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AN ANALOGUE OF THE CARTAN DECOMPOSITION FOR p-ADIC SYMMETRIC SPACES OF SPLIT p-ADIC REDUCTIVE GROUPS

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Let k be a nonarchimedean locally compact field of residue characteristic p, let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G, and H an open k-subgroup of the fixed points group of σ. We denote by G_k and H_k the groups of k-points of G and H. We obtain an analogue of the Cartan decomposition for the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ in the case where G is k-split and p is odd. More precisely, we obtain a decomposition of G_k as a union of $(\mathrm{H}_k, \mathrm{K})$-double cosets, where K is the stabilizer of a special point in the Bruhat–Tits building of G over k. This decomposition is related to the H_k-conjugacy classes of maximal σ-antiinvariant k-split tori in G. In a more general context, Benoist and Oh obtained a polar decomposition for any p-adic reductive symmetric space. In the case where G is k-split and p is odd, our decomposition makes more precise that of Benoist and Oh, and generalizes results of Offen for GL_n.

1. Introduction

Let k be a nonarchimedean locally compact field of odd residue characteristic. Let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and let H be an open k-subgroup of the fixed points group of σ. We denote by G_k and H_k the groups of k-points of G and H. Harmonic analysis on the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ is the study of the action of G_k on the space of complex square integrable functions on $\mathrm{H}_k\backslash\mathrm{G}_k$. This study is related to the classification of H_k-distinguished representations of G_k, that is representations having a nonzero space of H_k-invariant linear forms. Offen [2004] has investigated the harmonic analysis of spherical functions in some cases related to GL_n. Hironaka [1988] has described a Cartan decomposition for the pair (GL_n, O_n). Blanc and Delorme [2008] have studied H_k-distinguishedness for families of parabolically induced representations of G_k. Lagier [2008], and independently Kato and Takano

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[2008], have introduced the notion of relative cuspidality for irreducible H_k-distinguished representations of G_k and constructed "Jacquet maps" at the level of invariant linear forms. In this paper, we investigate the geometry of the reductive symmetric space $H_k\backslash G_k$.

Connected reductive groups can be considered as reductive symmetric spaces. Indeed, if G' is such a group, the map

$$\sigma : (x, y) \mapsto (y, x)$$

defines a k-involution of $G = G' \times G'$ whose fixed points group H is the diagonal image of G' in G, and the reductive symmetric space $H_k\backslash G_k$ naturally identifies with G'_k via the map $(x, y) \mapsto x^{-1}y$. Moreover, if K' is a subgroup of G'_k, and if we set $K = K' \times K'$, then this map induces a bijective correspondence:

$$\{(H_k, K)\text{-double cosets of } G_k\} \leftrightarrow \{K'\text{-double cosets of } G'_k\}.$$

In particular, if K' is the G'_k-stabilizer of a special point in the Bruhat–Tits building of G' over k, the decomposition of $H_k\backslash G_k$ into K-orbits corresponds to the Cartan decomposition of G'_k relative to K' [Bruhat and Tits 1972, Proposition 4.4.3].

In this paper, we obtain an analogue of the Cartan decomposition for $H_k\backslash G_k$ when the group G is k-split. In a more general context (k any nonarchimedean locally compact field of odd characteristic and G any connected reductive group over k), Benoist and Oh [2007] have obtained a polar decomposition for $H_k\backslash G_k$. In the case where k has odd residue characteristic and G is k-split, our decomposition is a refinement of Benoist–Oh's polar decomposition (see 4.14). This decomposition can be seen as a p-adic analogue of the Cartan decomposition for real reductive symmetric spaces [Flensted-Jensen 1978, Theorem 4.1]. It generalizes the decompositions obtained by Offen [2004, Proposition 3.1] for $G = GL_{2n}$ in what he called Cases 1 and 3.

Let $\{A^j \mid j \in J\}$ be a set of representatives of the H_k-conjugacy classes of maximal σ-antiinvariant k-split tori of G (called maximal (σ, k)*-split* tori in [Helminck 1994]; see also Definition 4.2). These tori, as well as related entities, have been studied in [Helminck 1994; Helminck and Helminck 1998; Helminck and Wang 1993]. In particular, the set J is finite and the A^j, $j \in J$, are all conjugate under G_k. Let S be a σ-stable maximal k-split torus of G containing a maximal (σ, k)-split torus A. For each $j \in J$, we choose $y_j \in G_k$ such that $y_j A y_j^{-1} = A^j$. Our main result is this:

Theorem 1.1 (see Theorem 4.13). *Assume* G *is k-split. Let* K *be the stabilizer in* G_k *of a special point in the apartment attached to* S. *Then*

$$G_k = \bigcup_{j \in J} H_k y_j S_k K. \tag{1-1}$$

If one compares with Offen's decompositions [2004, Proposition 3.1], one sees that in each of his Cases 1 and 3 (where $\mathrm{G} = \mathrm{GL}_{2n}$ for $n \geqslant 1$), the set J reduces to a single element and y_j can be chosen to be trivial. In general however, one cannot avoid having several non-H_k-conjugate maximal σ-antiinvariant k-split tori of G appearing in (1-1).

To prove Theorem 1.1, we make generous use of Bruhat–Tits theory [1972; 1984a]. First, let G be any connected reductive group over k, and let $\mathcal{B}$ be its Bruhat–Tits building. It is endowed with an action of σ. Then:

Proposition 1.2 (see Proposition 3.8). *$\mathcal{B}$ is the union of its σ-stable apartments.*

Note that in the case where $\mathrm{G} = \mathrm{G}' \times \mathrm{G}'$ and $\sigma(x, y) = (y, x)$ as above, the building $\mathcal{B}$ identifies with the product of two copies of the building of G' over k and the proposition simply says that two arbitrary points in the building of G' are always contained in a common apartment.

When G is k-split, we obtain the following refinement of the proposition above:

Proposition 1.3 (see Proposition 4.8). *Assume* G *is k-split, and let x be a special point of $\mathcal{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and the maximal σ-antiinvariant subtorus of* S *is a maximal (σ, k)-split torus of* G.

As we will see in 5.13, this is no longer true for nonsplit groups.

Summary. In Section 2, we recall the main properties of the Bruhat–Tits building attached to a connected reductive group defined over k. In Section 3, we study the set of all apartments containing a given σ-stable subset of the building, and we prove Proposition 1.2. In Section 4, we prove our main theorem for G a k-split group. In Section 5, we study in more detail the case of $\mathrm{G}_k = \mathrm{GL}_n(k)$ and $\sigma(g) =$ transpose of g^{-1}, and the case of $\mathrm{G}_k = \mathrm{GL}_n(k')$ with k' quadratic over k and $\mathrm{id} \neq \sigma \in \mathrm{Gal}(k'/k)$. When $n = 2$ and k' is totally ramified over k, the second case provides an example of a nonsplit group for which Proposition 1.3 is not satisfied.

2. The Bruhat–Tits building

Let k be a nonarchimedean nondiscrete locally compact field, and let ω be its normalized valuation. In this section, we recall the main properties of the Bruhat–Tits building attached to a connected reductive group defined over k. The reader may refer to [Bruhat and Tits 1972; 1984a] or to the more concise presentations [Landvogt 1995; Schneider and Stuhler 1997; Tits 1979].

If G is a linear algebraic group defined over k, the group of its k-points will be denoted by G_k or $\mathrm{G}(k)$, and its neutral component will be denoted by G°. If X is a subset of G, then $\mathrm{N}_\mathrm{G}(\mathrm{X})$ and $\mathrm{Z}_\mathrm{G}(\mathrm{X})$ denote respectively the normalizer and centralizer of X in G, and, given $g \in \mathrm{G}$, we write ${}^g\mathrm{X}$ for $g\mathrm{X}g^{-1}$.

