla'^H). \tag{5-5}$$

The proof is that of [Huang 2010a, Theorem 5.3] and will be omitted. (Actually, it is simpler, since $\mathscr{B}^H$ and $\mathscr{B}'^H$ have the same spectrum, so there is no complex conjugation involved.)

For simplicity, we set

$$\tau_{b,\nabla,H,(a,+\infty)} = \det{}'(d_{\bar{0}}^{\#} d_{\bar{0}}|_{\Omega^{\bar{0}}_{(a,+\infty)}(M,E)})^{-1} \cdot (\det{}'(d_{\bar{1}}^{\#} d_{\bar{1}}|_{\Omega^{\bar{1}}_{(a,+\infty)}(M,E)})).$$

Setting $\Delta'^H = (\nabla'^H)^{\#}\nabla'^H + \nabla'^H(\nabla'^H)^{\#}$, we then have

$$\Delta'^H = \Gamma\Delta^H\Gamma.$$

Lemma 5.4. $$\tau_{b,\nabla,H,(a,+\infty)} = \tau_{b,\nabla',H,(a,+\infty)}.$$

Proof. Applying (5-2) and using the fact that

$$\nabla'^H : \Omega^{\bar{k}}_{(a,+\infty)}(M, E, H) \cap \operatorname{im}(\nabla'^H)^{\#} \to \Omega^{\overline{k+1}}_{(a,+\infty)}(M, E, H) \cap \operatorname{im}\nabla'^H$$

is an isomorphism, we get

$$\begin{aligned}
\tau_{b,\nabla,H,(a,+\infty)} &= \prod_{k=0,1} \det{}'\big((\nabla^H)^{\#}\nabla^H|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E,H)}\big)^{(-1)^{k+1}} \\
&= \prod_{k=0,1} \det{}'\big(\Gamma\nabla'^H(\nabla'^H)^{\#}\Gamma|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E,H)}\big)^{(-1)^{k+1}} \\
&= \prod_{k=0,1} \det{}'\big(\nabla'^H(\nabla'^H)^{\#}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E,H)}\big)^{(-1)^{k}} \\
&= \prod_{k=0,1} \det{}'\big((\nabla'^H)^{\#}\nabla'^H|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E,H)}\big)^{(-1)^{k+1}} = \tau_{b,\nabla',H},
\end{aligned} \tag{5-6}$$

which completes the proof. □

Then for any $h \in \det H^{\bullet}(M, E, H)$, we have

$$\tau_{b,\nabla,H}(h) = \tau_{b,\nabla',H}(\alpha(h)). \tag{5-7}$$

Hence, by (5-5) and (5-7),

$$\tau_{b,\nabla,H}\left(\rho_{\mathrm{an}}(\nabla^H)\right) = \tau_{b,\nabla',H}\left(\rho_{\mathrm{an}}\nabla'^H\right). \tag{5-8}$$

Let

$$\tilde{\nabla} = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla' \end{pmatrix}, \quad \tilde{\nabla}^H = \begin{pmatrix} \nabla^H & 0 \\ 0 & \nabla'^H \end{pmatrix}.$$

Then, for any $a \geq 0$,

$$\begin{aligned}
\tau_{b,\tilde{\nabla},H,(a,+\infty)} &= \tau_{b,\nabla,H,(a,+\infty)} \cdot \tau_{b,\nabla',H,(a,+\infty)}, \\
\tau_{b,\tilde{\nabla},H}\big(\rho_{\mathrm{an}}(\tilde{\nabla}^H)\big) &= \tau_{b,\nabla,H}\big(\rho_{\mathrm{an}}(\nabla^H)\big) \cdot \tau_{b,\nabla',H}\big(\rho_{\mathrm{an}}(\nabla'^H)\big).
\end{aligned}$$

Combining the latter equality with (5-8) shows that

$$\tau_{b,\tilde{\nabla},H}\big(\rho_{\mathrm{an}}(\tilde{\nabla}^H)\big) = \tau_{b,\nabla,H}\big(\rho_{\mathrm{an}}(\nabla^H)\big)^2.$$

Hence, (5-1) is equivalent to the equality

$$\tau_{b,\tilde{\nabla},H}\big(\rho_{\mathrm{an}}(\tilde{\nabla}^H)\big) = e^{-4\pi i\left(\eta(\nabla^H)-\operatorname{rank} E\cdot\eta_{\mathrm{trivial}}\right)}. \tag{5-9}$$

By a slight modification of the deformation argument in [Braverman and Kappeler 2007a, Section 4.7] where the untwisted case was treated, we obtain (5-9). This concludes the proof of Theorem 5.1. □

6. On the Cappell–Miller analytic torsion

In this section, we briefly discuss the extension of the Cappell–Miller analytic torsion to the twisted de Rham complexes. Let $\dim M$ be odd.

