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Following the definition of parabolic rational functions and in view of the behavior of transcendental meromorphic functions, we give the definition of parabolic transcendental meromorphic functions. We discuss their dynamical behavior and prove the existence of conformal measures and invariant measures over their Julia sets, thus extending Denker and Urbański's work on parabolic rational functions. However, our method for proving the existence of the conformal measures differs in that we use the Perron–Frobenius–Ruelle operator.

1. Introduction and notations

Let $f(z)$ be a meromorphic function that is transcendental or rational with degree at least two. Let $f^n(z)$ be the n-th iterate of $f(z)$, let $\mathcal{F}(f)$ be the Fatou set of $f(z)$, and let $\hat{\mathcal{J}}(f)=\hat{\mathbb{C}}\setminus\mathcal{F}(f)$, which is the Julia set of $f(z)$. If f is transcendental, then $\infty\in\hat{\mathcal{J}}(f)$, and set $\mathcal{J}(f)=\hat{\mathcal{J}}(f)\setminus\{\infty\}$ and $\mathcal{J}_\infty(f)=\bigcup_{n=0}^{\infty} f^{-n}(\infty)$. If $\mathcal{J}_\infty(f)$ contains at least three points, then $\mathcal{J}(f)=\overline{\mathcal{J}_\infty(f)}$ and so f is analytic on $\mathcal{F}(f)$. $\mathcal{F}(f)$ is open and consists of at most a countable number of components, which are called Fatou components. Since $\mathcal{F}(f)$ is completely invariant, the image of every Fatou component under f is contained in a Fatou component. A Fatou component U is called periodic if $f^m(U)\subset U$ for some $m\geq 1$ and the least such m is called its period; U is preperiodic if $f^m(U)$ is periodic for some $m\geq 1$ but U is not periodic; U is wandering if $f^n(U)\cap f^m(U)=\varnothing$ for $m\neq n$. The periodic Fatou components are classified into five types: attracting domain, parabolic domain, Siegel disk, Herman ring and Baker domain. The Baker domain and wandering domain are possible only for transcendental meromorphic functions.

By $\mathrm{sing}(f^{-1})$ we mean the closure of the set of all finite critical and asymptotic values of $f(z)$ in the complex plane $\mathbb{C}$ and by $\widehat{\mathrm{sing}}(f^{-1})$ the closure of the set of all critical and asymptotic values of $f(z)$ in the extended complex plane $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Hence if $f(z)$ has multiple poles, then ∞ is a critical value of $f(z)$

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and $\infty \in \widehat{\text{sing}}(f^{-1})$. If ∞ is an asymptotic value of $f(z)$, then $\infty \in \widehat{\text{sing}}(f^{-1})$, but in any case, $\infty \notin \text{sing}(f^{-1})$. Then $\infty \notin \widehat{\text{sing}}(f^{-1})$ if and only if $f(z)$ has no multiple poles and no ∞ as an asymptotic value and ∞ is not a limit point of finite singular values of $f(z)$. We denote by $\mathcal{P}(f)$ the postsingular set defined to be the closure in $\hat{\mathbb{C}}$ of

$$\bigcup_{n=0}^{\infty} f^n\Big(\text{sing}(f^{-1}) \setminus \bigcup_{j=0}^{n-1} f^{-j}(\infty)\Big)$$

and set $\hat{\mathcal{P}}(f) = \mathcal{P}(f) \cup \widehat{\text{sing}}(f^{-1})$.

In [Zheng 2008] we proved that for a hyperbolic meromorphic function on the complex plane, the Hausdorff dimension of the radial Julia set $\mathcal{J}_r(f)$ is equal to the Poincaré exponent $s(f)$ of f over $\mathcal{J}(f)$. Actually, the proof showed that

$$\dim_h \mathcal{J}(f) = \dim_H \mathcal{J}_r(f) = s(f),$$

where $\dim_h \mathcal{J}(f)$ is the hyperbolic dimension of $\mathcal{J}(f)$. The first equality above was proved in [Rempe 2009] for the general case. For a hyperbolic meromorphic function on the Riemann sphere, the author proved that

$$\dim_h \mathcal{J}(f) = \dim_H \mathcal{J}(f) = \lambda(f) = s(f),$$

where $\lambda(f)$ is the exponent of conformal measure of f over $\mathcal{J}(f)$, and there exists the invariant Gibbs measure that is equivalent to the $\lambda(f)$-conformal measure which extends the results in [Kotus and Urbański 2002]. Here, we say a probability measure μ over $\mathcal{J}(f)$ is a s-conformal measure for f if $f^{\times}(z)^s$ is the Jacobian of f over $\mathcal{J}(f)$ with respect to μ, that is, for any Borel subset A of $\mathcal{J}(f)$ such that f is injective on A, we have

$$\mu(f(A)) = \int_A f^{\times}(z)^s \mathrm{d}\mu.$$

In this paper, we investigate parabolic meromorphic functions. The papers [Denker and Urbański 1991a; 1991b; Aaronson et al. 1993] are careful investigations of the Hausdorff dimension, conformal measure and invariant measure of parabolic rational functions. The definition of a parabolic rational function is clear: we know that a rational function f with degree at least two is called parabolic if $\hat{\mathcal{J}}(f) \cap \widehat{\text{sing}}(f^{-1}) = \varnothing$ and f has at least one rational indifferent periodic point. However, the transcendental case is more complicated.

Definition 1.1. Let f be a transcendental meromorphic function in $\mathbb{C}$. We say that f is *parabolic on the complex plane* if $\mathcal{P}(f) \cap \mathcal{J}(f)$ is finite and nonempty, each point in $\mathcal{P}(f) \cap \mathcal{J}(f)$ is a rational indifferent periodic point of f, and $\text{sing}(f^{-1})$ is contained in $\mathcal{F}(f)$. We say f is *parabolic on the Riemann sphere* (or *with respect to the spherical metric*) if f is parabolic on the complex plane and $\infty \notin \hat{\mathcal{P}}(f)$.

We denote by $\mathcal{P}(\mathbb{C})$ and $\mathcal{P}(\hat{\mathbb{C}})$ the set of all parabolic transcendental meromorphic functions on $\mathbb{C}$ and $\hat{\mathbb{C}}$, respectively. A rational function has only a finite number of rational indifferent periodic points, while a transcendental meromorphic function may have infinitely many rational indifferent periodic points. Since every rational indifferent periodic point must be in $\mathcal{P}(f)\cap\mathcal{J}(f)$, a parabolic meromorphic function on the complex plane has only finitely many rational indifferent periodic points. In Definition 1.1, we need to stress the condition that $\operatorname{sing}(f^{-1}) \subset \mathcal{F}(f)$: although a rational indifferent periodic point cannot be a critical point, it may be a critical value, and for transcendental case it may be an asymptotic value. This can be explained by considering the functions $z(z-1)^2$ and ze^z. The point 0 is a rational indifferent fixed point of $z(z-1)^2$, which is also a critical value, and of ze^z, which is also an asymptotic value. The functions $z(z-1)^2$ and ze^z satisfy the conditions for parabolicity on the complex plane (Definition 1.1) except for the requirement that $\operatorname{sing}(f^{-1}) \subset \mathcal{F}(f)$. Hence they are not parabolic on the complex plane. If $\infty \notin \hat{\mathcal{P}}(f)$, then f is of bounded type, that is, in class $\mathcal{B}$. Clearly, a parabolic meromorphic function on the Riemann sphere is in class $\mathcal{B}$, that is, $\mathcal{P}(\hat{\mathbb{C}}) \subset \mathcal{B}$.

Let f be a transcendental meromorphic function in class $\mathcal{S}$, so that $\operatorname{sing}(f^{-1})$ is finite. If $\operatorname{sing}(f^{-1}) \subset \mathcal{F}(f)$ (resp. $\widehat{\operatorname{sing}}(f^{-1}) \subset \mathcal{F}(f)$), then f is hyperbolic whenever it has no rational indifferent periodic points; otherwise it is parabolic on the complex plane (resp. on the Riemann sphere). This is because f has only attracting domains and/or parabolic domains. For a general case, see Theorem 3.1 and Theorem 3.2 below. A simple calculation yields that $\tan z$ is in $\mathcal{P}(\hat{\mathbb{C}})$.

In the papers cited above, Denker, Urbański, and Aaronson obtained the existence of a conformal measure and an invariant measure, and showed they are equivalent for parabolic rational functions. Using the results attained in [Zheng 2009] by developing Walters' theory, we extend some of the Denker and Urbański's results to the parabolic transcendental meromorphic function, and establish:

Theorem 1.2. *Let $f(z)$ be a parabolic meromorphic function on the Riemann sphere. Then $f(z)$ has a s-conformal measure μ_s and $P(f,s)=0$.*

Here $P(f,t)$ is the pressure of f at t, whose definition is given in Lemma 3.8. Applying a result from [Martens 1992] we determine conditions about the existence of μ_s-equivalent, f-invariant measure:

Theorem 1.3. *Let $f(z)$ be a parabolic meromorphic function on the Riemann sphere. Assume that s-conformal measure μ_s is atomless. Then $f(z)$ has a μ_s-equivalent, f-invariant measure if for some $a \in \mathcal{J}(f) \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$, where $\Omega = \mathcal{P}(f)\cap\mathcal{J}(f)$, we have*

$$\sum_{n=0}^{\infty} \mathcal{L}_s^n(\mathbb{1})(a) = \infty.$$

Here we say a measure m is f-invariant if $m(f^{-1}(A)) = m(A)$ for any Borel subset A of $\mathcal{J}(f)$. Actually, Ω is the set of all rational indifferent periodic points of $f(z)$ and $\mathcal{L}_s(\mathbb{1}) = \mathcal{L}_{-s\log f^\times}(\mathbb{1})$ is the Perron–Frobenius–Ruelle operator for $-s\log f^\times(z)$ over $\mathcal{J}(f)$ and please see the statements before Lemma 3.8 for its definition.