2.1. Let G be a connected reductive group defined over k, and let S be a maximal k-split torus of G. We denote by $X^*(S) = \mathrm{Hom}(S, GL_1)$ the group of algebraic characters, and by $X_*(S) = \mathrm{Hom}(GL_1, S)$ the group of cocharacters, of S. We define a map

$$X_*(S) \times X^*(S) \to \mathbb{Z} \tag{2-1}$$

as follows. If $\lambda \in X_*(S)$ and $\chi \in X^*(S)$, then $\chi \circ \lambda$ is an endomorphism of the multiplicative group GL_1, which corresponds to an endomorphism of the ring $\mathbb{Z}[t, t^{-1}]$. It is of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$. This integer n is denoted by $\langle \lambda, \chi \rangle$. The map (2-1) defines a perfect duality [Borel 1991, § 8.6].

2.2. Let N and Z denote the normalizer and centralizer of S in G. If we extend the map (2-1) by $\mathbb{R}$-linearity, there exists a unique group homomorphism

$$\nu : Z_k \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2-2}$$

such that the condition

$$\langle \nu(z), \chi \rangle = -\omega(\chi(z))$$

holds for any $z \in Z_k$ and any k-rational character $\chi \in X^*(Z)_k$ [Tits 1979, § 1.2]. According to [Landvogt 1995, Proposition 1.2], the kernel of (2-2) is the maximal compact subgroup of Z_k.

2.3. Let C denote the connected center of G and let $X_*(C)$ be the group of its algebraic cocharacters. It is a subgroup of the free abelian group $X_*(S)$. We denote by $\mathscr{A}$ the space

$$V = (X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})/(X_*(C) \otimes_{\mathbb{Z}} \mathbb{R}),$$

considered as an affine space on itself and by $\mathrm{Aff}(\mathscr{A})$ the group of its affine automorphisms. By making V act on $\mathscr{A}$ by translations, we can think of V as a subgroup of $\mathrm{Aff}(\mathscr{A})$. It is the kernel of the natural group homomorphism $\mathrm{Aff}(\mathscr{A}) \to GL(V)$ which associates to any affine automorphism its linear part.

2.4. The map (2-2) induces a homomorphism

$$Z_k \to \mathrm{Aff}(\mathscr{A}), \tag{2-3}$$

which we still denote by ν. Its image is contained in V. An important property of this homomorphism is that it extends to a homomorphism $N_k \to \mathrm{Aff}(\mathscr{A})$ [Tits 1979, § 1.2]. It does not extend in a unique way, but two homomorphisms extending (2-3) to N_k are conjugated by a *unique* element of $\mathrm{Aff}(\mathscr{A})$ [Landvogt 1995, Proposition 1.8].

2.5. The affine space $\mathscr{A}$ endowed with an action of N_k defined by a group homomorphism $\nu : \mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the homomorphism (2-3) is called the (reduced) *apartment* attached to S. It satisfies these conditions:

A1. $\mathscr{A}$ is an affine space on V;

A2. ν is a group homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the canonical homomorphism $\mathrm{Z}_k \to \mathrm{V}$.

It has the following uniqueness property: if $(\mathscr{A}', \nu')$ satisfies A1 and A2, there is a unique affine and N_k-equivariant isomorphism from $\mathscr{A}'$ to $\mathscr{A}$.

Remark 2.6. As in [Tits 1979], one obtains the *nonreduced* apartment $\mathscr{A}_{\mathrm{nr}}$ by replacing V by $\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbb{Z}} \mathbb{R}$. It is not as canonical as the reduced one: two homomorphisms extending the map $\nu_{\mathrm{nr}} : \mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ to N_k are conjugated by an element of $\mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ which is not necessarily unique [Landvogt 1995, Chapter 1, § 1; Tits 1979, § 1.2].

2.7. Let $\Phi = \Phi(\mathrm{G}, \mathrm{S})$ denote the set of roots of G relative to S. It is a subset of $\mathrm{X}^*(\mathrm{S})$. Therefore, any root $a \in \Phi$ can be seen as a linear form on $\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ which is trivial on the subspace $\mathrm{X}_*(\mathrm{C}) \otimes_{\mathbb{Z}} \mathbb{R}$, hence as a linear form on V [Landvogt 1995, Chapter 1, § 1].

For $a \in \Phi$, we denote by U_a the root subgroup associated to a, which is a unipotent subgroup of G normalized by Z [Borel 1991, Proposition 21.9], and by s_a the reflection corresponding to a, considered as an element of GL(V) — or, more precisely, of the quotient of $\nu(\mathrm{N}_k)$ by $\nu(\mathrm{Z}_k)$.

2.8. Let $a \in \Phi$ and $u \in \mathrm{U}_a(k) - \{1\}$. The intersection

$$\mathrm{U}_{-a}(k) u \mathrm{U}_{-a}(k) \cap \mathrm{N}_k \tag{2-4}$$

consists of a single element, called $m(u)$, whose image by ν is an affine reflection the linear part of which is s_a [Borel and Tits 1965, § 5]. The set $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$ is an affine hyperplane of $\mathscr{A}$, which is called a *wall* of $\mathscr{A}$.

A *chamber* of $\mathscr{A}$ is a connected component of the complementary in $\mathscr{A}$ of the union of its walls. Note that a chamber is open in $\mathscr{A}$.

A point $x \in \mathscr{A}$ is said to be *special* if, for all root $a \in \Phi$, there is a root $b \in \Phi \cap \mathbb{R}_+ a$ and an element $u \in \mathrm{U}_b(k) - \{1\}$ such that $x \in \mathscr{H}_{b,u}$ [Landvogt 2000, § 1.2.3; Tits 1979, § 1.9].

2.9. Let $\theta(a, u)$ denote the affine function $\mathscr{A} \to \mathbb{R}$ whose linear part is a and whose vanishing hyperplane is the wall $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$. We fix a base point in $\mathscr{A}$, so that $\mathscr{A}$ can be identified with the vector space V. For $r \in \mathbb{R}$, we set

$$\mathrm{U}_a(k)_r = \left\{u \in \mathrm{U}_a(k) - \{1\} \mid \theta(a, u)(x) \geqslant a(x) + r \text{ for all } x \in \mathscr{A}\right\} \cup \{1\}.$$

Thus we obtain a filtration of $U_a(k)$ by subgroups. If we change the base point in $\mathscr{A}$, this filtration is only modified by a translation of the indexation.

2.10. Let Ω be a nonempty subset of $\mathscr{A}$. We set

$$N_\Omega = \{n \in N_k \mid \nu(n)(x) = x \text{ for all } x \in \Omega\},$$

and we denote by U_Ω the subgroup of G_k generated by all the $U_a(k)_r$ such that the affine function $x \mapsto a(x) + r$ is nonnegative on Ω. According to [Landvogt 1995, § 12], this subgroup is compact in G_k, and we have $nU_\Omega n^{-1} = U_{\nu(n)(\Omega)}$ for $n \in N_k$. In particular, N_Ω normalizes U_Ω. The subgroup $P_\Omega = N_\Omega U_\Omega$ is open in G_k [Landvogt 1995, Corollary 12.12].

