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We show that the inclusion map of the generalized Thompson groups $F(n_i)$ is exponentially distorted in the Thompson–Stein groups $F(n_1, \ldots, n_k)$ for $k > 1$. One consequence is that F is exponentially distorted in $F(n_1, \ldots, n_k)$ for $k > 1$ whenever $n_i = 2^m$ for some m (whenever no i, m exist such that $n_i = 2^m$, there is no obviously "natural" inclusion map of F into $F(n_1, \ldots, n_k)$). This is the first known example in which the natural embedding of one of the Thompson-type groups into another is not quasi-isometric.

1. Introduction

In this paper, we use some of the motivating ideas behind the proofs of the metric properties developed in [Wladis 2009] to show that the inclusion map of the generalized Thompson groups $F(n_i)$ into $F(n_1, \ldots, n_k)$ is exponentially distorted for $k > 1$. A quasi-isometric embedding of a subgroup into a larger group induces a metric on the subgroup that is equivalent to subgroup metric. In contrast, when an embedding is not quasi-isometric, the subgroup distortion measures the extent to which this metric is distorted by the embedding map (for formal definitions, see Section 4).

We give here the first known example of the natural embedding of one Thompson-type group being distorted inside another. Burillo, Cleary and Stein [Burillo et al. 2001] showed that $F(n)$ is quasi-isometrically embedded into $F(m)$ for all $n, m \in \mathbb{N} - \{1\}$, and along with Taback, that F is quasi-isometrically embedded in Thompson's group T [Burillo et al. 2009]. Different methods have been used to show that $F^n \times \mathbb{Z}^m$ is quasi-isometrically embedded in F for all $m, n \in \mathbb{N}$ [Burillo 1999; Cleary and Taback 2003; Guba and Sapir 1999; Guba and Sapir 1997]. Since the development of the main theorem of this paper, Burillo and Cleary [2010] have

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used similar methods as those described here to prove that the canonical embeddings of Thompson's groups F and V are also distorted in the higher dimensional Thompson's group nV.

Robert Thompson introduced the three groups named after him in the early 1960s (see [McKenzie and Thompson 1973]). Denoted by $F \subset T \subset V$, they have provided many interesting group-theoretic counterexamples: T and V were the first known infinite, simple, finitely presented groups, and F was the first known example of a torsion-free infinite-dimensional FP_∞ group. For more information see [Cannon et al. 1996].

The groups $F(n_1, \dots, n_k)$, generalizing F, were first explored in depth by Melanie Stein [1992]. Related explorations of general classes in this family of groups, each of which can be considered to be a generalization of the Thompson groups, include [Higman 1974; Brown and Geoghegan 1984; Brown 1987; Brin and Guzmán 1998; Brin and Squier 2001; Bieri and Strebel 1985].

Definition 1.1. The *Thompson–Stein group* $F(n_1, \dots, n_k)$, where $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \{2, 3, 4, \dots\}$ are pairwise relatively prime, is the group of piecewise linear orientation-preserving homeomorphisms of the closed unit interval with finitely many breakpoints in $\mathbb{Z}[\frac{1}{n_1 \cdots n_k}]$ and slopes in the group $\langle n_1, n_2, \dots, n_k \rangle$ in each linear piece. We abbreviate $F(2)$ by F.

Stein [1992] explored the homological and simplicity properties of $F(n_1, \dots, n_k)$ and showed that they are of type FP_∞ and finitely presented, and gave a technique for computing infinite and finite presentations. In [Wladis 2009], using Stein's presentations, we developed the theory of tree-pair diagram representation for elements of $F(n_1, \dots, n_k)$, gave a unique normal form, and calculated sharp upper and lower bounds on the metric in terms of the number of leaves in the minimal tree-pair diagram representative. The proofs in this paper use the normal form results and some of the same motivating ideas behind the metric approximations used in our 2009 paper.

The results of this article hold for all groups of the form $F(n_1, \dots, n_k)$ that satisfy the condition $n_1 - 1 \,|\, n_j - 1$ for all $j \in \{1, \dots, k\}$; throughout this paper, when we refer to the group $F(n_1, \dots, n_k)$, this divisibility criterion will be implied. Groups not satisfying this criterion will have a significantly different group presentation, and therefore require alternate normal form and metric techniques than those presented here or in [Wladis 2009]. Much of the introductory material in this paper is summarized from that paper, where more detail can be found.

