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OSGOOD–HARTOGS-TYPE PROPERTIES OF POWER SERIES AND SMOOTH FUNCTIONS

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We study the convergence of a formal power series of two variables if its restrictions on curves belonging to a certain family are convergent. Also analyticity of a given C^∞ function f is proved when the restriction of f on analytic curves belonging to some family is analytic. Our results generalize two known statements: a theorem of P. Lelong and the Bochnak–Siciak theorem. The questions we study can be regarded as problems of Osgood–Hartogs type.

Introduction

Hartogs' theorem is a fundamental result in complex analysis: *A function f in $\mathbb{C}^n$, where $n > 1$, is holomorphic if it is holomorphic in each variable separately.* That is, f is holomorphic in $\mathbb{C}^n$ if for each axis it is holomorphic on every complex line parallel to this axis. In the last interpretation this statement leads to a number of questions described in an article by K. Spallek, P. Tworzewski, T. Winiarski [Spallek et al. 1990] in the following way: "Osgood–Hartogs-type problems ask for properties of 'objects' whose restrictions to certain 'test-sets' are well known". The article has a number of examples of such problems. Here are two classical examples: a theorem of P. Lelong and one proved independently by J. Bochnak and J. Siciak.

Theorem [Lelong 1951]. *A formal power series $g(x, y)$ converges in some neighborhood of the origin if there exists a set $E \subset \mathbb{C}$ of positive capacity such that, for each $s \in E$, the formal power series $g(x, sx)$ converges in some neighborhood of the origin (of a size possibly depending on s).*

Theorem [Bochnak 1970; Siciak 1970]. *Let $f \in C^\infty(D)$, where D is a domain in $\mathbb{R}^n$ containing 0. Suppose f is analytic on every line segment through 0. Then f is analytic in a neighborhood of 0 (as a function of n variables).*

In many articles the same two "objects" are usually considered: power series and functions of several variables. The test sets in many cases form a family of linear

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subspaces of lower dimension. For example, articles by S. S. Abhyankar, T. T. Moh [1970], N. Levenberg and R. E. Molzon, [1988], R. Ree [1949], A. Sathaye [1976], M. A. Zorn [1947] and others consider the convergence of formal power series of several variables provided the restriction of such a series on each element of a sufficiently large family of linear subspaces is convergent. T. S. Neelon [2009; 2006] proved that a formal power series is convergent if its restrictions to certain families of curves or surfaces parametrized by polynomial maps are convergent. The articles [Bochnak 1970; Neelon 2004; 2009; Siciak 1970], among others, prove that a function of several variables is highly smooth (or even analytic) if it is smooth enough on each of a sufficiently large set of linear or algebraic curves (or surfaces of lower dimension). The publication by E. Bierstone, P. D. Milman, A. Parusiński [Bierstone et al. 1991] provides an interesting example of a noncontinuous function in $\mathbb{R}^2$ that is analytic on every analytic curve.

In this article we also consider both: power series with complex coefficients and functions in a neighborhood of the origin in $\mathbb{R}^2$. As test sets we consider separately two families. They are derived the following way. First consider a nonlinear analytic curve $\Gamma = \{x, \gamma(x)\}$, with $\gamma(0) = 0$. One family, $\Im_1$, is a set of dilations of Γ: $\Im_1 = \{sx, s\gamma(x)\}, s \in \Lambda_1\}$, where $\Lambda_1 \subset \mathbb{R}$ is a closed subset of $\mathbb{C}$ of positive capacity. The other family, $\Im_2$, consists of curves Γ_θ, each of which is a rotation of Γ about the origin by an angle $\theta \in \Lambda_2$, where Λ_2 is a subset of $[0, 2\pi]$ of positive capacity. If f is C^∞ and its restriction on every curve of $\Im_1$ can be extended as an analytic function in a neighborhood of that curve, then f is real analytic in a neighborhood of the origin in the region covered by the curves of $\Im_1$. The same is true regarding $\Im_2$. (For precise statements see Theorems 2.1 and 2.2).