2.11. Let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. We denote by U^+ (U^-) the subgroup of G_k generated by the U_a for all $a \in \Phi^+$ ($a \in \Phi^-$). Then the group P_Ω has the following Iwahori decomposition [Landvogt 1995, Corollary 12.6; Bruhat and Tits 1972, § 7.1.4]:

$$P_\Omega = (U_\Omega \cap U^-) \cdot (U_\Omega \cap U^+) \cdot N_\Omega. \tag{2-5}$$

2.12. Bruhat and Tits [1972; 1984a] associate to the apartment $(\mathscr{A}, \nu)$ a G_k-set $\mathscr{B} = \mathscr{B}(G, k)$ containing $\mathscr{A}$, called the (reduced) *building* of G over k and satisfying the following conditions:

B1. The set $\mathscr{B}$ is the union of the $g \cdot \mathscr{A}$ for $g \in G_k$.

B2. The subgroup N_k is the stabilizer of $\mathscr{A}$ in G_k, and $n \cdot x = \nu(n)(x)$ for all $x \in \mathscr{A}$ and $n \in N_k$.

B3. For all $a \in \Phi$ and $r \in \mathbb{R}$, the subgroup $U_a(k)_r$ defined in 2.9 fixes the subset $\{x \in \mathscr{A} \mid a(x) + r \geqslant 0\}$ pointwise.

The building has the following uniqueness property: if $\mathscr{B}'$ is a G_k-set containing $\mathscr{A}$ and satisfying B1–B3, there is a unique G_k-equivariant bijection from $\mathscr{B}'$ to $\mathscr{B}$ [Tits 1979, § 2.1; Prasad and Yu 2002, § 1.9].

2.13. The subsets of $\mathscr{B}$ of the form $g \cdot \mathscr{A}$ with $g \in G_k$ are called *apartments*. According to B1, the building is the union of its apartments. For $g \in G_k$, the apartment $g \cdot \mathscr{A}$ can be naturally endowed with a structure of affine space and an action of ${}^gN_k$ by affine isomorphisms. Up to unique isomorphism, it is the apartment attached to the maximal k-split torus ${}^gS$ (see 2.5). This defines a unique G_k-equivariant map

$$S' \mapsto \mathscr{A}(S') \subseteq \mathscr{B} \tag{2-6}$$

between maximal k-split tori of G and apartments of $\mathscr{B}$, such that S maps to $\mathscr{A}$.

Note that the building $\mathscr{B}$ does not depend on the maximal k-split torus S. Indeed, let S$'$ be a maximal k-split torus of G, let $(\mathscr{A}', \nu')$ be the apartment attached to S$'$ and $\mathscr{B}'$ be the building of G over k relative to this apartment (see 2.12). If we identify $\mathscr{A}'$ with the unique apartment of $\mathscr{B}$ corresponding to S$'$ via (2-6), then $\mathscr{B}' = \mathscr{B}$.

2.14. The building has the following important properties [Bruhat and Tits 1972, § 7.4; Landvogt 1995, Chapter 4, § 13]:

(1) Let Ω be a nonempty subset of $\mathscr{A}$. Then P_Ω is the subgroup of G_k made of those elements fixing Ω pointwise.

(2) Let $g \in G_k$. There is $n \in N_k$ such that $g \cdot x = n \cdot x$ for any $x \in \mathscr{A} \cap g^{-1} \cdot \mathscr{A}$.

In particular, (1) together with B2 imply that $N_\Omega = N_k \cap P_\Omega$.

2.15. Let σ be a k-automorphism of G. There is a unique bijective map from $\mathscr{B}$ to itself, still denoted σ, such that

(1) the condition

$$\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$$

holds for any $g \in G_k$ and $x \in \mathscr{B}$; and

(2) the map σ permutes the apartments and, for any apartment $\mathscr{A}$, the restriction of σ to $\mathscr{A}$ is an affine isomorphism from $\mathscr{A}$ to $\sigma(\mathscr{A})$.

This makes (2-6) into a σ-equivariant map. In particular, an apartment is σ-stable if and only if its corresponding maximal k-split torus of G is σ-stable [Bruhat and Tits 1984a, § 4.2.12].

3. Existence of σ-stable apartments

From now on, k will be a nonarchimedean locally compact field of odd residue characteristic. Let G be connected reductive group defined over k and let σ be a k-involution on G. According to 2.15, the building $\mathscr{B}$ of G over k is endowed with an action of σ. In this section, we prove that, given $x \in \mathscr{B}$, there exists a σ-stable apartment containing x. We keep using notation of Section 2.

3.1. Let Ω be a nonempty σ-stable subset of $\mathscr{B}$ contained in some apartment, and let Ap(Ω) be the set of all apartments of $\mathscr{B}$ containing Ω. It is a nonempty set on which the group P_Ω acts transitively [Landvogt 1995, Corollary 13.7]. Because Ω is σ-stable, both P_Ω and Ap(Ω) are σ-stable. Note that the σ-stable apartments containing Ω are exactly the σ-fixed points in Ap(Ω).

3.2. Let us fix an apartment $\mathscr{A} \in \mathrm{Ap}(\Omega)$ and an element $u \in \mathrm{P}_\Omega$ such that $\sigma(\mathscr{A}) = u \cdot \mathscr{A}$. Let N denote the normalizer in G of the maximal k-split torus of G corresponding to $\mathscr{A}$. As σ is involutive, we have

$$\sigma(u)u \in \mathrm{P}_\Omega \cap \mathrm{N}_k = \mathrm{N}_\Omega. \tag{3-1}$$

The map $\rho : g \mapsto g \cdot \mathscr{A}$ induces a P_Ω-equivariant bijection between the homogeneous spaces $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ and $\mathrm{Ap}(\Omega)$. The automorphism

$$\theta : x \mapsto u^{-1}\sigma(x)u$$

of the group G_k stabilizes P_Ω and N_Ω. Indeed $\sigma(\mathrm{N}_k) = u\mathrm{N}_k u^{-1}$, and

$$\theta(\mathrm{N}_\Omega) = u^{-1}\sigma(\mathrm{P}_\Omega \cap \mathrm{N}_k)u = \mathrm{P}_\Omega \cap u^{-1}\sigma(\mathrm{N}_k)u = \mathrm{N}_\Omega.$$

Note that the condition (3-1) implies that $\theta \circ \theta$ is conjugation by some element of N_Ω. As N_Ω is θ-stable, the map

$$(\sigma, g\mathrm{N}_\Omega) \mapsto u\theta(g\mathrm{N}_\Omega), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$, making ρ into a σ-equivariant bijection. Note that this action differs from the natural action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ (which obviously has fixed points).

3.3. Let Ω be a nonempty σ-stable subset of $\mathscr{B}$ contained in some apartment.

Proposition 3.4. *Assume that Ω contains a point of a chamber of $\mathscr{B}$. Then Ω is contained in some σ-stable apartment.*

Proof. We describe the quotient $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ as a projective limit of finite σ-sets. According to [Cartier 1979, § 1.2], Example (f), the group G_k is locally compact and totally disconnected. Therefore we can choose a decreasing filtration $(\mathrm{Q}_i)_{i \geqslant 0}$ of the open subgroup P_Ω of G_k satisfying the following properties:

(A) The intersection of the Q_i is reduced to $\{1\}$.

(B) For any $i \geqslant 0$, the subgroup Q_i is compact open and normal in P_Ω.

Lemma 3.5. *Consider the decreasing filtration of P_Ω formed by the subgroups $\mathrm{P}_{\Omega,i} = \mathrm{N}_\Omega\mathrm{Q}_i \cap \theta(\mathrm{N}_\Omega\mathrm{Q}_i)$, for $i \geqslant 0$.*

(1) *The intersection of the $\mathrm{P}_{\Omega,i}$ is reduced to N_Ω.*

(2) *For any $i \geqslant 0$, the subgroup $\mathrm{P}_{\Omega,i}$ is θ-stable and of finite index in P_Ω.*

Proof. As N_Ω is θ-stable, it is contained in the intersection of the $\mathrm{P}_{\Omega,i}$. Let g be in this intersection. For any $i \geqslant 0$, there exist $n_i \in \mathrm{N}_\Omega$ and $q_i \in \mathrm{Q}_i$ such that $g = n_i q_i$. Because of (A) above, q_i converges to 1. Therefore n_i converges to a limit contained in the closed subgroup N_Ω, and this limit is g. This proves (1).

