OSGOOD–HARTOGS-TYPE PROPERTIES OF POWER SERIES AND SMOOTH FUNCTIONS

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We study the convergence of a formal power series of two variables if its restrictions on curves belonging to a certain family are convergent. Also analyticity of a given $C^\infty$ function $f$ is proved when the restriction of $f$ on analytic curves belonging to some family is analytic. Our results generalize two known statements: a theorem of P. Lelong and the Bochnak–Siciak theorem. The questions we study can be regarded as problems of Osgood–Hartogs type.

Introduction

Hartogs’ theorem is a fundamental result in complex analysis: A function $f$ in $\mathbb{C}^n$, where $n > 1$, is holomorphic if it is holomorphic in each variable separately. That is, $f$ is holomorphic in $\mathbb{C}^n$ if for each axis it is holomorphic on every complex line parallel to this axis. In the last interpretation this statement leads to a number of questions described in an article by K. Spallek, P. Tworzewski, T. Winiarski [Spallek et al. 1990] in the following way: “Osgood–Hartogs-type problems ask for properties of ‘objects’ whose restrictions to certain ‘test-sets’ are well known”. The article has a number of examples of such problems. Here are two classical examples: a theorem of P. Lelong and one proved independently by J. Bochnak and J. Siciak.

Theorem [Lelong 1951]. A formal power series $g(x, y)$ converges in some neighborhood of the origin if there exists a set $E \subset \mathbb{C}$ of positive capacity such that, for each $s \in E$, the formal power series $g(x, sx)$ converges in some neighborhood of the origin (of a size possibly depending on $s$).

Theorem [Bochnak 1970; Siciak 1970]. Let $f \in C^\infty(D)$, where $D$ is a domain in $\mathbb{R}^n$ containing 0. Suppose $f$ is analytic on every line segment through 0. Then $f$ is analytic in a neighborhood of 0 (as a function of $n$ variables).

In many articles the same two “objects” are usually considered: power series and functions of several variables. The test sets in many cases form a family of linear

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subspaces of lower dimension. For example, articles by S. S. Abhyankar, T. T. Moh [1970], N. Levenberg and R. E. Molzon, [1988], R. Ree [1949], A. Sathaye [1976], M. A. Zorn [1947] and others consider the convergence of formal power series of several variables provided the restriction of such a series on each element of a sufficiently large family of linear subspaces is convergent. T. S. Neelon [2009; 2006] proved that a formal power series is convergent if its restrictions to certain families of curves or surfaces parametrized by polynomial maps are convergent. The articles [Bochnak 1970; Neelon 2004; 2009; Siciak 1970], among others, prove that a function of several variables is highly smooth (or even analytic) if it is smooth enough on each of a sufficiently large set of linear or algebraic curves (or surfaces of lower dimension). The publication by E. Bierstone, P. D. Milman, A. Parusiński [Bierstone et al. 1991] provides an interesting example of a noncontinuous function in \( \mathbb{R}^2 \) that is analytic on every analytic curve.

In this article we also consider both: power series with complex coefficients and functions in a neighborhood of the origin in \( \mathbb{R}^2 \). As test sets we consider separately two families. They are derived the following way. First consider a nonlinear analytic curve \( \Gamma = \{ x, \gamma(x) \} \), with \( \gamma(0) = 0 \). One family, \( \mathcal{Z}_1 \), is a set of dilations of \( \Gamma \): \( \mathcal{Z}_1 = \{ sx, s\gamma(x) \}, s \in \Lambda_1 \} \), where \( \Lambda_1 \subset \mathbb{R} \) is a closed subset of \( \mathbb{C} \) of positive capacity. The other family, \( \mathcal{Z}_2 \), consists of curves \( \Gamma_\theta \), each of which is a rotation of \( \Gamma \) about the origin by an angle \( \theta \in \Lambda_2 \), where \( \Lambda_2 \) is a subset of \([0, 2\pi]\) of positive capacity. If \( f \) is \( C^\infty \) and its restriction on every curve of \( \mathcal{Z}_1 \) can be extended as an analytic function in a neighborhood of that curve, then \( f \) is real analytic in a neighborhood of the origin in the region covered by the curves of \( \mathcal{Z}_1 \). The same is true regarding \( \mathcal{Z}_2 \). (For precise statements see Theorems 2.1 and 2.2).

We start however with two results related to power series. First we prove a generalization of P. Lelong’s theorem. Namely, if \( g(x, y) \) is a formal power series and \( h(x), h(0) = 0 \), is a convergent power series such that the inhomogeneous dilations \( g(s^\sigma x, s^\tau h(x)) \) are convergent for sufficiently many \( s \) (\( \sigma, \tau \) are fixed), then \( g(x, y) \) is convergent (for the precise statement see Theorem 1.1). Theorem 1.2 is devoted to a reverse claim: if \( h(x) \) is a formal power series and \( g(s^\sigma x, s^\tau h(x)) \) converges for sufficiently many \( s \), then \( h(x) \) is convergent.