In the notation above, we have the twisted de Rham complex $\nabla^H : \Omega^{\bar{k}}(M, E) \to \Omega^{\overline{k+1}}(M, E)$ and the chirality operator $\Gamma : \Omega^{\bar{k}}(M, E) \to \Omega^{\overline{k+1}}(M, E)$, $k = 0, 1$. Define

$$d^\flat_{\bar{k}} = \Gamma d_{\bar{k}} \Gamma : \Omega^{\bar{k}}(M, E) \to \Omega^{\overline{k+1}}(M, E).$$

Then consider the non-self-adjoint Laplacian

$$\Delta^\flat_{\bar{k}} = (d_{\bar{k}} + d^\flat_{\bar{k}})^2 : \Omega^{\bar{k}}(M, E) \to \Omega^{\bar{k}}(M, E).$$

For any $a \geq 0$, let $\Omega^{\flat,\bar{k}}_{[0,a]}(M, E)$ $(\Omega^{\flat,\bar{k}}_{(a,+\infty)}(M, E))$ denote the span in $\Omega^{\bar{k}}(M, E)$ of the generalized eigensolutions of $\Delta^\flat_{\bar{k}}$ with generalized eigenvalues with absolute value in $[0, a]$ $((a, +\infty))$. Then we have the decomposition of the complex

$$(\Omega^\bullet(M, E), d) = (\Omega^{\flat,\bullet}_{[0,a]}(M, E), d) \oplus (\Omega^{\flat,\bullet}_{(a,+\infty)}(M, E), d).$$

The subcomplex $(\Omega^{\flat,\bullet}_{[0,a]}(M, E), d)$ is a $\mathbb{Z}_2$-graded finite dimensional complex. Then we can define the torsion element $\rho^\flat_{\Gamma_{[0,a]}} \otimes \rho^\flat_{\Gamma_{[0,a]}} \in \det H^\bullet(\Omega^{\flat,\bullet}_{[0,a]}(M, E), d)^2 \cong \det H^\bullet(M, E, H)^2$, where $\rho^\flat_{\Gamma_{[0,a]}}$ defined by [Huang 2010a, (2.22)]. On the other hand, for the subcomplex $(\Omega^{\flat,\bullet}_{(a,+\infty)}(M, E), d)$, the following zeta-regularized determinant is well defined (see (3-5)):

$$\det{}'(d^\flat_{\bar{k}} d_{\bar{k}}|_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)}) := \exp\big(-\zeta'(0, d^\flat_{\bar{k}} d_{\bar{k}}|_{\operatorname{im} d^\flat_{\bar{k}} \cap \Omega^{\flat,\bar{k}}_{(a,+\infty)}(M,E)})\big). \tag{6-1}$$

Considering the square of the graded determinant defined in [Huang 2010a, (2.38)], for the $\mathbb{Z}_2$-graded finite dimensional complex $\Omega^{\flat,\bullet}_{(a,c]}(M, E)$, $0 \leq a < c < \infty$, we find that

$$\det{}'(d^\flat_{\bar{0}} d_{\bar{0}}|_{\Omega^{\flat,\bar{0}}_{(a,c]}(M,E)}) \cdot \det{}'(d^\flat_{\bar{1}} d_{\bar{1}}|_{\Omega^{\flat,\bar{1}}_{(a,c]}(M,E)})^{-1} = \big(\mathrm{Det}_{\mathrm{gr}}(\mathcal{B}_{\bar{0}}|_{\Omega^{\flat,\bullet}_{(a,c]}(M,E)})\big)^2.$$

Then by [Huang 2010a, Proposition 2.7], we easily get:

Proposition 6.1. *The torsion element defined by*

$$\rho^{\flat}_{\Gamma_{[0,a]}} \otimes \rho^{\flat}_{\Gamma_{[0,a]}} \cdot \prod_{k=0,1} \left(\det{}'(d^{\flat}_{\bar{k}} d_{\bar{k}} |_{\Omega^{\bar{k}}_{(a,+\infty)}(M,E)})\right)^{(-1)^k} \in \det H^{\bullet}(M, E, H)^2 \tag{6-2}$$

is independent of the choice of $a \geq 0$.

Definition 6.2. The torsion element in $\det H^{\bullet}(M, E, H)^2$ defined by (6-2) is called the twisted Cappell–Miller analytic torsion for the twisted de Rham complex and is denoted by $\tau_{\nabla,H}$.

Next we study the torsion $\tau_{\nabla,H}$ under metric and flux deformations. Since the methods are the same as the cases in the twisted refined analytic torsion [Huang 2010a] and the twisted Burghelea–Haller analytic torsion above, we only briefly outline the results.