Question 1.4. *For* $f \in \mathcal{P}(\hat{\mathbb{C}})$, *is* $\dim_H \mathcal{J}(f)$ *always equal to* s?

We conjecture the answer is affirmative.

2. Conformal measures and expansiveness of covering maps

To discuss the existence of conformal measures of parabolic meromorphic functions, we need some results from [Zheng 2009] on the existence of conformal measures for covering maps. Let $(\hat{X}, d)$ be a compact metric space and X be an open and dense subset of $\hat{X}$ and X_0 an open and dense subset of X. For a point $x \in \hat{X}$, $B(x,\delta)$ is the ball centered at x with radius δ. $\mathscr{C}(\Lambda)$ will denote the set of all real-valued continuous functions on $\Lambda = \hat{X}$, X or X_0. Let $T : X_0 \to X$ be continuous and $\varphi \in \mathscr{C}(X_0)$.

Definition 2.1. An ordered pair (T,φ) is called *admissible* if:

(1a) For each $x \in X$, the set $T^{-1}(x)$ is at most countable.

(1b) T has the uniform covering property: there exists a $\delta > 0$ such that for each $x \in X$, $T^{-1}(B_X(x,\delta))$ can be written uniquely as a disjoint union of a finite or countable number of open subsets $A_i(x)$ $(1 \le i \le N \le \infty)$ of X_0 and for each i, T is a homeomorphism of $A_i(x)$ onto $B_X(x,\delta)$, where $B_X(x,\delta) = B(x,\delta)\cap X$. For simplicity, we will call $A_i(x)$ the injective component of T^{-1} over $B_X(x,\delta)$ and δ the injectivity radius.

(1c) The inverse of T is locally uniformly continuous: $\forall \varepsilon > 0$, $\exists \delta_0$ with $0 < \delta_0 < \delta$ such that for each $x \in X$ and each $y \in X_0$ with $T(y) = x$, once $d(x,x') < \delta_0$ for $x' \in X$, we have $d(T_y^{-1}(x), T_y^{-1}(x')) < \varepsilon$, where T_y^{-1} is the branch of the inverse of T which sends x to y. That is to say, every injective component of T^{-1} over $B_X(x,\delta_0)$ has diameter less than ε.

(1d) $\varphi \in \mathscr{C}(X_0)$ is summable on X, that is to say,
$$\sup\left\{\sum_{T(y)=x} \exp\varphi(y) : x \in X\right\} < +\infty.$$

(1e) For all $\varepsilon > 0$, there exists a $0 < \delta_1 < \delta$ such that for any pair $x, x' \in X$, once $d(x,x') < \delta_1$, we have
$$\sum_{T(y)=x} \left|\exp\varphi(T_y^{-1}(x)) - \exp\varphi(T_y^{-1}(x'))\right| < \varepsilon,$$

that is, $\sum_{T(y)=x} \left|\exp\varphi(T_y^{-1}(x)) - \exp\varphi(T_y^{-1}(x'))\right| \to 0$ uniformly as $d(x, x')$ goes to 0.

We now give a condition under which (1b) implies (1c).

Lemma 2.2 [Zheng 2009, Lemma 2.1 and following remark]. *Let T satisfy* (1b) *with $X = \hat{X}$. The inverse of T is locally uniformly continuous*, *that is*, *T satisfies* (1c), *if one of following statements holds*:

(1) *For arbitrary $\varepsilon > 0$, we have a $0 < \eta \le \varepsilon$ such that for each $x \in X$, $\partial B(x, \eta) \subset X_0$.*

(2) *all limit points of $T^{-1}(x)$ for each $x \in X$ lie in $X \setminus X_0$ and* (1) *holds only for $x \in X \setminus X_0$.*

We can define for a summable function φ on X_0 the Perron–Frobenius–Ruelle operator by setting

$$\mathcal{L}_\varphi(f)(x) := \sum_{T(y)=x} f(y) \exp\varphi(y) \quad \text{for } x \in X.$$

Obviously, $\mathcal{L}_\varphi(f)(x)$ is a bounded real-valued function on X when f is a bounded real-valued function on X_0. Sometimes, we write $\mathcal{L}_{\varphi,T}$ for $\mathcal{L}_\varphi$ to emphasize T. It is obvious that T^n is a continuous mapping of $T^{-n+1}X_0$ to X. Set

$$S_n\varphi(y) = \sum_{i=0}^{n-1} \varphi(T^i(y)) \quad \text{for } y \in T^{-n+1}X_0.$$

Noting that $T^{-n+1}X_0 \subseteq X_0$, we easily deduce that

$$\mathcal{L}^n_{\varphi,T}(f)(x) = \mathcal{L}_{S_n\varphi,T^n}(f)(x) = \sum_{T^n(y)=x} f(y)\exp(S_n\varphi(y)) \quad \text{for } x \in X. \tag{2-1}$$

(Here and throughout the paper we denote by $\mathcal{L}^n_{\varphi,T}$ the n-th iterate of $\mathcal{L}_{\varphi,T}$.) We want to get the desired probability measure on $\hat{X}$ through the dual operator of the $\mathcal{L}_\varphi$ over $\mathcal{M}(\hat{X})$, here $\mathcal{M}(\hat{X})$ denotes the set of all probability measures over $\hat{X}$.

Theorem 2.3. *Let (T, φ) be admissible.*

(1) *For each fixed positive integer N, $(T^N, S_N\varphi)$ is admissible.*

(2) *$\mathcal{L}_\varphi$ can be extended to a linear operator of $\mathcal{C}(\hat{X})$ to itself*, *which is still denoted by $\mathcal{L}_\varphi$.*

(3) *There exists a $\mu \in \mathcal{M}(\hat{X})$ such that* $\mathcal{L}^*_\varphi(\mu) = \lambda\mu$, $\lambda = \mathcal{L}^*_\varphi(\mu)(1) > 0$, *where $\mathcal{L}^*_\varphi$ is the dual operator of $\mathcal{L}_\varphi$, and the following statements hold*:

 (3a) *$\lambda\exp(-\varphi)$ is the Jacobian of T with respect to μ.*

 (3b) *μ is positively nonsingular and nonsingular for T, that is, $\mu \circ T \ll \mu$ and $\mu \circ T^{-1} \ll \mu$.*

With the exception of part (1), this theorem is a modification of the main result from [Walters 1978] (in which the expanding property is stressed; compare [Zheng 2009, Theorem 2.1 and following remark]). Obviously, the λ in Theorem 2.3 satisfies

$$\lambda^n = \mathcal{L}^{*n}_{\varphi}(\mu)(\mathbb{1}) = \mu(\mathcal{L}^n_{\varphi}(\mathbb{1}))$$

and therefore

$$\inf_{x\in X}\{\mathcal{L}^n_{\varphi}(\mathbb{1})(x)\} \le \lambda^n \le \sup_{x\in X}\{\mathcal{L}^n_{\varphi}(\mathbb{1})(x)\}. \tag{2-2}$$

Lemma 2.4. *Let T, φ, λ and μ be as in Theorem 2.3. Assume that there exist a sequence of positive number $\{K_n\}$ such that, for any $x, x' \in X$,*

$$e^{-K_n}\mathcal{L}^n_{\varphi}(\mathbb{1})(x') \le \mathcal{L}^n_{\varphi}(\mathbb{1})(x) \le e^{K_n}\mathcal{L}^n_{\varphi}(\mathbb{1})(x') \tag{2-3}$$

and $K_n/n \to 0$ as $n \to \infty$. Then

$$\log\lambda = \lim_{n\to\infty}\frac{1}{n}\log\mathcal{L}^n_{\varphi}(\mathbb{1})(x)$$

and the limit exists uniformly on X.

Indeed, the inequality (2-3) holds if

$$|S_n\varphi(y) - S_n\varphi(y')| \le K_n,$$

whenever y and y' are in a component of $T^{-n}(B_X(x,\delta))$, for any $x \in X$. This is proved by noting that $\hat{X}$ is compact and a finite number of such disks $B_X(x,\delta)$ cover X.

We have pointed out that the existence of a conformal measure does not require expansiveness; however the existence of an equivalent invariant measure seems to depend on this property, from Walters theory. In the second part of this section, we consider the expansiveness of a continuous map of $\hat{X}$ from X_0 preparing for the discussion of a parabolic meromorphic function on its Julia set. Since a transcendental meromorphic function is not a self-mapping of a compact metric space, this forces us to analyze carefully the definition of expansiveness.

Definition 2.5. A continuous map $T : \hat{X} \to \hat{X}$ of a compact metric space $(\hat{X}, d)$ is called *expansive* if there exists $\delta > 0$ such that we have $x = y$ if $d(T^n(x), T^n(y)) < \delta$ for all $n \ge 0$.