The results in this paper do not carry over in a routine way to dimensions greater than two. We intend to study corresponding problems for higher dimensions in future work.

1. On the convergence of a power series in two variables

Let \( \mathbb{C}[[x_1, x_2, \ldots, x_n]] \) denote the set of (formal) power series

\[
g(x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n \geq 0} a_{k_1 \ldots k_n} x_1^{k_1} \cdots x_n^{k_n}
\]
in \( n \) variables with complex coefficients. Let \( g(0) = g(0, \ldots, 0) \) denote the coefficient \( a_0, \ldots, 0 \). A power series equals 0 if all of its coefficients \( a_{k_1 \ldots k_n} \) are equal to 0. A power series \( g \in \mathbb{C}[[x_1, x_2, \ldots, x_n]] \) is said to be convergent if there is a constant \( C = C_g \) such that \( |a_{k_1 \ldots k_n}| \leq C^{k_1 + \cdots + k_n} \) for all \((k_1, \ldots, k_n) \neq (0, \ldots, 0)\). If \( g \) is convergent, then it represents a holomorphic function in some neighborhood of 0 in \( \mathbb{C}^n \). If \( g \in \mathbb{C}[[x_1, x_2, \ldots, x_n]] \) and \( s \in \mathbb{C}^n \), then \( g_s(t) := g(s_1 t, \ldots, s_n t) \) is well defined and belongs to \( \mathbb{C}[t] \). By [Zorn 1947], \( g \) is convergent if and only if \( g_s(t) \) is convergent for each \( s \in \mathbb{C}^n \). The partial derivatives of a power series are well defined even when it is divergent (not convergent). For example, if \( g \in \mathbb{C}[[x, y]] \) and if \( g = \sum a_{ij} x^i y^j \), then
\[
 g' = \frac{\partial g}{\partial y} = \sum j a_{ij} x^i y^{j-1}.
\]
Thus \( g' \neq 0 \) simply means that \( g \notin \mathbb{C}[[x]] \). If \( g \in \mathbb{C}[[x, y]] \), and if \( h \in \mathbb{C}[[x]] \) with \( h(0) = 0 \), then \( g(x, h(x)) \) is a well-defined element of \( \mathbb{C}[[x]] \).

As mentioned above, a lot of work has been done on the convergence of a power series with the assumption that the series is convergent after restriction to sufficiently many subspaces; see [Abhyankar and Moh 1970; Levenberg and Molzon 1988; Lelong 1951; Siciak 1970; 1982].

We consider substitution of a power series \( y = h(x) \) into an inhomogeneous dilation \( g(s^\sigma x, s^\tau y) \) of a series \( g(x, y) \), where \( \sigma, \tau \) are integers.

Let
\[
 Q := \{(\sigma, \tau) : \sigma, \tau \in \mathbb{Z}, (\sigma, \tau) \neq (0, 0)\}.
\]
Let \( \text{cap} E \) denote the (logarithmic) capacity of a closed set \( E \) in the complex plane. We now present our two main theorems.

**Theorem 1.1.** Let \( g \in \mathbb{C}[[x, y]] \) be a power series of two variables \( x, y \), let \( h \in \mathbb{C}[[x]] \) be a nonzero convergent power series with \( h(0) = 0 \), let \( E \) be a closed subset of \( \mathbb{C} \setminus \{0\} \) with \( \text{cap} E > 0 \), and let \( (\sigma, \tau) \) be a pair in the set \( Q \). Assume, in case \( \sigma \tau > 0 \), that \( h(x) \) is not a monomial of the form \( b_k x^k \) with \( \sigma k - \tau = 0 \). Suppose that \( g(s^\sigma x, s^\tau h(x)) \) is convergent for each \( s \in E \). Then \( g \) is convergent.

**Theorem 1.2.** Let \( g \in \mathbb{C}[[x, y]] \) be a power series with \( g' \neq 0 \), let \( h \in \mathbb{C}[[x]] \) be a nonzero power series with \( h(0) = 0 \), let \( E \) be a closed subset of \( \mathbb{C} \setminus \{0\} \) with \( \text{cap} E > 0 \), and let \( (\sigma, \tau) \) be a pair in the set \( Q \) with \( \sigma \tau > 0 \). Suppose that \( g(s^\sigma x, s^\tau h(x)) \) is convergent for each \( s \in E \). Then \( h \) is convergent.

The examples in Section 3 show that if any condition in these two theorems is dispensed with, the resulting statement is false. We now prove some auxiliary results.

The following theorem is a consequence of a result by B. Malgrange [1966]. We present an independent short proof.
**Theorem 1.3.** Let \( g \in \mathbb{C}[[x_1, \ldots, x_n, y]] \) with \( g'_y \neq 0 \), and let \( h \in \mathbb{C}[[x_1, \ldots, x_n]] \) with \( h(0) = 0 \). Suppose that \( g \) and \( g(x_1, \ldots, x_n, h(x_1, \ldots, x_n)) \) are convergent. Then \( h \) must be convergent.

