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TAKESHI TORII

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$K(n)$-LOCALIZATION OF THE $K(n+1)$-LOCAL E_{n+1}-ADAMS SPECTRAL SEQUENCES

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We construct a spectral sequence converging to the homotopy set of maps from a spectrum to the $K(n)$-localization of the $K(n+1)$-local sphere. We also construct a map of spectral sequences from the $K(n)$-local E_n-Adams spectral sequence to the preceding one. Then we compare the map on E_2-terms with a map induced by the inflation maps of continuous cohomology groups for Morava stabilizer groups. As an application we show that ζ_n in $\pi_{-1}(L_{K(n)}S^0)$ represented by the reduced norm map in the $K(n)$-local E_n-Adams spectral sequence has a nontrivial image under the map $\pi_*(L_{K(n)}S^0) \to \pi_*(L_{K(n)}L_{K(n+1)}S^0)$.

1. Introduction

The motivation of this note is toward understanding the relationship between the $K(n)$-local category and the $K(n+1)$-local category. For each prime number p, the stable homotopy category of p-local spectra has a filtration of full subcategories corresponding to the height filtration of the moduli space of formal groups [Morava 1985]. The n-th associated graded part of the filtration is equivalent to the $K(n)$-local category, that is, the Bousfield localization of the stable homotopy category with respect to the n-th Morava K-theory spectrum $K(n)$ [Hovey and Strickland 1999]. So it can be considered that the stable homotopy category of p-local spectra is built up from the $K(n)$-local categories for various n. In fact, the chromatic convergence theorem [Ravenel 1992] says that a p-local finite spectrum X is homotopy equivalent to the homotopy inverse limit of the chromatic tower $\cdots \to L_{n+1}X \to L_nX \to \cdots \to L_0X$, where L_n is the Bousfield localization functor with respect to the wedge of Morava K-theories $K(0) \vee K(1) \vee \cdots \vee K(n)$. This means that a p-local finite spectrum X can be recovered from $\{L_nX\}_{n\geq 0}$ through the chromatic tower. Furthermore, if the chromatic splitting conjecture is true, then it implies that the p-completion of a finite spectrum X is a direct summand of the product $\prod_n L_{K(n)}X$ [Hovey 1995]. This means that it is not necessary to reconstruct the tower but it is sufficient to know all $L_{K(n)}X$ to obtain

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some information of X. Since the chromatic splitting conjecture is concerned with the relationship among various chromatic pieces, it is important to understand the relationship between the $K(n)$-local category and the $K(n+1)$-local category.

Let E_n be the n-th Morava E-theory spectrum. The $K(n)$-local E_n-Adams spectral sequence $L_{K(n)}E_r^{s,t}(W)$ is a natural spectral sequence for any spectrum W,

$$L_{K(n)}E_2^{s,t}(W) = H_c^s(G_n;\, E_n^t(W)) \Longrightarrow [W, L_{K(n)}S^0]^{s+t},$$

which converges to $[W, L_{K(n)}S^0]^*$ strongly and conditionally; see [Devinatz and Hopkins 2004, Appendix A]. On the E_2-term, G_n is the n-th extended Morava stabilizer group, and $H_c^s(G_n;\, E_n^t(W))$ is a continuous cohomology group for the profinite group G_n with coefficients in the profinite module $E_n^t(W)$.

We construct a natural spectral sequence converging to $[W, L_{K(n)}L_{K(n+1)}S^0]^*$ by applying the $K(n)$-localization functor to the $K(n+1)$-local E_{n+1}-Adams resolution of $L_{K(n+1)}S^0$. Let $\mathbb{A} = L_{K(n)}E_{n+1}$ be the $K(n)$-localization of the $(n+1)$-st Morava E-theory E_{n+1}. We identify the E_2-term as a cohomology group based on the continuous cochain complex for G_{n+1} with coefficients in the topological module $\mathbb{A}^*(W)$. We call this spectral sequence the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence for W.

Theorem 4.7. *For any spectrum W, there is a natural spectral sequence*

$$L_{K(n)}L_{K(n+1)}E_2^{s,t}(W) = H_c^s(G_{n+1};\, \mathbb{A}^t(W)) \Longrightarrow [W, L_{K(n)}L_{K(n+1)}S^0]^{s+t},$$

which converges strongly and conditionally.

By the $K(n)$-localization of the $K(n+1)$-localization map $S^0 \to L_{K(n+1)}S^0$, we obtain a map $L_{K(n)}S^0 \to L_{K(n)}L_{K(n+1)}S^0$, which induces a map

$$[W, L_{K(n)}S^0]^* \to [W, L_{K(n)}L_{K(n+1)}S^0]^*$$

for any spectrum W. We construct in Theorem 6.2 a natural map of spectral sequences

$$\varphi_r(W) : L_{K(n)}E_r^{s,t}(W) \longrightarrow L_{K(n)}L_{K(n+1)}E_r^{s,t}(W),$$

which converges to the map $[W, L_{K(n)}S^0]^{s+t} \to [W, L_{K(n)}L_{K(n+1)}S^0]^{s+t}$. Furthermore, we give an interpretation of the map on E_2-terms. We construct a natural homomorphism

$$\theta(W) : H_c^*(G_n;\, E_n^*(W)) \longrightarrow H_c^*(G_{n+1};\, \mathbb{A}^*(W)),$$

which is obtained from some kind of inflation maps (see (7-1)).

Theorem 7.6. *The map $\varphi_2(W)$ coincides with $\theta(W)$ for any spectrum W.*

By the Hopkins–Miller theorem [Devinatz and Hopkins 2004, Theorem 6], we know that there is a nontrivial element $\zeta_n \in \pi_{-1}(L_{K(n)}S^0)$ which is represented by the reduced norm map of G_n in the E_2-term of the $K(n)$-local E_n-Adams spectral sequence. Let ω_n be the image of ζ_n under the map

$$\pi_*(L_{K(n)}S^0) \to \pi_*(L_{K(n)}L_{K(n+1)}S^0).$$

As an application of our results, we show the following theorem.

Theorem 8.1. *The image ω_n is nontrivial.*

The organization of the remaining sections is as follows: In Section 2 we review the results in [Torii 2010a]. We recall the construction of a commutative ring spectrum $\mathbb{B}$ which is an extension of both of E_n and E_{n+1}, and the action of the group $\mathbb{G} = G_n \times_\Gamma G_{n+1}$ on $\mathbb{B}$. In Section 3 we introduce a topology for $\mathbb{A}^*$-modules of certain type, and study modules of continuous maps from a topological space to such a topological $\mathbb{A}^*$-module. In particular, we show that the functor $\mathrm{Map}_c(T, \mathbb{A}^*(-))$ is a generalized cohomology theory for any compact space T. In Section 4 we construct the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence by applying the $K(n)$-localization functor to the $K(n+1)$-local E_{n+1}-Adams resolution of $L_{K(n+1)}S^0$, and prove Theorem 4.7. In Section 5 we define a cohomology of $\mathbb{G}$ with coefficients in $\mathbb{B}^*(W)$ for the purpose of connecting the cohomology of G_n and that of G_{n+1}. Then we show that the inflation map from the cohomology of G_{n+1} with coefficients in $\mathbb{A}^*(W)$ to the cohomology of $\mathbb{G}$ with coefficients in $\mathbb{B}^*(W)$ is an isomorphism for any spectrum W. In Section 6 we construct a map of spectral sequences from the $K(n)$-local E_n-Adams spectral sequence to the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence. In Section 7 we construct a homomorphism $\theta(W)$ from the cohomology group of G_n with coefficients in $E_n^*(W)$ to the cohomology group of G_{n+1} with coefficients in $\mathbb{A}^*(W)$ by using the cohomology of $\mathbb{G}$ with coefficients in $\mathbb{B}^*(W)$ constructed in Section 5. Then we identify this homomorphism with the map of spectral sequences on E_2-terms, and prove Theorem 7.6. In Section 8 we prove Theorem 8.1 as an application of the results obtained earlier.

2. The ring spectrum $\mathbb{B}$

In this section we review the results in [Torii 2010a]. We recall the construction of a commutative ring spectrum $\mathbb{B}$ and two ring spectrum maps $\Theta : E_{n+1} \to \mathbb{B}$ and $I : E_n \to \mathbb{B}$. Furthermore, we recall that the action of a profinite group $\mathbb{G}$ on $\mathbb{B}$ and the equivariance of Θ and I under the actions of $\mathbb{G}$.

Let p be a prime number, and let n be a positive integer. We fix a finite field $\boldsymbol{F}$ which contains the finite fields $\mathbb{F}_{p^n}$ and $\mathbb{F}_{p^{n+1}}$. Note that the minimal field satisfying the condition is $\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^{n+1}} \cong \mathbb{F}_{p^{n^2+n}}$. We denote by W the ring of Witt vectors with coefficients in $\boldsymbol{F}$. We define variants of the n-th Morava E-theory spectrum E_n and the $(n+1)$-st Morava E-theory spectrum E_{n+1} such that the coefficient rings are given by

$$E_n^* = W[\![w_1, \dots, w_{n-1}]\!][w^{\pm 1}], \quad E_{n+1}^* = W[\![u_1, \dots, u_n]\!][u^{\pm 1}].$$

There is an associated degree 0 formal group law F_n over E_n^0 since E_n is complex oriented and even-periodic. The formal group law F_n is a universal deformation of the Honda formal group law H_n of height n over $\boldsymbol{F}$. Note that we can take F_n as a p-typical formal group law. The Morava stabilizer group S_n is defined to be the group of automorphisms of H_n over $\boldsymbol{F}$. Then the extended Morava stabilizer group G_n is defined to be the semi-direct product $G_n = \Gamma \ltimes S_n$, where $\Gamma = \mathrm{Gal}(\boldsymbol{F}/\mathbb{F}_p)$ is the Galois group of $\boldsymbol{F}$ over the prime field $\mathbb{F}_p$. We can identify G_n with the group of automorphisms of the ring spectrum E_n in the stable homotopy category. Then $g = (\gamma, s) \in \Gamma \ltimes S_n = G_n$ induces a ring homomorphism $g^* : E_n^* \to E_n^*$. We denote by F_n^g the formal group law obtained from F_n by the coefficient change along g^*. Then there is a unique isomorphism $t(g) : F_n \to F_n^g$ of formal group laws which is a lifting of the isomorphism $s : H_n \to H_n^\gamma = H_n$. There are projections $G_n \to \Gamma$ and $G_{n+1} \to \Gamma$. We define a profinite group $\mathbb{G}$ to be the fiber product of G_n and G_{n+1} over Γ

$$\mathbb{G} = G_n \times_\Gamma G_{n+1}.$$

Let $K(n)$ be the n-th Morava K-theory spectrum at p. We denote by $\mathbb{A}$ the commutative ring spectrum $L_{K(n)}E_{n+1}$, the Bousfield localization of E_{n+1} with respect to $K(n)$. The coefficient ring of $\mathbb{A}$ is given by the following Lemma.

Lemma 2.1. *The coefficient ring $\mathbb{A}^*$ is isomorphic to $(E_{n+1}^*[u_n^{-1}])_{I_n}^\wedge$, the completion of the localization $E_{n+1}^*[u_n^{-1}]$ at the ideal $I_n = (p, u_1, \dots, u_{n-1})$. Hence $\mathbb{A}^*$ is a graded complete Noetherian regular local ring isomorphic to*

$$(W((u_n)))_p^\wedge[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]$$

with residue field $\boldsymbol{F}((u_n))[u^{\pm 1}]$.

Proof. There is a tower $\{M(J)\}_J$ of generalized Moore spectra of height n as in [Hovey and Strickland 1999, Proposition 4.2]. If $J = (p^{a_0}, v_1^{a_1}, \dots, v_{n-1}^{a_{n-1}})$, then $(E_{n+1} \wedge M(J))^* = E_{n+1}^*/(p^{a_0}, u_1^{a_1}, \dots, u_{n-1}^{a_{n-1}})$ since $v_i = u_i u^{p^i - 1}$ for $i = 1, \dots, n-1$. We set $X_{I_n}^\wedge = \mathrm{holim}_J\, X \wedge M(J)$ for a spectrum X. Since E_{n+1} is Landweber exact of height $(n+1)$, it satisfies the telescope conjecture at n in the sense of [Hovey 1997, Definition 1.5.2]. Then $L_{K(n)}E_{n+1} \simeq (E_{n+1}[v^{-1}])_{I_n}^\wedge$ by [Hovey 1997, Theorem 1.5.4], where v is a generalized v_n-element in E_{n+1}^* in the sense of [Hovey 1997, Definition 1.2.2]. We can take $v_n = u_n u^{p^n - 1} \in \pi_{2p^n-2}E_{n+1}$ as a generalized v_n-element. Since the sequence $p^{a_0}, u_1^{a_1}, \dots, u_{n-1}^{a_{n-1}}$ is regular in $E_{n+1}^*[v_n^{-1}] = E_{n+1}^*[u_n^{-1}]$, $(E_{n+1}[v_n^{-1}] \wedge M(J))^* = E_{n+1}^*[u_n^{-1}]/(p^{a_0}, u_1^{a_1}, \dots, u_{n-1}^{a_{n-1}})$ if $J = (p^{a_0}, v_1^{a_1}, \dots, v_{n-1}^{a_{n-1}})$. Then we see that $\mathbb{A}^* = (L_{K(n)}E_{n+1})^*$ is the completion of $E_{n+1}^*[u_n^{-1}]$ at the ideal $I_n = (p, u_1, \dots, u_{n-1})$: $\mathbb{A}^* \cong (E_{n+1}^*[u_n^{\pm 1}])_{I_n}^\wedge$. Since the sequence $p, u_1, \dots, u_{n-1}$ is regular in $E^*_{n+1}[u_n^{\pm 1}]$, and it generates a maximal ideal, $\mathbb{A}^*$ is a graded regular local ring with maximal ideal generated by $p, u_1, \dots, u_{n-1}$ and residue field $\boldsymbol{F}((u_n))[u^{\pm 1}]$.

The obvious ring homomorphism $W[\![u_n]\!] \to \mathbb{A}^*$ extends to $(W((u_n)))^\wedge_p \to \mathbb{A}^*$, since u_n is a unit in $\mathbb{A}^*$, and $\mathbb{A}^*$ is p-complete. Furthermore, since $\mathbb{A}^*$ is I_n-adically complete, the obvious ring homomorphism $(W((u_n)))^\wedge_p[u_1, \dots, u_{n-1}][u^{\pm 1}] \to \mathbb{A}^*$ extends to $(W((u_n)))^\wedge_p[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}] \to \mathbb{A}^*$. The ring

$$(W((u_n)))^\wedge_p[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]$$

is a graded complete regular local ring with maximal ideal generated by $p, u_1, \dots, u_{n-1}$ and residue field $\boldsymbol{F}((u_n))[u^{\pm 1}]$. Since the ring homomorphism

$$(W((u_n)))^\wedge_p[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}] \to \mathbb{A}^*$$

is continuous, and it induces an isomorphism on the associated graded rings, we obtain an isomorphism between $\mathbb{A}^*$ and $(W((u_n)))^\wedge_p[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]$. □

Since a complete local ring is Henselian, $\mathbb{A}^*$ is a Henselian ring by Lemma 2.1.