We start however with two results related to power series. First we prove a generalization of P. Lelong's theorem. Namely, if $g(x, y)$ is a formal power series and $h(x)$, $h(0) = 0$, is a convergent power series such that the inhomogeneous dilations $g(s^\sigma x, s^\tau h(x))$ are convergent for sufficiently many s (σ, τ are fixed), then $g(x, y)$ is convergent (for the precise statement see Theorem 1.1). Theorem 1.2 is devoted to a reverse claim: if $h(x)$ is a formal power series and $g(s^\sigma x, s^\tau h(x))$ converges for sufficiently many s, then $h(x)$ is convergent.

The results in this paper do not carry over in a routine way to dimensions greater than two. We intend to study corresponding problems for higher dimensions in future work.

1. On the convergence of a power series in two variables

Let $\mathbb{C}[[x_1, x_2, \dots, x_n]]$ denote the set of (formal) power series

$$g(x_1, \dots, x_n) = \sum_{k_1,\dots,k_n \geq 0} a_{k_1\dots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

in n variables with complex coefficients. Let $g(0) = g(0, \dots, 0)$ denote the coefficient $a_{0,\dots,0}$. A power series equals 0 if all of its coefficients $a_{k_1 \dots k_n}$ are equal to 0. A power series $g \in \mathbb{C}[[x_1, x_2, \dots, x_n]]$ is said to be convergent if there is a constant $C = C_g$ such that $|a_{k_1 \dots k_n}| \le C^{k_1+\dots+k_n}$ for all $(k_1, \dots, k_n) \neq (0, \dots, 0)$. If g is convergent, then it represents a holomorphic function in some neighborhood of 0 in $\mathbb{C}^n$. If $g \in \mathbb{C}[[x_1, x_2, \dots, x_n]]$ and $s \in \mathbb{C}^n$, then $g_s(t) := g(s_1 t, \dots, s_n t)$ is well defined and belongs to $\mathbb{C}[[t]]$. By [Zorn 1947], g is convergent if and only if $g_s(t)$ is convergent for each $s \in \mathbb{C}^n$. The partial derivatives of a power series are well defined even when it is divergent (not convergent). For example, if $g \in \mathbb{C}[[x, y]]$ and if $g = \sum a_{ij} x^i y^j$, then

$$g'_y = \frac{\partial g}{\partial y} = \sum j a_{ij} x^i y^{j-1}.$$

Thus $g'_y \neq 0$ simply means that $g \notin \mathbb{C}[[x]]$. If $g \in \mathbb{C}[[x, y]]$, and if $h \in \mathbb{C}[[x]]$ with $h(0) = 0$, then $g(x, h(x))$ is a well-defined element of $\mathbb{C}[[x]]$.

As mentioned above, a lot of work has been done on the convergence of a power series with the assumption that the series is convergent after restriction to sufficiently many subspaces; see [Abhyankar and Moh 1970; Levenberg and Molzon 1988; Lelong 1951; Siciak 1970; 1982].

We consider substitution of a power series $y = h(x)$ into an inhomogeneous dilation $g(s^\sigma x, s^\tau y)$ of a series $g(x, y)$, where σ, τ are integers.

Let

$$Q := \{(\sigma, \tau) : \sigma, \tau \in \mathbb{Z}, (\sigma, \tau) \neq (0, 0)\}.$$

Let cap E denote the (logarithmic) capacity of a closed set E in the complex plane.

We now present our two main theorems.

Theorem 1.1. *Let $g \in \mathbb{C}[[x, y]]$ be a power series of two variables x, y, let $h \in \mathbb{C}[[x]]$ be a nonzero convergent power series with $h(0) = 0$, let E be a closed subset of $\mathbb{C} \setminus \{0\}$ with* cap $E > 0$, *and let (σ, τ) be a pair in the set Q. Assume, in case $\sigma\tau > 0$, that $h(x)$ is not a monomial of the form $b_k x^k$ with $\sigma k - \tau = 0$. Suppose that $g(s^\sigma x, s^\tau h(x))$ is convergent for each $s \in E$. Then g is convergent.*

Theorem 1.2. *Let $g \in \mathbb{C}[[x, y]]$ be a power series with $g'_y \neq 0$, let $h \in \mathbb{C}[[x]]$ be a nonzero power series with $h(0) = 0$, let E be a closed subset of $\mathbb{C} \setminus \{0\}$ with* cap $E > 0$, *and let (σ, τ) be a pair in the set Q with $\sigma\tau > 0$. Suppose that $g(s^\sigma x, s^\tau h(x))$ is convergent for each $s \in E$. Then h is convergent.*

The examples in Section 3 show that if any condition in these two theorems is dispensed with, the resulting statement is false. We now prove some auxiliary results.