Now recall that $\theta \circ \theta$ is conjugation by some element of N_Ω. This implies that $\mathrm{P}_{\Omega,i}$ is θ-stable. As $\mathrm{P}_{\Omega,i}$ is open in P_Ω and contains N_Ω, the quotient $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$ can be identified with the quotient of U_Ω, which is compact, by some open subgroup. This gives (2). □

Because of Lemma 3.5(2), the map

$$(\sigma, g\mathrm{P}_{\Omega,i}) \mapsto u\theta(g\mathrm{P}_{\Omega,i}), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on the finite quotient $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$, which gives us a projective system $(\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i})_{i\geqslant 0}$ of finite σ-sets. Since P_Ω is complete, and thanks to Lemma 3.5(1), the natural σ-equivariant map from $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ to the projective limit of the $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$ is bijective.

Lemma 3.6. *Let $(\mathrm{X}_i)_{i\geqslant 0}$ be a projective system of finite σ-sets. For all $i \geqslant 0$, assume the transition maps $\varphi_i : \mathrm{X}_{i+1} \to \mathrm{X}_i$ to be surjective and X_i to have odd cardinality. Then the projective limit X has a σ-fixed point.*

Proof. For each $i \geqslant 0$, the set X_i^σ of σ-fixed points of X_i is nonempty, since X_i has odd cardinality. This defines a projective system $(\mathrm{X}_i^\sigma)_{i\geqslant 0}$ whose transition maps may not be surjective. For each $i \geqslant 0$, let Y_i denote the intersection in X_i of the images of the X_{i+n}^σ, for $n \geqslant 0$. Then Y_i is nonempty, and the transition maps $\varphi_i : \mathrm{Y}_{i+1} \to \mathrm{Y}_i$ are surjective. Therefore, the projective limit $\mathrm{Y} = \mathrm{X}^\sigma \subseteq \mathrm{X}$ of the system $(\mathrm{Y}_i)_{i\geqslant 0}$ is nonempty. □

Let p denote the residue characteristic of k.

Lemma 3.7. *Let K be a normal subgroup of finite index in P_Ω containing N_Ω. Then the index of K in P_Ω is a power of p.*

Proof. Let S be the maximal k-split torus associated to $\mathscr{A}$, let Φ be the set of roots of G relative to S and let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. According to (2-5), the group P_Ω has the Iwahori decomposition

$$\mathrm{P}_\Omega = (\mathrm{U}_\Omega \cap \mathrm{U}^-) \cdot (\mathrm{U}_\Omega \cap \mathrm{U}^+) \cdot \mathrm{N}_\Omega.$$

That Ω contains a point of a chamber of $\mathscr{B}$ implies that the group N_Ω is reduced to $\mathrm{Ker}(\nu)$, hence normalizes the groups $\mathrm{V}^+ = \mathrm{U}_\Omega \cap \mathrm{U}^+$ and $\mathrm{V}^- = \mathrm{U}_\Omega \cap \mathrm{U}^-$. The index of K in P_Ω can be decomposed as

$$(\mathrm{P}_\Omega : \mathrm{K}) = (\mathrm{P}_\Omega : \mathrm{V}^+\mathrm{K}) \cdot (\mathrm{V}^+\mathrm{K} : \mathrm{K}).$$

On the one hand, the index

$$(\mathrm{V}^+\mathrm{K} : \mathrm{K}) = (\mathrm{V}^+ : \mathrm{V}^+ \cap \mathrm{K})$$

is a power of p, since V^+ is a pro-p-group. On the other hand, the index

$$(\mathrm{P}_\Omega : \mathrm{V}^+\mathrm{K}) = (\mathrm{V}^- : \mathrm{V}^- \cap \mathrm{V}^+\mathrm{K})$$

is a power of p, since V^- is a pro-p-group. The result follows. □

According to Lemma 3.7, the cardinality of each $P_\Omega/P_{\Omega,i}$, with $i \geqslant 0$, is odd (recall that p is different from 2). Proposition 3.4 follows from Lemma 3.6. □

We now prove the first main result of this section.

Proposition 3.8. *For any $x \in \mathscr{B}$, there exists a σ-stable apartment containing x.*

Proof. Let x be a point in $\mathscr{B}$, and let y be a point of a chamber of $\mathscr{B}$ whose closure contains x. The set $\Omega = \{y, \sigma(y)\}$ is a σ-stable subset of $\mathscr{B}$ satisfying the conditions of Proposition 3.4. Hence we get a σ-stable apartment of $\mathscr{B}$ containing y. Such an apartment contains the closure of the chamber of y. In particular, it contains x. □

3.9. Let S be a σ-stable maximal k-split torus, and let N and Z denote the normalizer and centralizer of S in G. Let $X = X(S)$ denote the set of all $g \in G_k$ such that $g^{-1}\sigma(g) \in N_k$, let $\mathscr{A}$ denote the σ-stable apartment corresponding to S and, given $x \in \mathscr{A}$, let P_x denote the subgroup P_Ω (see 2.11) with $\Omega = \{x\}$.

Proposition 3.10. X *is a finite union of* (H_k, Z_k)*-double cosets and* $G_k = XP_x$.

Proof. Let us fix a minimal parabolic k-subgroup P of G containing the torus S. According to Helminck and Wang [1993, Proposition 6.8], the map $g \mapsto H_k g P_k$ induces a bijection between the (H_k, Z_k)-double cosets in X and the (H_k, P_k)-double cosets in G_k. The first part of the proposition then follows from [Helminck and Wang 1993, Corollary 6.16].

Note that we have $g \in X$ if and only if $g \cdot \mathscr{A}$ is σ-stable. For $g \in G_k$, we set $x' = g \cdot x$. According to Proposition 3.8, there is a σ-stable apartment $\mathscr{A}'$ containing x'. Let $g' \in X$ be such that $\mathscr{A}' = g' \cdot \mathscr{A}$. According to Property (2) in 2.14, there is $n \in N_k$ such that we have $g'^{-1} g \cdot x = n \cdot x$. Hence we get $g \in XN_kP_x$. As $XN_k = X$, we obtain the expected result. □

4. Decomposition of $H_k \backslash G_k$

In all this section, we assume that G is k-split. Let H be an open k-subgroup of the fixed points group G^σ. Equivalently, H is a k-subgroup of G^σ containing $(G^\sigma)^\circ$.

4.1. If T is a σ-stable torus in G, we write T^+ for the neutral component of $T \cap H$ and T^- for the neutral component of the subgroup $\{t \in T \mid \sigma(t) = t^{-1}\}$. The torus T is the almost direct product of T^+ and T^-, that is $T = T^+T^-$ and the intersection $T^+ \cap T^-$ is finite [Borel 1991, xi].

Definition 4.2 [Helminck and Wang 1993, § 4.4]. A σ-stable torus T of G is said to be (σ, k)*-split* if it is k-split and if $T = T^-$.

By Proposition 10.3 of the same reference, two arbitrary maximal (σ, k)-split tori of G are G_k-conjugated.

4.3. Let $\mathscr{D}\mathrm{G}$ denote the derived subgroup of G, and recall that C denotes the connected center of G. This latter subgroup is a k-split torus of G.

Lemma 4.4. *Let* T *be a k-split torus of* G.