This definition of an expansive self-mapping of a compact metric space is not suitable to the case when T is a continuous map from X_0 into $\hat{X}$, where X_0 is a dense open subset of $\hat{X}$. For example, if there exist a point $w \in \hat{X} \setminus X_0$ and two points $x, y \in X_0$ such that $T^n(x) \to w$ and $T^n(y) \to w$ as $n \to \infty$ and $T^n(x) \ne T^n(y)$, then $d(T^n(x), T^n(y)) \to 0$ as $n \to \infty$. Thus such a continuous map can never satisfy Definition 2.5; in particular, according to this definition, no transcendental meromorphic function is expansive over its Julia set, since its escape set to infinity is nonempty.

Neither is Definition 2.5 suitable to the case when T is an infinite-to-one continuous map from X_0 to $\hat{X}$. Indeed, take a point $a \in \hat{X}$ such that $T^{-1}(a)$ contains a countable sequence $x_n \to x \in \hat{X}$ and then $d(x_n, x_{n+1}) \to 0$.

Let us analyze Definition 2.5 a bit further. Assume T is expansive. Given $x \neq y$, there exist two possibilities: either $T^m(x) = T^m(y)$ and $T^{m-1}(x) \neq T^{m-1}(y)$, for some $m \geq 1$; or $T^n(x) \neq T^n(y)$ for each n. In the first case, we have

$$d(T^{m-1}(x), T^{m-1}(y)) > \delta, \tag{2-4}$$

that is, $y \notin T^{-m+1}(B(T^{m-1}(x), \delta))$ and $x \notin T^{-m+1}(B(T^{m-1}(y), \delta))$ with $T^m(x) = T^m(y)$. In the second case, we have $d(T^{n_k}(x), T^{n_k}(y)) > \delta$ for a increasing sequence of natural numbers $\{n_k\}$ with $n_k \to \infty$, that is, $y \notin T^{-n_k}(B(T^{n_k}(x), \delta))$ and $x \notin T^{-n_k}(B(T^{n_k}(y), \delta))$. If T is not homeomorphism, then $T^{-n_k}(B(T^{n_k}(y), \delta))$ may contain two disjoint components $A^j_{n_k}$ $(j = 1, 2)$ such that T^{n_k} maps $A^j_{n_k}$ onto $B(T^{n_k}(y), \delta)$, while the definition above of expansive maps does not allow x being in any component $A^j_{n_k}$. We note that the crucial point of expansiveness is in the component $A^j_{n_k}(y)$ which contains y and that $T^{n_k} : A^j_{n_k}(y) \to B(T^{n_k}(y), \delta)$ expands the distance. From this point of view, we can extend the above definition of expansive maps to the case when T is a continuous map from X_0 to $\hat{X}$, where X_0 is a dense open subset of $\hat{X}$. Generally, the component of the preimage of a set B by a map T containing y will be denoted by $T_y^{-1}(B)$.

Definition 2.6. A continuous map $T : X_0 \to \hat{X}$ is called *precisely expansive* if there exists $\delta > 0$ such that for $x \neq y$ in $\hat{X}$, one of the following statements holds:

(1) For some $s \geq 0$, at least one of $T^s(x)$ and $T^s(y)$ is in $\hat{X} \setminus X_0$ and $T^s(x) \neq T^s(y)$;

(2) For some $m \geq 1$ with $T^m(x) = T^m(y) \in \hat{X}$ but $T^{m-1}(x) \neq T^{m-1}(y)$, we have $y \notin T_x^{-m}(B(T^m(x), \delta))$ and $x \notin T_y^{-m}(B(T^m(y), \delta))$;

(3) For a sequence of natural numbers $\{n_k\}$ with $n_k < n_{k+1} \to \infty$,

$$y \notin T_x^{-n_k}(B(T^{n_k}(x), \delta)) \quad \text{and} \quad x \notin T_y^{-n_k}(B(T^{n_k}(y), \delta)).$$

We call this δ the expansive constant for T. Note that item (2) in Definition 2.6 implies the uniform covering property (1b) of T with injectivity radius at least $\delta/2$. Generally, we cannot require that $T^{m-1}(x)$ and $T^{m-1}(y)$ have a distance with positive infimum, but if $T^{-1}(a)$ is finite for each $a \in \hat{X}$, such a positive infimum for the distance exists; see (2-4).

Obviously, the property of precise expansiveness implies that two points x and y will coincide if for every n, $y \in T_x^{-n}(B(T^n(x), \delta))$ and $x \in T_y^{-n}(B(T^n(y), \delta))$.

A continuous map $T : \hat{X} \to \hat{X}$ is precisely expansive if it is expansive. (For such an expansive map T, the set $T^{-1}(x)$ is finite for each $x \in \hat{X}$.)

When one considers a homeomorphism $T : \hat{X} \to \hat{X}$, there exists an equivalent definition of expansiveness, namely, the existence of a generator. An open cover α of $\hat{X}$ is called a *one-sided generator* for T if $\bigcap_{n=0}^{\infty} T^{-n}\overline{A}_n$ contains at most one point for any choice of $\{A_n\}$ from α. We set $\overline{\alpha} = \{\overline{A} : A \in \alpha\}$. We will consider a similar result for a precisely expansive map.

If α and β are two sets of subsets of $\hat{X}$, we denote by $\alpha \vee \beta$ the set of all subsets with the form $A \cap B$, for all $A \in \alpha$ and $B \in \beta$. Further, we set

$$\operatorname{diam}\alpha = \sup\{\operatorname{diam} A : A \in \alpha\}.$$

Definition 2.7. A finite cover α of $\hat{X}$ is called a *one-sided generator* for a continuous map $T : X_0 \to \hat{X}$, if each element of $\bigvee_{n=0}^{\infty} T^{-n}\overline{\alpha}$ has at most one point. Equivalently, the cover α is a one-sided generator for T if and only if

$$\operatorname{diam} \bigvee_{j=0}^{n} T^{-j}\overline{\alpha} \to 0 \quad \text{as } n \to \infty.$$

Theorem 2.8. *A continuous map $T : X_0 \to \hat{X}$ is precisely expansive if and only if there exists a one-sided generator for T and T has the uniform covering property* (1b) *with a fixed injectivity radius.*

Proof. Suppose that T is precisely expansive with expansive constant δ. The uniform covering property (1b) of T follows from (2) in Definition 2.6. Therefore, we only need to prove the existence of a one-sided generator.

Take a finite cover α of $\hat{X}$ by open balls with radius $\delta/2$. Let

$$E = \bigcap_{n=0}^{\infty} T_0^{-n}(A_n)$$

be an element of $\bigvee_{n=0}^{\infty} T^{-n}\overline{\alpha}$, where each A_n lies in $\overline{\alpha}$ and $T_0^{-n}(A_n)$ is a component of $T^{-n}(A_n)$. Suppose that $x, y \in E$. Then for every n, we have $x, y \in T_0^{-n}(A_n)$ and $T^n(x), T^n(y) \in A_n = \bar{B}(x_n, \delta/2)$ for a point $x_n \in \hat{X}$. Obviously, $A_n \subset B(T^n(x), \delta)$ and $A_n \subset B(T^n(y), \delta)$. From this it follows that $y \in T_x^{-n}(B(T^n(x), \delta))$ and $x \in T_y^{-n}(B(T^n(y), \delta))$. Then $x = y$, which shows that E contains at most one point. We have proved that α is a one-sided generator.

Now suppose that there exists a one-sided generator α for T and T has the uniform covering property (1b) with injective radius δ. Let η be a positive number less than the Lebesgue number of α and δ. Given two distinct points $x, y \in \hat{X}$, we assume that $\forall\, n$, $T^n(x), T^n(y) \in X_0$. Suppose that (3) in Definition 2.6 does not hold for η and therefore, there exists a $m \geq 1$ such that for all $n \geq m$, we have $y \in T_x^{-n}(B(T^n(x), \eta))$ and $x \in T_y^{-n}(B(T^n(y), \eta))$ and $T^m(y) \in T^m T_x^{-n}(B(T^n(x), \eta))$

and $T^m(x) \in T^m T_y^{-n}(B(T^n(y), \eta))$ and clearly, $T^m(y) \in T_{T^m(x)}^{-n+m}(B(T^n(x), \eta))$ and $T^m(x) \in T_{T^m(y)}^{-n+m}(B(T^n(y), \eta))$. Since η is less than the Lebesgue number of α, $B(T^n(x), \eta) \subset A_{n-m}$ for some $A_{n-m} \in \alpha$ and for $n \geq m$. Thus $T^m(x)$ and $T^m(y)$ are in an element of $\bigvee_{n=0}^{\infty} T^{-n}\alpha$. It follows that $T^m(x) = T^m(y)$.

Now we can assume that $T^m(x) = T^m(y) \in \hat{X}$ but $T^{m-1}(x) \neq T^{m-1}(y)$. It follows from the uniform covering property (1b) of T that

$$T^{m-1}(y) \notin T_{T^{m-1}(x)}^{-1}(B(T^m(x), \delta)) \quad \text{and} \quad T^{m-1}(x) \notin T_{T^{m-1}(y)}^{-1}(B(T^m(y), \delta)).$$

Obviously, $y \notin T_x^{-m}(B(T^m(x), \delta))$ and $x \notin T_y^{-m}(B(T^m(y), \delta))$. Therefore, T is precisely expansive. $\square$

3. Dynamical properties of parabolic meromorphic functions

A meromorphic function is a map from the complex plane $\mathbb{C}$ into the extended complex plane $\hat{\mathbb{C}}$. In this section, we consider two metrics: the euclidean metric d on $\mathbb{C}$ and the spherical metric d_∞ on $\hat{\mathbb{C}}$. The metric space $(\mathbb{C}, d)$ is noncompact, but the metric space $(\hat{\mathbb{C}}, d_\infty)$ is compact. And $(\mathbb{C}, d_\infty)$ is a subspace of $(\hat{\mathbb{C}}, d_\infty)$. We are in $(\mathbb{C}, d_\infty)$ and $(\hat{\mathbb{C}}, d_\infty)$ to consider the situation of conformal measures. Set $B(a, \delta) = \{z : d(z, a) < \delta\}$ for $a \in \mathbb{C}$ and $B_\infty(a, \delta) = \{z : d_\infty(z, a) < \delta\}$ for $a \in \hat{\mathbb{C}}$.