2. Representing elements using tree-pair diagrams

The proofs in this paper depend heavily on the representation of elements of $F(m)$ and $F(n_1, \dots, n_k)$ by tree-pair diagrams; see [Wladis 2007; 2009] for more details.

Definition 2.1. An *n-ary caret*, or caret of type n, is a graph which has $n+1$ vertices joined by n edges: one vertex has degree n (the parent) and the rest have degree 1 (the children).

An *$(n_1, \dots, n_k)$-ary tree* is a graph formed by joining a finite number of carets by identifying the child vertex of one caret with the parent vertex of another so that every caret in the tree has a type in $\{n_1, \dots, n_k\}$. An *$(n_1, \dots, n_k)$-ary tree-pair diagram* is an ordered pair of $(n_1, \dots, n_k)$-ary trees with the same number of leaves.

If a vertex in a tree has degree 1, it is referred to as a *leaf*.

An $(n_1, \dots, n_k)$-ary tree represents a subdivision of [0, 1] using the following recursive process, which assigns a subinterval of [0, 1] to each leaf in the tree: the root vertex represents the interval [0, 1]; for a given n-ary caret in the tree with parent vertex representing $[a, b]$, the n child vertices represent the subintervals $\left[a, a+\frac{1}{n}\right], \left[a+\frac{1}{n}, a+\frac{2}{n}\right], \dots, \left[b-\frac{1}{n}, b\right]$ respectively.

Every element of $F(n_1, \dots, n_k)$ can be represented by an $(n_1, \dots, n_k)$-ary tree-pair diagram and vice versa. We number the leaves in a tree beginning with zero, in increasing order from left to right; a leaf's placement in this order is determined by the relative position of the subinterval within [0, 1] which it represents. Once the leaves of each tree in a tree-pair diagram are numbered, then the element of $F(n_1, \dots, n_k)$ which it represents is the map which takes the subinterval of [0, 1] represented by the ith leaf in the domain tree to the subinterval of [0, 1] represented by the ith leaf in the range tree. Because every element of $F(n_1, \dots, n_k)$ is a piecewise linear map with fixed endpoints, it can be determined solely by the ordered subintervals in the domain and range. For example, the element given in Figure 1 is just the map $\left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\} \to \left\{\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right]\right\}$.

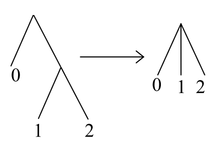


Figure 1. An example element of $F(2, 3)$.

Equivalence and minimality of tree-pair diagrams. We will analyze properties of $F(m)$ and $F(n_1, \dots, n_k)$ by identifying each group element with an equivalence class of tree-pair diagrams, so we must have criteria for equivalence. And because our metric is based on using a minimal tree-pair diagram representative for an element, we also give minimality criteria.

Definition 2.2. Two trees are *equivalent* if they represent the same subdivision of the unit interval; two tree-pair diagrams are *equivalent* if they represent the same element of $F(n_1, \dots, n_k)$.

An *exposed caret pair* in a tree-pair diagram is a pair of carets of the same type, one in each tree, such that all the child vertices of each caret are leaves, and both sets of leaves have identical leaf index numbers. Exposed caret pairs can be canceled in a tree-pair diagram to produce an equivalent tree-pair diagram with fewer leaves. Analogously, we can add a pair of identical carets to a tree-pair diagram to the leaves with the same index number in each tree and obtain an equivalent tree-pair diagram.

Definition 2.3. An $(n_1, \dots, n_k)$-ary tree-pair diagram is *minimal* if it has the smallest number of leaves of any tree-pair diagram in the equivalence class representing a given element of $F(n_1, \dots, n_k)$. In $F(m)$, a tree-pair diagram is minimal if and only if it contains no exposed caret pairs.

Definition 2.4. For any given $j \in \{1, \dots, k\}$, the n_j*-valence* of a leaf $l \in T$ is the number of n_j-ary carets which have an edge on the path from the root vertex to l; it is denoted by $v_{n_j}(l)$. If we refer to just the valence of l, or $\boldsymbol{v}(l)$, this refers to the vector $\langle v_{n_1}(l), \dots, v_{n_k}(l)\rangle$.