The following theorem is a consequence of a result by B. Malgrange [1966]. We present an independent short proof.

Theorem 1.3. *Let $g \in \mathbb{C}[[x_1, \dots, x_n, y]]$ with $g'_y \neq 0$, and let $h \in \mathbb{C}[[x_1, \dots, x_n]]$ with $h(0) = 0$. Suppose that g and $g(x_1, \dots, x_n, h(x_1, \dots, x_n))$ are convergent. Then h must be convergent.*

Proof. Let $f \in \mathbb{C}[[x_1, \dots, x_n, y]]$ be defined by

$$f(x_1, \dots, x_n, y) = g(x_1, \dots, x_n, y) - g(x_1, \dots, x_n, h(x_1, \dots, x_n)).$$

Then f is convergent and $f(x_1, \dots, x_n, h(x_1, \dots, x_n)) = 0$. Fix $s = (s_1, \dots, s_n) \in \mathbb{C}^n$. Let $f_s(t, y) \in \mathbb{C}[[t, y]]$ be defined by $f_s(t, y) = f(s_1 t, \dots, s_n t, y)$. Then $f_s(t, y)$ is convergent and $f_s(t, h_s(t)) = 0$. By the Weierstrass preparation theorem (see [Griffiths and Harris 1978, p. 8], for example), there is a nonnegative integer k such that $f_s(t, y) = t^k P(t, y) Q(t, y)$, where $P(t, y) = y^m + a_1(t) y^{m-1} + \cdots + a_m(t)$ is a polynomial in y with coefficients being convergent power series in t, and $Q(t, y)$ is a convergent power series with $Q(0, 0) \neq 0$. Hence $P(t, h_s(t)) = 0$. By [Fuks 1963, Theorem 4.12, p. 73] there is a positive integer r such that $P(t^r, y)$ splits into linear factors in y:

$$P(t^r, y) = (y - u_1(t)) \cdots (y - u_m(t)),$$

where the $u_j(t)$ are convergent power series. Thus

$$0 = P(t^r, h_s(t^r)) = (h_s(t^r) - u_1(t)) \cdots (h_s(t^r) - u_m(t)).$$

It follows that $h_s(t^r) = u_j(t)$ for some j. Therefore $h_s(t)$ is convergent. Since $h_s(t)$ is convergent for each $s \in \mathbb{C}^n$, the series $h(x_1, \dots, x_n)$ must be convergent. □

Let E be a closed bounded set in the complex plane. The *transfinite diameter* of E is defined as

$$d_\infty(E) = \lim_n \Big(\max \Big\{ \prod_{i<j} |z_i - z_j|^{2/n(n-1)} : z_1, \dots, z_n \in E \Big\} \Big).$$

For a probability measure μ on the compact set E, the *logarithmic potential* of μ is

$$p_\mu(z) = \lim_{N \to \infty} \int \min\Big(N, \log \frac{1}{|z - \zeta|}\Big) \, d\mu(\zeta),$$

and the *capacity* of E is defined by

$$\operatorname{cap} E = \exp(- \min_{\mu(E)=1} \sup_{z \in \mathbb{C}} p_\mu(z)).$$

It turns out that $d_\infty(E) = \operatorname{cap} E$ [Ahlfors 1973, pp. 23–28]. It follows from the definition of the transfinite diameter that, for $E_1 \supset E_2 \supset \cdots$,

$$E = \bigcap E_n \implies \operatorname{cap} E = \lim(\operatorname{cap} E_n),$$

and from the definition of the capacity that, if $E_1 \subset E_2 \subset \cdots$,

$$E = \bigcup E_n \implies \operatorname{cap} E = \lim(\operatorname{cap} E_n). \tag{1}$$