Lemma 2.2 [Milne 1980, Proposition I.4.4]. *Let R be a Henselian ring with residue field k. Then the functor $S \mapsto S \otimes_R k$ induces an equivalence between the category of finite étale R-algebras and the category of finite étale k-algebras.*

Let $\bar{F}_{n+1}$ be the formal group law over $\boldsymbol{F}((u_n))$ obtained from F_{n+1} by the reduction $E^0_{n+1} \to \boldsymbol{F}((u_n))$. Then the height of $\bar{F}_{n+1}$ is n. Since the isomorphism classes of formal group laws over a separably closed field are classified by their height, there is an isomorphism between $\bar{F}_{n+1}$ and the height n Honda formal group law H_n over the separable closure $\boldsymbol{F}((u_n))^{\mathrm{sep}}$. In [Torii 2003, §2.3] we have constructed an extension field L of $\boldsymbol{F}((u_n))$, where L is the minimal extension such that there is an isomorphism between $\bar{F}_{n+1}$ and H_n. The extension L is Galois over $\boldsymbol{F}((u_n))$ with Galois group isomorphic to S_n. There is a sequence of finite Galois extensions of $\boldsymbol{F}((u_n))$

$$\boldsymbol{F}((u_n)) = L(-1) \to L(0) \to L(1) \to \cdots \tag{2-1}$$

such that $L = \bigcup_i L(i)$. We denote by $S_n(i)$ the Galois group for $\boldsymbol{F}((u_n)) \to L(i)$. Then $S_n(i)$ is a finite quotient group of S_n of order $(p^n - 1)p^{ni}$, and $S_n = \varprojlim_i S_n(i)$. The action of G_{n+1} on E^0_{n+1} induces an action on the residue field $\boldsymbol{F}((u_n))$ of $\mathbb{A}^0$. By [Torii 2003, §2.4], there is an action of $\mathbb{G}$ on L, which is an extension of the action of G_{n+1} on $\boldsymbol{F}((u_n))$ and the action of S_n on L as Galois group. Note that $L(i)$ is stable under the action of $\mathbb{G}$ for all i.

By Lemma 2.2, the sequence of Galois extensions (2-1) induces a sequence of graded commutative rings

$$\mathbb{A}^* = \mathbb{B}(-1)^* \to \mathbb{B}(0)^* \to \mathbb{B}(1)^* \to \cdots .$$

The ring $\mathbb{B}(i)^*$ is an even-periodic graded complete Noetherian regular local ring with residue field $L(i)[u^{\pm 1}]$. Furthermore, $\mathbb{A}^* \to \mathbb{B}(i)^*$ is a Galois extension of graded commutative rings with Galois group $S_n(i)$ in the sense of [Chase et al. 1965; Greither 1992]. Let $\mathbb{B}(\infty)^*$ be the direct limit of the sequence: $\mathbb{B}(\infty)^* = \varinjlim_i \mathbb{B}(i)^*$. Then we define a graded commutative ring $\mathbb{B}^*$ to be the completion of $\mathbb{B}(\infty)^*$ at the ideal $I_n = (p, u_1, \ldots, u_{n-1})$

$$\mathbb{B}^* = (\mathbb{B}(\infty)^*)^{\wedge}_{I_n}.$$

By Lemma 2.2, there is a unique lifting of the action of $\mathbb{G}$ on $\mathbb{B}^*$ and $\mathbb{B}(i)^*$ for $0 \le i \le \infty$ compatible with canonical inclusions.

By the $\mathbb{A}^*$-algebra structures, we can regard $\mathbb{B}^*$ and $\mathbb{B}(i)^*$ for $0 \le i \le \infty$ as Landweber exact even-periodic graded commutative rings. We denote the corresponding commutative ring spectra by $\mathbb{B}$ and $\mathbb{B}(i)$ for $0 \le i \le \infty$, respectively. Hence we obtain a sequence of commutative ring spectra

$$\mathbb{A} = \mathbb{B}(-1) \to \mathbb{B}(0) \to \mathbb{B}(1) \to \cdots .$$

Then we have $\mathbb{B}(\infty) = \operatorname{hocolim}_{\overrightarrow{i}} \mathbb{B}(i)$ and $\mathbb{B} = L_{K(n)}\mathbb{B}(\infty)$. We define a ring spectrum map $\Theta : E_{n+1} \to \mathbb{B}$ to be the composition

$$\Theta : E_{n+1} \longrightarrow L_{K(n)}E_{n+1} = \mathbb{A} \longrightarrow \mathbb{B}.$$

By [Torii 2003, §2.3], the formal group law induced by the ring homomorphism $E_n^0 \to \boldsymbol{F} \hookrightarrow L$ is isomorphic to the formal group law induced by the ring homomorphism $E_{n+1}^0 \to \boldsymbol{F}((u_n)) \hookrightarrow L$. By the universality of the formal group law F_n associated with E_n, there exists a ring homomorphism $E_n^* \to \mathbb{B}^*$ and an isomorphism Φ between the formal group laws F_n and F_{n+1} over $\mathbb{B}^0$

$$\Phi : F_{n+1} \xrightarrow{\cong} F_n.$$

Note that $\mathbb{B}^0$ is the minimal extension ring of both of E_n^0 and E_{n+1}^0 such that there exists an isomorphism between F_n and F_{n+1}. Since E_n and $\mathbb{B}$ are even-periodic Landweber exact commutative ring spectra, the ring homomorphism $E_n^* \to \mathbb{B}^*$ extends to a ring spectrum map

$$I : E_n \longrightarrow \mathbb{B}.$$

By the projection $\mathbb{G} \to G_n$, we can consider that $\mathbb{G}$ acts on E_n as automorphisms of commutative ring spectrum in the stable homotopy category. Also, by the projection $\mathbb{G} \to G_{n+1}$, we can consider that $\mathbb{G}$ acts on E_{n+1} as automorphisms of commutative ring spectrum.

Proposition 2.3 [Torii 2010a, §4]. *The profinite group $\mathbb{G}$ acts on the commutative ring spectrum $\mathbb{B}$ in the stable homotopy category. The ring spectrum maps $I : E_n \to \mathbb{B}$ and $\Theta : E_{n+1} \to \mathbb{B}$ are equivariant with respect to the actions of $\mathbb{G}$.*

Remark 2.4 [Torii 2010b]. The ring spectrum $\mathbb{B}$ supports a commutative S-algebra structure and the group $\mathbb{G}$ acts on $\mathbb{B}$ in the category of commutative S-algebras. Let $T = L_{K(n)}S^0 \otimes_{\mathbb{Z}_p} W$ be the commutative S-algebra obtained from $L_{K(n)}S^0$ by adjoining a primitive $(p^m - 1)$-st root of unity, where m is the dimension of $\boldsymbol{F}$ over $\mathbb{F}_p$. Then there is an equivalence $\mathbb{B} \simeq L_{K(n)}(E_n \overset{\wedge}{_T} \mathbb{A})$ of commutative S-algebras. In particular, when $\boldsymbol{F} = \mathbb{F}_{p^{n^2+n}}$, there is an equivalence $\mathbb{B} \simeq L_{K(n)}(E'_n \wedge E'_{n+1})$ of commutative S-algebras, where E'_n and E'_{n+1} are the standard Morava E-theory spectra so that $\pi_0 E'_n/I_n = \mathbb{F}_{p^n}$ and $\pi_0 E'_{n+1}/I_{n+1} = \mathbb{F}_{p^{n+1}}$. In this case

$$\mathrm{Gal}(\boldsymbol{F}/\mathbb{F}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \times \mathrm{Gal}(\mathbb{F}_{p^{n+1}}/\mathbb{F}_p) \quad \text{and} \quad \mathbb{G} \cong G'_n \times G'_{n+1},$$

where $G'_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes S_n$ and $G'_{n+1} = \mathrm{Gal}(\mathbb{F}_{p^{n+1}}/\mathbb{F}_p) \ltimes S_{n+1}$ are the standard extended Morava stabilizer groups.

3. Mapping space $\mathrm{Map}_c(T, \mathbb{A}^*(W))$

To interpret the E_2-term of the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence which will be constructed in Section 4 below as a cohomology group of G_{n+1}, we need to give an appropriate topology for $\mathbb{A}^*$-cohomology groups. In this section we introduce a topology for $\mathbb{A}^*$-modules of certain type, and study modules of continuous maps from a topological space to such an $\mathbb{A}^*$-module.

For a topological space T, and a topological module M, denote by $\mathrm{Map}_c(T, M)$ the module of continuous maps from T to M. Recall the fact that a surjection between profinite groups has a continuous section of topological spaces [Serre 1994, Proposition I.1.2.1]. This implies that $\mathrm{Map}_c(T, -)$ gives an exact functor from the category of profinite modules to that of abelian groups. The coefficient ring E^*_{n+1} is a graded complete Noetherian local ring with maximal ideal $I_{n+1} = (p, u_1, \ldots, u_n)$. Since E^*_{n+1}/I^r_{n+1} is a graded finite ring for each r, E^*_{n+1} is a graded profinite ring. Let N be a finitely generated E^*_{n+1}-module. Then N is a graded profinite abelian group. In this case there is an easy description for $\mathrm{Map}_c(T, N)$ as follows.

Lemma 3.1. *If N is a finitely generated E^*_{n+1}-module, there is a natural isomorphism*

$$\mathrm{Map}_c(T, N) \cong \mathrm{Map}_c(T, E^*_{n+1}) \otimes_{E^*_{n+1}} N.$$

Proof. Since N is finitely generated, there is an exact sequence of profinite modules $N^1 \to N^0 \to N \to 0$, where N^i is finitely generated free for $i = 0, 1$. This induces two exact sequences $\mathrm{Map}_c(T, N^1) \to \mathrm{Map}_c(T, N^0) \to \mathrm{Map}_c(T, N) \to 0$ and $\mathrm{Map}_c(T, E^*_{n+1}) \otimes N^1 \to \mathrm{Map}_c(T, E^*_{n+1}) \otimes N^0 \to \mathrm{Map}_c(T, E^*_{n+1}) \otimes N \to 0$. Since N^i is finitely generated free, we have $\mathrm{Map}_c(T, N^i) \cong \mathrm{Map}_c(T, E^*_{n+1}) \otimes N^i$ for $i = 0, 1$. Hence we obtain that $\mathrm{Map}_c(T, N) \cong \mathrm{Map}_c(T, E^*_{n+1}) \otimes N$. $\square$

Corollary 3.2. *For an ideal I of E^*_{n+1} and a finitely generated E^*_{n+1}-module N, there is a natural isomorphism*

$$\mathrm{Map}_c(T, N/IN) \cong \mathrm{Map}_c(T, N)/I\mathrm{Map}_c(T, N).$$

By Lemma 3.1, it is fundamental to understand $\mathrm{Map}_c(T, E^*_{n+1})$. Recall that a module over a (graded) regular local ring is called profree if it is isomorphic to the completion at the maximal ideal of some free module (see [Hovey and Strickland 1999, Theorem A.9] for equivalent conditions of profree modules).

Proposition 3.3. *For a topological space T, $\mathrm{Map}_c(T, E^*_{n+1})$ is a profree E^*_{n+1}-module.*

Proof. Put $P = \mathrm{Map}_c(T, E^*_{n+1})$. We have $P \cong \varprojlim_r \mathrm{Map}_c(T, E^*_{n+1}/I^r_{n+1})$, since $E^*_{n+1} \cong \varprojlim_r E^*_{n+1}/I^r_{n+1}$. Then $P \cong \varprojlim_r P/I^r_{n+1}P$ by Corollary 3.2. This shows that P is L-complete by [Hovey and Strickland 1999, Theorem A.6(a)]. Since $p, u_1, \dots, u_n$ is a regular sequence on E^*_{n+1},

$$0 \to E^*_{n+1}/I_k \xrightarrow{u_k} E^*_{n+1}/I_k \to E^*_{n+1}/I_{k+1} \to 0$$

is an exact sequence of profinite modules for $k = 0, 1, \dots, n$. By applying the functor $\mathrm{Map}_c(T, -)$, we obtain an exact sequence

$$0 \to P/I_kP \xrightarrow{u_k} P/I_kP \to P/I_{k+1}P \to 0$$

for $k = 0, 1, \dots, n$ by Corollary 3.2. Hence $p, u_1, \dots, u_n$ is a regular sequence on P, and P is profree by [Hovey and Strickland 1999, Theorem A.9]. $\square$

Recall that $\mathbb{A} = L_{K(n)}E_{n+1}$ and $\mathbb{A}^* \cong E^*_{n+1}[u_n^{-1}]^\wedge_{I_n} = \varprojlim_r E^*_{n+1}/I^r_n[u_n^{-1}]$ by Lemma 2.1. We denote by J_n the ideal of $\mathbb{A}^*$ generated by $p, u_1, \dots, u_{n-1}$, that is, $J_n = I_n\mathbb{A}^* \subset \mathbb{A}^*$. Then we have $\mathbb{A}^*/J^r_n = E^*_{n+1}/I^r_n[u_n^{-1}]$. Note that $\mathbb{A}^*/J^r_n$ is a graded ring of formal Laurent series over an Artinian local ring. To introduce a topology for $\mathbb{A}^*$-modules of certain type, we first consider the case of such a ring.

Definition 3.4. Let R be a (graded) Artinian local ring. Then the ring $R[\![a]\!]$ of formal power series is a Noetherian local ring. Note that the topology of $R[\![a]\!]$ coincides with the (a)-adic topology since the maximal ideal of R is nilpotent. We give the ring $R((a)) = R[\![a]\!][a^{-1}]$ of formal Laurent series a $R[\![a]\!]$-linear topology such that $R[\![a]\!]$ is an open submodule. Then $R((a))$ is a union of open submodules

$a^r R[\![a]\!]$ for $r \in \mathbb{Z}$: $R((a)) = \bigcup_{r\in\mathbb{Z}} a^r R[\![a]\!]$. For an $R[\![a]\!]$-module N, we give the (a)-adic topology on N. The localization $N[a^{-1}]$ is an $R((a))$-module. Let N' be the image of the localization map $N \to N[a^{-1}]$. Then N' is an $R[\![a]\!]$-submodule of $N[a^{-1}]$. We give an $R[\![a]\!]$-linear topology on $N[a^{-1}]$ such that N' is an open submodule. Then $N[a^{-1}]$ is a union of open submodules $a^r N'$ for $r \in \mathbb{Z}$: $N[a^{-1}] = \bigcup_{r\in\mathbb{Z}} a^r N'$.