Theorem 2.5 [Wladis 2009]. *The $(n_1, \dots, n_k)$-ary trees T and S are equivalent if and only if $L(T) = L(S)$ and $\boldsymbol{v}(l_i) = \boldsymbol{v}(k_i)$ for all leaves $l_i \in T$, $k_i \in S$.*

Tree-pair diagram composition. To find ba for $b, a \in F(n_1, \dots, n_k)$, $b = (T_-, T_+)$ and $a = (S_-, S_+)$, we need to make S_+ equivalent to T_-. This is accomplished by adding carets to T_- and S_+ (and by extension to the leaves with the same index numbers in T_+ and S_- respectively) until the valence of all leaves of both T_- and S_+ are the same. If we let T_-^*, T_+^*, $S_-^* S_+^*$ denote T_-, T_+, S_-, S_+, respectively, after this addition of carets; then the (possibly nonminimal) product is (S_-^*, T_+^*) (see Figure 2). The process of tree-pair diagram composition always terminates; see [Wladis 2009].

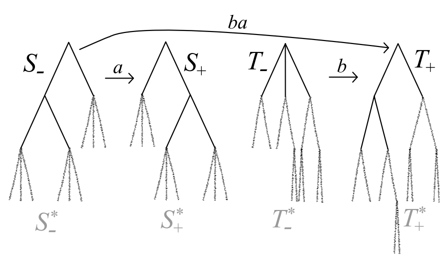


Figure 2. Composition of two elements of $F(2, 3)$. Solid lines indicate the carets present in the original elements a and b, and dotted lines indicate carets that must be added during composition. The tree-pair diagram representative of ba is the pair which contains the domain tree of a and the range tree of b, with both hatched and solid line carets included.

3. The metric in $F(n)$ and $F(n_1, \dots, n_k)$

Standard presentations. Stein [1992] gave a method for finding the finite presentations for the groups $F(n_1, \dots, n_k)$; in [Wladis 2009] we computed the exact finite presentations explicitly. For the sake of simplicity, we give the presentation for $F(2, 3)$ only here. For presentations for $F(n_1, \dots, n_k)$ more generally, see [Wladis 2009].

Theorem 3.1 [Stein 1992; Wladis 2009]. *Thompson's group $F(2, 3)$ admits the infinite presentation with generators $x_0, y_0, z_0, x_1, y_1, z_1, \dots$ and relators*

$$\gamma_j x_i = x_i \gamma_{j+1} \quad \text{and} \quad \gamma_j z_i = z_i \gamma_{j+2} \quad \text{when } i < j \text{ for } \gamma = x, y, z;$$
$$y_{i+1} z_i = y_i x_{i+1} x_i \quad \text{and} \quad x_i z_{i+1} z_i = z_i x_{i+2} x_{i+1} x_i \quad \text{for all } i.$$

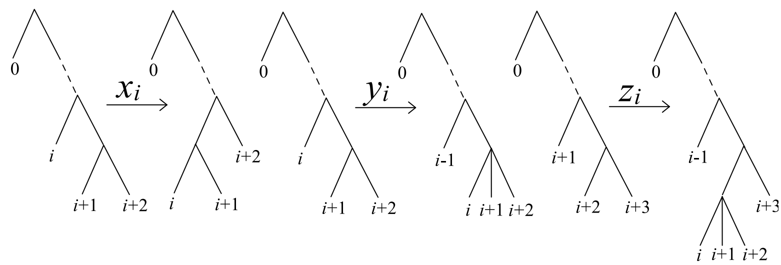


Figure 3. Infinite generators for $F(2, 3)$.

Theorem 3.2 [Stein 1992; Wladis 2009]. *$F(2, 3)$ admits the finite presentation with generators $\{x_0, x_1, y_0, y_1\}$ and relators*

$$x_2 x_0 = x_0 x_3, \quad y_2 x_0 = x_0 y_3, \quad x_1 z_0 = z_0 x_3, \quad y_1 z_0 = z_0 y_3,$$
$$x_3 x_1 = x_1 x_4, \quad y_3 x_1 = x_1 y_4, \quad x_2 z_1 = z_1 x_4, \quad y_2 z_1 = z_1 y_4,$$
$$x_0 z_1 z_0 = z_0 x_2 x_1 x_0, \qquad x_1 z_2 z_1 = z_1 x_3 x_2 x_1,$$

where

$$x_3 = x_1^{-1} x_2 x_1, \quad y_3 = x_1^{-1} y_2 x_1, \quad z_0 = y_1^{-1} y_0 x_1 x_0,$$
$$x_4 = x_2^{-1} x_3 x_2, \quad y_4 = x_2^{-1} y_3 x_2, \quad z_1 = y_2^{-1} y_1 x_2 x_1, \quad z_2 = y_3^{-1} y_2 x_3 x_2.$$