(1) *There is a k-subtorus* T' *of* C *such that the groups* $\mathrm{T}\cdot\mathscr{D}\mathrm{G}$ *and* $\mathrm{T}'\cdot\mathscr{D}\mathrm{G}$ *are equal.*

(2) *If* T *is* (σ, k)*-split, any* T' *satisfying* (1) *is* (σ, k)*-split.*

(3) *Assume that* $\mathscr{D}\mathrm{G}$ *is contained in* H *and* T *is* (σ, k)*-split. Then any* T' *satisfying* (1) *is* (σ, k)*-split and has the same dimension as* T.

Proof. We set $\tilde{\mathrm{G}} = \mathrm{G}/\mathscr{D}\mathrm{G}$ and, for any k-subgroup K of G, we write $\tilde{\mathrm{K}}$ for the image of K in $\tilde{\mathrm{G}}$. According to [Borel 1991, Proposition 14.2], the group G is the almost direct product of C and $\mathscr{D}\mathrm{G}$, which means that G is equal to the product $\mathrm{C}\cdot\mathscr{D}\mathrm{G}$ and that the intersection $\mathrm{C}\cap\mathscr{D}\mathrm{G}$ is finite. This implies that $\tilde{\mathrm{C}} = \tilde{\mathrm{G}}$. Let f denote the k-rational map $\mathrm{C}\to\tilde{\mathrm{C}}$. It is surjective with finite kernel. Hence $\tilde{\mathrm{G}}$ is a k-split torus, and we denote by $\tilde{\sigma}$ the involutive k-automorphism of $\tilde{\mathrm{G}}$ induced by σ. We now prove each conclusion claim in the lemma.

(1) By [Borel 1991, Proposition 8.2(c)], the neutral component of the inverse image $f^{-1}(\tilde{\mathrm{T}})$ is a k-split subtorus of C which we denote by T'. It has finite index in $f^{-1}(\tilde{\mathrm{T}})$. The image $f(\mathrm{T}')$ is then a subtorus of finite index in the connected group $\tilde{\mathrm{T}}$, so that $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$.

(2) Assume that T is (σ, k)-split, and let T' satisfy (1). Let us consider the map $t\mapsto t\sigma(t)$ from T' to itself. As $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$ is a $(\tilde{\sigma}, k)$-split torus, the image of this map is a connected k-subgroup contained in the kernel of f, which is finite.

(3) Assume that $\mathscr{D}\mathrm{G}$ is contained in H and T is (σ, k)-split. Then the map $\mathrm{T}\to\tilde{\mathrm{T}}$ has finite kernel, which implies that T and $\tilde{\mathrm{T}}$ have the same dimension. Now let T' satisfy (1). According to (2), such a torus is (σ, k)-split, and it has the same dimension as $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$. □

4.5. Let S be a σ-stable maximal (k-split) torus of G, let $\mathscr{A}$ be the apartment corresponding to S and let Φ be the set of roots of G relative to S. Let $x\in\mathscr{A}$ be a special point (see 2.8), and write U_x for U_Ω (see 2.11) with $\Omega = \{x\}$. Let $a\in\Phi$ be a σ-invariant root, which means that $a\circ\sigma = a$.

Lemma 4.6. *Assume that* $\mathrm{U}_{-a}(k)$ *is contained in* $\{g\in\mathrm{G}_k \mid \sigma(g) = g^{-1}\}$. *Then there are* $n\in\mathrm{N}_k$ *and* $c\in\mathrm{U}_x$ *such that* $n = c^{-1}\sigma(c)$ *and* $\nu(n)$ *is the affine reflection of* $\mathscr{A}$ *which let* x *invariant and whose linear part is* s_a.

Proof. We fix a base point in the apartment $\mathscr{A}$, so that it can be identified with the vector space V. For any $b \in \Phi$, this defines a filtration of the group $U_b(k)$ (see 2.9). For $u \in U_b(k) - \{1\}$, we denote by $\varphi_b(u)$ the greatest real number $r \in \mathbb{R}$ such that $u \in U_b(k)_r$. Let us choose $w \in U_{-a}(k) - \{1\}$ such that x is contained in the wall $\mathscr{H}_{-a,w}$. Thus $\nu(m(w))$ is the affine reflection of $\mathscr{A}$ which fixes x and whose linear part is s_a, and we can set

$$n = m(w) \in N_k.$$

Moreover $\theta(-a, w)$, which is the unique affine function from $\mathscr{A}$ to $\mathbb{R}$ whose linear part is $-a$ and whose vanishing hyperplane is $\mathscr{H}_{-a,w}$, vanishes on x. Therefore it is equal to

$$y \mapsto -a(y) + a(x),$$

which implies that $\varphi_{-a}(w) = a(x)$. According to B3 (see 2.12), it follows that w fixes x.

The group $U_{-a}(k)$ is isomorphic to the additive group of k. Thus, for $r \in \mathbb{R}$, the subgroup $U_{-a}(k)_r$ corresponds through this isomorphism to a nontrivial sub-$\mathcal{O}$-module of k, where $\mathcal{O}$ denotes the ring of integers of k [Landvogt 1995, Proposition 7.7]. Therefore, there is a unique element $v \in U_{-a}(k)$ such that $w = v^2$ and $\varphi_{-a}(v) = \varphi_{-a}(w)$, hence $v \in U_x$.

The map $U_a(k) \times U_a(k) \to G_k$ defined by $(u, u') \mapsto uwu'$ is injective and the intersection given by (2-4) consists of a single element, which is n. If we choose $u, u' \in U_a(k)$ such that $uwu' = n$, then the element

$$\sigma(u')^{-1} w \sigma(u)^{-1} = \sigma(n)^{-1}$$

is contained in the intersection (2-4). Hence $\sigma(n)^{-1}$ is equal to n, and the uniqueness property implies that $u' = \sigma(u)^{-1}$. Moreover, according to [Landvogt 1995, Lemma 7.4(ii)], the real numbers $\varphi_a(u)$ and $\varphi_a(\sigma(u))$ are both equal to $-\varphi_{-a}(w)$. This implies that u and $\sigma(u)$ are contained in U_x. Since v is σ-antiinvariant and $w = v^2$, we get the expected result by choosing $c = (uv)^{-1}$. □

Remark 4.7. Note that $\sigma(c) \in U_x$. Indeed we have $\sigma(v) = v^{-1} \in U_x$ and $\sigma(u) \in U_x$. Hence $n = c^{-1}\sigma(c) \in N_k \cap U_\Omega$, which is contained in N_Ω with $\Omega = \{x, \sigma(x)\}$.

Let $\mathscr{B}$ denote the building of G over k.

Proposition 4.8. *Let x be a special point of $\mathscr{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and such that* S^- *is a maximal (σ, k)-split torus of* G.

Remark 4.9. In 5.13, we give an example of a *nonsplit* k-group G such that Proposition 4.8 does not hold.

Proof. Let $\mathscr{A}$ be a σ-stable apartment containing x (see Proposition 3.8) and let S be the corresponding maximal k-split torus of G. Assume that $\mathscr{A}$ has been chosen such that the dimension of the (σ, k)-split torus S^- is maximal. If it is a maximal (σ, k)-split torus of G, then Proposition 4.8 is proved. Assume that this is not the case, and let A be a maximal (σ, k)-split torus of G containing S^-. The dimension of A is greater than $\dim S^-$ (if not, the containment $S^- \subseteq A$ would imply that $S^- = A$). Let G' be the neutral component of the centralizer of S^- in G. It is a k-split connected reductive subgroup of G containing S and A, which is naturally endowed with a nontrivial action of σ. Let C' denote the connected center of G'.