For an $R[\![a]\!]$-module N, the localization map $N \to N[a^{-1}]$ induces a map $\mathrm{Map}_c(T, N)[a^{-1}] \to \mathrm{Map}_c(T, N[a^{-1}])$ of $R((a))$-modules. The following lemma gives a sufficient condition that this map is an isomorphism.

Lemma 3.5. *Let R be a (graded) Artinian local ring with finite residue field, and let T be a compact space. For an $R[\![a]\!]$-module N, there is a natural isomorphism*

$$\mathrm{Map}_c(T, N[a^{-1}]) \cong \mathrm{Map}_c(T, N')[a^{-1}],$$

where N' is the image of the localization map $N \to N[a^{-1}]$. Furthermore, if N is (a)-torsion free or finitely generated, then there is a natural isomorphism

$$\mathrm{Map}_c(T, N[a^{-1}]) \cong \mathrm{Map}_c(T, N)[a^{-1}].$$

Proof. Since $N[a^{-1}]$ is a union of open submodules $a^r N'$ for $r \in \mathbb{Z}$, any continuous map from T to $N[a^{-1}]$ factors through $a^r N'$ for some r. Hence

$$\mathrm{Map}_c(T, N')[a^{-1}] \xrightarrow{\cong} \mathrm{Map}_c(T, N[a^{-1}]).$$

If N is (a)-torsion free, then $N' = N$. Assume that N is finitely generated. Let K be the kernel of the surjection $N \to N'$. Since $N[a^{-1}] \cong N'[a^{-1}]$, $K[a^{-1}] = 0$. Since K is finitely generated, there is a positive integer m such that $a^m K = 0$. Since $R[\![a]\!]$ is profinite, $\mathrm{Map}_c(T, -)$ is an exact functor on the category of finitely generated $R[\![a]\!]$-modules. Then the exact sequence $0 \to K \to N \to N' \to 0$ induces an exact sequence $0 \to \mathrm{Map}_c(T, K) \to \mathrm{Map}_c(T, N) \to \mathrm{Map}_c(T, N') \to 0$. The fact that $a^m K = 0$ implies $a^m \mathrm{Map}_c(T, K) = 0$. Hence $\mathrm{Map}_c(T, K)[a^{-1}] = 0$. So we obtain that $\mathrm{Map}_c(T, N)[a^{-1}] \cong \mathrm{Map}_c(T, N')[a^{-1}]$. $\square$

We define a topology for $\mathbb{A}^*$-modules of the form $\varprojlim_r N/I_n^r[u_n^{-1}]$ for some E_{n+1}^*-module N.

Definition 3.6. For an $\mathbb{A}^*/J_n^r$-module M, since $\mathbb{A}^*/J_n^r$ is a graded ring of formal Laurent series over an Artinian local ring, we give a topology on M as in Definition 3.4. For an E_{n+1}^*-module N, we define an $\mathbb{A}^*$-module $\mathbb{A}^*N$ by

$$\mathbb{A}^*N = N[u_n^{-1}]^{\wedge}_{I_n} = \varprojlim_r N/I_n^r N[u_n^{-1}].$$

Then $N/I_n^r[u_n^{-1}]$ is an $\mathbb{A}^*/J_n^r$-module. We give $\mathbb{A}^*N = \varprojlim_r N/I_n^r N[u_n^{-1}]$ a topology by using the inverse limit topology.

Note that there is an isomorphism $\mathbb{A}^*E^*_{n+1} \cong \mathbb{A}^*$. If N is a finitely generated E^*_{n+1}-module, then $N[u_n^{-1}]$ is finitely generated over the Noetherian ring $E^*_{n+1}[u_n^{-1}]$. Then the completion of $N[u_n^{-1}]$ at the ideal I_n is given by the tensor product with $\mathbb{A}^*$. Hence there is a natural isomorphism $\mathbb{A}^*N \cong \mathbb{A}^* \otimes_{E^*_{n+1}} N$ for any finitely generated E^*_{n+1}-module N, and the functor $\mathbb{A}^*(-)$ is exact on the category of finitely generated E^*_{n+1}-modules.

In the rest of this section we study the functor $\mathrm{Map}_c(T, \mathbb{A}^*(-))$ with T compact.

Lemma 3.7. *If T is a compact space and N is a finitely generated E^*_{n+1}-module, then there is a natural isomorphism of $\mathbb{A}^*$-modules*

$$\mathrm{Map}_c(T, \mathbb{A}^*N) \cong \mathbb{A}^*\mathrm{Map}_c(T, N).$$

Proof. Since $\mathbb{A}^*N = \varprojlim_r N/I_n^r N[u_n^{-1}]$, we have

$$\mathrm{Map}_c(T, \mathbb{A}^*N) \cong \varprojlim_r \mathrm{Map}_c(T, N/I_n^r N[u_n^{-1}]).$$

By Lemma 3.5 and Corollary 3.2,

$$\mathrm{Map}_c(T, N/I_n^r N[u_n^{-1}]) \cong \mathrm{Map}_c(T, N)/I_n^r \mathrm{Map}_c(T, N)[u_n^{-1}].$$

Hence $\mathrm{Map}_c(T, \mathbb{A}^*N)$ is isomorphic to $\varprojlim_r \mathrm{Map}_c(T, N)/I_n^r\mathrm{Map}_c(T, N)[u_n^{-1}] = \mathbb{A}^*\mathrm{Map}_c(T, N)$. □

The basic case is when $N = E^*_{n+1}$:

Proposition 3.8. *For any compact space T, $\mathrm{Map}_c(T, \mathbb{A}^*)$ is a profree $\mathbb{A}^*$-module.*

Proof. By Proposition 3.3, $\mathrm{Map}_c(T, E^*_{n+1})$ is profree over E^*_{n+1}, and is thus a direct summand of some product $\prod_\alpha E^*_{n+1}$ by [Hovey and Strickland 1999, Proposition A.13]. Hence it is sufficient to show that $\mathbb{A}^*(\prod_\alpha E^*_{n+1})$ is profree over $\mathbb{A}^*$. For $k = 0, 1, \dots, n-1$, we put $M = E^*_{n+1}/I_k$ and $N = E^*_{n+1}/I_{k+1}$. Let K_r be the kernel of the map $M/I_n^r M \xrightarrow{u_k} M/I_n^r M$, and let L_r be the kernel of the map $M/I_n^r M \to N/I_n^r N$. Then there are exact sequences $0 \to K_r \to M/I_n^r M \to L_r \to 0$ and $0 \to L_r \to M/I_n^r M \to N/I_n^r N \to 0$. Since E^*_{n+1} is regular, the canonical map $K_{r+1} \to K_r$ is 0. Then

$$\varprojlim_r((\prod_\alpha K_r)[u_n^{-1}]) = \varprojlim_r{}^1 ((\prod_\alpha K_r)[u_n^{-1}]) = 0.$$

Hence we obtain $\varprojlim_r((\prod_\alpha M/I_n^r M)[u_n^{-1}]) \xrightarrow{\cong} \varprojlim_r((\prod_\alpha L_r)[u_n^{-1}])$, and

$$0 = \varprojlim_r{}^1 ((\prod_\alpha M/I_n^r M)[u_n^{-1}]) \cong \varprojlim_r{}^1 ((\prod_\alpha L_r)[u_n^{-1}]).$$

This implies that the sequence

$$0 \to \varprojlim_r((\prod_\alpha M/I_n^r)[u_n^{-1}]) \xrightarrow{u_k} \varprojlim_r((\prod_\alpha M/I_n^r M)[u_n^{-1}]) \longrightarrow \varprojlim_r((\prod_\alpha N/I_n^r N)[u_n^{-1}]) \to 0$$

is exact. This shows that $p, u_1, \dots, u_{n-1}$ is a regular sequence on $\mathbb{A}^*(\prod_\alpha E^*_{n+1})$. Therefore $\mathbb{A}^*(\prod_\alpha E^*_{n+1})$ is profree $\mathbb{A}^*$-module by [Hovey and Strickland 1999, Theorem A.9]. □

The map from T to the one point space $*$ induces a ring homomorphism $\mathbb{A}^* = \mathrm{Map}_c(*, \mathbb{A}^*) \to \mathrm{Map}_c(T, \mathbb{A}^*)$. Then the composition with the commutative MU^*-algebra structure map $MU^* \to \mathbb{A}^*$ gives $\mathrm{Map}_c(T, \mathbb{A}^*)$ a commutative MU^*-algebra structure. Since a profree module over $\mathbb{A}^*$ is Landweber exact, we obtain the following corollary

Corollary 3.9. *If T is a compact space, then $\mathrm{Map}_c(T, \mathbb{A}^*)$ is Landweber exact.*

We have a similar description for $\mathrm{Map}_c(T, \mathbb{A}^*N)$ as in Lemma 3.1 when T is a compact space and N is a finitely generated E^*_{n+1}-module as follows.

Proposition 3.10. *If T is a compact space and N is a finitely generated E^*_{n+1}-module, then there is a natural isomorphism of $\mathbb{A}^*$-modules*

$$\mathrm{Map}_c(T, \mathbb{A}^*N) \cong \mathrm{Map}_c(T, \mathbb{A}^*) \underset{\mathbb{A}^*}{\otimes} \mathbb{A}^*N.$$

For the proof of Proposition 3.10, we prepare the following (well-known) lemmas.

Lemma 3.11 ([Lam 1999, Proposition 4.4]). *Let R be a (graded) ring. If M is a finitely presented module over R, then $(\prod_\alpha R) \otimes_R M \cong \prod_\alpha M$.*

Proof. Since M is finitely presented, there is an exact sequence $M^1 \to M^0 \to M \to 0$, where M^i is finitely generated free for $i = 0, 1$. Then there are two exact sequences $(\prod_\alpha R) \otimes M^1 \to (\prod_\alpha R) \otimes M^0 \to (\prod_\alpha R) \otimes M \to 0$ and $\prod_\alpha M^1 \to \prod_\alpha M^0 \to \prod_\alpha M \to 0$. Since M^i is finitely generated free, $(\prod_\alpha R) \otimes M^i \cong \prod_\alpha M^i$ for $i = 0, 1$. Hence we obtain $(\prod_\alpha R) \otimes M \cong \prod_\alpha M$. □

Lemma 3.12. *If F is a profree $\mathbb{A}^*$-module and M is a finitely generated $\mathbb{A}^*$-module, then $F \otimes_{\mathbb{A}^*} M$ is J_n-adically complete.*

Proof. Since F is profree, it is a direct summand of some product $\prod_\alpha \mathbb{A}^*$ by [Hovey and Strickland 1999, Proposition A.13]. Since a direct summand of complete module is complete, it is sufficient to show that $(\prod_\alpha \mathbb{A}^*) \otimes M$ is complete. By Lemma 3.11, $(\prod_\alpha \mathbb{A}^*) \otimes M \cong \prod_\alpha M$, and $\prod_\alpha M$ is complete. □

Proof of Proposition 3.10. By Lemma 3.1, $\mathrm{Map}_c(T, N) \cong \mathrm{Map}_c(T, E^*_{n+1}) \otimes_{E^*_{n+1}} N$. Then we see that $\mathbb{A}^*\mathrm{Map}_c(T, N)$ is the completion of $\mathbb{A}^*\mathrm{Map}_c(T, E^*_{n+1}) \otimes_{\mathbb{A}^*} \mathbb{A}^*N$ at the ideal J_n. By Lemma 3.12, we see that $\mathbb{A}^*\mathrm{Map}_c(T, E^*_{n+1}) \otimes_{\mathbb{A}^*} \mathbb{A}^*N$ is J_n-adically complete. Hence we obtain

$$\mathbb{A}^*\mathrm{Map}_c(T, N) \cong \mathbb{A}^*\mathrm{Map}_c(T, E^*_{n+1}) \otimes_{\mathbb{A}^*} \mathbb{A}^*N. \qquad \square$$

Let $\mathcal{S}$ be the stable homotopy category, and let $\mathcal{K}$ be the $K(n)$-local stable homotopy category. For a $K(n)$-local spectrum $X \in \mathcal{K}$, we define $\Lambda''(X)$ to be the full subcategory of the comma category $(\mathcal{S} \downarrow X)$, whose objects are maps $X'' \to X$ from finite spectra X'' of type at least n. Then $\Lambda''(X)$ is an essentially small filtered category (see [Hovey and Strickland 1999, §9] and [Hovey et al. 1997, §2.3]). For a spectrum $W \in \mathcal{S}$, we set $\Lambda(W) = \Lambda''(L_{K(n)}W)$. The following lemma gives a sufficient condition that we can describe a generalized cohomology group of W in terms of cohomology groups of W_λ for $\lambda \in \Lambda(W)$.

Lemma 3.13. *Let R be a $K(n)$-local commutative ring spectrum. Suppose that the coefficient ring R^* is even-periodic and R^0 is a linearly compact Noetherian ring. Then there is a natural isomorphism*

$$R^*(W) \cong \varprojlim_\lambda R^*(W_\lambda)$$

for any $W \in \mathcal{S}$, where the inverse limit is taken over $\lambda \in \Lambda(W)$.

Proof. For $W \in \mathcal{S}$, we set $F^*(W) = \varprojlim_\lambda R^*(W_\lambda)$. Note that $R^*(W) \cong R^*(L_{K(n)}W)$ for any $W \in \mathcal{S}$ since R is $K(n)$-local. Then it is sufficient to show that $R^*(X) \cong F^*(X)$ for any $X \in \mathcal{K}$. By the assumption of the coefficient ring R^*, the functor $R^*(-)$ on the category of finite spectra takes values in the category of linearly compact R^*-modules and continuous maps. Then $F^*(-)$ is a cohomology theory on $\mathcal{S}$ by [Hovey et al. 1997, Proposition 2.3.16] and [Hovey and Strickland 1999, Proposition 9.2]. There is a natural transformation $R^*(-) \to F^*(-)$ of cohomology theories, which induces an isomorphism

$$R^*(X'') \xrightarrow{\cong} F^*(X'')$$

for any finite spectrum X'' of type at least n. Since $L_{K(n)}F(n)$ is a graded weak generator of $\mathcal{K}$ for any finite spectrum $F(n)$ of type n ([Hovey and Strickland 1999, Theorem 7.3]), we obtain that $R^*(X) \xrightarrow{\cong} F^*(X)$ for any $X \in \mathcal{K}$. □

Definition 3.14. For a finite spectrum X of type at least n, $E_{n+1}^*(X)$ is annihilated by a power of I_n, and $\mathbb{A}^*(X) \cong E_{n+1}^*(X)[u_n^{-1}]$ is a module over $\mathbb{A}^*/J_n^r = E_{n+1}^*/I_n^r[u_n^{-1}]$ for some r. We give a topology on $\mathbb{A}^*(X)$ as in Definition 3.6. For a spectrum W, $\mathbb{A}^*(W) \cong \varprojlim_\lambda \mathbb{A}^*(W_\lambda)$ by Lemma 3.13, where W_λ are finite spectra of type at least n. We give a topology on $\mathbb{A}^*(W)$ by the inverse limit topology.