The standard presentations for F (see [Brown 1987]) are:

Infinite: $\{x_0, x_1, x_2, \cdots \mid x_j x_i = x_i x_{j+1} \text{ for } i < j\}$

Finite: $\{x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2]\}$

The metric. It is well known that the metric in F and $F(n)$ is quasi-isometric to the number of carets (or equivalently to the number of leaves) in the minimal tree-pair diagram representative of a given group element. However, this does not hold for $F(n_1, \dots, n_k)$ when $k > 1$; it is this fact which will be exploited to show that F is distorted in $F(n_1, \dots, n_k)$.

Notation 3.3. The notation $|x|_{F(n)}$ and $|x|_{F(n_1,\dots,n_k)}$ will be used to represent the length of the element x in $F(n)$ and $F(n_1, \dots, n_k)$ respectively, with respect to the standard finite generating set.

Notation 3.4. The notation $L(T)$, $L(T_-, T_+)$, and $L(x)$ denotes the number of leaves in the tree T, in either tree of the tree-pair diagram (T_-, T_+), and in either tree of the minimal tree-pair diagram for x respectively.

We note that both trees in a tree-pair diagram have the same number of leaves.

Theorem 3.5 [Fordham and Cleary 2009; Burillo et al. 2001]. *For $x \in F(n)$, $|x|_{F(n)}$ is quasi-isometric to $L(x)$* (see Definition 4.1 for formal definition).

Theorem 3.6 [Wladis 2009]. *There exist fixed $B, C \in \mathbb{N}$ such that*

$$\log_B L(x) \leq |w|_{F(n_1,\dots,n_k)} \leq CL(w) \quad \textit{for all } x \in F(n_1, \dots, n_k).$$

These bounds are sharp.

Normal form. A unique normal form exists for $F(n_1, \dots, n_k)$ with respect to the standard infinite presentations. This normal form essentially provides an algorithm for converting a tree-pair diagram into an algebraic expression in the normal form and vice versa. For the main proofs of this paper, we will introduce several elements for which we will give both an algebraic expression in the normal form and a tree-pair diagram representative. To understand the proofs that follow, one need only consider the tree-pair diagrams, and one need not see explicitly how the algebraic expression comes from the tree-pair diagram representative, so for the sake of space and simplicity of presentation, we have omitted a full explanation of how to write out the normal form for a given element in $F(n_1, \dots, n_k)$; however, full details on this algorithm can be found in [Wladis 2009].

4. Quasi-isometry and subgroup distortion

A quasi-isometrically embedded subgroup has a metric that is equivalent to the induced metric within the larger group. In contrast, an embedding which is not quasi-isometric can be said to be distorted, and the type of this distortion measures the extent to which the metric is distorted by the embedding map.

Definition 4.1. The groups X and Y are *quasi-isometric* if there exist fixed $c_1, c_2 > 0$ and an embedding $f : X \to Y$ such that

$$\frac{1}{c_1}|x|_X - c_2 \leq |f(x)|_Y \leq c_1|x|_X + c_2,$$

where $|x|_X$ and $|x|_Y$ are the lengths of $x \in X$ and $x \in Y$ respectively, with respect to a fixed finite generating set. When $X \subset Y$, the embedding f will be assumed to be the inclusion map, so we often omit explicit mention of the embedding itself.

Let $x \in X \subset Y$. The *distortion function* is defined by

$$D(r) = \frac{1}{r} \max \left\{ |x|_X, |x|_Y \;\middle|\; |x|_Y < r \right\}.$$

For finitely generated groups, the distortion function is bounded if and only if the inclusion map of X into Y is a quasi-isometric embedding. When $D(r)$ is a function that grows without bound as $r \to \infty$, then we say that X is distorted in Y; the function type of $D(r)$ determines the type of the distortion (i.e. we say that a subgroup with exponential $D(r)$ is exponentially distorted). We use the notation $\sim$ to denote quasi-isometry. The property of quasi-isometry is transitive: whenever $X \sim Y$ and $Y \sim Z$, $X \sim Z$.

***F* is exponentially distorted in $F(n_1, \dots, n_k)$.** We begin by proving that the inclusion map of $F(n_i)$ is exponentially distorted in $F(n_1, \dots, n_k)$ whenever there exists $j \in \{1, \dots, k\}$ such that $n_i - 1 \mid n_j - 1$ by constructing a distorted element in $F(n_i)$ explicitly. In the next section, we generalize this result to all $i \in \{1, \dots, l\}$.