**Proof.** Let \( f \in \mathbb{C}[[x_1, \ldots, x_n, y]] \) be defined by

\[
f(x_1, \ldots, x_n, y) = g(x_1, \ldots, x_n, y) - g(x_1, \ldots, x_n, h(x_1, \ldots, x_n)).
\]

Then \( f \) is convergent and \( f(x_1, \ldots, x_n, h(x_1, \ldots, x_n)) = 0 \). Fix \( s = (s_1, \ldots, s_n) \in \mathbb{C}^n \). Let \( f_s(t, y) \in \mathbb{C}[t, y] \) be defined by \( f_s(t, y) = f(s_1t, \ldots, s_nt, y) \). Then \( f_s(t, y) \) is convergent and \( f_s(t, h_s(t)) = 0 \). By the Weierstrass preparation theorem (see [Griffiths and Harris 1978, p. 8], for example), there is a nonnegative integer \( k \) such that \( f_s(t, y) = t^k P(t, y)Q(t, y) \), where \( P(t, y) = y^m + a_1(t)y^{m-1} + \cdots + a_m(t) \) is a polynomial in \( y \) with coefficients being convergent power series in \( t \), and \( Q(t, y) \) is a convergent power series with \( Q(0, 0) \neq 0 \). Hence \( P(t, h_s(t)) = 0 \). By [Fuks 1963, Theorem 4.12, p. 73] there is a positive integer \( r \) such that \( P(t^r, y) \) splits into linear factors in \( y \):

\[
P(t^r, y) = (y - u_1(t)) \cdots (y - u_m(t)),
\]

where the \( u_j(t) \) are convergent power series. Thus

\[
0 = P(t^r, h_s(t^r)) = (h_s(t^r) - u_1(t)) \cdots (h_s(t^r) - u_m(t)).
\]

It follows that \( h_s(t^r) = u_j(t) \) for some \( j \). Therefore \( h_s(t) \) is convergent. Since \( h_s(t) \) is convergent for each \( s \in \mathbb{C}^n \), the series \( h(x_1, \ldots, x_n) \) must be convergent.

Let \( E \) be a closed bounded set in the complex plane. The **transfinite diameter** of \( E \) is defined as

\[
d_\infty(E) = \lim_n \left( \max \left\{ \prod_{i<j} |z_i - z_j|^{2/(n(n-1))} : z_1, \ldots, z_n \in E \right\} \right).
\]

For a probability measure \( \mu \) on the compact set \( E \), the **logarithmic potential** of \( \mu \) is

\[
p_\mu(z) = \lim_{N \to \infty} \int \min\left( N, \log \frac{1}{|z - \xi|} \right) d\mu(\xi),
\]

and the **capacity** of \( E \) is defined by

\[
\text{cap } E = \exp(- \min_{\mu(E) = 1} \sup_{z \in \mathbb{C}} p_\mu(z)).
\]

It turns out that \( d_\infty(E) = \text{cap } E \) [Ahlfors 1973, pp. 23–28]. It follows from the definition of the transfinite diameter that, for \( E_1 \supset E_2 \supset \cdots \),

\[
E = \bigcap E_n \implies \text{cap } E = \lim(\text{cap } E_n).
\]
and from the definition of the capacity that, if \( E_1 \subset E_2 \subset \cdots \),

\[
E = \bigcup E_n \implies \text{cap } E = \lim \text{cap } E_n.
\]

If \( E \) is a closed set, its capacity can be defined by

\[
\text{cap } E = \lim_n \text{cap} (E \cap \{|x| \leq n\}).
\]

**Lemma 1.4** (Bernstein inequality). Let \( E \) be a compact set in the complex plane with \( \text{cap } E > 0 \). Then there exists a positive constant \( C = C_E \), depending only on \( E \), such that for each positive integer \( n \) and each polynomial \( P(z) = \sum a_k z^k \in \mathbb{C}[z] \) of degree \( n \), each coefficient \( a_k \), \( 0 \leq k \leq n \), of \( P(z) \) satisfies

\[
|a_k| \leq C^n \max_{z \in E} |P(z)|.
\]

Proposition 4.6 in [Neelon 2009] can be used to prove this statement. Also (we thank Nessim Sibony for pointing this out to us) this lemma follows from considerations in [Sibony 1985]. We present here an independent short proof.

**Proof.** Without loss of generality we assume that \( \max_{z \in E} |P(z)| = 1 \). Let \( \Omega \) be the unbounded component of the complement of \( E \) in \( \mathbb{C} \). It is known that \( \Omega \) has a Green’s function with a pole at \( \infty \) [Ahlfors 1966; 1973, pp. 25–27]. The Green’s function is harmonic in \( \Omega \), 0 on \( \partial \Omega \), and its asymptotic behavior at \( \infty \) is

\[
u(z) = \log |z| - \log \alpha + o(1),
\]

where \( \alpha := \text{cap } E \). On applying the maximum principle to the subharmonic function \( \log |P(z)| - (n+\epsilon)\nu(z) \), we obtain \( |P(z)| \leq e^{\nu(z)} \) for \( z \in \Omega \). Choose an \( R > 1 \) so that \( E \subset \{|z| < R\} \). Set \( C = \max_{|z|=R} e^{\nu(z)} \). Then \( |P(z)| \leq C^n \) if \( |z| = R \), and

\[
|a_k| = \left| \frac{1}{2\pi i} \int_{|z|=R} \frac{P(z)}{z^{k+1}} \, dz \right| \leq R^{-k} \max_{|z|=R} |P(z)| \leq C^n.
\]

This proves the lemma.