If E is a closed set, its capacity can be defined by

$$\operatorname{cap} E = \lim_n \operatorname{cap}(E \cap \{|x| \le n\}).$$

Lemma 1.4 (Bernstein inequality). *Let E be a compact set in the complex plane with $\operatorname{cap} E > 0$. Then there exists a positive constant $C = C_E$, depending only on E, such that for each positive integer n and each polynomial $P(z) = \sum a_k z^k \in \mathbb{C}[z]$ of degree n, each coefficient a_k, $0 \le k \le n$, of $P(z)$ satisfies*

$$|a_k| \le C^n \max_{z \in E} |P(z)|.$$

Proposition 4.6 in [Neelon 2009] can be used to prove this statement. Also (we thank Nessim Sibony for pointing this out to us) this lemma follows from considerations in [Sibony 1985]. We present here an independent short proof.

Proof. Without loss of generality we assume that $\max_{z \in E} |P(z)| = 1$. Let Ω be the unbounded component of the complement of E in $\mathbb{C}$. It is known that Ω has a Green's function with a pole at ∞ [Ahlfors 1966; 1973, pp. 25–27]. The Green's function is harmonic in Ω, 0 on $\partial\Omega$, and its asymptotic behavior at ∞ is

$$u(z) = \log|z| - \log\alpha + o(1),$$

where $\alpha := \operatorname{cap} E$. On applying the maximum principle to the subharmonic function $\log|P(z)| - (n+\epsilon)u(z)$, we obtain $|P(z)| \le e^{nu(z)}$ for $z \in \Omega$. Choose an $R > 1$ so that $E \subset \{z : |z| < R\}$. Set $C = \max_{|z|=R} e^{u(z)}$. Then $|P(z)| \le C^n$ if $|z| = R$, and

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z|=R} \frac{P(z)}{z^{k+1}}\, dz \right| \le R^{-k} \max_{|z|=R} |P(z)| \le C^n.$$

This proves the lemma. □

Proof of Theorem 1.1. We assume that $a_{00} = g(0,0) = 0$, that E is bounded, that $\gcd(\sigma, \tau) = 1$, that $\sigma \ge 0$, and, in case $\sigma = 0$, that $\tau = -1$. This causes no loss of generality. Indeed, if E is unbounded, set $E_n = \{s \in E : n \ge |s| \ge 1/n\}$. Since $\lim \operatorname{cap} E_n = \operatorname{cap} E > 0$, the set E_n has positive capacity when n is sufficiently large. On replacing E by E_n, we obtain that $0 \notin E$ and E is bounded. If $d := \gcd(\sigma, \tau) > 1$, we can replace (σ, τ) by $(\sigma/d, \tau/d)$, and E by the set $\{s \in \mathbb{C} : s^d \in E\}$. Finally, if $\sigma < 0$, or if $(\sigma, \tau) = (0, 1)$, we can replace (σ, τ) by $(-\sigma, -\tau)$, and E by $\{s \in \mathbb{C} : s^{-1} \in E\}$.

Let

$$h(x) = \sum_{i=1}^{\infty} b_i x^i.$$

Then

$$h(x)^j = \sum_{k=j}^{\infty} c_{jk} x^k,$$

where

$$c_{jk} = \sum_{l_1+\cdots+l_j=k} b_{l_1} \cdots b_{l_j}.$$

Note that $c_{jk} = 0$ for $k < j$. Hence

$$g(s^\sigma x, s^\tau h(x)) = \sum_{i,j,k} a_{ij} c_{jk} s^{\sigma i + \tau j} x^{i+k} = \sum_{p=1}^{\infty} \left(\sum_{q=-\tau^- p}^{(\sigma+\tau^+)p} d_{pq} s^q \right) x^p,$$