Definition 4.2. We say that a tree is *balanced* if $\boldsymbol{v}(l_i) = \boldsymbol{v}(l_j)$ for all leaves $l_i, l_j \in T$.

Theorem 4.3. *$F(n_i)$ is exponentially distorted in $F(n_1, \dots, n_k)$ for $k > 1$ whenever there exists n_j such that $j \in \{1, \dots, k\}$, $i \neq j$, and $n_i - 1 \mid n_j - 1$.*

Proof. For the sake of readability, we will restrict all the explicit details of this proof to the canonical embedding of F into $F(2, 3)$ since this is the simplest case. However, this proof holds for all $F(n_i)$ that meet the stated conditions of the theorem; at key points in this proof, we will indicate what adjustments need to be made to generalize the results to the general case.

We will show that $w = y_0^{-n} x_0 y_0^n$ is such that $|w|_F \geq \frac{1}{A} 3^n$ for some $A \in \mathbb{N}$ by showing that $L(w) \geq \frac{1}{A} 3^n$. We consider the product of the representative tree-pair diagrams given in Figure 4.

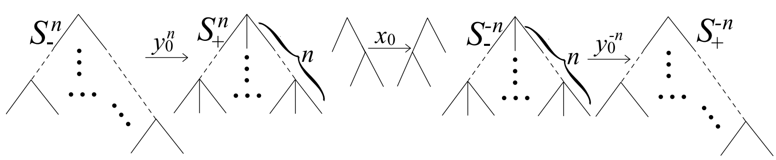


Figure 4. The product $w = y_0^{-n} x_0 y_0^n$.

In order to perform this composition, a binary caret must be added to every leaf in S_-^n and S_+^n, to produce $(S_-^n)^1$ and $(S_+^n)^1$ respectively. Then a second binary caret must be added to the leaves with index numbers $3^n, \dots, 2 \cdot 3^n - 1$ in both $(S_-^n)^1$ and $(S_+^n)^1$ to produce $(S_-^n)^2$ and $(S_+^n)^2$ respectively. Then a balanced n-level ternary tree (identical to S_+^n) must be added to each leaf of T_- and T_+. And finally, a binary caret must be added to each leaf in S_-^{-n} and S_+^{-n} to produce $(S_-^{-n})^1$ and $(S_+^{-n})^1$ respectively, and then another binary caret must be added to the leaves with index numbers $0, \ldots, 3^n - 1$ in $(S_-^{-n})^1$ and $(S_+^{-n})^1$ to produce $(S_-^{-n})^2$ and $(S_+^{-n})^2$ respectively. It is clear then that $\big((S_-^n)^2, (S_+^{-n})^2\big)$ is a tree-pair diagram for w whose number of leaves is $2 \cdot 3^n - 1$. However, $\big((S_-^n)^2, (S_+^{-n})^2\big)$ may not be minimal. In fact, there exist exposed caret pairs in $\big((S_-^n)^2, (S_+^{-n})^2\big)$, but not enough to significantly reduce the number of leaves in the tree-pair diagram; to see this, we list the leftmost leaf index number of every exposed caret in $\big((S_-^n)^2, (S_+^{-n})^2\big)$:

$(S_-^n)^2 : 0, 2, 4, \ldots, 3^n - 3,$ (even)

$\mathbf{3^n}, \mathbf{3^n{+}2}, \mathbf{3^n{+}4}, \ldots, \mathbf{2 \cdot 3^n{-}1}, 2 \cdot 3^n + 1, 2 \cdot 3^n + 3, \ldots, 3 \cdot 3^n - 2$ (odd)

$(S_+^n)^2 : 0, 2, 4, \ldots, 3^n - 3, \mathbf{3^n{-}1}, \mathbf{3^n{+}1}, \mathbf{3^n{+}3}, \ldots, \mathbf{2 \cdot 3^n{-}2},$ (even)

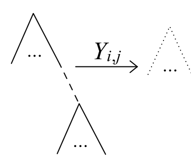
$2 \cdot 3^n + 1, 2 \cdot 3^n + 3, 2 \cdot 3^n + 5, \ldots, 3 \cdot 3^n - 2$ (odd)

It is clear that all exposed carets with leftmost leaf number in bold cannot cancel, because these leaves in the domain tree have odd index numbers and these leaves in the range tree have even index numbers. So

$$L(w) \geq (2 \cdot 3^n - 2) - (3^n - 1) = 3^n + 1,$$

and because the metric in F is quasi-isometric to the number of leaves in the minimal tree-pair diagram representative of an element, there exists $A \in \mathbb{N}$ such that $|w|_F \geq \frac{1}{A} 3^n$. However, clearly $|w|_{F(2,3)} \leq 2n + 1$.