We begin with basic dynamical properties of parabolic meromorphic functions.

Theorem 3.1. *Let f be a parabolic meromorphic function on $\mathbb{C}$ and in Class $\mathcal{B}$. Then it has finitely many and at least one parabolic domain and at most finitely many attracting domains without other types of stable domains and furthermore, $\mathcal{P}(f)$ is bounded.*

Proof. Clearly, $f(z)$ has at least one but only finitely many rational indifferent periodic points, and the number of its parabolic domains is finite and positive. Notice that $f(z)$ is in Class $\mathcal{B}$ and if $f(z)$ has a Baker domain U, then $\{f^n\}$ in U has a finite limit point. By Theorem 2.2 of [Zheng 2003], the limit point is in $\mathcal{P}(f) \cap \mathcal{J}(f)$ and so it is a rational indifferent periodic point. A contradiction is derived as every $f^n(z)$ is analytic at it. This implies that $f(z)$ has no Baker domains at all. By Theorem 2.1 of the same reference, all limit points of $\{f^n\}$ in a wandering domain are in $\mathcal{P}(f) \cap \mathcal{J}(f)$ and if a limit point is finite and not prepoles, then there exist infinitely many limit points. Thus $f(z)$ has no wandering domains. Since the boundaries of Siegel disks and Herman rings are contained in $\mathcal{P}(f) \cap \mathcal{J}(f)$, $f(z)$ therefore has no Siegel disks and Herman rings. Obviously, $f(z)$ may have attracting domains. Suppose that $f(z)$ has infinitely many attracting domains. Since every cycle of attracting domains contains at least a singular value, we take a singular value from every cycle of attracting domains to form a sequence of singular values which has a finite limit point, and clearly the limit point is in $\mathcal{J}(f)$. This implies that $\operatorname{sing}(f^{-1}) \cap \mathcal{J}(f) \neq \varnothing$. A contradiction is derived.

It is obvious that $\mathcal{P}(f)$ is bounded. □

Theorem 3.2. *Let f be a transcendental meromorphic function satisfying the parabolic condition on the complex plane in Definition 1.1, except for* $\operatorname{sing}(f^{-1}) \subset \mathcal{F}(f)$. *If $\mathcal{P}(f)$ is bounded, then it has finitely many and at least one parabolic domain and at most finitely many attracting domains without other types of stable domains.*

Proof. From the proof of Theorem 3.1, it is sufficient to prove that the number of attracting domains is finite. Suppose that $f(z)$ has infinitely many attracting domains. Let $\{a_n\}$ be the sequence of all distinct attracting periodic points of f and let E be the set of all limit points of $\{a_n\}$. It is clear that $E \subset \mathcal{J}(f)$. Since every a_n is in the derived set of $\mathcal{P}(f)$, we have $E \subset \mathcal{P}(f)$, and every point in E is a rational indifferent periodic point of $f(z)$. Hence E is finite and we write $E = \{b_1, b_2, \dots, b_q\}$. Obviously, $f(E) \subseteq E$. We choose a $\delta > 0$ and a $\eta > \delta$ such that $f(B(b_j, \delta)) \subset B(f(b_j), \eta)$ $(j = 1, \dots, q)$ and f is univalent on each $B(b_j, \delta)$ and $\{B(b_j, \eta)\}$ are disjoint. For all $n \geq N$, we have $a_n \in \bigcup_{j=1}^{q} B(b_j, \delta)$. We can take a cycle of attracting periodic points $\{a, f(a), \dots, f^{p-1}(a)\}$ in $\bigcup_{j=1}^{q} B(b_j, \delta)$. Assume that $a \in B(b_1, \delta)$ and $f(a) \in B(f(b_1), \eta)$ so that $f(a) \in B(f(b_1), \delta)$. Thus $\{a, f(a), \dots, f^{p-1}(a)\} \subset \bigcup_{j=0}^{m-1} B(f^j(b_1), \delta)$, where m is the period of b_1, and $p = km$ for a positive integer k. This implies that in $B(b_1, \delta)$, $f^{km}(a) = a$. However, it is impossible for sufficiently small δ in view of the expansiveness in a neighborhood of rational indifferent periodic cycles. □

The following describes equivalently the function in $\mathcal{P}(\hat{\mathbb{C}})$.

Theorem 3.3. *A meromorphic function is parabolic on the Riemann sphere if and only if it has finitely many and at least one parabolic domain and at most finitely many attracting domains without other types of stable domains and* $\widehat{\operatorname{sing}}(f^{-1}) \subset \mathcal{F}(f)$.

Proof. We just need to prove the "only if". That $\widehat{\operatorname{sing}}(f^{-1}) \subset \mathcal{F}(f)$ implies that $\infty \notin \widehat{\operatorname{sing}}(f^{-1})$ and $\operatorname{sing}(f^{-1})$ is bounded. Since $f(z)$ has only finitely many attracting and parabolic domains without other types of stable domains, $\bigcup_{n=0}^{\infty} f^n(\operatorname{sing}(f^{-1})) \subset \mathcal{F}(f)$ and the limit points of $\bigcup_{n=0}^{\infty} f^n(\operatorname{sing}(f^{-1}))$ on $\mathcal{J}(f)$ are rational indifferent periodic points of f. Thus f is parabolic on the Riemann sphere. □

Denker and Urbański [1991a] investigated such properties of parabolic rational functions as the convergent speed of backward orbits of points in a small neighborhood of rational indifferent periodic points and expansive property over the Julia set, which we attempt to extend to transcendental case. The local properties of rational indifferent periodic points, for example, the Fatou's flower theorem, can be directly transferred to transcendental case. For convenience, we collect some of them.

Let $f(z)$ be a parabolic meromorphic function on the Riemann sphere and let Ω be the set of all rational indifferent periodic points of $f(z)$. The following result is basic.

Lemma 3.4. *For every $\theta > 0$ there exists $\delta = \delta(\theta) > 0$ such that for every $a \in \hat{\mathcal{J}}(f) \setminus B(\Omega, \theta)$, we have $B_\infty(a, 2\delta) \cap \mathcal{P}(f) = \varnothing$. In particular, all analytic branches of the inverse of f^n are well defined on $B_\infty(a, 2\delta)$ and $B_\infty(f(a), 2\delta)$ for every $n = 1, 2, \dots$.*

The dynamical behavior in a neighborhood of a rational indifferent periodic point was discussed in [Denker and Urbański 1991a] in view of the Fatou's Flower Theorem. Some of their results are extracted as follows.

Lemma 3.5. *Let ω be a rational indifferent periodic point of a meromorphic function $f(z)$ with period p and $(f^p)'(\omega) = 1$. Then there exists $0 < \eta < 1$ such that*

$$|(f_\omega^{-p})'(z)| < 1 \quad \text{and} \quad |f_\omega^{-p}(z) - \omega| < |z - \omega|$$

for every $z \in B(\omega, \eta) \cap \mathcal{J}(f) \setminus \{\omega\}$, where f_ω^{-p} is the branch of the inverse of f^p sending ω to ω. And the branch f_ω^{-np} of f^{-np} sending ω to ω is well defined and is an analytical homemophism from $B(\omega, \eta) \cap \mathcal{J}(f)$ into $B(\omega, \eta) \cap \mathcal{J}(f)$.

We stress that f_ω^{-np} is not conformal on $B(\omega, \eta) \cap \mathcal{J}(f)$ (the definition of conformality can be found in [Zheng 2009]), as it has no bounded distortions over there. f^{np} is not expanding near ω.

Lemma 3.6. *Let $f(z)$ be a meromorphic function which is precisely expansive from $\mathcal{J}(f)$ to $\hat{\mathcal{J}}(f)$. Then $\widehat{\operatorname{sing}}(f^{-1}) \subset \mathcal{F}(f)$ and f^n is precisely expansive from $\hat{\mathcal{J}}(f) \setminus \bigcup_{j=0}^{n-1} f^{-j}(\infty)$ to $\hat{\mathcal{J}}(f)$.*

Proof. Suppose that $\widehat{\operatorname{sing}}(f^{-1}) \cap \hat{\mathcal{J}}(f) \neq \varnothing$. From this intersecting set, take a point a. Let δ be an arbitrary small fixed positive number. If a is a critical value of $f(z)$, for a $0 < \eta < \delta$ we have a component U of $f^{-1}(B_\infty(a, \eta))$ with $\operatorname{diam}_\infty U < \delta$ such that $f : U \to B_\infty(a, \eta)$ has covering number at least 2. There exist two distinct points z_1 and z_2 in U such that $f(z_1) = f(z_2)$. This contradicts the precisely expansive property of f. Assume that a is an asymptotic value and U is a tract of f over $B_\infty(a, \eta)$. Then there exists a sequence of points $\{z_n\}$ such that $z_n \to \infty$ and $f(z_n) = b \in B_\infty(a, \eta)$. Thus for all sufficiently large n, $d_\infty(z_n, z_{n+1}) < \delta$ and this contradicts the precisely expansive property of f. It is obvious that f^n is precisely expansive. $\square$

We remark that the condition $\widehat{\operatorname{sing}}(f^{-1}) \subset \mathcal{F}(f)$ may not imply that f is parabolic or hyperbolic, but it does when f is rational.