**Proof of Theorem 1.1.** We assume that \( a_{00} = g(0, 0) = 0 \), that \( E \) is bounded, that \( \gcd(\sigma, \tau) = 1 \), that \( \sigma \geq 0 \), and, in case \( \sigma = 0 \), that \( \tau = -1 \). This causes no loss of generality. Indeed, if \( E \) is unbounded, set \( E_n = \{s \in E : n \geq |s| \geq 1/n\} \). Since \( \lim \text{cap } E_n = \text{cap } E > 0 \), the set \( E_n \) has positive capacity when \( n \) is sufficiently large. On replacing \( E \) by \( E_n \), we obtain that \( 0 \notin E \) and \( E \) is bounded. If \( d := \gcd(\sigma, \tau) > 1 \), we can replace \( (\sigma, \tau) \) by \( (\sigma/d, \tau/d) \), and \( E \) by the set \( \{s \in \mathbb{C} : s^d \in E\} \). Finally, if \( \sigma < 0 \), or if \( (\sigma, \tau) = (0, 1) \), we can replace \( (\sigma, \tau) \) by \( (-\sigma, -\tau) \), and \( E \) by \( \{s \in \mathbb{C} : s^{-1} \in E\} \).

Let

\[
h(x) = \sum_{i=1}^{\infty} b_i x^i.
\]
Then
\[ h(x)^j = \sum_{k=j}^{\infty} c_{jk} x^k, \]
where
\[ c_{jk} = \sum_{l_1 + \cdots + l_j = k} b_{l_1} \cdots b_{l_j}. \]

Note that \( c_{jk} = 0 \) for \( k < j \). Hence
\[ g(s^\sigma x, s^\tau h(x)) = \sum_{i,j,k} a_{ij} c_{jk} s^{\sigma i + \tau j} x^{i+k} = \sum_{p=1}^{\infty} \left( \sum_{q=-\tau^-}^{(\sigma + \tau^+)} d_{pq} s^q \right) x^p, \]
where \( \tau^+ = \max(0, \tau) \), \( \tau^- = -\min(0, \tau) \), and
\[ d_{pq} = \sum_{\sigma i + \tau j = q} a_{ij} c_{j,p-i}. \]

For each \( p \geq 1 \) and each \( q \in \mathbb{Z} \), the sum (2) contains only a finite number of nonzero terms. Let \( u_p(s) = \sum_q d_{pq} s^q \). Then \( s^{\tau^-} u_p(s) \) is a polynomial in \( s \) of degree at most \( (\sigma + |\tau|) p \), and \( g(s^\sigma x, s^\tau h(x)) = \sum u_p(s) x^p \). For \( s \in E \), since \( g(s^\sigma x, s^\tau h(x)) \) is convergent, its coefficients \( u_p(s) \) satisfy \( |u_p(s)| \leq C^p \) for some positive constant \( C \), possibly depending on \( s \), and \( p = 1, 2, \ldots \). Set, for \( n = 1, 2, \ldots \),
\[ E_n = \{ s \in E : |u_p(s)| \leq n^p \text{ for all } p > 0 \}. \]
The sequence \((E_n)\) is an increasing sequence of closed sets. Since \( \lim \sup \cap E_n = \text{cap} E > 0 \), the set \( E_n \) has positive capacity for some \( n \). On replacing \( E \) by \( E_n \), we obtain \( |u_p(s)| \leq n^p \) for \( s \in E \) and \( p = 1, 2, \ldots \). The polynomial \( s^{\tau^-} u_p(s) \) is of degree at most \( (\sigma + |\tau|) p \), and satisfies
\[ |s^{\tau^-} u_p(s)| \leq M^{\tau^-} n^p, \quad s \in E, \]
where \( M = \max_E |s| \). By Lemma 1.4, the coefficients of the above mentioned polynomial satisfy \( |d_{pq}| \leq C_E^{(\sigma + |\tau|) p} M^{\tau^-} n^p \), where \( C_E \) is the constant in Lemma 1.4, depending only on \( E \). Set \( C = C_E^{\sigma + |\tau|} M^{\tau^-} n \). Then
\[ |d_{pq}| \leq C^p. \]