To generalize this proof for $F(n_i)$ in $F(n_1, \ldots, n_k)$, we begin by defining the element $Y_{i,j}$ as the element with tree-pair diagram of the form given on the right. (In the case $i = 1$, we simply have $Y_{i,j} = (y_j)_0$.)



We define Z_i as the element with the tree-pair diagram given in Figure 5. We consider the product

$$w_{i,j,n} = Y_{i,j}^{-n} Z_i Y_{i,j}^n$$

given in that figure the same way that we considered $y_0^n x_0 y_0^{-n}$ for F in $F(2, 3)$ in Figure 4. After adding all carets to each tree-pair diagram in Figure 5, as necessary in order for composition to take place, the resulting diagram $\big((S_-^n)^2, (S_+^{-n})^2\big)$ for $w_{i,j,n}$ will have exposed carets whose leftmost leaf index numbers are as follows,

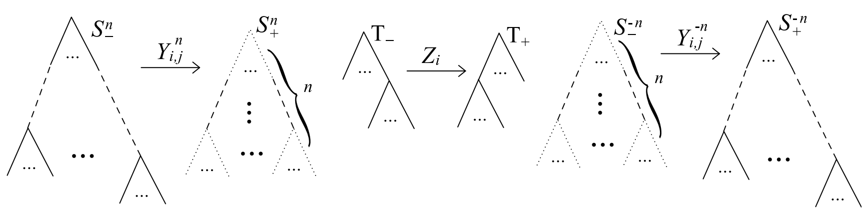


Figure 5. The product $Y_{i,j}^{-n} Z_i Y_{i,j}^n$. Solid carets are n_i-ary and dotted carets are n_j-ary (also in inset above).

where $c = \lfloor n_j^n / n_i \rfloor$ and $*$ denotes "not divisible by n_i":

$$
\begin{array}{lll}
(S_-^n)^2: & 0,\, n_i,\, 2n_i,\, 3n_i,\, \dots,\, (c-1)n_i, & \text{(divisible by } n_i) \\
& \boldsymbol{(n_i-1)n_j^n},\, \boldsymbol{(n_i-1)n_j^n+n_i},\, \boldsymbol{(n_i-1)n_j^n+2n_i},\, \dots, & (*) \\
& \boldsymbol{(2n_i-1)n_j^n-(c+2)n_i},\, 2(n_i-1)n_j^n-(c+1)n_i,\, \dots,\, (2n_i-1)n_j^n-n_i & (*) \\
(S_+^n)^2: & 0,\, n_i,\, 2n_i,\, 3n_i,\, \dots,\, (c-1)n_i,\, \boldsymbol{cn_i},\, \dots,\, \boldsymbol{(n_j^n-1)n_i}, & \text{(divisible by } n_i) \\
& 2(n_i-1)n_j^n-(c+1)n_i,\, 2(n_i-1)n_j^n-cn_i,\, \dots,\, (2n_i-1)n_j^n-n_i & (*)
\end{array}
$$

Because n_i and n_j are relatively prime, the carets with leftmost leaf numbers in bold will not cancel. Thus

$$
\begin{aligned}
L(w_{i,j,n}) &\geq [(2n_i-1)n_j^n-(c+2)n_i]-[cn_i] \\
&= (2n_i-1)n_j^n-(2c-2)n_i \\
&> (2n_i-3)n_j^n-2n_i \\
&> n_j^n-2n_i,
\end{aligned}
$$

the last inequality being a consequence of $cn_i < n_j^n$ and $n_i \geq 2$. However, if we let $A = |Y_{i,j}|_{F(n_1,\dots,n_k)}$ and $B = |Z_i|_{F(n_1,\dots,n_k)}$, we have

$$
\begin{aligned}
|w_{i,j,n}|_{F(n_1,\dots,n_k)} &\leq |Y_{i,j}^{-n}|_{F(n_1,\dots,n_k)} + |Z_i|_{F(n_1,\dots,n_k)} + |Y_{i,j}^{n}|_{F(n_1,\dots,n_k)} \\
&\leq An + B + An = 2An + B. \qquad \square
\end{aligned}
$$