For a compact space T and a finite spectrum X of type at least n,

$$\mathrm{Map}_c(T, \mathbb{A}^*(X)) \cong \mathrm{Map}_c(T, \mathbb{A}^*) \otimes_{\mathbb{A}^*} \mathbb{A}^*(X)$$

by Proposition 3.10, and $\mathrm{Map}_c(T, \mathbb{A}^*)$ is profree by Proposition 3.8. To study the functor $\mathrm{Map}_c(T, \mathbb{A}^*(-))$ on the stable homotopy category $\mathcal{S}$, we consider the following functor. Let F be a profree $\mathbb{A}^*$-module. We define a functor $H_F(-)$ from the stable homotopy category $\mathcal{S}$ to the category of $\mathbb{A}^*$-modules by

$$H_F(W) = \varprojlim_\lambda F \otimes_{\mathbb{A}^*} \mathbb{A}^*(W_\lambda),$$

where the inverse limit is taken over $\lambda \in \Lambda(W)$.

Lemma 3.15. *The functor $H_F(-)$ is a cohomology theory on $\mathcal{S}$.*

Proof. Since F is a direct summand of some product $\prod_\alpha \mathbb{A}^*$ by [Hovey and Strickland 1999, Proposition A.13], it is sufficient to show that the functor $Z \mapsto \varprojlim_\lambda(\prod_\alpha \mathbb{A}^*) \otimes_{\mathbb{A}^*} \mathbb{A}^*(W_\lambda)$ is a cohomology theory. Since $\mathbb{A}^*(W_\lambda)$ is finitely presented, $(\prod_\alpha \mathbb{A}^*) \otimes_{\mathbb{A}^*} \mathbb{A}^*(W_\lambda) \cong \prod_\alpha \mathbb{A}^*(W_\lambda)$ by Lemma 3.11. Hence

$$\varprojlim_\lambda(\prod_\alpha \mathbb{A}^*) \otimes_{\mathbb{A}^*} \mathbb{A}^*(W_\lambda) \cong \prod_\alpha \mathbb{A}^*(W),$$

and $\prod_\alpha \mathbb{A}^*(W)$ is a cohomology theory. This completes the proof. □

The following theorem will be used to identify the E_2-term of the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence to the continuous cohomology group of G_{n+1} in Section 4 below.

Theorem 3.16. *For any compact space T, the functor* $\mathrm{Map}_c(T, \mathbb{A}^*(-))$ *is a cohomology theory.*

Proof. By Proposition 3.10, there is a natural isomorphism

$$\mathrm{Map}_c(T, \mathbb{A}^*(W)) \cong \varprojlim_\lambda \mathrm{Map}_c(T, \mathbb{A}^*) \underset{\mathbb{A}^*}{\otimes} \mathbb{A}^*(W_\lambda).$$

But $\mathrm{Map}_c(T, \mathbb{A}^*)$ is profree by Proposition 3.8. Therefore the theorem follows from Lemma 3.15. □

4. Construction of the spectral sequence

We set $\widehat{\mathbb{S}} = L_{K(n)} L_{K(n+1)} S^0$. In this section we construct a spectral sequence which converges strongly and conditionally to $[W, \widehat{\mathbb{S}}]^*$ for any spectrum W by applying the $K(n)$-localization functor to the $K(n+1)$-local E_{n+1}-Adams resolution of $L_{K(n+1)} S^0$. Then we describe the E_2-term in terms of the continuous cohomology group of G_{n+1} with coefficients in $\mathbb{A}^*(W)$.

Let $E_n^{\wedge s}$ be the $K(n)$-localization of the smash product of s-copies of E_n

$$E_n^{\wedge s} = L_{K(n)}(\overbrace{E_n \wedge \cdots \wedge E_n}^{s}).$$

The commutative ring spectrum structure on E_n gives $E_n^{\wedge \bullet+1} = \{E_n^{\wedge s+1}\}_{s \geq 0}$ a cosimplicial $K(n)$-local commutative ring spectrum structure with augmentation

$L_{K(n)}S^0 \xrightarrow{\varepsilon} E_n^{\wedge \bullet+1}$. Then the associated cochain complex

$$* \to L_{K(n)}S^0 \xrightarrow{\varepsilon} E_n \xrightarrow{d} E_n^{\wedge 2} \xrightarrow{d} E_n^{\wedge 3} \xrightarrow{d} \cdots \tag{4-1}$$

is a $K(n)$-local E_n-Adams resolution of $L_{K(n)}S^0$ in the sense of [Miller 1981; Devinatz and Hopkins 2004]. We denote the sequence (4-1) by $\mathrm{Res}(E_n; L_{K(n)}S^0)$. There is an associated diagram of exact triangles

$$\begin{tikzcd} L_{K(n)}S^0 = Y^0 \arrow[dr, "j"'] & & Y^1 \arrow[ll, "i"'] \arrow[dr, "j"'] & & Y^2 \arrow[ll, "i"'] \arrow[dr, "j"'] & & Y^3 \arrow[ll, "i"'] & \cdots, \\ & E_n \arrow[ur, "k"'] & & \Sigma^{-1}E_n^{\wedge 2} \arrow[ur, "k"'] & & \Sigma^{-2}E_n^{\wedge 3} \arrow[ur, "k"'] & & \end{tikzcd} \tag{4-2}$$

in the $K(n)$-local stable homotopy category, where k has degree -1 and $jk = d$. We denote by $\mathrm{Ad}(E_n; L_{K(n)}S^0)$ the diagram of exact triangles (4-2).

For any spectrum W, by applying the functor $[W, -]^*$ to $\mathrm{Ad}(E_n; L_{K(n)}S^0)$ we obtain a $K(n)$-local E_n-Adams spectral sequence

$$L_{K(n)}E_r^{s,t}(W) \Longrightarrow [W, L_{K(n)}S^0]^{s+t}$$

with $L_{K(n)}E_2^{s,t}(W) \cong H_c^s(G_n; E_n^t(W))$. This spectral sequence converges strongly and conditionally. Furthermore, since $L_{K(n)}S^0$ is $K(n)$-local E_n-nilpotent [Devinatz and Hopkins 2004, Proposition A.3], the filtration (4-2) has the following property: There exists $N > 0$ such that $Y^{s+N} \to Y^s$ is null for all $s \geq 0$. This property implies that there exist positive integers $r(n)$ and $s(n)$, which do not depend on W, such that $L_{K(n)}E_{r(n)}^{s,*}(W) = 0$ for $s > s(n)$.

By applying the $K(n)$-localization functor to $\mathrm{Ad}(E_{n+1}; L_{K(n+1)}S^0)$, we obtain the following diagram $L_{K(n)}\mathrm{Ad}(E_{n+1}, L_{K(n+1)}S^0)$ of exact triangles

$$\begin{tikzcd} \widehat{\mathbb{S}} = Z^0 \arrow[dr, "j"'] & & Z^1 \arrow[ll, "i"'] \arrow[dr, "j"'] & & Z^2 \arrow[ll, "i"'] \arrow[dr, "j"'] & & Z^3 \arrow[ll, "i"'] & \cdots. \\ & L_{K(n)}E_{n+1} \arrow[ur, "k"'] & & \Sigma^{-1}L_{K(n)}E_{n+1}^{\wedge 2} \arrow[ur, "k"'] & & \Sigma^{-2}L_{K(n)}E_{n+1}^{\wedge 3} \arrow[ur, "k"'] & & \end{tikzcd} \tag{4-3}$$

For any spectrum W, applying the functor $[W, -]^*$ to $L_{K(n)}\mathrm{Ad}(E_{n+1}, L_{K(n+1)}S^0)$, we obtain a spectral sequence

$$L_{K(n)}L_{K(n+1)}E_r^{s,t}(W) \Longrightarrow [W, \widehat{\mathbb{S}}]^{s+t}.$$

We call this spectral sequence the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence.

Lemma 4.1. *The spectral sequence* $L_{K(n)}L_{K(n+1)}E_r^{s,t}(W) \Longrightarrow [W, \widehat{\mathbb{S}}]^{s+t}$ *converges conditionally and strongly for any spectrum* W.

Proof. There exists $N > 0$ such that $Y^{s+N} \to Y^s$ is null for all $s \geq 0$. Applying the $K(n)$-localization functor, we see that $Z^{s+N} \to Z^s$ is also null for all $s \geq 0$. This implies that the filtration of $[W, \widehat{\mathbb{S}}]^*$ is finite. Hence the spectral sequence converges strongly by [Boardman 1999, Definition 5.2]. Also, we obtain that $\varprojlim_n [W, Z^n]^* = \varprojlim^1_n [W, Z^n]^* = 0$. Hence the spectral sequence converges conditionally by [Boardman 1999, Definition 5.10]. □

Remark 4.2. Note that there exist positive integers r_0 and s_0, which do not depend on W, such that $L_{K(n)}L_{K(n+1)}E_{r_0}^{s,*}(W) = 0$ for $s > s_0$.

In the rest of this section we identify the E_2-term of the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence $L_{K(n)}L_{K(n+1)}E_r^{s,t}(W)$ with the continuous cohomology group of G_{n+1} with coefficients in $\mathbb{A}^*(W)$. Let $C(s) = E_{n+1}^{\wedge s+1}$. The E_1-term of the spectral sequence is given by $E_1^{s,t} = [W, L_{K(n)}C(s)]^t$. There is an isomorphism $C(s)^* \cong \mathrm{Map}_c(G_{n+1}^s, E_{n+1}^*)$ (see [Devinatz and Hopkins 2004, §2]). Then we see that $C(s)^*$ is profree over E_{n+1}^* by Proposition 3.3. The following lemma gives a similar description for $L_{K(n)}C(s)^*$.

Lemma 4.3. *For* $s \geq 0$, *we have* $L_{K(n)}C(s)^* \cong \mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*)$.

Proof. There is a tower $\{M(J)\}_J$ of generalized Moore spectra of type n as in [Hovey and Strickland 1999, Proposition 4.2] such that $L_{K(n)}W \simeq \operatorname{holim}_J L_n W \wedge M(J)$ for any spectrum W [Hovey and Strickland 1999, Proposition 7.10(e)]. Since $C(s)$ is Landweber exact of height $(n+1)$, we obtain that $L_{K(n)}C(s)^* \cong \mathbb{A}^*C(s)^*$. Then $\mathbb{A}^*C(s)^* \cong \mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*)$ by Lemma 3.7, since

$$C(s)^* \cong \mathrm{Map}_c(G_{n+1}^s, E_{n+1}^*).$$ □

Corollary 4.4. *For* $s \geq 0$, $L_{K(n)}C(s)^*$ *is Landweber exact and profree over* $\mathbb{A}^*$.

Proof. This follows from Proposition 3.8 and Corollary 3.9. □

Then we obtain a description for the E_1-term $[W, L_{K(n)}C(s)]^*$ as a module of continuous maps from G_{n+1}^s to $\mathbb{A}^*(W)$.

Proposition 4.5. *For any spectrum* W, *there is a natural isomorphism*

$$[W, L_{K(n)}C(s)]^* \cong \mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*(W)).$$

Proof. By Lemma 4.3 and Corollary 4.4, $L_{K(n)}C(s)^* \cong \mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*)$ is Landweber exact. Then there is a natural isomorphism

$$[W, L_{K(n)}C(s)]^* \cong \mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*) \otimes_{\mathbb{A}^*} \mathbb{A}^*(W)$$

for any finite spectrum W. By Proposition 3.10, the right hand side is isomorphic to $\mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*(W))$. Since $\mathrm{Map}_c(G_{n+1}^s, \mathbb{A}^*)$ is even concentrated, there is a unique extension to a cohomology theory for any spectra by [Hovey and Strickland 1999, Theorem 2.8]. Obviously, $[-, L_{K(n)}C(s)]^*$ is such an extension. On the other hand, $\mathrm{Map}_c(G^s_{n+1}, \mathbb{A}^*(-))$ is also an extension by Theorem 3.16. Therefore $[W, L_{K(n)}C(s)]^* \cong \mathrm{Map}_c(G^s_{n+1}, \mathbb{A}^*(W))$ for any spectrum W. □

For a topological group G and a topological G-module M, denote by $C^*_c(G; M)$ the continuous cochain complex of G with coefficients in M. Define $H^*_c(G; M)$ to be the cohomology group of $C^*_c(G; M)$, and call it the continuous cohomology of G with coefficients in M. Let $[W, C(*)]^t$ be the cochain complex associated with the cosimplicial abelian group $[W, C(\bullet)]^t$. Then there is a natural isomorphism $[W, C(*)]^t \cong C^*_c(G_{n+1}, E^t_{n+1}(W))$ of cochain complexes [Devinatz and Hopkins 2004, §4]. By Proposition 4.5, this implies a natural isomorphism $[W, L_{K(n)}C(*)]^t \cong C^*_c(G_{n+1}, \mathbb{A}^t(W))$ of cochain complexes. Hence we obtain the following corollary.

Corollary 4.6. *For any spectrum W, there is a natural isomorphism*

$$H^s([W, L_{K(n)}C(*)]^t) \cong H^s_c(G_{n+1}; \mathbb{A}^t(W)).$$

As a summary we obtain the following theorem.

Theorem 4.7. *For any spectrum W, there is a natural spectral sequence*

$$L_{K(n)}L_{K(n+1)}E^{s,t}_r(W)$$

which converges strongly and conditionally to $[W, \widehat{\mathbb{S}}]^*$:

$$L_{K(n)}L_{K(n+1)}E^{s,t}_2(W) \Longrightarrow [W, \widehat{\mathbb{S}}]^{s+t}.$$

The E_2-term is given by

$$L_{K(n)}L_{K(n+1)}E^{s,t}_2(W) \cong H^s_c(G_{n+1}; \mathbb{A}^t(W)).$$

Furthermore, there exist positive integers r_0 and s_0 such that

$$L_{K(n)}L_{K(n+1)}E^{s,*}_{r_0}(W) = 0$$

for $s > s_0$, where r_0 and s_0 do not depend on W.