Let
\[ g_q(x, y) = \sum_{\sigma i + \tau j = q} a_{ij} x^i y^j, \]
and let \( \phi_q(x) = g_q(x, h(x)) \), for \( q \in \mathbb{Z} \). Then \( g_q \in \mathbb{C}[[x, y]] \) in general, and it is a polynomial when \( \sigma, \tau > 0 \). It is straightforward to verify that

\[
\phi_q(x) = g_q(x, h(x)) = \sum_{p=1}^{\infty} d_{pq} x^p.
\]

The series \( \phi_q(x) \) is convergent because of (3). Choose a positive number \( r < 1/C \), where \( C \) is the constant in (3), so that \( h(x) \) converges in a neighborhood of the closed ball \( \{ x \in \mathbb{C} : |x| \leq r \} \) and \( h(x) \neq 0 \) when \( 0 < |x| \leq r \). Let \( m = \min_{|x|=r} |h(x)| \). Then \( m > 0 \). For \( x \in \mathbb{C} \), \( |x| \leq r \),

\[
|\phi_q(x)| \leq \sum |d_{pq}| |x|^p \leq \sum (Cr)^p = \frac{1}{1 - Cr}.
\]

We now consider two cases, depending on whether \( \sigma \tau \) is positive.

Case (i): \( \sigma > 0, \tau > 0 \). Let

\[
\Omega_q = \{(i, j) : i, j \in \mathbb{Z}, i, j \geq 0, \sigma i + \tau j = q \}.
\]

Let \( \omega_q \) be the cardinality of \( \Omega_q \). It is clear that \( \omega_q \leq q + 1 \). Fix a \( q \geq 1 \) so that \( \omega_q > 0 \). Let \( (\lambda, \mu) \) be the element of \( \Omega_q \) so that \( \mu \) is the minimum. Then

\[
\Omega_q = \{(\lambda - k \tau, \mu + k \sigma) : k = 0, 1, \ldots, \omega_q - 1 \},
\]

and

\[
g_q(x, y) = x^\lambda y^\mu \sum_{k=0}^{\omega_q-1} a_{\lambda-k \tau, \mu+k \sigma} (x^{-\tau} y^\sigma)^k.
\]

Let

\[
\psi_q(t) = \sum_{k=0}^{\omega_q-1} a_{\lambda-k \tau, \mu+k \sigma} t^k,
\]

so that \( g_q(x, y) = x^\lambda y^\mu \psi_q(x^{-\tau} y^\sigma) \), and

\[
\psi_q(x^{-\tau} h(x)^\sigma) = x^{-\lambda} h(x)^{-\mu} \phi_q(x).
\]

Let \( u(x) = x^{-\tau} h(x)^\sigma \), \( S = \{ x \in \mathbb{C} : |x| = r \} \), and \( F = u(S) \). Since \( h(x) \) is not a monomial of the form \( b_\kappa x^k \) with \( \sigma k - \tau = 0 \), the function \( u(x) \) is a nonconstant meromorphic function, hence \( F \) has positive capacity. For \( t = x^{-\tau} h(x)^\sigma \in F \), we obtain, by (7), that

\[
|\psi_q(t)| \leq \frac{r^{-\lambda} m^{-\mu}}{1 - Cr} \leq \frac{(1 + r^{-1} + m^{-1})^{\lambda + \mu}}{1 - Cr}.
\]

The summand 1 in the right-hand side of the above inequality is included to ensure that the numerator is greater than 1 as needed later. Hence \( |\psi_q(t)| \leq L^q \) on \( F \),
Thus for $g$:

\[ |a_{ij}| \leq L^q C_F^{\alpha_q} \leq (L + C_F)^{2q} \leq (L + C_F)^{2(\sigma + \tau)(i+j)}, \]

or $|a_{ij}| \leq K^{i+j}$, where $K = (L + C_F)^{2(\sigma + \tau)}$. The number $K$ does not depend on $q$. It follows that

\[ |a_{ij}| \leq K^{i+j}, \text{ if } \sigma i + \tau j \geq 1. \]

This proves that $g$ is convergent.

**Case (ii)**: $\sigma \geq 0, \tau \leq 0$. In this case the set $\Omega_q$ in (6) can be written as

\[ \Omega_q = \{(\lambda - k\tau, \mu + k\sigma) : k = 0, 1, 2, \ldots \}, \]

where $(\lambda, \mu)$ is the element in $\Omega_q$ with least value of $\mu$ when $\sigma > 0$, and $(\lambda, \mu) = (0, -q)$ when $(\sigma, \tau) = (0, -1)$. Let

\[ \psi_q(t) = \sum_{k=0}^{\infty} a_{\lambda+k|\tau|, \mu+k\sigma} t^k. \]