$F(n_i)$ is exponentially distorted in $F(n_1,\dots,n_k)$. We now extend the results of the last two pages to all n_i such that $i \in \{1,\dots,k\}$. We will again do this by explicitly constructing a product in $F(n_1,\dots,n_k)$ that produces an element in $F(n_i)$ so that the number of leaves in the product is logarithmic with respect to the number of factors in $F(n_1,\dots,n_k)$. Without the added condition that $n_i-1 \mid n_j-1$ for some $j \in \{1,\dots,k\}$, this product will have to be more complex than the one constructed in the last section; however, the underlying structure will be similar. We begin by defining elements of $F(n_1,\dots,n_k)$ which will occur in our product. As in the previous section, for the sake of clarity we give our detailed proof for the embedding of $F(3)$ into $F(2,3)$, including notes indicating how this can be generalized for any $F(n_i)$ into $F(n_1,\dots,n_k)$ that meet the conditions of Theorem 4.5.

Notation 4.4. For a fixed $i \in \{1,\dots,k\}$ we define A_i, Z_i, λ_i as follows:

$i=2$	arbitrary i
$A_2 = x_0 y_0^{-1}$ (see Figure 9)	A_i has the form seen in Figure 7, left
$Z_2 = y_1 z_1 y_3^{-1} y_1^{-1}$ (see Figure 9)	Z_i has the form seen in Figure 7, middle
$\lambda_2 = x_0 y_1^{-1}$ (see Figure 6)	λ_i has the form seen in Figure 7, right

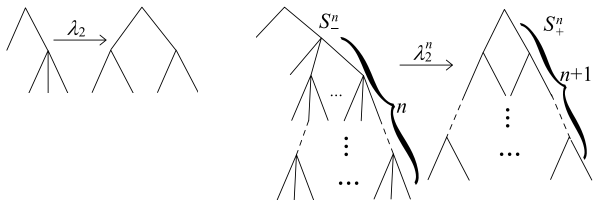


Figure 6. The elements λ_2 and λ_2^n in $F(2,3)$. Level i from the top in S_-^n has 2^{i-2} ternary carets.

For readability, the theorem and proof that follow are restricted to the case $F(2,3)$, which is illustrated in Figure 6. However, the proof can be extended to all cases by using the generalized elements given in Figure 7. Particular examples of more complicated λ_i can be seen in Figure 8.

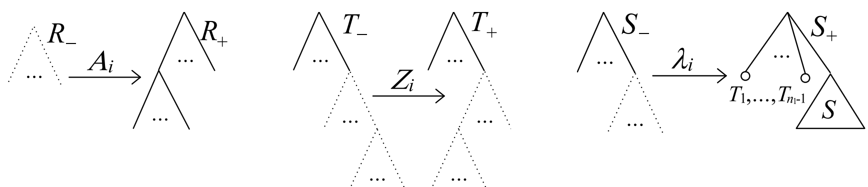


Figure 7. The elements A_i, Z_i, and λ_i in $F(n_1,\ldots,n_k)$. Solid carets are n_1-ary and dotted carets are n_i-ary. On the right, S is a balanced n_1-ary tree where $L(S) \le n_i$, while $T_1,\ldots,T_{n_1-1}$ are (possibly empty) n_1-ary subtrees of $D(S)$ levels or less, chosen as needed in order to make $L(S_-) = L(S_+)$. For simplicity, we fill in the subtrees $T_1,\ldots,T_{n_1-1}$ from left to right, but this is not strictly necessary. For specific examples, see Figure 8.

Theorem 4.5. *The canonical embedding of $F(n_i)$ is exponentially distorted in $F(n_1,\ldots,n_k)$ for all $i \in \{1,\ldots,k\}$.*

Proof. We will establish this by showing that the product $W_{2,n} = (\lambda_2^n A_2)^{-1} z_2 (\lambda_2^n A_2)$ is an element of $F(3)$, and that it has a minimal tree-pair diagram representative whose number of leaves is of the order B^n for some fixed $B > 1$. All of the following steps generalize in a straightforward way to show the same result for $F(n_i)$ in $F(n_1,\ldots,n_k)$ by simply replacing all the elements A_2, λ_2, Z_2 with their general formulations.