where $\tau^+ = \max(0, \tau)$, $\tau^- = -\min(0, \tau)$, and

$$d_{pq} = \sum_{\sigma i + \tau j = q} a_{ij} c_{j,p-i}. \tag{2}$$

For each $p \geq 1$ and each $q \in \mathbb{Z}$, the sum (2) contains only a finite number of nonzero terms. Let $u_p(s) = \sum_q d_{pq} s^q$. Then $s^{\tau^- p} u_p(s)$ is a polynomial in s of degree at most $(\sigma + |\tau|)p$, and $g(s^\sigma x, s^\tau h(x)) = \sum u_p(s) x^p$. For $s \in E$, since $g(s^\sigma x, s^\tau h(x))$ is convergent, its coefficients $u_p(s)$ satisfy $|u_p(s)| \leq C_s^p$ for some positive constant C_s, possibly depending on s, and $p = 1, 2, \ldots$. Set, for $n = 1, 2, \ldots$,

$$E_n = \{s \in E : |u_p(s)| \leq n^p \text{ for all } p > 0\}.$$

The sequence (E_n) is an increasing sequence of closed sets. Since $\lim \operatorname{cap} E_n = \operatorname{cap} E > 0$, the set E_n has positive capacity for some n. On replacing E by E_n, we obtain $|u_p(s)| \leq n^p$ for $s \in E$ and $p = 1, 2, \ldots$. The polynomial $s^{\tau^- p} u_p(s)$ is of degree at most $(\sigma + |\tau|)p$, and satisfies

$$|s^{\tau^- p} u_p(s)| \leq M^{\tau^- p} n^p, \quad s \in E,$$

where $M = \max_E |s|$. By Lemma 1.4, the coefficients of the above mentioned polynomial satisfy $|d_{pq}| \leq C_E^{(\sigma+|\tau|)p} M^{\tau^- p} n^p$, where C_E is the constant in Lemma 1.4, depending only on E. Set $C = C_E^{\sigma+|\tau|} M^{\tau^-} n$. Then

$$|d_{pq}| \leq C^p. \tag{3}$$

Let

$$g_q(x, y) = \sum_{\sigma i + \tau j = q} a_{ij} x^i y^j, \tag{4}$$

and let $\phi_q(x) = g_q(x, h(x))$, for $q \in \mathbb{Z}$. Then $g_q \in \mathbb{C}[\![x, y]\!]$ in general, and it is a polynomial when $\sigma, \tau > 0$. It is straightforward to verify that

$$\phi_q(x) = g_q(x, h(x)) = \sum_{p=1}^{\infty} d_{pq} x^p. \tag{5}$$

The series $\phi_q(x)$ is convergent because of (3). Choose a positive number $r < 1/C$, where C is the constant in (3), so that $h(x)$ converges in a neighborhood of the closed ball $\{x \in \mathbb{C} : |x| \le r\}$ and $h(x) \ne 0$ when $0 < |x| \le r$. Let $m = \min_{|x|=r} |h(x)|$. Then $m > 0$. For $x \in \mathbb{C}$, $|x| \le r$,

$$|\phi_q(x)| \le \sum |d_{pq}||x|^p \le \sum (Cr)^p = \frac{1}{1 - Cr}.$$

We now consider two cases, depending on whether $\sigma\tau$ is positive.

Case (i): $\sigma > 0$, $\tau > 0$. Let

$$\Omega_q = \{(i, j) : i, j \in \mathbb{Z},\ i, j \ge 0,\ \sigma i + \tau j = q\}. \tag{6}$$

Let ω_q be the cardinality of Ω_q. It is clear that $\omega_q \le q + 1$. Fix a $q \ge 1$ so that $\omega_q > 0$. Let (λ, μ) be the element of Ω_q so that μ is the minimum. Then

$$\Omega_q = \{(\lambda - k\tau,\ \mu + k\sigma) : k = 0, 1, \dots, \omega_q - 1\},$$

and

$$g_q(x, y) = x^\lambda y^\mu \sum_{k=0}^{\omega_q - 1} a_{\lambda - k\tau,\, \mu + k\sigma} (x^{-\tau} y^\sigma)^k.$$