Theorem 3.7. *A parabolic meromorphic function f on the Riemann sphere is precisely expansive over $\hat{\mathcal{J}}(f)$.*

Proof. Take two distinct points x and y in $\mathcal{J}(f)$. Assume without any loss of generality that $f^n(x) \neq \infty$ and $f^n(y) \neq \infty$ for every n, or $f^n(x) = f^n(y) = \infty$ for some n. According to Definition 2.6, we need to treat the two cases, as follows.

(I) For some m, $f^m(x) = f^m(y) = a \in \hat{\mathcal{J}}(f)$ but $f^{m-1}(x) \neq f^{m-1}(y)$. Take a number Θ such that $0 < \Theta < \operatorname{dist}(\operatorname{sing}(f^{-1}), \mathcal{J}(f))$. If $a \notin B(\Omega, \Theta)$, then f^m is univalent from $f_x^{-m}(B_\infty(a, \delta))$ onto $B_\infty(a, \delta)$ with $\delta = \delta(\Theta)$ (by Lemma 3.4), so $y \notin f_x^{-m}(B_\infty(a, \delta))$. If $a \in B(\Omega, \Theta)$, then f is univalent from $f^{-1}_{f^{m-1}(x)}(B_\infty(a, \delta))$ onto $B_\infty(a, \delta)$ so that $f^{m-1}(y) \notin f^{-1}_{f^{m-1}(x)}(B_\infty(a, \delta))$ and furthermore, $y \notin f_x^{-m+1} \circ f^{-1}_{f^{m-1}(x)}(B_\infty(a, \delta)) = f_x^{-m}(B_\infty(a, \delta))$.

(II) for each n, $f^n(x) \neq f^n(y)$. If for a sequence of positive integers $\{n_k\}$ tending to ∞ such that $f^{n_k}(x) \notin B(\Omega, \Theta)$, then $f_x^{-n_k}$ is a single-valued function over $B_\infty(f^{n_k}(x), \delta)$ and therefore $\operatorname{diam} f_x^{-n_k}(B_\infty(f^{n_k}(x), \delta)) \to 0$ as $k \to \infty$. This implies that for all sufficiently large k, $y \notin f_x^{-n_k}(B_\infty(f^{n_k}(x), \delta)$. Now assume that for all $n \geq N$, $f^n(x) \in B(\Omega, \Theta)$ and $f^n(y) \in B(\Omega, \Theta)$. When Θ is sufficiently small, we have for some m, $f^m(x), f^m(y) \in \Omega$. Then $f^m(x)$ and $f^m(y)$ are distinct rational indifferent periodic points of $f(z)$ so that $B_\infty(f^n(x), \delta)$ is disjoint from $B_\infty(f^n(y), \delta)$ for all $n \geq m$. Obviously, $y \notin f_x^{-n}(B_\infty(f^n(x), \delta))$ and $x \notin f_y^{-n}(B_\infty(f^n(y), \delta))$. □

That a rational function is parabolic if and only if it is expansive with at least one rational indifferent periodic point is proved in [Denker and Urbański 1991a]. In view of Theorem 2.8, Theorem 3.7 implies that $f(z)$ has a one-sided generator over $\hat{\mathcal{J}}(f)$. Actually, we can also use the existence of a one-sided generator to show the precisely expansive property of a parabolic meromorphic function on the Riemann sphere, as in view of Lemma 3.5, for each n, $f^n(z)$ has the uniformly covering property (1b) over $\hat{\mathcal{J}}(f)$ with a fixed injectivity radius.

In what follows, let us discuss the existence of conformal measures of parabolic meromorphic functions on the Riemann sphere. We shall use the results in Section 2 to attain our purpose. Let $f(z)$ be a parabolic meromorphic function on the Riemann sphere and let d_∞ be the Riemann spherical metric. Hence $(\hat{\mathcal{J}}(f), d_\infty)$ is a compact metric space. Consider the continuous map $f : \mathcal{J}(f) \to \hat{\mathcal{J}}(f)$ under the Riemann spherical metric. This map is not conformal, so we cannot use Theorem 3.1 of [Zheng 2009] to achieve our purpose. We take a different approach.

Define the *pressure* $P(f, t)$ for a parabolic meromorphic function f over $\hat{\mathcal{J}}(f)$ as follows. Define $\varphi_t : \mathcal{J}(f) \to \mathbb{R}$ by $\varphi_t(z) = -t \log f^\times(z)$, and set $\mathcal{L}_t = \mathcal{L}_{\varphi_t}$.

Thus, for a fixed value $a \in \hat{\mathcal{J}}(f)$ and $g \in \mathscr{C}(\mathcal{J}(f))$, we have

$$\mathscr{L}_t(g)(a) = \sum_{f(z)=a} \frac{g(z)}{f^{\times}(z)^t}.$$

Obviously, $\mathscr{L}_t^n(\mathbb{1})(a) = \sum_{f^n(z)=a} (f^n)^{\times}(z)^{-t}$. Set

$$P_a(f,t) = \limsup_{n\to\infty} \frac{1}{n} \log \mathscr{L}_t^n(\mathbb{1})(a);$$

then the pressure is

$$\hat{P}(f,t) = \sup\{P_a(f,t) : a \in \hat{\mathcal{J}}(f)\}.$$

Lemma 3.8. *Let f be a parabolic meromorphic function on the Riemann sphere.*

(1) $\hat{P}(f,t) \geq 0$.

(2) $P_a(f,t) = P_b(f,t)$ *whenever* $a, b \in \hat{\mathcal{J}}(f) \setminus \Omega$.

We write $P(f,t)$ for $P_a(f,t)$ for $a \in \hat{\mathcal{J}}(f) \setminus \Omega$.

Proof. (1) Take a point $a \in \Omega$ with period p. For each n, $(f^{np})^{\times}(a) = 1$ and $\mathscr{L}_t^{np}(\mathbb{1})(a) > 1$. This implies that $\hat{P}(f,t) \geq 0$.

(2) Assume that $P_a(f,t) < \infty$. For arbitrarily small $\varepsilon > 0$ and for all sufficiently large n, we have

$$\begin{aligned} e^{n(P_a(f,t)+\varepsilon)} &\geq \mathscr{L}_t^n(\mathbb{1})(a) \\ &= \sum_{f^n(z)=a} (f^n)^{\times}(z)^{-t} \\ &= \sum_{f^p(w)=a} \sum_{f^{n-p}(z)=w} (f^p)^{\times}(w)^{-t} (f^{n-p})^{\times}(z)^{-t} \\ &\geq (f^p)^{\times}(w)^{-t} \sum_{f^{n-p}(z)=w} (f^{n-p})^{\times}(z)^{-t}, \end{aligned}$$

where $w \in f^{-p}(a)$. Since $a, b \notin \Omega$, we have a $\delta > 0$ such that $B_\infty(a, 2\delta) \cap \mathcal{P}(f) = \varnothing$ and $B_\infty(b, 2\delta) \cap \mathcal{P}(f) = \varnothing$. We can choose a p such that $f^{-p}(a) \cap B_\infty(b, \delta) \neq \varnothing$ and therefore by the Koebe distortion theorem for the Riemann spherical metric, for an absolute constant K we have

$$e^{n(P_a(f,t)+\varepsilon)} \geq (f^p)^{\times}(w)^{-t} \sum_{f^{n-p}(z)=b} K^{-t} (f^{n-p})^{\times}(z)^{-t}.$$

This yields that $P_a(f,t) + \varepsilon \geq P_b(f,t)$ and so $P_a(f,t) \geq P_b(f,t)$. The same argument implies that $P_b(f,t) \geq P_a(f,t)$. Hence $P_b(f,t) = P_a(f,t)$. □

Set $\tau(f) = \inf\{t \geq 0 : P(f,t) < \infty\}$ and $s(f) = \inf\{t \geq 0 : P(f,t) \leq 0\}$. We call $s(f)$ the *Poincaré exponent*.

Lemma 3.9. *Let f be a parabolic meromorphic function on the Riemann sphere.*

(I) $\tau(f) \le s(f) \le 2$.

(II) *$P(f,t)$ is strictly decreasing and convex in $t \in (\tau(f), +\infty)$.*

(III) *If $t \ge s(f)$, then $\varphi_t(z) = -t \log f^\times(z)$ is summable, and $P(f,s) = 0 = \hat{P}(f,t)$.*

Proof. (I) Take a point $a \in \mathcal{J}(f)$ and $r > 0$ such that $B_\infty(a, 2r) \cap \hat{\mathcal{P}}(f) = \varnothing$. Let f_z^{-n} be the analytic branch of f^{-n} over $B_\infty(a,r)$ sending a to z with $f^n(z) = a$. Set $U(z) = f_z^{-n}(B_\infty(a,r))$. By the Koebe covering theorem for the spherical metric, we have

$$U(z) \supseteq B_\infty(z, Kr(f_z^{-n})^\times(a))$$

for an absolute constant K. Thus, noting that $U(z)$ is disjoint for distinct $z \in f^{-n}(a)$, we have

$$\sum_{f^n(z)=a} \pi(Kr(f_z^{-n})^\times(a))^2 \le \sum_{f^n(z)=a} \text{spherical area of}(U(z)) \le \pi,$$

and furthermore, using $(f_z^{-n})^\times(a) = (f^n)^\times(z))^{-1}$, we obtain

$$\sum_{f^n(z)=a} \frac{1}{((f^n)^\times(z))^2} \le (Kr)^{-2}.$$

This implies that $P(f,2) = P_a(f,2) \le 0$ and hence $s(f) \le 2$.