5. The cohomology group $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$

In this section we introduce a cohomology group $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ of $\mathbb{G}$ with coefficients in $\mathbb{B}^*(W)$ for a spectrum W. Then we show that $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ is naturally isomorphic to the continuous cohomology group $H^*_c(G_{n+1}; \mathbb{A}^*(W))$ of G_{n+1} with coefficients in $\mathbb{A}^*(W)$. The cohomology group $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ will be used to connect the E_2-term of the $K(n)$-local E_n-Adams spectral sequence for W and the E_2-term of the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence for W in Section 7 below.

First we introduce a topology for modules of continuous maps from a profinite group to an $\mathbb{A}^*$-module of certain type. Then we study a continuous cohomology group of a profinite group with coefficients in such a topological module of mappings.

Definition 5.1. Let G be a profinite group. Suppose that $M = \varprojlim_\lambda \mathbb{A}^* N_\lambda$ with the inverse limit topology, where $\{N_\lambda\}_{\lambda\in\Lambda}$ is a cofiltered system of finitely generated E_{n+1}^*-modules. By Lemma 3.7, there is an isomorphism

$$\mathrm{Map}_c(G, M) \cong \varprojlim_\lambda \mathbb{A}^* \mathrm{Map}_c(G, N_\lambda).$$

We give a topology on $\mathbb{A}^*\mathrm{Map}_c(G, N_\lambda)$ as in Definition 3.6. Then we give a topology on $\mathrm{Map}_c(G, M)$ by the inverse limit topology. For any spectrum W, $\mathbb{A}^*(W) \cong \varprojlim_\lambda \mathbb{A}^* E_{n+1}^*(W_\lambda)$ by Lemma 3.13, where W_λ are finite spectra of type at least n. We give a topology on $\mathrm{Map}_c(G, \mathbb{A}^*(W))$ as above.

The following lemma shows that the mapping spaces have an expected adjunction property.

Lemma 5.2. *Let G and H be profinite groups. Suppose that $M = \varprojlim_\lambda \mathbb{A}^* N_\lambda$ with the inverse limit topology, where $\{N_\lambda\}_{\lambda\in\Lambda}$ is a cofiltered system of finitely generated E_{n+1}^*-modules. Then there is an isomorphism*

$$\mathrm{Map}_c(G, \mathrm{Map}_c(H, M)) \cong \mathrm{Map}_c(G \times H, M).$$

Proof. We have

$$\mathrm{Map}_c(G, \mathrm{Map}_c(H, M) = \varprojlim_\lambda \mathrm{Map}_c(G, \mathrm{Map}_c(H, \mathbb{A}^* N_\lambda),$$
$$\mathrm{Map}_c(G \times H, M) = \varprojlim_\lambda \mathrm{Map}_c(G \times H, \mathbb{A}^* N_\lambda).$$

Hence it is sufficient to show that the lemma holds when $M = \mathbb{A}^* N$ with finitely generated N. Suppose that N is a finitely generated E_{n+1}^*-module. Let N_r be the image of the localization map $N/I_n^r N \to N/I_n^r N[u_n^{-1}]$, and let $L_r = \mathrm{Map}_c(H, N_r)$. Note that N_r and L_r are (u_n)-torsion free. By Lemma 3.5, $\mathrm{Map}_c(H, \mathbb{A}^* N) = \varprojlim_r L_r[u_n^{-1}]$. Then $\mathrm{Map}_c(G, \mathrm{Map}_c(H, \mathbb{A}^* N)) = \varprojlim_r \mathrm{Map}_c(G, L_r[u_n^{-1}])$. Again by Lemma 3.5, we have $\mathrm{Map}_c(G, L_r[u_n^{-1}]) = \mathrm{Map}_c(G, L_r)[u_n^{-1}]$. The fact that N_r is a profinite module implies that $\mathrm{Map}_c(G, L_r) = \mathrm{Map}_c(G \times H, N_r)$. By Lemma 3.5, we obtain $\varprojlim_r \mathrm{Map}_c(G \times H, N_r)[u_n^{-1}] = \mathrm{Map}_c(G \times H, \mathbb{A}^* N)$. □

Corollary 5.3. *Let G and H be profinite groups. For any spectrum W, there is a natural isomorphism*

$$\mathrm{Map}_c(G, \mathrm{Map}_c(H, \mathbb{A}^*(W)))) \cong \mathrm{Map}_c(G \times H, \mathbb{A}^*(W)).$$

Suppose that a profinite group G continuously acts on a topological module M from the right. For $q > 0$, we define a right G-action on $\mathrm{Map}_c(G, M)$ by

$$\varphi^g(h_1, \ldots, h_q) = \varphi(h_1 g^{-1}, \ldots, h_q g^{-1})^g,$$

where $\varphi \in \mathrm{Map}_c(G^q, M)$ and $g, h_1, \ldots, h_q \in G$. Then $\mathrm{Map}_c(G^q, M)$ is a topological G-module. The following proposition shows that the coinduced module $\mathrm{Map}_c(G^q, M)$ is acyclic with respect to $H_c^*(G; -)$.

Proposition 5.4. *Let G be a profinite group. Suppose that $M = \varprojlim_\lambda \mathbb{A}^* N_\lambda$ with the inverse limit topology, where $\{N_\lambda\}_{\lambda\in\Lambda}$ is a cofiltered system of finitely generated E_{n+1}^*-modules. Furthermore, suppose that G continuously acts on M. For $p > 0$ and $q > 0$, we have $H_c^p(G; \mathrm{Map}_c(G^q, M)) = 0$, and $H_c^0(G; \mathrm{Map}_c(G^q, M)) = \mathrm{Map}_c(G^q, M)^G$.*

Proof. Set

$$C_c^{-1}(G; \mathrm{Map}_c(G^q, M)) = \mathrm{Map}_c(G^q, M)^G, \quad C^{p,q} = C_c^p(G; \mathrm{Map}_c(G^q, M)).$$

Then $C^{p,q} \cong \mathrm{Map}_c(G^q \times G^{p+1}, M)^G$ by Lemma 5.2. The boundary map $d^p : C^{p,q} \to C^{p+1,q}$ is given by

$$d^p f(h_1, \ldots, h_q; g_0, \ldots, g_{p+1}) \\ = \sum_{i=0}^{p+1} (-1)^i f(h_1, \ldots, h_q; g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{p+1}).$$

We define $s^p : C^{p,q} \to C^{p-1,q}$ by

$$s^p f(h_1, \ldots, h_q; g_0, \ldots, g_{p-1}) = f(h_1, \ldots, h_q; h_q, g_0, \ldots, g_{p-1}).$$

Then we can verify that $s^{p+1} d^p(f) + d^{p-1} s^p(f) = f$ for any $f \in C^{p,q}$. □

Corollary 5.5. *Let $p > 0$ and $q > 0$. Then $H_c^p(G_{n+1}; \mathrm{Map}_c(G_{n+1}^q, \mathbb{A}^*(W))) = 0$ and $H_c^0(G_{n+1}; \mathrm{Map}_c(G_{n+1}^q, \mathbb{A}^*(W))) = \mathrm{Map}_c(G_{n+1}^q, \mathbb{A}^*(W))^{G_{n+1}}$ for any spectrum W.*

Next we define a cohomology group $\boldsymbol{H}_c^*(\mathbb{G}; \mathbb{B}^*(W))$. For this purpose, we introduce a topology on $\mathbb{B}(i)^*(W)$.

Definition 5.6. For a spectrum W, $\mathbb{B}(i)^*(W)$ is a product of finite many copies of $\mathbb{A}^*(W)$ since $\mathbb{B}(i)^*$ is finitely generated free over $\mathbb{A}^*$. We give a topology on $\mathbb{B}(i)^*(W)$ by the product topology.

Recall that the group $\mathbb{G} = G_{n+1} \times_\Gamma G_n$ acts on the cohomology theory $\mathbb{B}^*(-)$ as multiplicative cohomology operations by Proposition 2.3. For $i \geq -1$, we set $\mathbb{G}(i) = G_{n+1} \times_\Gamma G_n(i)$, where $G_n(i) = \Gamma \ltimes S_n(i)$. Then $\mathbb{G}(i)$ acts on $\mathbb{B}(i)^*(W)$ naturally and continuously. Note that we can write $\mathbb{B}(i)^*(W) = \varprojlim_\lambda \mathbb{A}^* N_\lambda$ with finitely generated E^*_{n+1}-modules N_λ since $\mathbb{B}(i)^*$ is finitely generated free over $\mathbb{A}^*$. Then $\mathrm{Map}_c(\mathbb{G}(i)^{p+1}, \mathbb{B}(i)^*(W))$ is a topological module for any $p \geq 0$ as in Definition 5.1.

Definition 5.7. For a spectrum W, we define a cochain complex $\boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ by

$$\boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W)) = \varprojlim_\lambda \varinjlim_i C^*_c(\mathbb{G}(i); \mathbb{B}(i)^*(W_\lambda)),$$

where the inverse limit is taken over $\lambda \in \Lambda(W)$. Then we define a cohomology group $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ of $\mathbb{G}$ with coefficients in $\mathbb{B}^*(W)$ to be the cohomology group of $\boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W))$

$$\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W)) = H^*(\boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W))).$$

Note that both of $\boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ and $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ are not functors of $\mathbb{B}^*(W)$ in spite of their notation.

For a continuous cochain complex $C^*_c(G_{n+1}; \mathbb{A}^*(W))$ of G_{n+1} with coefficients in $\mathbb{A}^*(W)$, there is an isomorphism

$$C^*_c(G_{n+1}; \mathbb{A}^*(W)) \cong \varprojlim_\lambda C^*_c(G_{n+1}; \mathbb{A}^*(W_\lambda)).$$

The canonical maps $\mathbb{A}^*(W_\lambda) \to \mathbb{B}(i)^*(W_\lambda)$ and the projections $\mathbb{G}(i) \to G_{n+1}$ define a cochain map

$$C^*_c(G_{n+1}; \mathbb{A}^*(W)) \longrightarrow \boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W)).$$

We call the induced map on cohomology groups an inflation map

$$H^*_c(G_{n+1}; \mathbb{A}^*(W)) \longrightarrow \boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W)). \tag{5-1}$$

In the rest of this section we prove the following theorem.

Theorem 5.8. *The inflation map $H^*_c(G_{n+1}; \mathbb{A}^*(W)) \to \boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ is an isomorphism for any spectrum W.*

By definition, $\boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W))$ is the cohomology group of the inverse limit of the cochain complexes $\varinjlim_i C^*_c(\mathbb{G}(i); \mathbb{B}(i)^*(W_\lambda))$. For the cohomology group of the inverse limit of cochain complexes $\{C^*_\lambda\}_{\lambda\in\Lambda}$, we have a spectral sequence to describe it in terms of the cohomology groups of C^*_λ under suitable circumstances.

Lemma 5.9. *Let $\{C^*_\lambda\}_{\lambda\in\Lambda}$ be a system of cochain complexes indexed by a small category Λ. We assume that $\varprojlim^j_\lambda C^*_\lambda = 0$ for $j > 0$. Then there is a spectral sequence*

$$E_2^{s,t} = \varprojlim^s_\lambda H^t(C^*_\lambda) \Longrightarrow H^{s+t}(\varprojlim_\lambda C^*_\lambda).$$

Proof. Let $\prod^* C^*_\lambda$ be the double complex associated to the cosimplicial replacement [Bousfield and Kan 1972, XI.5] of $\{C^*_\lambda\}$. Then we have two spectral sequences

$$\varprojlim{}^s_\lambda\ H^t(C^*_\lambda) \implies H^{s+t}(\textstyle\prod^* C^*_\lambda),$$
$$H^s(\varprojlim{}^t_\lambda\ C^*_\lambda) \implies H^{s+t}(\textstyle\prod^* C^*_\lambda).$$

By the assumption, the second spectral sequence collapses to give $H^*(\varprojlim_\lambda C^*_\lambda) \cong H^*(\prod^* C^*_\lambda)$. Hence the first spectral sequence gives the desired one. $\square$

The next lemma gives a sufficient condition for all the higher inverse limits to vanish.

Lemma 5.10. *Let F be a profree $\mathbb{A}^*$-module. Then $\varprojlim^j_\lambda\ F \otimes_{\mathbb{A}^*} \mathbb{A}^*(W_\lambda) = 0$ for $j > 0$.*

Proof. Since F is a direct summand of some product of (suspensions of) $\mathbb{A}^*$ by [Hovey and Strickland 1999, Proposition A.13], we may assume that $F = \prod_\alpha \mathbb{A}^*$. For a finite spectrum W_λ, $F \otimes \mathbb{A}^*(W_\lambda) \cong \prod_\alpha \mathbb{A}^*(W_\lambda)$ since $\mathbb{A}^*(W_\lambda)$ is a finitely presented $\mathbb{A}^*$-module. Then we have $\varprojlim^j_\lambda\ \prod_\alpha \mathbb{A}^*(W_\lambda) \cong \prod_\alpha \varprojlim^j_\lambda\ \mathbb{A}^*(W_\lambda)$. The lemma follows from the fact that $\varprojlim^j_\lambda\ \mathbb{A}^*(W_\lambda) = 0$ for $j > 0$ since $\mathbb{A}^*(W_\lambda)$ is a linearly compact $\mathbb{A}^*$-module for all λ. $\square$

By Proposition 3.8, $\mathrm{Map}_c(G^{q+1}_{n+1}; \mathbb{A}^*)$ and $\mathrm{Map}_c(\mathbb{G}(i)^{q+1}, \mathbb{B}(i)^*)$ are profree $\mathbb{A}^*$-modules. Then the completion of $\varinjlim_i C^*_c(\mathbb{G}(i); \mathbb{B}(i)^*)$ at I_n is also a profree $\mathbb{A}^*$-module. By Lemma 5.10, we obtain that $\varprojlim^j_\lambda C^*_c(G_{n+1}; \mathbb{A}^*(W_\lambda)) = 0$ and $\varprojlim^j_\lambda\ \varinjlim_i C^*_c(\mathbb{G}(i); \mathbb{B}(i)^*(W_\lambda)) = 0$ for $j > 0$. Hence, by Lemma 5.9, we obtain two spectral sequences

$$\begin{aligned}
{}_I E_2^{s,t} &= \varprojlim{}^s_\lambda\ H^t_c(G_{n+1}; \mathbb{A}^*(W_\lambda)) &&\implies H^*_c(G_{n+1}; \mathbb{A}^*(W)),\\
{}_{II} E_2^{s,t} &= \varprojlim{}^s_\lambda\ \varinjlim_i H^t_c(\mathbb{G}(i); \mathbb{B}(i)^*(W_\lambda)) &&\implies \boldsymbol{H}^*_c(\mathbb{G}; \mathbb{B}^*(W)).
\end{aligned}$$

The system of cochain maps

$$\{C^*_c(G_{n+1}; \mathbb{A}^*(W_\lambda))\}_\lambda \longrightarrow \{\varinjlim_i C^*_c(\mathbb{G}(i); \mathbb{B}(i)^*(W_\lambda))\}_\lambda$$

induces a morphism of spectral sequences

$$f_r : {}_I E_r^{*,*} \longrightarrow {}_{II} E_r^{*,*} \tag{5-2}$$

which converges to the inflation map (5-1).