Then $g_q(x, y) = x^\lambda y^\mu \psi_q(x|\tau|, y^\sigma)$. The formal power series $\psi_q(t)$ satisfies $\phi_q(x) = x^\lambda h(x)^\mu \psi_q(x|\tau|, h(x)^\sigma)$. Since $x^\lambda h(x)^\mu$ and $\phi_q(x)$ are convergent, the series

\[ \alpha(x) := \psi_q(x|\tau|, h(x)^\sigma) \]

has to be convergent. Write $x|\tau| h(x)^\sigma = cx^v + \cdots$, $c \neq 0$. There is a power series $\beta(x)$, also convergent in a neighborhood of $\{ |x| \leq r \}$, such that $x^\lambda h(x)^\sigma = \beta(x)^v$. Reducing $r$ if necessary, we assume that $\beta(x)$ is univalent in a neighborhood of $\{ |x| \leq r \}$. Note that the reduction in the value of $r$ is independent of $q$. The set $\{ \beta(x) : |x| < r \}$ contains an open disc $\{ z \in \mathbb{C} : |z| < \delta \}$. The series $\beta(x)$ has an inverse $\gamma(z)$, convergent in $\{ z \in \mathbb{C} : |z| < \delta \}$, such that $\gamma(\beta(x)) = x$ and $\gamma(\beta(x)) = z$. Now $\psi_q(z^v)$ is convergent in $\{ |z| < \delta \}$, so $\psi_q(t)$ is convergent in $\{ |t| < \delta^v \}$. Let $t \in \mathbb{C}$ with $|t| < \delta^v$. Then $t = z^v$ for some $z$ with $|z| < \delta$, and $z = \beta(x)$ for some $x$ with $|x| < r$. Hence

\[ |\psi_q(t)| = |\psi_q(\beta(x)^v)| = |\alpha(x)| \leq \max_{|x|=r} |\alpha(x)|. \]

Thus

\[ \sup_{|t| < \delta^v} |\psi_q(t)| \leq \max_{|x|=r} \left| \frac{\phi_q(x)}{x^\lambda h(x)^\mu} \right| \leq \frac{r^{-\lambda} m^{-\mu}}{1 - Cr}. \]
By the Cauchy estimates, the coefficients of \( \psi_q \) satisfy
\[
|a_{\lambda+k|\tau|, \mu+k\sigma}| \leq \frac{r^{-\lambda}m^{-\mu}}{1-Cr}\delta^{-kv} \leq \frac{(1+r^{-1}+m^{-1}+\delta^{-v})^{\lambda+\mu+k}}{1-Cr}.
\]

The summand 1 in the right-hand side of the above inequality is included to ensure that the numerator is greater than 1 as needed later. It follows that, for \((i, j) \in \Omega_q\),
\[
|a_{ij}| \leq \left(\frac{1+r^{-1}+m^{-1}+\delta^{-v}}{1-Cr}\right)^{i+j}.
\]

The number \( K := (1+r^{-1}+m^{-1}+\delta^{-v})/(1-Cr) \) does not depend on \( q \). Therefore, \(|a_{ij}| \leq K^{i+j} \) for all \((i, j)\). This proves that \( g \) is convergent. \( \square \)

Proof of Theorem 1.2. This proof and the proof of Theorem 1.1 share the discussion through Equation (5). Note that the convergence of \( h \) has not been used in the derivation of (5). We define polynomials \( g_q(x, y) \) by (4). Then \( g_q(x, h(x)) \) are convergent by (3) and (5). Since \( g_q'(x, y) \neq 0 \), \( \partial g_q/\partial y \neq 0 \) for some \( q \). It follows from Theorem 1.3 that \( h(x) \) is convergent. \( \square \)

For \( h \in \mathbb{C}[x] \) with \( h(0) = 0 \), let \( h_s(x) = s^{-1}h(sx) \).

**Corollary 1.5.** Let \( g \in \mathbb{C}[x, y] \) be a power series, let \( h \in \mathbb{C}[x] \) be a nonzero and nonlinear power series with \( h(0) = 0 \), and let \( E \) be a closed subset of \( \mathbb{R} \setminus \{0\} \) with cap \( E > 0 \). Suppose that \( g(x, h_s(x)) \) is convergent for each \( s \in E \). Then \( g \) is convergent.

**Proof.** If \( g_y' = 0 \) then the statement holds. Suppose \( g_y' \neq 0 \). For \( s \neq 0 \), \( g(x, h_s(x)) \) is convergent if and only if \( g(s^{-1}x, h_s(s^{-1}x)) = g(s^{-1}x, s^{-1}h(x)) \) is convergent. By Theorem 1.2, \( h \) is convergent. Then \( g \) is convergent by Theorem 1.1. \( \square \)

For \( f \in \mathbb{C}[x, y] \) and \( \theta \in [0, 2\pi] \), write
\[
f_\theta(x, y) = f(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta).
\]

**Theorem 1.6.** Let \( f \in \mathbb{C}[x, y] \) be a power series, let \( h \in \mathbb{C}[x] \) be a convergent power series with \( h(0) = 0 \), and let \( E \) be a closed subset of \([0, 2\pi]\) with cap \( E > 0 \). Suppose that \( f_\theta(x, h(x)) \) is convergent for each \( \theta \in E \). Then \( f \) is convergent.