Lemma 4.10. *There is $a \in \Phi(G', S)$ such that the corresponding root subgroup U'_a is not contained in* H, *and such a root is σ-invariant.*

Proof. Assume that $U'_a \subseteq H$ for each root $a \in \Phi(G', S)$. Then the derived subgroup $\mathscr{D}G'$, which is generated by the U'_a for $a \in \Phi(G', S)$, is contained in H [Humphreys 1975, Theorem 27.5(e)]. According to Lemma 4.4(iii), there exists a (σ, k)-subtorus A' of C' such that $A \cdot \mathscr{D}G' = A' \cdot \mathscr{D}G'$ and $\dim(A) = \dim(A')$. The subgroup generated by C' and S is a k-torus of G'. As G' is k-split, S is a maximal torus of G', hence it contains C'. Therefore S^- contains A' which has the same dimension as A, and this dimension is greater than $\dim S^-$. This gives us a contradiction.

Now let a be a root in $\Phi(G', S)$ such that U'_a is not contained in H. The root a and its conjugate $a \circ \sigma$ coincide on S^+ and are both trivial on S^-. As S is the almost direct product of S^+ and S^- (see 4.1), they are equal. Therefore a is σ-invariant. This ends the proof of Lemma 4.10. □

Let $a \in \Phi(G', S)$ as in Lemma 4.10. If we think of a as a root in $\Phi(G, S)$, then U_a is σ-stable and is not contained in H. Moreover:

Lemma 4.11. $U_a(k)$ *is contained in* $\{g \in G_k \mid \sigma(g) = g^{-1}\}$.

Proof. As G is k-split, U_a is k-isomorphic to the additive group. Thus the action of σ on $U_a(k)$ corresponds to an involutive automorphism of the k-algebra $k[t]$. It has the form $t \mapsto \lambda t$ for some $\lambda \in k^\times$ with $\lambda^2 = 1$. As U_a is not contained in H, we have $\lambda = -1$. This gives us the expected result. □

According to Lemma 4.6, there are $n \in N_k$ and $c \in U_x$ such that $n = c^{-1}\sigma(c)$ and $\nu(n)$ is the affine reflection of $\mathscr{A}$ which let x invariant and whose linear part is s_a. For any $t \in S$, note that

$$\sigma(ctc^{-1}) = cn\sigma(t)n^{-1}c^{-1} = cs_a(\sigma(t))c^{-1}.$$

Let $\mathscr{A}'$ denote the apartment $c \cdot \mathscr{A}$ and let $\mathrm{S}' = {}^c\mathrm{S}$ be the corresponding maximal k-split torus of G. Then $\mathscr{A}'$ contains x and is σ-stable. Moreover, since the root a is trivial on S^- and s_a fixes the kernel of a pointwise, the conjugate ${}^c(\mathrm{S}^-)$ is a (σ, k)-split subtorus of S'. Thus S'^- has dimension not smaller than $\dim \mathrm{S}^-$.

Now let S_a denote the maximal k-split torus in the set of all $t \in \mathrm{S}$ such that $s_a(t) = t^{-1}$. Since a is σ-invariant, such a torus is σ-stable. It is also one-dimensional and its intersection with $\mathrm{Ker}(a)$ is finite. Therefore ${}^c\mathrm{S}_a$ is a nontrivial (σ, k)-split subtorus of S' which is not contained in ${}^c(\mathrm{S}^-)$. Thus the dimension of S'^-, which contains ${}^c(\mathrm{S}_a\mathrm{S}^-)$, is greater than $\dim \mathrm{S}^-$, which contradicts the maximality property of $\mathscr{A}$. This ends the proof of Proposition 4.8. □

4.12. Let A be a maximal (σ, k)-split torus of G, let S be a σ-stable maximal k-split torus of G containing A and let $\mathscr{A}$ denote the apartment corresponding to S. Let $\{\mathrm{A}^j \mid j \in \mathrm{J}\}$ be a set of representatives of the H_k-conjugacy classes of maximal (σ, k)-split tori in G. According to [Helminck and Wang 1993], the set J is finite. Let $x \in \mathscr{A}$ be a special point and write K for its stabilizer in G_k.

Theorem 4.13. *For $j \in \mathrm{J}$, let $y_j \in \mathrm{G}_k$ such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. We have*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. By Proposition 4.8, for any $g \in \mathrm{G}_k$, there is a σ-stable maximal k-split torus S' of G such that the apartment corresponding to it contains $g \cdot x$ and such that S'^- is a maximal (σ, k)-split torus of G. Let $j \in \mathrm{J}$ be such that S'^- is H_k-conjugate to A^j. According to Helminck and Helminck [1998, Lemma 2.2], there is $h \in \mathrm{H}_k$ such that $\mathrm{S}' = {}^{hy_j}\mathrm{S}$. Hence $g \cdot x$ is contained in $hy_j \cdot \mathscr{A}$. According to Property (2) in 2.14, there exists $n \in \mathrm{N}_k$ such that $g \cdot x = hy_j n \cdot x$. Therefore G_k is the union of the $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As x is special, we have $\mathrm{N}_k\mathrm{K} = \mathrm{S}_k\mathrm{K}$ and we get the expected result. □

4.14. In the case where G is not necessarily k-split, we have the following result. For each j, let $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ be the quotient of the normalizer of A^j in G_k by its centralizer, and likewise with G_k replaced by H_k. According to [Helminck and Wang 1993], the group $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ is the Weyl group of a root system. For $j \in \mathrm{J}$, let $\mathscr{N}_j \subseteq \mathrm{N}_{\mathrm{G}_k}(\mathrm{A}^j)$ be a set of representatives of

$$\mathrm{W}_{\mathrm{H}_k}(\mathrm{A}^j) \backslash \mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j),$$

and let $y_j \in \mathrm{G}_k$ be such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. Let P be a minimal parabolic k-subgroup of G containing S and such that $\mathrm{P} \cap \sigma(\mathrm{P})$ is a Levi component of P [Helminck and Wang 1993, § 4]. Let ϖ be a uniformizer of k, and write Λ for the lattice made of the images of ϖ by the various algebraic cocharacters of A and Λ^- for the subset of antidominant elements of Λ relative to P. Then one can derive from Proposition 3.10 the existence of a compact subset Q of G_k such that

$$G_k = \bigcup_{j\in J} \bigcup_{n\in\mathcal{N}_j} H_k n y_j \Lambda^- Q. \tag{4-1}$$

Benoist and Oh [2007] have obtained a similar decomposition of G_k, with a weaker condition on the base field k (they assume k to have odd characteristic).

Remark 4.15. In the split case, starting from Theorem 4.13, one can obtain a sharper result than the decomposition (4-1).

Let us mention that the question of the disjointness of the various components appearing in the decomposition (4-1) has been investigated in [Lagier 2008].

5. Examples

Let k be a nonarchimedean locally compact field of odd residue characteristic. Let $\mathcal{O}$ be its ring of integers and $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$.

5.1. We now consider the k-split reductive group $G = GL_n$, $n \geqslant 1$, endowed with the k-involution $\sigma : g \mapsto {}^t g^{-1}$, where ${}^t g$ denotes the transpose of g. We set $K = GL_n(\mathcal{O})$ and $H = G^\sigma$, and write S for the diagonal torus of G. This case has been explicitly investigated by Hironaka [1988] from a different point of view.

We start with the following lemma.

Lemma 5.2. *Let* V *be a finite dimensional k-vector space and* B *a symmetric bilinear form on* V. *Then any free $\mathcal{O}$-submodule of finite rank of* V *has a basis which is orthogonal relative to* B.