We show that this morphism of spectral sequences is an isomorphism from the E_2-terms onward. For this purpose, it is sufficient to show that the inflation map $H^*_c(G_{n+1}; \mathbb{A}^*(W)) \to H^*_c(\mathbb{G}(i); \mathbb{B}(i)^*(W))$ is an isomorphism for $i \geq 0$. We shall construct two acyclic resolutions $I^*(W)$ and $J^*(i, W)$ of $\mathbb{A}^*(W)$ with respect to $H^*_c(G_{n+1}; -)$ so that

$$I^*(W)^{G_{n+1}} \cong C^*_c(G_{n+1}; \mathbb{A}^*(W)) \quad \text{and} \quad J^*(i, W)^{G_{n+1}} \cong C^*_c(\mathbb{G}(i); \mathbb{B}(i)^*(W)).$$

We shall enlarge the complexes $C_c^*(G_{n+1};\mathbb{A}^*(W))$ and $C_c^*(\mathbb{G}(i);\mathbb{B}(i)^*(W))$ to double complexes $C_c^*(G_{n+1};\,I^*(W))$ and $C_c^*(G_{n+1};\,J(i,W))$. We shall construct a map of double complexes $C_c^*(G_{n+1};\,I^*(W))\to C_c^*(G_{n+1};\,J(i,W))$, which induces the inflation map $H_c^*(G_{n+1};\mathbb{A}^*(W))\to H_c^*(\mathbb{G}(i);\mathbb{B}(i)^*(W))$. Then we shall show that the map of double complexes induces an isomorphism on cohomology groups.

First, we construct an acyclic resolution $I^*(W)$ of $\mathbb{A}^*(W)$. We set

$$I^q(W)=\mathrm{Map}_c(G_{n+1}^{q+1},\mathbb{A}^*(W))$$

the topological $\mathbb{A}^*$-module of all continuous maps from G_{n+1}^{q+1} to $\mathbb{A}^*(W)$. Define a map $d^q: I^q(W)\to I^{q+1}(W)$ by

$$d^q(f)(g_0,\ldots,g_{q+1})=\sum_{j=0}^{q+1}(-1)^j f(g_0,\ldots,g_{j-1},g_{j+1},\ldots,g_{q+1}).$$

Then $I^*(W)=\{I^q(W),d^q\}_{q\ge -1}$ forms an augmented cochain complex satisfying $I^{-1}(W)=\mathbb{A}^*(W)$. The group G_{n+1} acts on the cochain complex $I^*(W)$ and

$$I^*(W)^{G_{n+1}}\cong C_c^*(G_{n+1};\mathbb{A}^*(W)).$$

Lemma 5.11. *For $p>0$ and $q\ge 0$, we have*

$$H_c^p(G_{n+1};\,I^q(W))=0\quad\textit{and}\quad H_c^0(G_{n+1};\,I^q(W))=C_c^q(G_{n+1};\mathbb{A}^*(W)).$$

The sequence $0\to\mathbb{A}^*(W)\overset{d^{-1}}{\to} I^0(W)\overset{d^1}{\to} I^1(W)\overset{d^2}{\to}\cdots$ is a split exact sequence of topological $\mathbb{A}^*$-modules. Hence $I^*(W)$ is an acyclic resolution of $\mathbb{A}^*(W)$ with respect to $H_c^*(G_{n+1};-)$.

Proof. Since $I^q(W)=\mathrm{Map}_c(G_{n+1}^{q+1},\mathbb{A}^*(W))$, the first assertion is a consequence of Corollary 5.5. We define $s^q: I^q(W)\to I^{q-1}(W)$ by $s^q(f)(g_0,\ldots,g_{q-1})=f(e,g_0,\ldots,g_{q-1})$. Then we can verify that $\{s^q\}_{q\ge 0}$ gives a desired splitting. □

Next we construct another acyclic resolution $J^*(i,W)$ of $\mathbb{A}^*(W)$. We set

$$J^q(i,W)=\mathrm{Map}_c(\mathbb{G}(i)^{q+1},\mathbb{B}(i)^*(W))^{S_n(i)}.$$

the topological $\mathbb{A}^*$-module of all $S_n(i)$-equivariant continuous maps from $\mathbb{G}(i)^{q+1}$ to $\mathbb{B}(i)^*(W)$. Define a map $d^q: J^q(i,W)\to J^{q+1}(i,W)$ by

$$d^q f(g_0,\ldots,g_{p+1})=\sum_{j=0}^{p+1}(-1)^j f(g_0,\ldots,g_{j-1},g_j,\ldots,g_{p+1}).$$

Then $J^*(i,W)=\{J^q(i,W),d^q\}_{q\ge -1}$ forms an augmented cochain complex with $J^{-1}(i,W)=\mathbb{A}^*(W)$. The group G_{n+1} acts on $J^*(i,W)$ and

$$J^*(i,W)^{G_{n+1}}\cong C_c^*(\mathbb{G}(i);\mathbb{B}(i)^*(W)).$$

We compare $J^*(i,W)$ with $I^*(W)$. Let $D^* = C^*(S_n(i); \mathbb{B}(i)^*)$ be the cochain complex of $S_n(i)$ with coefficients in $\mathbb{B}(i)^*$. Since $\mathbb{A}^* \to \mathbb{B}(i)^*$ is a Galois extension with Galois group $S_n(i)$, there is an isomorphism $D^q \cong \mathbb{B}(i)^{*\otimes(q+1)}$. Then the differential $d^q : D^q \to D^{q+1}$ corresponds to $d^q : \mathbb{B}(i)^{*(q+1)} \to \mathbb{B}(i)^{*(q+2)}$ given by

$$d^q(b_0 \otimes \cdots \otimes b_q) = \sum_{j=0}^{q} (-1)^j b_0 \otimes \cdots \otimes b_{j-1} \otimes 1 \otimes b_j \otimes \cdots \otimes b_q$$

for $b_0, \ldots, b_q \in \mathbb{B}(i)^*$. Since $\mathbb{G}(i) \cong G_{n+1} \times S_n(i)$ as an $S_n(i)$-space, and D^q is a finitely generated free $\mathbb{A}^*$-module, we see that $J^q(i,W) \cong I^q(W) \otimes D^q$. Then the differential $d^q : J^q(i,W) \to J^{q+1}(i,W)$ corresponds to

$$d^q : I^q(i,W) \otimes \mathbb{B}(i)^{*\otimes(q+1)} \to I^{q+1}(i,W) \otimes \mathbb{B}(i)^{*\otimes(q+2)}$$

given by

$$\begin{aligned} &d^q(f \otimes b_0 \otimes \cdots \otimes b_q)(g_0, \ldots, q_{q+1}) \\ &= \sum_{j=0}^{q+1} (-1)^j f(g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_{q+1}) \otimes b_0 \otimes \cdots \otimes b_{j-1} \otimes 1 \otimes b_j \otimes \cdots \otimes b_q. \end{aligned}$$

Proposition 5.12. *For $p > 0$ and $q \geq 0$, we have*

$$H_c^p(G_{n+1}; J^q(i,W)) = 0 \quad \text{and} \quad H_c^0(G_{n+1}; J^q(i,W)) = C_c^q(G_{n+1}; \mathbb{A}^*(W)).$$

The sequence $0 \to \mathbb{A}^*(W) \stackrel{d^{-1}}{\to} J^0(i,W) \stackrel{d^0}{\to} J^1(i,W) \stackrel{d^2}{\to} \cdots$ is a split exact sequence of topological $\mathbb{A}^*$-modules. Hence $J^*(i,W)$ is an acyclic resolution of $\mathbb{A}^*(W)$ with respect to $H_c^*(G_{n+1}; -)$.

Proof. Let $M = \mathrm{Map}(S_n(i)^q, \mathbb{B}(i)^*(W))$. We have an isomorphism $J^q(i,W) \cong \mathrm{Map}_c(G_{n+1}^{q+1}, M)$ of topological G_{n+1}-modules. Since M is a product of finite many copies of $\mathbb{A}^*(W)$, we can write $M = \varprojlim_\lambda \mathbb{A}^* N_\lambda$ with finitely generated N_λ. Then the first assertion follows from Proposition 5.4. There is a continuous map $\varepsilon : \mathbb{B}^*(i) \to \mathbb{A}^*$ of topological $\mathbb{A}^*$-modules such that $\varepsilon \circ \eta = 1$, where $\eta : \mathbb{A}^* \to \mathbb{B}^*(i)$ is the unit. Define a map $s^q : I^q(i,W) \otimes \mathbb{B}(i)^{*\otimes(q+1)} \to I^{q-1}(i,W) \otimes \mathbb{B}(i)^{*\otimes q}$ by

$$s^q(f \otimes b_0 \otimes \cdots \otimes b_q)(g_0, \ldots, g_{q-1}) = f(e, g_0, \ldots, g_{q-1}) \otimes \varepsilon(b_0) b_1 \otimes \cdots \otimes b_q.$$

Then we can verify that $\{s^q\}_{q \geq 0}$ gives a desired splitting. □

We consider the double complexes $C_c^*(G_{n+1}; I^*(W))$ and $C_c^*(G_{n+1}; J^*(i,W))$. The canonical inclusion $\mathbb{A}^*(W) \to \mathbb{B}(i)^*(W)$ and the projection $\mathbb{G}(i) \to G_{n+1}$ induce a cochain map $I^*(W) \to J^*(i,W)$, which is equivariant under the actions of G_{n+1}. Hence we obtain a map of double complexes

$$C_c^*(G_{n+1}; I^*(W)) \longrightarrow C_c^*(G_{n+1}; J^*(i,W)). \tag{5-3}$$

We denote by $\mathrm{Tot}^* C^{*,*}$ the total cochain complex of a double complex $C^{*,*}$.

Lemma 5.13. *The cochain map*

$$\mathrm{Tot}^* C_c^*(G_{n+1};\, I^*(W)) \to \mathrm{Tot}^* C_c^*(G_{n+1};\, J^*(i,W))$$

is a quasi-isomorphism.

Proof. This follows from the fact that the map (5-3) induces an isomorphism on cohomology groups on the second index by Lemma 5.11 and Proposition 5.12. □

Since the invariant subcomplex $I^*(W)^{G_{n+1}}$ is isomorphic to $C_c^*(G_{n+1};\, \mathbb{A}^*(W))$, there is a cochain map

$$C_c^*(G_{n+1};\, \mathbb{A}^*(W)) \longrightarrow \mathrm{Tot}^* C_c^*(G_{n+1};\, I^*(W)).$$

Since the invariant subcomplex $J^*(i,W)^{G_{n+1}}$ is isomorphic to $C_c^*(\mathbb{G}(i);\, \mathbb{B}(i)^*(W))$, there is a cochain map

$$C_c^*(\mathbb{G}(i);\, \mathbb{B}(i)^*(W)) \longrightarrow \mathrm{Tot}^* C_c^*(G_{n+1};\, J^*(i,W)).$$

Then we obtain the commutative diagram of cochain complexes

$$\begin{tikzcd} C_c^*(G_{n+1};\, \mathbb{A}^*(W)) \arrow[r] \arrow[d] & C_c^*(\mathbb{G}(i);\, \mathbb{B}(i)^*(W)) \arrow[d] \\ \mathrm{Tot}^* C_c^*(G_{n+1};\, I^*(W)) \arrow[r] & \mathrm{Tot}^* C_c^*(G_{n+1};\, J^*(i,W)), \end{tikzcd} \tag{5-4}$$

where the top horizontal arrow induces the inflation map

$$H_c^*(G_{n+1};\, \mathbb{A}^*(W)) \longrightarrow H_c^*(\mathbb{G}(i);\, \mathbb{B}(i)^*(W)).$$

Lemma 5.14. *The vertical arrows in the diagram* (5-4) *are quasi-isomorphisms.*

Proof. By Lemma 5.11, the cohomology group of $C_c^*(G_{n+1};\, I^*(W))$ on the first index is isomorphic to $C_c^*(G_{n+1};\, \mathbb{A}^*(W))$. Hence the left vertical arrow is a quasi-isomorphism. By Proposition 5.12, the cohomology group of $C_c^*(G_{n+1};\, J^*(i,W))$ on the first index is isomorphic to $C_c^*(\mathbb{G}(i);\, \mathbb{B}(i)^*(W))$. Hence the right vertical arrow is a quasi-isomorphism. □

Corollary 5.15. *The inflation map $H_c^*(G_{n+1};\, \mathbb{A}^*(W)) \longrightarrow H_c^*(\mathbb{G}(i);\, \mathbb{B}(i)^*(W))$ is an isomorphism for any spectrum W and any $i \geq 0$.*

Proof of Theorem 5.8. Corollary 5.15 implies that the morphism (5-2) of spectral sequences is an isomorphism from the E_2-terms onward. Hence the inflation map (5-1) is an isomorphism. □

Remark 5.16. Let Λ be an essentially small cofiltered category. For a system $\{N_\lambda\}_{\lambda\in\Lambda}$ of finitely generated twisted E_{n+1}^*-G_{n+1}-modules, we set $M=\varprojlim_\lambda \mathbb{A}^*N_\lambda$ and $\mathbb{B}^*M=\varprojlim_\lambda \mathbb{B}^*\otimes_{\mathbb{A}^*}\mathbb{A}^*N_\lambda$. By the same method as above, we can define $\boldsymbol{H}_c^*(\mathbb{G};\mathbb{B}^*M)$ and show that there is an isomorphism

$$H_c^*(G_{n+1};M)\xrightarrow{\cong}\boldsymbol{H}_c^*(\mathbb{G};\mathbb{B}^*M).$$

6. Morphism of spectral sequences

In this section we construct a natural morphism of spectral sequences from the $K(n)$-local E_n-Adams spectral sequence to the $K(n)$-localization of the $K(n+1)$-local E_{n+1}-Adams spectral sequence.