(II) Take a point $a \in \mathcal{J}(f)$ and a $\delta > 0$ such that $B_\infty(a,\delta) \cap \mathcal{P}(f) = \varnothing$. Then there exists an integer N such that for $n \ge N$

$$d_\infty(f^n(x), f^n(y)) \ge \lambda C^n d_\infty(x,y),$$

with $C > 1$ and $\lambda > 0$, whenever x and y are in an injective component of $f^{-n}(B_\infty(a,\delta))$ and $(f^n)^\times(w) > \lambda C^n, \forall w \in f^{-n}(a)$. This easily implies that $P(f,t) = P_a(f,t)$ is strictly decreasing and convex in t. (See the proof of Theorem 2.3 of [Zheng 2008]).

(III) For arbitrary $t > s(f)$, $P(f,t) < 0$. For a fixed $a \in \hat{\mathcal{J}}(f) \setminus \Omega$, $\mathcal{L}_t^n(\mathbb{1})(a) \to 0$ as $n \to \infty$ and hence for $n \ge m$, $\mathcal{L}_t^n(a) < 1$. Take $z_j (1 \le j \le q)$ such that $\hat{\mathcal{J}}(f) \subset \bigcup_{j=1}^q B_\infty(z_j, \delta/2)$, where δ is chosen such that for each j, $B_\infty(z_j, 2\delta) \cap \operatorname{sing}(f^{-1}) = \varnothing$. Take a positive integer N such that for arbitrary pair of j and i, $B_\infty(z_j,\delta) \cap f^{-N+1}(z_i) \ne \varnothing$. For a P, we have $\mathcal{L}_t^{PN}(\mathbb{1})(a) < 1$. This implies that $\mathcal{L}_t^N(\mathbb{1})(b) < 1$ for some $b \in \hat{\mathcal{J}}(f) \setminus \Omega$. Then $b \in B_\infty(z_{j_0}, \delta/2)$ for some j_0. We find $M = M(j_0)$ disks $B_\infty(b_i, \eta)$ $(1 \le i \le M)$ covering the $B_\infty(z_{j_0}, \delta/2)$ such that each disk $B_\infty(b_i, 2\eta)$ does not intersect $\operatorname{sing}(f^{-N})$. In view of the Koebe distortion theorem, we have

$$\mathcal{L}_t^N(\mathbb{1})(z_{j_0}) \le K^{Mt} \mathcal{L}_t^N(\mathbb{1})(b) < K^{Mt},$$

where K is an absolute constant.

For each $j \in \{1, 2, \dots, q\}$, $f^{-N+1}(z_{j0}) \cap B_\infty(z_j, \delta/2) \neq \varnothing$, from which we take a point w^j_{j0}. We have

$$\begin{aligned}\mathcal{L}_t^N(\mathbb{1})(z_{j0}) &= \sum_{f^N(z)=z_{j0}} (f^N)^\times(z)^{-t} = \sum_{f^{N-1}(w)=z_{j0}} (f^{N-1})^\times(w)^{-t} \sum_{f(z)=w} f^\times(z)^{-t} \\ &\geq (f^{N-1})^\times(w^j_{j0})^{-t} \sum_{f(z)=w^j_{j0}} f^\times(z)^{-t} \\ &= (f^{N-1})^\times(w^j_{j0})^{-t} \mathcal{L}_t(\mathbb{1})(w^j_{j0});\end{aligned}$$

equivalently,

$$\begin{aligned}\mathcal{L}_t(\mathbb{1})(w^j_{j0}) &\leq (f^{N-1})^\times(w^j_{j0})^t \mathcal{L}_t^N(\mathbb{1})(z_{j0}) \\ &< (f^{N-1})^\times(w^j_{j0})^t K^{Mt}.\end{aligned}$$

Set

$$C = \max\{(f^{N-1})^\times(w^v_j) K^{M(j)} : 1 \leq j, v \leq q\}.$$

For each $w \in \hat{\mathcal{J}}(f)$ we have $w \in B_\infty(z_j, \delta/2)$, so $w \in B_\infty(w^j_{j0}, \delta)$ for some j. By the Koebe distortion theorem,

$$\mathcal{L}_t(\mathbb{1})(w) \leq L^t \mathcal{L}_t(\mathbb{1})(w^j_{j0}) < L^t C^t$$

for an absolute constant $L > 0$. This yields that φ_t is summable. Letting t approach $s(f)$ from above, we have

$$\mathcal{L}_s(\mathbb{1})(w) \leq L^s C^s.$$

We have proved that $\varphi_s = -s \log f^\times(z)$ with $s = s(f)$ is summable on $\hat{\mathcal{J}}(f)$ so that $P(f, s) \leq 0$. This immediately implies that $P(f, s) = 0$.

Now we prove that $\hat{P}(f, t) = 0$. For $t > s(f)$, we know that $P(f, t) < 0$. Therefore, we want to calculate $P_a(f, t) = 0$ for $a \in \Omega$. It suffices to prove that $\mathcal{L}_t^n(\mathbb{1})(a)$ is uniformly bounded in n and t for $a \in \Omega$. Assume without loss that the period of a is 1. We take $\eta > 0$ such that $B_\infty(w, \eta) \cap \mathcal{P}(f) = \varnothing$ for $w \in f^{-1}(a) \setminus \{a\}$ and $B_\infty(\infty, \eta) \cap \mathcal{P}(f) = \varnothing$. We can take finitely many w_j, for $1 \leq j \leq q$, such that $w_j \in f^{-1}(a) \setminus \{a\}$ and $\{B_\infty(w_j, \eta/2)\}$ together with $B_\infty(\infty, \eta/2)$ form a covering of $f^{-1}(a) \setminus \{a\}$. By the Koebe distortion theorem, for some c with $P(f, t) < c < 0$, we have for $n \geq N$

$$\mathcal{L}_t^n(\mathbb{1})(w) \leq e^{nc} \quad \text{for } w \in f^{-1}(a) \setminus \{a\}.$$

Set

$$\sum_{\substack{f(w)=a \\ w \neq a}} f^\times(w)^{-t} = K_t.$$

We have

$$\begin{aligned}
\text{(3-1)}\qquad \mathcal{L}_t^n(\mathbb{1})(a) &= \sum_{f^n(z)=a} (f^n)^\times(z)^{-t} \\
&= (f^n)^\times(a)^{-t} + \sum_{\substack{f^n(z)=a\\ z\neq a}} (f^n)^\times(z)^{-t} \\
&= 1 + \sum_{f(w)=a} \sum_{\substack{f^{n-1}(z)=w\\ z\neq a}} f^\times(w)^{-t}(f^{n-1})^\times(z)^{-t} \\
&\le 1 + e^{(n-1)c} \sum_{\substack{f(w)=a\\ w\neq a}} f^\times(w)^{-t} + \sum_{\substack{f^{n-1}(z)=a\\ z\neq a}} (f^{n-1})^\times(z)^{-t} \\
&\le 1 + K_t e^{(n-1)c} + \sum_{f^{n-1}(z)=a, z\neq a} (f^{n-1})^\times(z)^{-t} \\
&\le K_t(1 + e^c + \cdots + e^{(n-1)c}) + \sum_{\substack{f^N(z)=a\\ z\neq a}} (f^N)^\times(z)^{-t} \\
&< K_t \frac{1}{1-e^c} + \sum_{\substack{f^N(z)=a\\ z\neq a}} (f^N)^\times(z)^{-t}.
\end{aligned}$$

This implies that $\hat{P}(f,t) = \max_{a\in\Omega} P_a(f,t) = 0$.

For the case when $s = s(f)$, for arbitrarily small $\varepsilon > 0$ it follows from the above implication that there exists $N = N(\varepsilon)$ such that

$$\mathcal{L}_s^n(\mathbb{1})(a) \le K_s \frac{e^{n\varepsilon}-1}{e^\varepsilon - 1} + \sum_{\substack{f^N(z)=a\\ z\neq a}} (f^N)^\times(z)^{-s}.$$

This implies that $\hat{P}(f,s) \le \varepsilon$ and hence $\hat{P}(f,s) = 0$. □

The next result reflects the expansiveness of a parabolic meromorphic function over $\hat{\mathcal{J}}(f)$. Its idea comes from [Rippon and Stallard 1999].