Proof. Let Λ be a free $\mathcal{O}$-submodule of finite rank of V. The proof goes by induction on the rank of Λ. If B is null, then the result is trivial. If not, we denote by B_Λ the restriction of B to $\Lambda \times \Lambda$. Its image is of the form $\mathfrak{p}^m$ for some integer $m \in \mathbb{Z}$. If ϖ is a uniformizer of k, then the form $B^0_\Lambda = \varpi^{-m} B_\Lambda$ has image $\mathcal{O}$ on $\Lambda \times \Lambda$. Therefore, it defines a nontrivial bilinear form

$$\bar{B}^0_\Lambda : \Lambda/\mathfrak{p}\Lambda \times \Lambda/\mathfrak{p}\Lambda \to \mathcal{O}/\mathfrak{p}.$$

Let $e \in \Lambda$ be a vector whose reduction modulo $\mathfrak{p}$ is not isotropic relative to $\bar{B}^0_\Lambda$, which means that $B^0_\Lambda(e, e)$ is a unit of $\mathcal{O}$. Then Λ is the direct sum of $\mathcal{O}e$ and $\Lambda \cap ke^\perp$, where $ke^\perp$ denotes the orthogonal of ke in V. Indeed, it follows from the decomposition

$$x = \frac{B(e,x)}{B(e,e)} e + \left(x - \frac{B(e,x)}{B(e,e)} e\right), \quad \text{for any } x \in \Lambda.$$

As $\Lambda \cap ke^{\perp}$ is a free $\mathcal{O}$-submodule of finite rank of V whose rank is smaller than the rank of Λ, we conclude by induction. □

We introduce the set Y of all $g \in \mathrm{G}_k$ such that ${}^t gg \in \mathrm{S}_k$. Using Lemma 5.2, we get the following decomposition of G_k.

Proposition 5.3. *We have* $\mathrm{G}_k = \mathrm{YK}$.

Proof. We make G_k act on the quotient G_k/K, which can be identified to the set of all $\mathcal{O}$-lattices (that is, cocompact free $\mathcal{O}$-submodules) of the k-vector space $\mathrm{V} = k^n$. Let B denote the symmetric bilinear form on V making the canonical basis of V into an orthonormal basis. According to Lemma 5.2, for any $g \in \mathrm{G}_k$, the $\mathcal{O}$-lattice Λ corresponding to the class $g\mathrm{K}$ has a basis which is orthogonal relative to B. This means that there exists $u \in \mathrm{K}$ such that the element $g' = gu^{-1} \in g\mathrm{K}$ maps the canonical basis of V to an orthogonal basis of Λ. Therefore we have $g' \in \mathrm{Y}$; thus $g \in \mathrm{YK}$. □

We now investigate the maximal (σ, k)-split tori of G. Note that S is a maximal (σ, k)-split torus of G.

Proposition 5.4. *The map* $g \mapsto {}^g\mathrm{S}$ *induces a bijection between* $(\mathrm{H}_k, \mathrm{N}_k)$*-double cosets of* Y *and* H_k*-conjugacy classes of maximal* (σ, k)*-split tori of* G.

Proof. One easily checks that this map is well defined and injective. For $g \in \mathrm{G}_k$, the conjugate ${}^g\mathrm{S}$ is a maximal (σ, k)-split torus of G if and only if $g^{-1}\sigma(g) \in \mathrm{S}_k$, which amounts to saying that $g \in \mathrm{Y}$ and proves surjectivity. □

Let $\mathcal{Q}$ denote the set of all equivalence classes of nondegenerate quadratic forms on k^n. For $a = \mathrm{diag}(a_1, \dots, a_n) \in \mathrm{S}_k$ we denote by Q_a the diagonal quadratic form $a_1\mathrm{X}_1^2 + \cdots + a_n\mathrm{X}_n^2$. Note that the map $a \mapsto \mathrm{Q}_a$ induces a surjective map from S_k to $\mathcal{Q}$.

We write H^0 and H^1 for the set of σ-fixed points and the first set of nonabelian cohomology of σ, respectively.

Proposition 5.5. (1) *The map* $g \mapsto {}^t gg$ *induces an injection* ι *from the set of* $(\mathrm{H}_k, \mathrm{N}_k)$*-double cosets of* Y *to* $\mathrm{H}^1(\mathrm{N}_k)$.

(2) *Given* $a \in \mathrm{S}_k$*, the class of* a *in* $\mathrm{H}^1(\mathrm{N}_k)$ *is in the image of* ι *if and only if* $\mathrm{Q}_a \sim \mathrm{X}_1^2 + \cdots + \mathrm{X}_n^2$.

Proof. We have an exact sequence

$$\mathrm{H}_k \to \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{G}_k),$$

where the map from $\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$ to $\mathrm{H}^1(\mathrm{N}_k)$ is induced by $g \mapsto {}^t gg$. As the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of Y is a subset of $\mathrm{H}_k\backslash\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$, we get the first assertion. To obtain the second one, it is enough to remark that $\mathrm{H}^1(\mathrm{G}_k)$ canonically identifies with $\mathcal{Q}$. □

Remark 5.6. Recall from [Serre 1970, IV.2.3] that for $a, b \in \mathrm{S}_k$, the quadratic forms $\mathrm{Q}_a, \mathrm{Q}_b$ are equivalent if and only if they have the same discriminant and the same Hasse invariant.

Proposition 5.7. *Let* $\{a^j \mid j \in \mathrm{J}\} \subseteq \mathrm{S}_k$ *form a set of representatives of* $\mathrm{Im}(\iota)$ *in* $\mathrm{H}^1(\mathrm{N}_k)$. *For* $j \in \mathrm{J}$, *we choose* $y_j \in \mathrm{Y}$ *such that* ${}^t y_j y_j = a^j$. *Then,*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. Propositions 5.3 and 5.4 imply that G_k is the union of the components $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As $\mathrm{N}_k \mathrm{K} = \mathrm{S}_k \mathrm{K}$, we get the expected result. □

Example 5.8. In the case where $n = 2$, we give an explicit description of $\mathrm{Im}(\iota)$. Let ϖ denote a uniformizer of $\mathcal{O}$ and $\xi \in \mathcal{O}^\times$ a nonsquare unit of $\mathcal{O}$, so that $\{1, \xi, \varpi, \xi\varpi\}$ is a set of representatives of $k^\times$ modulo $k^{\times 2}$. The set of elements of $k^\times$ which are represented by the quadratic form $\mathrm{Q}_1 = \mathrm{X}^2 + \mathrm{Y}^2$ depends on the image of p in $\mathbb{Z}/4\mathbb{Z}$. If $p \equiv 1 \bmod 4$, all elements of $k^\times$ are represented by Q_1. If $p \equiv 3 \bmod 4$, an element of $k^\times$ is represented by Q_1 if and only if its normalized valuation if even. We set

$$\mathrm{J} = \begin{cases} \{1, \xi, \varpi, \xi\varpi\} & \text{if } p \equiv 1 \bmod 4, \\ \{1, \xi\} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

For each $j \in \mathrm{J}$, set $a^j = \mathrm{diag}(j, j)$. Then the elements a^j form a set of representatives of $\mathrm{Im}(\iota)$ in $\mathrm{H}^1(\mathrm{N}_k)$.

5.9. We now consider the connected reductive k-group $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_n$, where k' is a quadratic extension of k, endowed with the involutive k-automorphism σ of G induced by the nontrivial element of $\mathrm{Gal}(k'/k)$. This case has been explicitly investigated by Offen [2004] when k'/k is unramified.

We set $\mathrm{H} = \mathrm{G}^\sigma$, so that we have $\mathrm{G}_k = \mathrm{GL}_n(k')$ and $\mathrm{H}_k = \mathrm{GL}_n(k)$. We denote by S the diagonal torus of G and by K the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}')$ of G_k, where $\mathcal{O}'$ denotes the ring of integers of k'. Note that S is σ-invariant.