Let BP be the Brown–Peterson spectrum at p. We denote by $BP^{\wedge s}$ the smash product of s copies of BP:

$$BP^{\wedge s}=\overbrace{BP\wedge\cdots\wedge BP}^{s}.$$

The commutative ring spectrum structure on BP makes $BP^{\wedge\bullet+1}=\{BP^{\wedge s+1}\}_{s\geq 0}$ a cosimplicial object in the p-local stable homotopy category with augmentation $S^0_{(p)}\xrightarrow{\varepsilon}BP^{\wedge\bullet+1}$. Then the associated cochain complex

$$* \to S^0_{(p)}\xrightarrow{\varepsilon}BP\xrightarrow{d}BP^{\wedge 2}\xrightarrow{d}BP^{\wedge 3}\xrightarrow{d}\cdots \tag{6-1}$$

is a p-local BP-Adams resolution of $S^0_{(p)}$ in the sense of [Miller 1981; Devinatz and Hopkins 2004]. We denote by $\mathrm{Res}(BP;S^0_{(p)})$ the sequence (6-1). Then $\mathrm{Res}(BP;S^0_{(p)})$ gives us a diagram of exact triangles

$$\begin{tikzcd}[column sep=small] S^0_{(p)}=X^0 \arrow[dr, "j"] & & X^1 \arrow[ll, "i"'] \arrow[dr, "j"] & & X^2 \arrow[ll, "i"'] \arrow[dr, "j"] & & X^3 \arrow[ll, "i"'] & \cdots, \\ & BP \arrow[ur, "k"] & & \Sigma^{-1}BP^{\wedge 2} \arrow[ur, "k"] & & \Sigma^{-2}BP^{\wedge 3} \arrow[ur, "k"] & & \end{tikzcd} \tag{6-2}$$

where k has degree -1 and $jk=d$. We denote by $\mathrm{Ad}(BP;S^0_{(p)})$ the diagram of exact triangles (6-2).

By applying the $K(n)$-localization functor to the augmented cosimplicial commutative ring spectrum $S^0_{(p)}\xrightarrow{\varepsilon}BP^{\wedge\bullet+1}$, we obtain an augmented cosimplicial $K(n)$-local commutative ring spectrum $L_{K(n)}S^0\xrightarrow{\varepsilon}L_{K(n)}BP^{\wedge\bullet+1}$, and the associated augmented cochain complex

$$* \to L_{K(n)}S^0\xrightarrow{\varepsilon}L_{K(n)}BP\xrightarrow{d}L_{K(n)}BP^{\wedge 2}\xrightarrow{d}L_{K(n)}BP^{\wedge 3}\xrightarrow{d}\cdots. \tag{6-3}$$

We denote by $L_{K(n)}\mathrm{Res}(BP;\, S^0_{(p)})$ the sequence (6-3).

Proposition 6.1. *The sequence $L_{K(n)}\mathrm{Res}(BP;\, S^0_{(p)})$ is a $K(n)$-local E_n-Adams resolution of $L_{K(n)}S^0$.*

Proof. To prove the proposition, it suffices to show that $L_{K(n)}BP^{\wedge s}$ is E_n-injective for $s > 0$ and the sequence (6-3) is E_n-exact. By [Hovey and Sadofsky 1999, Theorem B], $L_{K(n)}BP$ is a coproduct of (suspensions of) $L_{K(n)}E(n)$'s in the $K(n)$-local category. Since $L_{K(n)}E(n)$ is a direct summand of E_n, $L_{K(n)}BP$ is E_n-injective. Hence $L_{K(n)}BP^{\wedge s}$ is E_n-injective for $s > 0$. To prove that the sequence (6-3) is E_n-exact, it is sufficient to show that the sequence (6-3) smashing with E_n is a split exact sequence. There is a canonical ring spectrum map $\eta : L_{K(n)}BP \to E_n$. Then the following map

$$L_{K(n)}(E_n \wedge BP^{\wedge s+1}) \xrightarrow{1\wedge\eta\wedge 1^{\wedge s}} L_{K(n)}(E_n \wedge E_n \wedge BP^{\wedge s}) \xrightarrow{m\wedge 1^{\wedge s}} L_{K(n)}(E_n \wedge BP^{\wedge s})$$

for $s \geq 0$ gives a splitting, where m is the multiplication of E_n. □

The $K(n)$-localization functor gives a map of cosimplicial objects $BP^{\bullet+1} \to E_n^{\bullet+1}$ covering the map $S^0_{(p)} \to L_{K(n)}S^0$. This induces a map

$$L_{K(n)}\mathrm{Res}(BP;\, S^0_{(p)}) \to \mathrm{Res}(E_n;\, L_{K(n)}S^0)$$

of cochain complexes and a map $L_{K(n)}\mathrm{Ad}(BP;\, S^0) \to \mathrm{Ad}(E_n;\, L_{K(n)}S^0)$ of diagrams of exact triangles. By Proposition 6.1, the map

$$L_{K(n)}\mathrm{Res}(BP;\, S^0_{(p)}) \to \mathrm{Res}(E_n;\, L_{K(n)}S^0)$$

is a cochain homotopy equivalence. Hence $L_{K(n)}\mathrm{Ad}(BP;\, S^0) \to \mathrm{Ad}(E_n;\, L_{K(n)}S^0)$ is an equivalence of diagram of exact triangles in an appropriate sense.

The canonical ring spectrum map $BP \to E_{n+1}$ induces a map of diagrams of exact triangles

$$\mathrm{Ad}(BP;\, S^0_{(p)}) \longrightarrow L_{K(n+1)}\mathrm{Ad}(BP;\, S^0_{(p)}) \xrightarrow{\simeq} \mathrm{Ad}(E_{n+1};\, L_{K(n+1)}S^0).$$

By applying the $K(n)$-localization functor to this map, we obtain a map of diagrams of exact triangles

$$L_{K(n)}\mathrm{Ad}(BP;\, S^0_{(p)}) \longrightarrow L_{K(n)}\mathrm{Ad}(E_{n+1};\, L_{K(n+1)}S^0).$$

Then this map of exact triangles implies the following theorem.

Theorem 6.2. *For any spectrum W, there is a natural morphism of spectral sequences*

$$\varphi_r(W) : L_{K(n)}E_r^{s,t}(W) \longrightarrow L_{K(n)}L_{K(n+1)}E_r^{s,t}(W),$$

which converges to $[W, L_{K(n)}S^0]^* \to [W, \widehat{\mathbb{S}}]^*$.

7. The inflation map

In Section 6 we constructed a natural morphism

$$\varphi_r(W) : L_{K(n)} E_r^{*,*}(W) \to L_{K(n)} L_{K(n+1)} E_r^{*,*}(W)$$

of spectral sequences for any spectrum W. In this section we construct a natural map $\theta(W) : H_c^*(G_n; E_n^*(W)) \to H_c^*(G_{n+1}; \mathbb{A}^*(W))$ by using the cohomology group $\boldsymbol{H}_c^*(\mathbb{G}; \mathbb{B}^*(W))$ in Section 5. Then we show that $\theta(W)$ coincides with $\varphi_2(W)$.

For a spectrum W, define cochain complexes $C_{BP}^{*,*}(W)$ and $L_{K(n)} C_{BP}^{*,*}(W)$ by

$$\begin{aligned} C_{BP}^{s,*}(W) &= [W, BP^{\wedge s+1}]^*, \\ L_{K(n)} C_{BP}^{s,*}(W) &= [W, L_{K(n)}(BP^{\wedge s+1})]^*. \end{aligned}$$

The ring spectrum maps $BP \to L_{K(n)} BP \to E_n$ induce cochain maps

$$C_{BP}^{*,*}(W) \to L_{K(n)} C_{BP}^{*,*}(W) \to C_c^*(G_n; E_n^*(W)).$$

We shall describe the cochain map $C_{BP}^{*,*}(W) \to C_c^*(G_n; E_n^*(W))$ in terms of formal group laws. The universal deformation F_n over E_n^0 induces a graded ring homomorphism $BP_* \to E_{n*}$. Recall that, for $g = (\gamma, s) \in \Gamma \ltimes S_n = G_n$, there is a unique isomorphism $t(g) : F_n \to F_n^g$ over E_n^0, which is a lifting of the isomorphism $s : H_n \to H_n^\gamma = H_n$ over $\boldsymbol{F}$. For $g, h \in G_n$, we set $t(g, h) = t(h) \circ t(g)^{-1} : F_n^g \to F_n^h$. For a sequence $\boldsymbol{g} = (g_0, g_1, \dots, g_s)$ of elements in G_n, we define a graded ring homomorphism

$$t(\boldsymbol{g}) : BP_*(BP)^{\otimes(s+1)} \longrightarrow E_{n*}$$

to be the map representing the following string of isomorphisms of formal group laws

$$F_n \xrightarrow{t(g_0)} F_n^{g_0} \xrightarrow{t(g_0, g_1)} F_n^{g_1} \xrightarrow{t(g_1, g_2)} \cdots \xrightarrow{t(g_{s-1}, g_s)} F_n^{g_s}.$$

For a spectrum W, we denote by $\mathrm{ev}(\boldsymbol{g}) : C_c^s(G_n; E_n^*(W)) \to E_n^*(W)$ the evaluation map at $\boldsymbol{g} = (g_0, g_1, \dots, g_s)$. If W is a finite spectrum, we denote its S-dual by DW. Then there are natural isomorphisms $BP^{-*}(W) \cong BP_*(DW)$ and $E_n^{-*}(W) \cong E_{n*}(DW) \cong BP_*(DW) \otimes_{BP_*} E_{n*}$. In particular, we have

$$C_{BP}^{s,-*}(W) \cong BP_*(DW) \otimes_{BP_*} BP_*(BP)^{\otimes s}.$$

Lemma 7.1. *Let W be a finite spectrum. For a sequence $\boldsymbol{g} = (g_0, g_1, \dots, g_s)$ of elements in G_n, the composition* $C_{BP}^{s,-*}(W) \longrightarrow C_c^s(G_n; E_n^{-*}(W)) \xrightarrow{\mathrm{ev}(\boldsymbol{g})} E_n^{-*}(W)$

is given by

$$\begin{aligned} BP_*(DW)\otimes_{BP_*} BP_*(BP)^{\otimes s} &\xrightarrow{\psi\otimes 1^{\otimes s}} BP_*(DW)\otimes_{BP_*} BP_*(BP)^{\otimes (s+1)} \\ &\xrightarrow{1\otimes t(\boldsymbol{g})} BP_*(DW)\otimes_{BP_*} E_{n*}, \end{aligned}$$

where ψ is the $BP_*(BP)$-comodule structure map of $BP_*(DW)$.

Proof. For $g\in G_n$, the ring spectrum map $g: E_n\to E_n$ induces a map $g^{-*}: E_n^{-*}(W)\to E_n^{-*}(W)$. This map g^{-*} is given by the composition

$$\begin{aligned} BP_*(DW)\otimes_{BP_*} E_{n*} &\xrightarrow{\psi\otimes 1} BP_*(DW)\otimes_{BP_*} BP_*(BP)\otimes_{BP_*} E_{n*} \\ &\xrightarrow{1\otimes t(g)\otimes g_*} BP_*(DW)\otimes_{BP_*} E_{n*}. \end{aligned}$$

Next we consider the map $g_0\wedge\cdots\wedge g_s: E_n^{\wedge s+1}\to E_n^{\wedge s+1}$. This induces a map $(g_0\wedge\cdots\wedge g_s)^{-*}: (E_n^{\wedge s+1})^{-*}(W)\to (E_n^{\wedge s+1})^{-*}(W)$. Note that there is a natural isomorphism $(E_n^{\wedge s+1})^{-*}(W)\cong BP_*(DW)\otimes_{BP_*}\pi_* E_n^{\wedge s+1}$ since $\pi_* E_n^{\wedge s+1}$ is Landweber exact. Then $(g_0\wedge\cdots\wedge g_s)^{-*}$ is given by

$$\begin{aligned} BP_*(DW)\otimes_{BP_*}\pi_* E_n^{\wedge s+1} &\xrightarrow{\psi\otimes 1} BP_*(DW)\otimes_{BP_*} BP_*(BP)\otimes_{BP_*}\pi_* E_n^{\wedge s+1} \\ &\xrightarrow{1\otimes t(g_0)\otimes \pi_*(g_0\wedge\cdots\wedge g_s)} BP_*(DW)\otimes_{BP_*} E_{n*}\otimes_{E_{n*}}\pi_* E_n^{\wedge s+1} \\ &\cong BP_*(DW)\otimes_{BP_*}\pi_* E_n^{\wedge s+1}. \end{aligned}$$

The lemma follows from the fact that the composition

$$C_{BP}^{s,-*}(W)\longrightarrow C_c^s(G_n; E_n^{-*}(W))\xrightarrow{\mathrm{ev}(\boldsymbol{g})} E_n^{-*}(W)$$

is induced by the map $BP^{\wedge s+1}\to E_n^{\wedge s+1}\xrightarrow{g_0\wedge\cdots\wedge g_s} E_n^{\wedge s+1}\xrightarrow{m} E_n$, where m is the multiplication map of the ring spectrum E_n. $\square$

Next we construct a cochain map $C_c^*(G_n; E_n^*(W))\longrightarrow \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W))$, which induces a map $H_c^*(G_n; E_n^*(W))\longrightarrow \boldsymbol{H}_c^*(\mathbb{G}; \mathbb{B}^*(W))$.

Lemma 7.2. *The ring spectrum map $I: E_n\to\mathbb{B}$ and the projection $\mathbb{G}\to G_n$ induce a cochain map $C_c^*(G_n; E_n^*(W))\longrightarrow \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W))$ for any spectrum W.*

Proof. There are isomorphisms

$$\begin{aligned} C_c^*(G_n; E_n^*(W)) &\cong \varprojlim_\lambda \varinjlim_i C_c^*(G(i), E_n^*(W_\lambda)), \\ \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W)) &\cong \varprojlim_\lambda \varinjlim_i C_c^*(\mathbb{G}(i), \mathbb{B}(i)^*(W_\lambda)). \end{aligned}$$

Then the canonical maps $E_n^*(W_\lambda)\to\mathbb{B}(i)^*(W_\lambda)$ and the projections $\mathbb{G}(i)\to G_n(i)$ induce the desired cochain map. $\square$

Remark 7.3. Let Λ be an essentially small cofiltered category. For a system $\{N_\lambda\}_{\lambda\in\Lambda}$ of finitely generated twisted E_n^*-G_n-modules annihilated by a power of the ideal I_n, we set $N = \varprojlim_\lambda N_\lambda$ and $\mathbb{B}^* N = \varprojlim_\lambda \mathbb{B}^* \otimes_{E_n^*} N_\lambda$. By the same method as above, we can obtain a cochain map $C_c^*(G_n; N) \to \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^* N)$.