**Proof.** Let \( g(x, y) = f((x+y)/2, -i(x-y)/2) \). Then \( f(x, y) = g(x+iy, x-iy) \) and \( f_\theta(x, y) = g(e^{i\theta}(x+iy), e^{-i\theta}(x-iy)) \). Let \( \phi_\theta(x) = f_\theta(x, h(x)) = g(e^{i\theta}(x + ih(x)), e^{-i\theta}(x - ih(x))) \). Then \( \phi_\theta(x) \) is convergent for \( \theta \in E \). The \( x \) terms of the two series \( x \pm ih(x) \) cannot both be zero. Say, the \( x \) term of \( x + ih(x) \) is nonzero. So \( x + ih(x) \) has an inverse \( \psi(x) \) which is a convergent power series such that \( \psi(x) + ih(\psi(x)) = x \). Set \( \psi(x) = e^{i\theta} \omega(x) \). Then \( \phi_\theta(\psi(x)) = g(e^{i\theta}x, e^{-i\theta}\omega(x)) \) is convergent for \( \theta \in E \). It follows that \( g(sx, s^{-1}\omega(x)) \) is convergent for each \( s \) in the set \( \{e^{i\theta} : \theta \in E\} \), which has positive capacity. By Theorem 1.1, \( g \) is convergent. Therefore \( f \) is convergent. \( \square \)
2. Analytic functions in $\mathbb{R}^2$

Suppose that $f(x, y), \phi(x), q(x)$ are $C^\infty$ functions defined near the origin with $\phi(0) = 0$. Let $\hat{f}, \hat{\phi}, \hat{q}$ denote the Taylor series at 0 of those functions. Then $\hat{f}$ lies in $\mathbb{C}[x, y]$ and $\hat{\phi}, \hat{q}$ lie in $\mathbb{C}[x]$. By the chain rule, $f(x, \phi(x)) = q(x)$ implies $\hat{f}(x, \hat{\phi}(x)) = \hat{q}(x)$. We consider here complex-valued analytic functions of real variables. If $I$ is an interval and if $\Gamma = \{(t, \gamma(t)) : t \in I\}$ is a curve, the dilation by $s$ of $\Gamma$ is

$$\Gamma_s = \{(st, s\gamma(t))\} = \{(t, \gamma_{1/s}(t))\}, \quad \gamma_s(t) = s^{-1}\gamma(st).$$

**Theorem 2.1.** Let $f$ be a $C^\infty$ function defined in an open set $\Omega \subset \mathbb{R}^2$ containing the origin, let $\Gamma = \{(t, \phi(t))\}$ be a nonlinear analytic curve in $\mathbb{R}^2$ passing through or ending at the origin, and let $E$ be a closed subset of $\mathbb{R} \setminus \{0\}$ of positive capacity. Suppose that for each $s \in E$, there is a real analytic function $F_s$ defined in a neighborhood $Q_s$ of $\Gamma_s \cap \Omega$ in $\mathbb{R}^2$ such that $f$ and $F_s$ coincide on $\Gamma_s \cap \Omega$. Then there is a neighborhood $U$ of the origin, and a real analytic function $F$ defined on $U$ that coincides with $f$ on $U \cap \Lambda$, where $\Lambda := \bigcup_{s \in E} \Gamma_s$.

**Proof.** Without loss of generality we assume that $\phi(0) = 0$. Since $f$ and $F_s$ coincide on $\Gamma_s$, we have

$$f(x, \phi_{1/s}(x)) = F_s(x, \phi_{1/s}(x)).$$

Let $g, h$ denote the Taylor series of $f, \phi$ respectively. Then (9) implies

$$g(x, h_{1/s}(x)) = F_s(x, h_{1/s}(x)).$$

Hence $g(x, h_{1/s}(x))$ is convergent for $s \in E$. By Corollary 1.5, $g$ is convergent. Thus $g$ represents a real analytic function $F$ in some neighborhood $U$ of the origin that satisfies $F(x, h_{1/s}(x)) = F_s(x, h_{1/s}(x))$. It follows that the real analytic function $F$ coincides with $f$ on $U \cap \Lambda$. $\square$

Note that $f$ does not need to be analytic in a neighborhood of the origin.

If $\Gamma = \{(t, \phi(t)) : t \in I\}$ is a curve, its rotation by $\theta$ is

$$\Gamma_{\theta} = \{(t \cos \theta + \phi(t) \sin \theta, -t \sin \theta + \phi(t) \cos \theta) : t \in I\}.$$

**Theorem 2.2.** Let $f$ be a $C^\infty$ function defined in an open set $\Omega \subset \mathbb{R}^2$ containing the origin, let $\Gamma = \{(t, \phi(t))\}$ be an analytic curve in $\mathbb{R}^2$ passing through or ending at the origin, and let $E$ be a closed subset of $[0, 2\pi]$ of positive capacity. Suppose that for each $\theta \in E$, there is a real analytic function $F_{\theta}$ defined in a neighborhood $Q_{\theta}$ of $\Gamma_{\theta} \cap \Omega$ in $\mathbb{R}^2$ such that $f$ and $F_{\theta}$ coincide on $\Gamma_{\theta} \cap \Omega$. Then there is an analytic function $F$ defined in some neighborhood $U$ of the origin that coincides with $f$ on $U \cap \Lambda$, where $\Lambda := \bigcup_{\theta \in E} \Gamma_{\theta}$.
Proof. The proof is similar to that of Theorem 2.1. Let
\[ g_\theta(x, y) := g(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta). \]
Then \( g_\theta(x, h(x)) \) is convergent for each \( \theta \in E \). By Theorem 1.6, \( g \) is convergent.