As usual, N and Z denote the normalizer and centralizer of S in G. Let $\mathfrak{S}_n$ denote the group of permutation matrices in G_k, so that N_k is the semidirect product of $\mathfrak{S}_n$ by Z_k. Note that S_k (resp. Z_k) is the subgroup of all diagonal matrices of G_k with entries in k (resp. in k').

Lemma 5.10. $\mathrm{H}^1(\mathrm{N}_k)$ *can be identified with the set of conjugacy classes of elements of* $\mathfrak{S}_n$ *of order* 1 *or* 2.

Proof. According to Hilbert's Theorem 90, the group $\mathrm{H}^1(\mathrm{Z}_k)$ is trivial. Therefore we have an exact sequence

$$1 \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k). \tag{5-1}$$

As σ acts trivially on $\mathrm{N}_k/\mathrm{Z}_k \simeq \mathfrak{S}_n$, the set $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$ can be identified to the set of $\mathfrak{S}_n$-conjugacy classes of $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathfrak{S}_n)$, that is, to the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order 1 or 2. This proves that $\mathrm{H}^1(\mathrm{N}_k)$ can be naturally embedded in the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order $\leqslant 2$.

Now two elements $w, w' \in \mathfrak{S}_n$ define the same class in $\mathrm{H}^1(\mathrm{N}_k)$ if and only if they are conjugate in $\mathfrak{S}_n$, thus if and only if $w\mathrm{Z}_k$ and $w'\mathrm{Z}_k$ define the same class in $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$. Therefore (5-1) is a bijection. □

Proposition 5.11. (1) *The number of* H_k*-conjugacy classes of* σ*-stable maximal* k*-split tori in* G_k *is* $[n/2]+1$.

(2) *There is a unique* H_k*-conjugacy class of maximal* (σ, k)*-split tori in* G_k.

Proof. (1) Let X denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$. Then the map $g \mapsto {}^g\mathrm{S}$ defines an injective map from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of X to $\mathrm{H}^1(\mathrm{N}_k)$. Therefore we are reduced to proving that this map is surjective, and the first assertion will follow from Lemma 5.10. For $n = 2$, let τ denote the nontrivial element of $\mathfrak{S}_2$ and choose an element $a \in k'$ which is not in k. Then the element

$$u = \begin{pmatrix} a & \sigma(a) \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(k') \tag{5-2}$$

satisfies the relation $u^{-1}\sigma(u) = \tau$. For an arbitrary integer $n \geqslant 2$, let $w \in \mathfrak{S}_n$ have order $\leqslant 2$. Then there is an integer $0 \leqslant i \leqslant [n/2]$ such that w is conjugate to the element

$$\tau_i = \mathrm{diag}(\tau, \dots, \tau, 1, \dots, 1) \in \mathrm{GL}_n(k'),$$

where $\tau \in \mathrm{GL}_2(k')$ appears i times and $1 \in \mathrm{GL}_1(k')$ appears $n-2i$ times. Thus

$$u_i = \mathrm{diag}(u, \dots, u, 1, \dots, 1) \in \mathrm{GL}_n(k') \tag{5-3}$$

satisfies the relation $u_i^{-1}\sigma(u_i) = \tau_i$. Therefore any 1-cocycle in N_k is G_k-cohomologous to the neutral element $1 \in \mathrm{G}_k$, which proves the first assertion.

(2) For any $0 \leqslant i \leqslant [n/2]$, the dimension of the (σ, k)-split torus $({}^{u_i}\mathrm{S})^-$ is equal to i. According to (1), the map

$$\mathrm{H}_k g \mathrm{N}_k \mapsto \text{class of } g^{-1}\sigma(g) \text{ in } \mathrm{H}^1(\mathrm{N}_k)$$

is a bijection from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of X to $\mathrm{H}^1(\mathrm{N}_k)$, and the elements of this latter set are the classes of the τ_i for $0 \leqslant i \leqslant [n/2]$. This gives us the expected result. □

Proposition 5.12. *For* $0 \leqslant i \leqslant [n/2]$, *let* u_i *denote the element defined by* (5-2) *and* (5-3). *Then*

$$\mathrm{G}_k = \bigcup_{i=0}^{[n/2]} \mathrm{H}_k u_i \mathrm{Z}_k \mathrm{K}.$$

Proof. According to the proof of Proposition 5.11, the set X is the union of the double cosets $\mathrm{H}_k u_i \mathrm{N}_k$ with $0 \leqslant i \leqslant [n/2]$. The result then follows from Proposition 3.10 and from the fact that $\mathrm{N}_k\mathrm{K} = \mathrm{Z}_k\mathrm{K}$. □

5.13. We now give an example (due to Bertrand Lemaire) of a nonsplit k-group such that Proposition 4.8 does not hold. We set $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_2$, where k' is now a *ramified* quadratic extension of k. The k-involution σ is still induced by the nontrivial element of $\mathrm{Gal}(k'/k)$ and we set $\mathrm{H} = \mathrm{GL}_2$. Let $\mathcal{B}'$ (resp. $\mathcal{B}$) denote the building of G (resp. H) over k.

Bruhat and Tits [1984b] give a description of the faces of $\mathcal{B}$ in terms of hereditary $\mathcal{O}$-orders of $\mathrm{M}_2(k)$. More precisely, there is a bijective correspondence

$$\mathrm{F} \mapsto \mathcal{M}_\mathrm{F}$$

between the faces of $\mathcal{B}$ and the hereditary $\mathcal{O}$-orders of $\mathrm{M}_2(k)$, such that the stabilizer of F in $\mathrm{GL}_2(k)$ in the normalizer of $\mathcal{M}_\mathrm{F}$ in $\mathrm{GL}_2(k)$. For $x \in \mathcal{B}$, we will denote by $\mathcal{M}_x$ the hereditary order corresponding to the face of $\mathcal{B}$ which contains x. We have a similar correspondence between faces of $\mathcal{B}'$ and hereditary $\mathcal{O}'$-orders of $\mathrm{M}_2(k')$. Moreover, since k' is tamely ramified over k, there is a bijective correspondence j from the set $\mathcal{B}'^\sigma$ of σ-fixed points of $\mathcal{B}'$ to $\mathcal{B}$ such that, for any $x \in \mathcal{B}'^\sigma$, we have

$$\mathcal{M}_{j(x)} = \mathcal{M}_x \cap \mathrm{M}_2(k).$$

Let q denote the cardinality of the residue field of k. As k' is totally ramified over k, any vertex of $\mathcal{B}$ has exactly $q+1$ neighbors in $\mathcal{B}$, and likewise for $\mathcal{B}'$. Let x be a σ-invariant point of $\mathcal{B}'$. Recall that, according to Proposition 3.8, it is contained in a σ-stable apartment.

- If $j(x)$ is in a chamber of $\mathcal{B}$, then x has $q+1$ neighbors in $\mathcal{B}'$ but only two σ-fixed ones. Thus x has non-σ-fixed neighbors.
- If $j(x)$ is a vertex of $\mathcal{B}$, then x has $q+1$ neighbors in $\mathcal{B}'$ as in $\mathcal{B}$. Therefore any neighbor of x in $\mathcal{B}'$ is σ-invariant, which implies that any σ-stable apartment containing x is σ-invariant. For instance, this is the case of the vertex x corresponding to the $\mathcal{O}'$-order $\mathrm{M}_2(\mathcal{O}')$, as its image $j(x)$ corresponds to the maximal $\mathcal{O}$-order $\mathrm{M}_2(\mathcal{O}') \cap \mathrm{M}_2(k) = \mathrm{M}_2(\mathcal{O})$. For such a special point, Proposition 4.8 does not hold.

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