Let

$$\psi_q(t) = \sum_{k=0}^{\omega_q - 1} a_{\lambda - k\tau,\, \mu + k\sigma} t^k,$$

so that $g_q(x, y) = x^\lambda y^\mu \psi_q(x^{-\tau} y^\sigma)$, and

$$\psi_q(x^{-\tau} h(x)^\sigma) = x^{-\lambda} h(x)^{-\mu} \phi_q(x). \tag{7}$$

Let $u(x) = x^{-\tau} h(x)^\sigma$, $S = \{x \in \mathbb{C} : |x| = r\}$, and $F = u(S)$. Since $h(x)$ is not a monomial of the form $b_k x^k$ with $\sigma k - \tau = 0$, the function $u(x)$ is a nonconstant meromorphic function, hence F has positive capacity. For $t = x^{-\tau} h(x)^\sigma \in F$, we obtain, by (7), that

$$|\psi_q(t)| \le \frac{r^{-\lambda} m^{-\mu}}{1 - Cr} \le \frac{(1 + r^{-1} + m^{-1})^{\lambda+\mu}}{1 - Cr}. \tag{8}$$

The summand 1 in the right-hand side of the above inequality is included to ensure that the numerator is greater than 1 as needed later. Hence $|\psi_q(t)| \le L^q$ on F, where

$$L = \frac{1 + r^{-1} + m^{-1}}{1 - Cr},$$

for $\lambda + \mu \le q$. By Lemma 1.4, the coefficients of ψ_q are bounded by $L^q C_F^{\omega_q - 1}$. Thus for $(i, j) \in \Omega_q$,

$$|a_{ij}| \le L^q C_F^{\omega_q - 1} \le (L + C_F)^{2q} \le (L + C_F)^{2(\sigma+\tau)(i+j)},$$

or $|a_{ij}| \le K^{i+j}$, where $K = (L + C_F)^{2(\sigma+\tau)}$. The number K does not depend on q. It follows that

$$|a_{ij}| \le K^{i+j}, \ \text{ if } \sigma i + \tau j \ge 1.$$

This proves that g is convergent.

Case (ii): $\sigma \ge 0$, $\tau \le 0$. In this case the set Ω_q in (6) can be written as

$$\Omega_q = \{(\lambda - k\tau,\ \mu + k\sigma) : k = 0, 1, 2, \dots\},$$

where (λ, μ) is the element in Ω_q with least value of μ when $\sigma > 0$, and $(\lambda, \mu) = (0, -q)$ when $(\sigma, \tau) = (0, -1)$. Let

$$\psi_q(t) = \sum_{k=0}^{\infty} a_{\lambda + k|\tau|,\, \mu + k\sigma} t^k.$$

Then $g_q(x, y) = x^\lambda y^\mu \psi_q(x^{|\tau|} y^\sigma)$. The formal power series $\psi_q(t)$ satisfies $\phi_q(x) = x^\lambda h(x)^\mu \psi_q(x^{|\tau|} h(x)^\sigma)$. Since $x^\lambda h(x)^\mu$ and $\phi_q(x)$ are convergent, the series

$$\alpha(x) := \psi_q(x^{|\tau|} h(x)^\sigma)$$

has to be convergent. Write $x^{|\tau|} h(x)^\sigma = c x^\nu + \cdots$, $c \ne 0$. There is a power series $\beta(x)$, also convergent in a neighborhood of $\{|x| \le r\}$, such that $x^{|\tau|} h(x)^\sigma = \beta(x)^\nu$. Reducing r if necessary, we assume that $\beta(x)$ is univalent in a neighborhood of $\{|x| \le r\}$. Note that the reduction in the value of r is independent of q. The set $\{\beta(x) : |x| < r\}$ contains an open disc $\{z \in \mathbb{C} : |z| < \delta\}$. The series $\beta(x)$ has an inverse $\gamma(z)$, convergent in $\{z \in \mathbb{C} : |z| < \delta\}$, such that $\gamma(\beta(x)) = x$ and $\beta(\gamma(z)) = z$. Now $\psi_q(z^\nu)$ is converge nt in $\{|z| < \delta\}$, so $\psi_q(t)$ is convergent in $\{|t| < \delta^\nu\}$. Let $t \in \mathbb{C}$ with $|t| < \delta^\nu$. Then $t = z^\nu$ for some z with $|z| < \delta$, and $z = \beta(x)$ for some x with $|x| < r$. Hence

$$|\psi_q(t)| = |\psi_q(\beta(x)^\nu)| = |\alpha(x)| \le \max_{|x|=r} |\alpha(x)|.$$