It is clear that $|W_{2,n}|_{F(2,3)} < 4n+8$ while $|W_{2,n}|_{F(3)} \sim L(W_{2,n})$. Straightforward computation of the product $W_{2,n}$, illustrated in Figure 9, shows that we must do the following:

(i) Add n levels of binary carets to each leaf in the trees T_- and T_+ of Z_2.

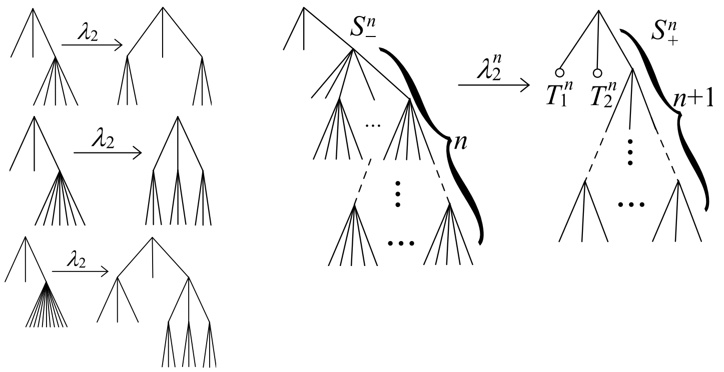


Figure 8. More complex examples of $\lambda_i \in F(n_1, \ldots, n_k)$. Left column: The element λ_2 in $F(3, 5)$, $F(3, 7)$ and $F(3, 11)$. Right: the element λ_2^n in $F(3, 5)$; level i from the top in S_-^n has 3^{i-2} quinary (5-ary) carets, and T_1^n, T_2^n are ternary subtrees.

(ii) Add a ternary caret to the 2^n rightmost leaves of S_+^n and S_-^{-n} (and by extension to the 2^n rightmost leaves of S_-^n and S_+^{-n}), and then add a ternary caret to the rightmost 2^n leaves of these added ternary carets in S_+^n (and S_-^n respectively) and to the leftmost 2^n leaves of these added ternary carets in S_-^{-n} (and S_+^{-n} respectively).

We can then see that the (not necessarily minimal) tree-pair diagram of the resulting product $\lambda_2^{-n} Z_2 \lambda_2^n$ has $3 \cdot 2^{n+1}$ leaves, and the only nonternary carets in each tree are the root carets. Conjugating this product by A_2 then produces a tree-pair diagram for $W_{2,n}$ with $(3 \cdot 2^{n+1} + 1)$ leaves consisting entirely of ternary carets (so clearly $W_{2,n} \in F(3)$).

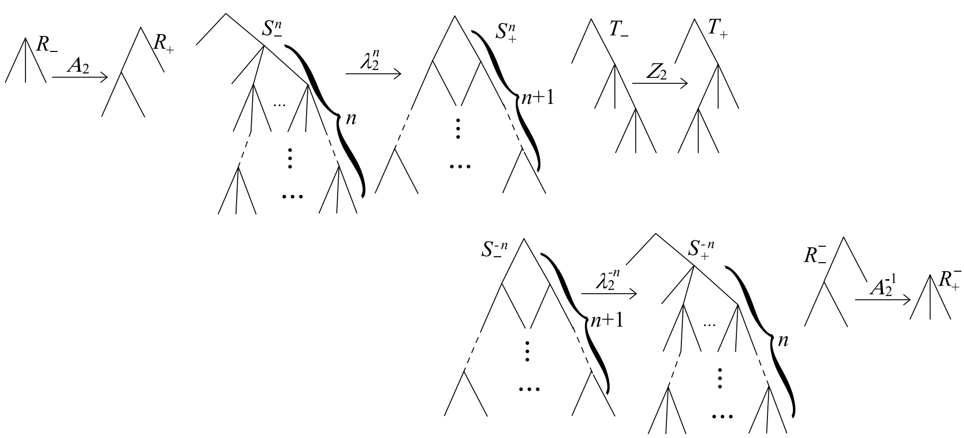


Figure 9. The product $(\lambda_2^n A_2)^{-1} Z_2 (\lambda_2^n A_2)$.

Now we need only show that a significant number of these leaves will not cancel. Using a similar argument to that in the proof of Theorem 4.3 where we tracked the leaf numbers and their divisors, it is easy to show that less than 2^{n+1} leaves will cancel, so we can conclude that $L(W_{2,n}) \geq 2^{n+1}$. □

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