Recall that in Section 5 we defined a cochain map $C_c^*(G_{n+1}; \mathbb{A}^*(W)) \longrightarrow \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W))$, which induces an isomorphism of cohomology groups

$$H_c^*(G_{n+1}; \mathbb{A}^*(W)) \xrightarrow{\cong} \boldsymbol{H}_c^*(\mathbb{G}; \mathbb{B}^*(W))$$

by Theorem 5.8. We define a map

$$\theta(W) : H_c^*(G_n; E_n^*(W)) \longrightarrow H_c^*(G_{n+1}; \mathbb{A}^*(W)) \tag{7-1}$$

by the composition

$$H_c^*(G_n; E_n^*(W)) \longrightarrow \boldsymbol{H}_c^*(\mathbb{G}; \mathbb{B}^*(W)) \xleftarrow{\cong} H_c^*(G_{n+1}; \mathbb{A}^*(W)),$$

where the first map is induced by the cochain map in Lemma 7.2.

In the rest of this section we compare $\theta(W)$ to $\varphi_2(W)$. The ring spectrum maps $BP \to L_{K(n)}BP \to L_{K(n)}E_{n+1} = \mathbb{A}$ induce cochain maps

$$C_{BP}^{*,*}(W) \to L_{K(n)}C_{BP}^{*,*}(W) \to C_c^*(G_{n+1}; \mathbb{A}^*(W)).$$

We consider the following diagram of cochain complexes

$$\begin{array}{ccc} C_{BP}^{*,*}(W) & \longrightarrow & C_c^*(G_{n+1}; \mathbb{A}^*(W)) \\ \downarrow & & \downarrow \\ C_c^*(G_n; E_n^*(W)) & \longrightarrow & \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W)). \end{array} \tag{7-2}$$

This diagram is not commutative but we shall show that it is cochain homotopy commutative for finite spectra W by constructing a natural cochain homotopy.

Lemma 7.4. *If W is a finite spectrum, then the diagram* (7-2) *is cochain homotopy commutative.*

Proof. Let $\pi : \mathbb{G} \to G_n$ be the projection. For $g, h \in \mathbb{G}$, we have an isomorphism of formal group laws $t(\pi(g), \pi(h)) : F_n^{\pi(g)} \to F_n^{\pi(h)}$ over E_n^0. If we regard $t(\pi(g), \pi(h))$ as a power series over $\mathbb{B}^0$, then we obtain an isomorphism of formal group laws $t(g, h) : F_n^g \to F_n^h$ over $\mathbb{B}^0$. In the same way we obtain an isomorphism of formal group laws $u(g, h) : F_{n+1}^g \to F_{n+1}^h$ over $\mathbb{B}^0$. Recall that there is an isomorphism of formal group laws $\Phi : F_{n+1} \to F_n$ over $\mathbb{B}^0$. For a sequence $\boldsymbol{g} = (g_0, g_1, \ldots, g_s)$ of elements in $\mathbb{G}$, consider the following diagram of formal groups laws and isomorphisms over $\mathbb{B}^0$

$$\begin{tikzcd}
F_{n+1} \arrow[r, "u(g_0)"] & F_{n+1}^{g_0} \arrow[r, "u(g_0,g_1)"] & F_{n+1}^{g_1} \arrow[r] & \cdots \arrow[r] & F_{n+1}^{g_i} \arrow[d, "\Phi^{g_i}"] & & & \\
& & & & F_n^{g_i} \arrow[r, "t(g_i,g_{i+1})"] & F_n^{g_{i+1}} \arrow[r] & \cdots \arrow[r] & F_n^{g_s}.
\end{tikzcd}$$

This diagram induces a graded ring homomorphism $T_i(\boldsymbol{g}) : BP_*(BP)^{\otimes(s+2)} \to \mathbb{B}_*$. We fix an isomorphism between $\mathbb{B}^{-*}(W)$ and $BP_*(DW) \otimes_{BP_*} \mathbb{B}_*$, where $\mathbb{B}_*$ is a BP_*-module through the graded ring homomorphism $BP_* \to \mathbb{B}_*$ classifying the p-typical formal group law F_{n+1}. We define a map $C_{BP}^{s+1,-*}(W) \to \mathbb{B}^{-*}(W)$ by

$$\begin{aligned} BP_*(DW) \otimes_{BP_*} BP_*(BP)^{\otimes(s+1)} &\xrightarrow{\psi\otimes 1^{\otimes(s+1)}} BP_*(DW) \otimes_{BP_*} BP_*(BP)^{\otimes(s+2)} \\ &\xrightarrow{1\otimes T_i(\boldsymbol{g})} BP_*(DW) \otimes_{BP_*} \mathbb{B}_*. \end{aligned}$$

This map extends to a map

$$S_i : C_{BP}^{s+1,*}(W) \longrightarrow \varinjlim_i \mathrm{Map}_c(\mathbb{G}(i)^{s+1}, \mathbb{B}(i)^*(W))^{\mathbb{G}(i)} = \boldsymbol{C}_c^s(\mathbb{G}; \mathbb{B}^*(W)).$$

We shall verify that $\sum_{i=0}^{s}(-1)^i S_i$ is a desired cochain homotopy. First note that the map $E_n^{-*}(W) \to \mathbb{B}^{-*}(W) \cong BP_*(DW) \otimes_{BP_*} \mathbb{B}_*$ is given by

$$\begin{aligned} BP_*(DW) \otimes_{BP_*} E_{n*} &\xrightarrow{\psi\otimes 1} BP_*(DW) \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_{n*} \\ &\xrightarrow{1\otimes\Phi\otimes I_*} BP_*(DW) \otimes_{BP_*} \mathbb{B}_*, \end{aligned}$$

where $\Phi : BP_*(BP) \to \mathbb{B}_*$ is the graded ring homomorphism classifying the isomorphism $\Phi : F_{n+1} \to F_n$, and $I_* : E_{n*} \to \mathbb{B}_*$ is the induced map by the ring spectrum map I. Let a^* be the cochain map $C_{BP}^{*,*}(W) \to C_c^*(G_n; E_n^*(W)) \to \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W))$ and let b^* be the cochain map $C_{BP}^{*,*}(W) \to C_c^*(G_{n+1}; E_{n+1}^*(W)) \to \boldsymbol{C}_c^*(\mathbb{G}; \mathbb{B}^*(W))$. We see that $\mathrm{ev}(\boldsymbol{g}) \circ a^s$ is given by

$$\begin{aligned} BP_*(DW) \otimes_{BP_*} BP_*(BP)^{\otimes s} &\xrightarrow{\psi\otimes 1^{\otimes s}} BP_*(DW) \otimes_{BP_*} BP_*(BP)^{\otimes(s+1)} \\ &\xrightarrow{1\otimes U(\boldsymbol{g})} BP_*(DW) \otimes_{BP_*} \mathbb{B}_*, \end{aligned}$$

where $U(\boldsymbol{g})$ is the graded ring homomorphism classifying the following string of isomorphisms of formal group laws

$$F_{n+1} \xrightarrow{t(g_0)\circ\Phi} F_n^{g_0} \xrightarrow{t(g_0,g_1)} F_n^{g_1} \xrightarrow{t(g_1,g_2)} \cdots \xrightarrow{t(g_{s-1},g_s)} F_n^{g_s}.$$

In the cosimplicial module $C^{\bullet,*}_{BP}(W)$, the map $d_i : C^{s,-*}_{BP}(W) \to C^{s+1,-*}_{BP}(W)$ is given by

$$d_i = \begin{cases} \psi \otimes 1^{\otimes s} & \text{if } i = 0, \\ 1 \otimes 1^{\otimes(i-1)} \otimes \Delta \otimes 1^{\otimes(s-i)} & \text{if } 1 \le i \le s, \\ 1 \otimes 1^{\otimes s} \otimes \eta_L & \text{if } i = s+1, \end{cases}$$

where $\Delta : BP_*(BP) \to BP_*(BP)^{\otimes 2}$ is the comultiplication, and $\eta_L : BP_* \to BP_*(BP)$ is the left unit. Then we see that

$$\begin{aligned} S_0 \circ d_0 &= a^s, \\ S_i \circ d_j &= d_j \circ S_{i-1} && \text{for } 0 \le j < i \le s, \\ S_{i-1} \circ d_i &= S_i \circ d_i && \text{for } 0 < i \le s, \\ S_i \circ d_j &= d_{j-1} \circ S_i && \text{for } 0 \le i < j-1 \le s, \\ S_s \circ d_{s+1} &= b^s. \end{aligned}$$

This implies that

$$\sum_{i=0}^{s}(-1)^i S_i \circ \sum_{j=0}^{s+1}(-1)^j d_j + \sum_{j=0}^{s}(-1)^j d_j \circ \sum_{i=0}^{s-1}(-1)^i S_i = a^s - b^s.$$

This completes the proof. $\square$

For a spectrum W, we have a similar diagram of cochain complexes

$$\text{(7-3)} \qquad \begin{array}{ccc} L_{K(n)} C^{*,*}_{BP}(W) & \longrightarrow & C^*_c(G_{n+1}; \mathbb{A}^*(W)) \\ \big\downarrow & & \big\downarrow \\ C^*_c(G_n; E^*_n(W)) & \longrightarrow & \boldsymbol{C}^*_c(\mathbb{G}; \mathbb{B}^*(W)). \end{array}$$

When W is a finite spectrum, we let $S(W) : C^{*,*}_{BP}(W) \to \boldsymbol{C}^{*-1}_c(\mathbb{G}; \mathbb{B}^*(W))$ be the cochain homotopy constructed in the proof of Lemma 7.4. Then $S(W)$ extends to a cochain homotopy $L_{K(n)}S(W) : L_{K(n)}C^{*,*}_{BP}(W) \to \boldsymbol{C}^{*-1}_c(\mathbb{G}; \mathbb{B}^*(W))$, which makes the diagram (7-3) homotopy commutative.

Proposition 7.5. *For any spectrum W, the diagram* (7-3) *is cochain homotopy commutative.*

Proof. Since the cochain homotopy $L_{K(n)}S(W)$ is natural for finite spectra W, we obtain a cochain homotopy

$$\varprojlim_\lambda L_{K(n)} S(W_\lambda) : \\ \varprojlim_\lambda L_{K(n)} C^{*,*}_{BP}(W_\lambda) \longrightarrow \varprojlim_\lambda \boldsymbol{C}^{*-1}_c(\mathbb{G}; \mathbb{B}^*(W_\lambda)) = \boldsymbol{C}^{*-1}_c(\mathbb{G}; \mathbb{B}^*(W)),$$

where the inverse limits are taken over $\lambda \in \Lambda(W)$. Then the composition with the cochain map $L_{K(n)}C^{*,*}_{BP}(W) \longrightarrow \varprojlim_{\lambda} L_{K(n)}C^{*,*}_{BP}(W_\lambda)$ makes the diagram (7-3) cochain homotopy commutative. □

Theorem 7.6. *The map*

$$\theta(W) : H^*_c(G_n;\, E^*_n(W)) \to H^*_c(G_{n+1};\, E^*_{n+1}(W))$$

coincides with the map $\varphi_2(W)$ for any spectrum W.

Proof. In the diagram (7-3) the left vertical arrow is a quasi-isomorphism by Proposition 6.1. So is the right vertical arrow, by Theorem 5.8. The theorem follows because the top horizontal arrow induces the map $\varphi_2(W)$ and the bottom horizontal arrow induces the map $\theta(W)$. □

8. Nontriviality of the image of ζ_n

In this section we prove Theorem 8.1 as an application of the results in this note. By the Hopkins–Miller theorem [Devinatz and Hopkins 2004, Theorem 6], we know that there exists a nontrivial element $\zeta_n \in \pi_{-1}(L_{K(n)}S^0)$, which is represented by the reduced norm map of G_n in the E_2-term of the $K(n)$-local E_n-Adams spectral sequence. The $K(n)$-localization of the $K(n+1)$-localization map $S^0 \to L_{K(n+1)}S^0$ induces a map $L_{K(n)}S^0 \to L_{K(n)}L_{K(n+1)}S^0$. In this section we show that the image of ζ_n under the map $\pi_*(L_{K(n)}S^0) \to \pi_*(L_{K(n)}L_{K(n+1)}S^0)$ is nontrivial as an application of Theorems 4.7 and 5.8.

By Theorem 6.2, we have a morphism of spectral sequences

$$\varphi_r = \varphi_r(S^0) : L_{K(n)}E^{*,*}_r(S^0) \longrightarrow L_{K(n)}L_{K(n+1)}E^{*,*}_r(S^0),$$

which converges to $\pi_*(L_{K(n)}S^0) \to \pi_*(L_{K(n)}L_{K(n+1)}S^0)$. Then φ_2 is identified with the inflation map

$$\theta = \theta(S^0) : H^*_c(G_n;\, E^*_n) \longrightarrow H^*_c(G_{n+1};\, \mathbb{A}^*)$$

by Theorem 5.8. The reduced norm map of G_n defines an element $z_n \in H^1_c(G_n;\, E^0_n)$ which represents $\zeta_n \in \pi_{-1}(L_{K(n)}S^0)$. We set $w_n = \theta(z_n) \in H^1_c(G_{n+1};\, \mathbb{A}^0)$, and denote by ω_n the image of ζ_n under the map $\pi_*(L_{K(n)}S^0) \to \pi_*(L_{K(n)}L_{K(n+1)}S^0)$. Then w_n is a permanent cycle and it represents ω_n.

Theorem 8.1. $\omega_n \in \pi_{-1}(L_{K(n)}L_{K(n+1)}S^0)$ *is nontrivial.*

Proof. In [Torii 2003] we constructed a map

$$\theta' : H^*_c(G_n;\, \boldsymbol{F}[w^{\pm 1}]) \longrightarrow H^*_c(G_{n+1};\, \boldsymbol{F}((u_n))[u^{\pm 1}]).$$

Then there exists a commutative diagram

$$\begin{array}{ccc} H_c^*(G_n; E_n^*) & \xrightarrow{\theta} & H_c^*(G_{n+1}; \mathbb{A}^*) \\ \pi\downarrow & & \downarrow\pi \\ H_c^*(G_n; \boldsymbol{F}[w^{\pm 1}]) & \xrightarrow{\theta'} & H_c^*(G_{n+1}; \boldsymbol{F}((u_n))[u^{\pm 1}]), \end{array}$$

where the vertical arrows π are canonical reduction maps. In [Torii 2005] we calculated the image of $\theta' : H_c^1(G_n; \boldsymbol{F}[w^{\pm 1}]) \to H_c^1(G_{n+1}; \boldsymbol{F}((u_n))[u^{\pm 1}])$, and we showed that $\theta'(\pi(z_n))$ is nontrivial. This implies that $\theta(z_n) \in H_c^1(G_{n+1}; \mathbb{A}^0)$ is nontrivial. Since $\theta(z_n)$ is a permanent cycle and lies in the 1-line of the spectral sequence, it represents a nontrivial element in $\pi_{-1}(L_{K(n)} L_{K(n+1)} S^0)$. □

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TAKESHI TORII
DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA 700-8530
JAPAN
torii@math.okayama-u.ac.jp

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