Thus

$$\sup_{|t|<\delta^\nu} |\psi_q(t)| \le \max_{|x|=r} \left| \frac{\phi_q(x)}{x^\lambda h(x)^\mu} \right| \le \frac{r^{-\lambda} m^{-\mu}}{1 - Cr}.$$

By the Cauchy estimates, the coefficients of ψ_q satisfy

$$|a_{\lambda+k|\tau|,\,\mu+k\sigma}| \le \frac{r^{-\lambda}m^{-\mu}}{1-Cr}\delta^{-k\nu} \le \frac{(1+r^{-1}+m^{-1}+\delta^{-\nu})^{\lambda+\mu+k}}{1-Cr}.$$

The summand 1 in the right-hand side of the above inequality is included to ensure that the numerator is greater than 1 as needed later. It follows that, for $(i, j) \in \Omega_q$,

$$|a_{ij}| \le \left(\frac{1+r^{-1}+m^{-1}+\delta^{-\nu}}{1-Cr}\right)^{i+j}.$$

The number $K := (1+r^{-1}+m^{-1}+\delta^{-\nu})/(1-Cr)$ does not depend on q. Therefore, $|a_{ij}| \le K^{i+j}$ for all (i, j). This proves that g is convergent. □

Proof of Theorem 1.2. This proof and the proof of Theorem 1.1 share the discussion through Equation (5). Note that the convergence of h has not been used in the derivation of (5). We define polynomials $g_q(x, y)$ by (4). Then $g_q(x, h(x))$ are convergent by (3) and (5). Since $g'_y(x, y) \ne 0$, $\partial g_q/\partial y \ne 0$ for some q. It follows from Theorem 1.3 that $h(x)$ is convergent. □

For $h \in \mathbb{C}[[x]]$ with $h(0) = 0$, let $h_s(x) = s^{-1}h(sx)$.

Corollary 1.5. *Let $g \in \mathbb{C}[[x, y]]$ be a power series, let $h \in \mathbb{C}[[x]]$ be a nonzero and nonlinear power series with $h(0) = 0$, and let E be a closed subset of $\mathbb{R} \setminus \{0\}$ with* cap $E > 0$. *Suppose that $g(x, h_s(x))$ is convergent for each $s \in E$. Then g is convergent.*

Proof. If $g'_y = 0$ then the statement holds. Suppose $g'_y \ne 0$. For $s \ne 0$, $g(x, h_s(x))$ is convergent if and only if $g(s^{-1}x, h_s(s^{-1}x)) = g(s^{-1}x, s^{-1}h(x))$ is convergent. By Theorem 1.2, h is convergent. Then g is convergent by Theorem 1.1. □

For $f \in \mathbb{C}[[x, y]]$ and $\theta \in [0, 2\pi]$, write

$$f_\theta(x, y) = f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Theorem 1.6. *Let $f \in \mathbb{C}[[x, y]]$ be a power series, let $h \in \mathbb{C}[[x]]$ be a convergent power series with $h(0) = 0$, and let E be a closed subset of* $[0, 2\pi]$ *with* cap $E > 0$. *Suppose that $f_\theta(x, h(x))$ is convergent for each $\theta \in E$. Then f is convergent.*