**Corollary 2.3.** Let \( f \) be a \( C^\infty \) function defined in a neighborhood of \( 0 \) in \( \mathbb{R}^2 \), and let \( \Gamma = \{(t, \phi(t))\} \) be an analytic curve passing through or ending at the origin in \( \mathbb{R}^2 \). Suppose that for each \( \theta \in [0, 2\pi] \), the restriction of \( f \) to \( \Gamma_\theta \) extends to a real analytic function \( F_\theta \) in a neighborhood \( Q_\theta \) of the origin. Then \( f \) is analytic in a neighborhood of the origin.

**Remark 2.4.** We can see from the proofs that in Theorem 2.1, Theorem 2.2, and Corollary 2.3 the hypothesis on \( f \) can be weakened to \( f \) having a Taylor series at the origin in the sense that there are numbers \( a_{ij} \) such that for each positive integer \( n \),
\[ f(x, y) - \sum_{i+j \leq n} a_{ij} x^i y^j = o((x^2 + y^2)^{n/2}). \]

3. Examples

Here we show that the restrictions in our main theorems are necessary.

**Example 3.1.** P. Lelong [1951] proved that if \( E \) is a set with \( \text{cap} \ E = 0 \) then one can find a divergent power series \( g(x, y) \) such that for all \( s \in E, g(x, sx) \) is convergent. For completeness we present here a construction of such an example. Since \( \text{cap} \ E = 0 \), there is a sequence of positive numbers \( (\delta_n) \) with \( \lim \delta_n = 0 \), and a sequence of polynomials \( (P_n(x)) \) with \( \max_{x \in E} |P(x)| \leq \delta_n^n \), where
\[ P_n(x) = \sum_{j=0}^n b_{nj} x^{n-j}, \]
with \( b_{n0} = 1 \). Let
\[ a_{ij} = \delta_{i+j}^{-i-j} b_{i+j,l} \quad \text{and} \quad g(x, y) = \sum a_{ij} x^i y^j. \]
Then
\[ g(x, sx) = \sum \delta_n^{-n} P_n(s)x^n. \]
For \( s \in E \) we have \( |\delta_n^{-n} P_n(s)| \leq 1 \), so \( g(x, sx) \) is convergent. Note that \( a_{0j} = \delta_j^{-j} \), which obviously implies that \( g \) is divergent, since \( \lim \delta_j = 0 \).

**Example 3.2.** We show that the condition in Theorem 1.1 that \( h(x) \) is not a monomial of the form \( b_k x^k \) with \( \sigma k - r = 0 \) cannot be dispensed with. Let \( \sigma, k \) be positive integers, and \( \phi \in \mathbb{C}[[x]] \) a divergent series with \( \phi(0) = 0 \). Let \( g(x, y) = \phi(x^k) - \phi(y) \) and \( h(x) = x^k \). Then \( g \) is divergent; but \( g(s^\sigma x, s^{\sigma k} h(x)) = 0 \) for each \( s \in \mathbb{C} \).
Example 3.3. We show that the hypothesis in Theorem 1.1 that $h(x)$ is convergent cannot be dispensed with when $\sigma \tau \leq 0$. (By Theorem 1.2 that hypothesis can be dispensed with when $\sigma \tau > 0$.) The example also shows that Theorem 1.2 fails for $\sigma \tau \leq 0$.

Suppose that $\tau \leq 0$, $\sigma > 0$. Let $u(x) = x + \cdots$ be a divergent series. Let $h(x), \phi(x)$ be the series satisfying $\phi(u(x)) = x$ and $x^{\lvert \tau \rvert} h(x)^\sigma = u(x^{\sigma + \lvert \tau \rvert})$. Then $\phi, h$ are divergent. Let $f(x, y) = \phi(x^{\lvert \tau \rvert} y^\sigma)$. Then $f$ is divergent; but

$$f(s^\sigma x, s^\tau h(x)) = x^{\sigma + \lvert \tau \rvert} \quad \text{for each } s \in \mathbb{C} \setminus \{0\}.$$

Now we consider the case where $\sigma = 0$, $\tau = 1$. Let $h(x) = x + \cdots$ be a divergent series, and let $\phi(x)$ be the series satisfying $h(x)\phi(x) = x^2$. Then $\phi$ is divergent. Let $f(x, y) = \phi(x) y$. Then $f$ is divergent; but $f(x, s h(x)) = sx^2$ for each $s \in \mathbb{C}$.

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