Theorem 6.3 (Metric independence). *Let M be a closed odd dimensional manifold, E be a complex vector bundle over M with flat connection ∇ and H be a closed odd-degree differential form on M. Then the torsion $\tau_{\nabla,H}$ is independent of the choice of the Riemannian metric g.*

Proof. By the definition of $\tau_{\nabla,H}$ and the observation on the determinants, this theorem follows easily from Proposition 2.4 and Equations (3.18) and (4.14) of [Huang 2010a]. □

Theorem 6.4 (Flux representative independence). *Let M be a closed odd dimensional manifold and E be a complex vector bundle over M with flat connection ∇. Suppose H and H' are closed differential forms on M of odd degrees representing the same de Rham cohomology class, and let B be an even form so that $H' = H - dB$. Then we have $\tau_{\nabla,H'} = \det(\varepsilon_B)\tau_{\nabla,H}$.*

Proof. From the above observation, this follows easily from Lemmas 4.6 and 4.7 of [Huang 2010a]. □

From the definition in (6-2), we see that the twisted Cappell–Miller analytic torsion is closely related to the twisted refined analytic torsion $\rho_{\mathrm{an}}(\nabla^H)$. Explicitly:

Theorem 6.5 (compare [Huang 2010b, Theorem 4.5]). *In* $\det H^{\bullet}(M, E, H)^2$,

$$\rho_{\mathrm{an}}(\nabla^H) \otimes \rho_{\mathrm{an}}(\nabla^H) = \tau_{\nabla,H} e^{-2\pi i\left(\eta(\nabla^H) - \operatorname{rank} E \cdot \eta_{\mathrm{trivial}}\right)}. \tag{6-3}$$

Proof. The twisted refined analytic torsion [Huang 2010a, (4.15)] is defined by

$$\rho_{\mathrm{an}}(\nabla^H) = \mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}^H_{0,(\lambda,\infty)}) \cdot \rho_{\Gamma_{[0,\lambda]}} \cdot e^{i\pi(\operatorname{rank} E)\eta_{\mathrm{trivial}}}.$$

By [Huang 2010a, (5.31)], we have

$$\begin{aligned}(6\text{-}4)\quad \rho_{\mathrm{an}}(\nabla^H)\otimes\rho_{\mathrm{an}}(\nabla^H) &= \rho_{\Gamma_{[0,\lambda]}}\otimes\rho_{\Gamma_{[0,\lambda]}}\cdot\exp(2\xi_\lambda(\nabla^H,g^M,\theta))\\ &\quad\cdot\exp\Bigl(-2i\pi\eta_\lambda(\nabla^H)-i\pi\sum_{k=0,1}(-1)^k d^-_{\bar{k},\lambda}+2i\pi(\operatorname{rank}E)\eta_{\mathrm{trivial}}\Bigr),\end{aligned}$$

where $\eta_\lambda(\nabla^H)$, $\xi_\lambda(\nabla^H,g^M,\theta)$, and $d^-_{\bar{k},\lambda}$ are defined in equations (3.17), (3.18), and (3.19) of [Huang 2010a]. By (6-2) and (6-4), we find that

$$\begin{aligned}(6\text{-}5)\quad \rho_{\mathrm{an}}(\nabla^H)\otimes\rho_{\mathrm{an}}(\nabla^H) &= \tau_{\nabla,H}\,\exp\Bigl(-2i\pi\eta_\lambda(\nabla^H)-i\pi\sum_{k=0,1}(-1)^k d^-_{\bar{k},\lambda}+2i\pi(\operatorname{rank}E)\eta_{\mathrm{trivial}}\Bigr).\end{aligned}$$

From [Huang 2010a, (5.28)], we get

$$(6\text{-}6)\qquad 2\eta_\lambda(\nabla^H)+\sum_{k=0,1}(-1)^k d^-_{\bar{k},\lambda}\equiv 2\eta(\nabla^H)\ \mathrm{mod}\ 2\mathbb{Z}.$$

Then (6-5) and (6-6) imply (6-3). □

Theorem 5.1 and Theorem 6.5 give the relation between the twisted Burghelea–Haller analytic torsion $\tau_{b,\nabla,H}$ and the twisted Cappell–Miller analytic torsion $\tau_{\nabla,H}$ if there is a nondegenerate symmetric bilinear form the bundle E.

Corollary 6.6. *If there is a nondegenerate symmetric bilinear form on E and* $\dim M$ *is odd, we have*

$$\tau_{b,\nabla,H}(\tau_{\nabla,H})=\pm 1.$$

Remark 6.7. Almost at the same time of the preprint [Su 2010] of this paper, Huang [2010b] defined and studied the twisted Cappell–Miller torsion both for holomorphic and analytic cases.

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Received January 25, 2010. Revised September 23, 2010.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 250 No. 2 April 2011