Lemma 3.10. *Let $f(z)$ be a parabolic meromorphic function on $\mathbb{C}$ and in class $\mathcal{B}$. There exists $c > 0$ such that for each n and $z \in \mathcal{J}(f) \setminus \bigcup_{j=0}^{n-1} f^{-j}(\infty)$, we have*

$$\text{(3-2)}\qquad |(f^n)'(z)| > c\,\frac{|f^n(z)|+1}{|z|+1}.$$

Let M_m be the set of all points $z \in \mathcal{J}(f) \setminus \mathcal{J}_\infty(f)$ for which there exists a sequence $\{s_k\}$ with $s_k \in [km, (k+1)m]$ and $f^{s_k}(z) \notin B(\Omega, \theta)$ for some constant $\theta > 0$. There *exist constants* $c > 0$ *and* $\lambda > 1$ *such that*

$$|(f^n)'(z)| > c\lambda^n \frac{|f^n(z)|+1}{|z|+1} \quad \text{for } z \in M_m. \tag{3-3}$$

Proof. Assume without loss of generality that $\{z : |z| < 1\} \subset \mathcal{F}(f)$. In view of Theorem 3.1, take a $R > 1$ such that $\mathcal{P}(f) \subset B(0, R)$ and $|f^n(0)| < R$ for all $n \in \mathbb{N}$. In view of the result in [Rippon and Stallard 1999] (compare [Zheng 2003]), we have

$$|(f^n)'(z)| > \frac{|f^n(z)|(\log|f^n(z)| - \log R)}{4|z|} \tag{3-4}$$

for $z \in \mathcal{J}(f) \setminus \mathcal{J}_\infty(f)$; furthermore, for $z \in \mathcal{J}(f) \setminus \mathcal{J}_\infty(f)$ with $|f^n(z)| \geq e^2 R$, we have

$$|(f^n)'(z)| > \frac{|f^n(z)|+1}{4(|z|+1)}.$$

We first prove (3-2) for $n = 1$. Since $d(\mathcal{J}(f), \operatorname{sing}(f^{-1})) > 0$, we can take a positive number $A \geq 1$ such that

$$B\left(z, \frac{|z|+1}{A}\right) \cap \operatorname{sing}(f^{-1}) = \varnothing$$

for any $z \in \mathcal{J}(f)$ and

$$B(0,1) \not\subseteq f^{-1}\left(B\left(f(z), \frac{|f(z)|+1}{A}\right)\right)$$

for $z \in \mathcal{J}(f) \setminus f^{-1}(\infty)$ with $|f(z)| < e^2 R$. Then for a fixed $z \in \mathcal{J}(f) \setminus f^{-1}(\infty)$, we have

$$B\left(f(z), \frac{|f(z)|+1}{A}\right) \cap \operatorname{sing}(f^{-1}) = \varnothing$$

and f_z^{-1} is a single-valued analytic branch on $B(f(z), (|f(z)|+1)/A)$ tending $f(z)$ to z. Let U be the component of $f^{-1}(B(f(z), (|f(z)|+1)/A))$ containing z. Then $f : U \to B(f(z), (|f(z)|+1)/A) = B$ (say) is univalent and U is simply connected. In view of the hyperbolic metric principle, we have

$$\lambda_U(z) = \lambda_B(f(z))|f'(z)| = \frac{2A|f'(z)|}{|f(z)|+1}.$$

For $z \in \mathcal{J}(f) \setminus f^{-1}(\infty)$ with $|f(z)| < e^2 R$, $B(0,1) \not\subseteq U$. If $0 \notin U$, then $|z|\lambda_U(z) \geq \frac{1}{4}$; If $0 \in U$, then for a with $|a| \leq 1$, $|z-a|\lambda_U(z) \geq \frac{1}{4}$. Therefore, we always have $(|z|+1)\lambda_U(z) \geq \frac{1}{4}$. These imply that

$$|f'(z)| \geq \frac{1}{8A} \frac{|f(z)|+1}{|z|+1}. \tag{3-5}$$

This proves (3-2) for $n = 1$ with $c = \frac{1}{8A}$.

Suppose (3-2) is not true. Then there exist a sequence of positive integers $\{m_k\}$ and a sequence of points $z_k \in \mathcal{J}(f) \setminus \bigcup_{j=0}^{m_k-1} f^{-j}(\infty)$ such that

$$\varepsilon_k = \frac{|(f^{m_k})'(z_k)|(|z_k|+1)}{|(f^{m_k})(z_k)|+1} \to 0 \quad \text{as } k \to \infty.$$

We can take a positive number C such that for $z \in \mathcal{J}(f) \setminus B(\Omega, \theta)$,

$$B\left(z, 2\frac{|z|+1}{C}\right) \cap \hat{\mathcal{P}}(f) = \varnothing. \tag{3-6}$$

If $f^{m_k}(z_k) \notin B(\Omega, \theta)$, a single-valued analytic branch g_k of f^{-m_k} sending $f^{m_k}(z_k)$ to z_k exists on $B(f^{m_k}(z_k), 2(|f^{m_k}(z_k)|+1)/C)$. By the Koebe covering theorem, we have

$$\begin{aligned} g_k\left(B\left(f^{m_k}(z_k), \frac{|f^{m_k}(z_k)|+1}{C}\right)\right) &\supseteq B\left(z_k, \frac{|f^{m_k}(z_k)|+1}{4C}|g_k'(f^{m_k}(z_k))|\right) \\ &= B\left(z_k, \frac{|z_k|+1}{4C\varepsilon_k}\right) \\ &\supseteq B\left(0, \frac{|z_k|+1}{4C\varepsilon_k} - |z_k|\right). \end{aligned} \tag{3-7}$$

Now assume that $f^j(z_k) \in B(\Omega, \theta)$, $p_k \le j \le m_k$ and $f^{p_k-1}(z_k) \notin B(\Omega, \theta)$. By Lemma 3.5, we have $|(f^{m_k-p_k})'(f^{p_k}(z_k))| \ge 1$ and so for a positive constant a,

$$|(f^{m_k-p_k})'(f^{p_k}(z_k))| \ge a\frac{|f^{m_k}(z_k)|+1}{|f^{p_k}(z_k)|+1}.$$

Thus

$$\begin{aligned} |f'(f^{p_k-1}(z_k))||(f^{p_k-1})'(z_k)| &= |(f^{p_k})'(z_k)| \\ &= \frac{|(f^{m_k})'(z_k)|}{|(f^{m_k-p_k})'(f^{p_k}(z_k))|} \\ &\le \frac{|(f^{m_k})'(z_k)|}{|f^{m_k}(z_k)|+1}\frac{|f^{p_k}(z_k)|+1}{a} \\ &= \frac{\varepsilon_k(|f^{p_k}(z_k)|+1)}{a(|z_k|+1)} \end{aligned}$$

and in view of (3-5), we have

$$|f'(f^{p_k-1}(z_k))| \ge \frac{1}{8A}\frac{|f^{p_k}(z_k)|+1}{|f^{p_k-1}(z_k)|+1}.$$

Combining the above two inequalities implies that

$$|(f^{p_k-1})'(z_k)| \le \frac{\varepsilon_k 8A(|f^{p_k-1}(z_k)|+1)}{a(|z_k|+1)}. \tag{3-8}$$

Since $f^{p_k-1}(z_k) \notin B(\Omega, \theta)$, from (3-8) we have

$$h_k\left(B\left(f^{p_k-1}(z_k), \frac{|f^{p_k-1}(z_k)|+1}{C}\right)\right) \supseteq B\left(0, \frac{a(|z_k|+1)}{32AC\varepsilon_k} - |z_k|\right),$$

where h_k is the analytic branch of f^{-p_k+1} which sends $f^{p_k-1}(z_k)$ to z_k. This together with (3-7) shows the existence of a sequence of positive integers $\{n_k\}$ such that

$$f_{z_k}^{-n_k}\left(B\left(f^{n_k}(z_k), \frac{|f^{n_k}(z_k)|+1}{C}\right)\right) \supseteq B\left(0, \frac{a(|z_k|+1)}{32AC\varepsilon_k} - |z_k|\right).$$

But this gives

$$\frac{a(|z_k|+1)}{32AC\varepsilon_k} - |z_k| \to +\infty$$

as $k \to \infty$, and a contradiction is derived. We have proved (3-2).

Now we prove (3-3). Let $z \in M_m$. In view of (3-4), there exists an $R_0 > R$ such that

$$|(f^n)'(z)| > 2(1+c^{-1})\frac{|f^n(z)|+1}{|z|+1}, \text{ for } n \in \mathbb{N}, |f^n(z)| > R_0 \tag{3-9}$$

where c is the constant in (3-2). Using the same argument as in the proof of (3-2), we can also attain (3-9) for $n \geq N \geq m$, $z \in (\mathcal{J}(f) \setminus \bigcup_{j=0}^{n-1} f^{-j}(\infty)) \cap B(0, R_0)$ with $f^n(z) \notin B(\Omega, \theta)$.

For any $0 \leq p < 2N$, we treat two cases. If $|f^{2N+p}(z)| > R_0$, from (3-9) we have

$$|(f^{2N+p})'(z)| > 2\frac{|f^{2N+p}(z)|+1}{|z|+1};$$

If $|f^{2N+p}(z)| \leq R_0$, for some $N \leq N_1 \leq 2N+p$ we have either $|f^{N_1}(z)| \leq R_0$ and $f^{N_1}(z) \notin B(\Omega, \theta)$ or $|f^{N_1}(z)| > R_0$. Therefore from (3-2) and (3-9) we have

$$\begin{aligned}|(f^{2N+p})'(z)| &= |(f^{2N+p-N_1})'(f^{N_1}(z))||(f^{N_1})'(z)| \\ &> c\frac{|f^{2N+p}(z)|+1}{|f^{N_1}(z)|+1}2c^{-1}\frac{|f^{N_1}(z)|+1}{|z|+1} \\ &= 2\frac{|f^{2N+p}(z)|+1}{|z|+1}.\end{aligned}$$

For $n \geq 2N$, we write $n = 2qN + p$ with $0 \leq p < 2N$ and thus

$$|(f^n)'(z)| > 2^q\frac{|f^n(z)|+1}{|z|+1} > \frac{1}{2}(2^{\frac{1}{2N}})^n\frac{|f^n(z)|+1}{|z|+1}.$$

For $1 \leq n < 2N$, we use (3-2). □

The next result confirms the existence of a measure that becomes s-conformal.

Lemma 3.11. *Let $f(z)$ be a parabolic meromorphic function on the Riemann sphere. Then (f, φ_s) is admissible over $\hat{\mathcal{J}}(f)$.*

Proof. We check the conditions in Definition 2.1. Obviously, for f, (1a) and (1b) hold, and (1d) holds by virtue of Lemma 3.9(III). In view of Lemma 2.2, (1c) is true for f. We state (1e) for (f, φ_s) as follows: for all $\varepsilon > 0$, there exists $\delta_1 \in (0, \delta)$ such that for any pair $a, b \in \hat{\mathcal{J}}(f)$, the condition $d_\infty(a, b) < \delta_1$ implies

$$\sum_{f(z)=a} \left|\exp \varphi_s(f_z^{-1}(a)) - \exp \varphi_s(f_z^{-1}(b))\right| < \varepsilon;$$

that is,

$$\sum_{f(z)=a} \left| \frac{1}{f^\times(z)^s} - \frac{1}{f^\times(z')^s} \right| < \varepsilon, \tag{3-10}$$

where $z' = f_z^{-1}(b)$. From Lemma 3.2 of [Zheng 2009], noting that φ_s is summable, (3-10) follows from

$$\left| 1 - \frac{f^\times(z)^s}{f^\times(z')^s} \right| \le C_s d_\infty(a, b), \tag{3-11}$$

whenever $d_\infty(a, b) < \delta$. And (3-11) can be proved via the same argument used in the proof of Lemma 3.1 of [Zheng 2008] and the inequality (3-2). ☐

Now we are in the position to prove Theorem 1.2, which, as we recall, states that any f in $\mathcal{P}(\hat{\mathbb{C}})$ has a s-conformal measure with $P(f, s) = 0$.

Proof of Theorem 1.2. In view of Lemma 3.11 and Theorem 2.3, there exists a probability measure μ with $\mathcal{L}_s^*(\mu) = \lambda\mu$, $\lambda = \mathcal{L}_s^*(\mu)(\mathbb{1})$, satisfying the conditions in Theorem 2.3(3). We calculate λ using (2-2), and obtain

$$\lambda^n = \mu(\mathcal{L}_s^n(\mathbb{1})) \le \sup\{\mathcal{L}_s^n(\mathbb{1})(x) : x \in \hat{\mathcal{J}}(f)\}.$$

Using the same argument as in the proof of (3-1), for arbitrarily small $\varepsilon > 0$, we have for $n \ge N$

$$\sup\{\mathcal{L}_s^n(\mathbb{1})(x) : x \in \hat{\mathcal{J}}(f)\} \le Kne^{n\varepsilon},$$

so $\log \lambda \le 0$.

If $\mu(\{a\}) > 0$ for a point $a \in \hat{\mathcal{J}}(f)$, then $\lambda^n \ge \mu(\{a\})\mathcal{L}_s^n(\mathbb{1})(a)$ and so $\log \lambda \ge P(f, s) = 0$. Now assume that μ is atomless and we can find a disk $B_\infty(a, \eta)$ with $\mu(B_\infty(a, \eta)) > 0$ which does not intersect $\mathcal{P}(f)$. Thus

$$\begin{aligned}\lambda^n &\ge \mu(B_\infty(a, \eta)) \inf\{\mathcal{L}_s^n(\mathbb{1})(x) : x \in B_\infty(a, \eta) \cap \hat{\mathcal{J}}(f)\}\\ &\ge \mu(B_\infty(a, \eta))K^{-s}\mathcal{L}_s^n(\mathbb{1})(a)\end{aligned}$$

so that $\log \lambda \ge P(f, s) = 0$. Therefore, we have proved that $\lambda = 1$ and μ is a s-conformal measure of $f(z)$ over $\hat{\mathcal{J}}(f)$. ☐

In what follows, we consider the existence of a f-invariant measure equivalent to the s-conformal measure μ_s. We cannot get such an invariant measure from Walters' result. Therefore, we will complete our discussion in light of the results of Martens.

Lemma 3.12 [Martens 1992, Proposition 2.6]. *Let μ be a σ-finite Borel measure on a σ-compact space X and $f : X \to X$ a measurable map. Then f has a μ-equivalent, σ-finite invariant measure $\mathfrak{m}$, if the following statements hold*:

(1) *There exist a countable collection of pairwise disjoint Borel sets $G = \{I_j : j \in \mathbb{N}\}$ of X such that each I_j is σ-compact, $0 < \mu(I_j) < \infty$, $\mu(X \setminus \bigcup_{j=1}^{\infty} I_j) = 0$ and for all pair I_i and I_j, for some $n \geq 0$, $\mu(f^{-n}(I_i) \cap I_j) > 0$.*

(2) *There exists a σ-finite measure ν having properties that for each $I \in G$ there exists a $K > 0$ such that $K^{-1} \leq \nu(I) \leq K$, $\sup_{n\geq 0} \nu(f^{-n}(I)) < \infty$, and*

$$\frac{1}{K}\frac{\mu(A)}{\mu(I)} \leq \frac{\nu(f^{-n}(A))}{\nu(f^{-n}(I))} \leq K\frac{\mu(A)}{\mu(I)}.$$

for all measurable sets $A \subset I$ and all $n \in \mathbb{N}$.

(3) $\sum_{n=0}^{\infty} \nu(f^{-n}(\hat{I})) = \infty$ *for some $\hat{I} \in G$.*

Actually, $\mathfrak{m}$ is a weakly convergent limit of $\{Q_n(\nu)\}$ on each $I \in G$, where

$$Q_n(\nu) = \frac{\sum_{j=0}^{n-1} f_*^i \nu}{\sum_{j=0}^{n-1} f_*^i \nu(\hat{I})}$$

and for a Borel measurable map g, $g_*\nu = \nu \circ g^{-1}$.

Let $f(z)$ be a parabolic meromorphic function in $\mathcal{P}(\hat{\mathbb{C}})$ and let μ_s be the s-conformal measure determined in Theorem 1.2. Assume that μ_s is atomless. Set $X_0 = \mathcal{J}(f) \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$ and $X = \hat{\mathcal{J}}(f) \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$. Then $\mu_s(X_0) = \mu_s(X) = 1$ and we can construct a countable collection of disjoint Borel sets $G = \{I_j : j \in \mathbb{N}\}$ of X such that for each j, $I_j \subset B_\infty(a_j, \delta_j)$ and $B_\infty(a_j, 2\delta_j) \cap \hat{\mathcal{P}}(f) = \varnothing$ for some $a_j \in I_j$ and which satisfies (1) in Lemma 3.12. In view of the Koebe distortion theorem for the spherical metric and the definition of s-conformal measure, we easily prove (2) in Lemma 3.12 for f and G with respect to μ_s and $\nu = \mu_s$. Therefore, the crucial point is in (3) in Lemma 3.12. We have

$$\begin{aligned}\mu_s(f^{-n}(I_j)) &= \sum_{f^n(z)=a_j} \mu_s(f_z^{-n}(I_j)) = \sum_{f^n(z)=a_j} \int_{I_j} (f_z^{-n})^\times(w)^s \mu_s(w) \\ &\geq \sum_{f^n(z)=a_j} K^{-s} (f_z^{-n})^\times(a_j)^s \mu_s(I_j) = K^{-s}\mu_s(I_j)\mathcal{L}_s^n(\mathbb{1})(a_j)\end{aligned}$$

and

$$\mu_s(f^{-n}(I_j)) \leq K^s \mu_s(I_j)\mathcal{L}_s^n(\mathbb{1})(a_j),$$

where K is the Koebe distortion constant. Thus we have

$$K^{-s}\mu_s(I_j)\sum_{n=0}^{\infty}\mathscr{L}_s^n(\mathbb{1})(a_j) \le \sum_{n=0}^{\infty}\mu_s(f^{-n}(I_j)) \le K^s\mu_s(I_j)\sum_{n=0}^{\infty}\mathscr{L}_s^n(\mathbb{1})(a_j).$$

In view of Lemma 3.12, $f(z)$ has an f-invariant, σ-finite measure m which is equivalent to μ_s if $\sum_{n=0}^{\infty}\mathscr{L}_s^n(\mathbb{1})(a)=\infty$ for some $a\in\mathcal{J}(f)\setminus\bigcup_{n=0}^{\infty}f^{-n}(\Omega)$.

In view of the statements above, we have actually proved Theorem 1.3.

On the other hand, assume that $f(z)$ has an f-invariant, σ-finite measure m_s which is equivalent to μ_s. Take an $a\in\mathcal{J}(f)\setminus\bigcup_{n=0}^{\infty}f^{-n}(\Omega)$, and $B_\infty(a,2\delta)\cap\mathcal{P}(f)=\varnothing$ for some $\delta>0$. Set $I=B_\infty(a,\delta)\cap\mathcal{J}(f)$. Then $\mu_s(I)>0$, and $m_s(I)>0$ and for each n, $m_s(f^{-n}(I))=m_s(I)$. This implies that

$$\sum_{n=0}^{\infty}m_s(f^{-n}(I))=\infty.$$

Then if the Radon–Nikodym derivative $dm_s/d\mu_s$ of m_s with respect to μ_s is bounded, we have $\sum_{n=0}^{\infty}\mathscr{L}_s^n(\mathbb{1})(a)=\infty$.

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