Proof. Let $g(x, y) = f((x+y)/2, -i(x-y)/2)$. Then $f(x, y) = g(x+iy, x-iy)$ and $f_\theta(x, y) = g(e^{i\theta}(x+iy), e^{-i\theta}(x-iy))$. Let $\phi_\theta(x) = f_\theta(x, h(x)) = g(e^{i\theta}(x+ih(x)), e^{-i\theta}(x-ih(x)))$. Then $\phi_\theta(x)$ is convergent for $\theta \in E$. The x terms of the two series $x \pm ih(x)$ cannot both be zero. Say, the x term of $x+ih(x)$ is nonzero. So $x+ih(x)$ has an inverse $\psi(x)$ which is a convergent power series such that $\psi(x)+ih(\psi(x)) = x$. Set $\psi(x) - ih(\psi(x)) = \omega(x)$. Then $\phi_\theta(\psi(x)) = g(e^{i\theta}x, e^{-i\theta}\omega(x))$ is convergent for $\theta \in E$. It follows that $g(sx, s^{-1}\omega(x))$ is convergent for each s in the set $\{e^{i\theta} : \theta \in E\}$, which has positive capacity. By Theorem 1.1, g is convergent. Therefore f is convergent. □

2. Analytic functions in $\mathbb{R}^2$

Suppose that $f(x, y), \phi(x), q(x)$ are C^∞ functions defined near the origin with $\phi(0) = 0$. Let $\hat{f}, \hat{\phi}, \hat{q}$ denote the Taylor series at 0 of those functions. Then $\hat{f}$ lies in $\mathbb{C}[[x, y]]$ and $\hat{\phi}, \hat{q}$ lie in $\mathbb{C}[[x]]$. By the chain rule, $f(x, \phi(x)) = q(x)$ implies $\hat{f}(x, \hat{\phi}(x)) = \hat{q}(x)$. We consider here complex-valued analytic functions of real variables. If I is an interval and if $\Gamma = \{(t, \gamma(t)) : t \in I\}$ is a curve, the dilation by s of Γ is

$$\Gamma_s = \{(st, s\gamma(t))\} = \{(t, \gamma_{1/s}(t))\}, \quad \gamma_s(t) = s^{-1}\gamma(st).$$

Theorem 2.1. *Let f be a C^∞ function defined in an open set $\Omega \subset \mathbb{R}^2$ containing the origin, let $\Gamma = \{(t, \phi(t))\}$ be a nonlinear analytic curve in $\mathbb{R}^2$ passing through or ending at the origin, and let E be a closed subset of $\mathbb{R} \setminus \{0\}$ of positive capacity. Suppose that for each $s \in E$, there is a real analytic function F_s defined in a neighborhood Q_s of $\Gamma_s \cap \Omega$ in $\mathbb{R}^2$ such that f and F_s coincide on $\Gamma_s \cap \Omega$. Then there is a neighborhood U of the origin, and a real analytic function F defined on U that coincides with f on $U \cap \Lambda$, where $\Lambda := \bigcup_{s\in E} \Gamma_s$.*

Proof. Without loss of generality we assume that $\phi(0) = 0$. Since f and F_s coincide on Γ_s, we have

$$f(x, \phi_{1/s}(x)) = F_s(x, \phi_{1/s}(x)). \tag{9}$$

Let g, h denote the Taylor series of f, ϕ respectively. Then (9) implies

$$g(x, h_{1/s}(x)) = F_s(x, h_{1/s}(x)).$$

Hence $g(x, h_{1/s}(x))$ is convergent for $s \in E$. By Corollary 1.5, g is convergent. Thus g represents a real analytic function F in some neighborhood U of the origin that satisfies $F(x, h_{1/s}(x)) = F_s(x, h_{1/s}(x))$. It follows that the real analytic function F coincides with f on $U \cap \Lambda$. □

Note that f does not need to be analytic in a neighborhood of the origin.

If $\Gamma = \{(t, \phi(t) : t \in I\}$ is a curve, its rotation by θ is

$$\Gamma_\theta = \{(t \cos\theta + \phi(t) \sin\theta, -t \sin\theta + \phi(t) \cos\theta) : t \in I\}.$$

Theorem 2.2. *Let f be a C^∞ function defined in an open set $\Omega \subset \mathbb{R}^2$ containing the origin, let $\Gamma = \{(t, \phi(t))\}$ be an analytic curve in $\mathbb{R}^2$ passing through or ending at the origin, and let E be a closed subset of $[0, 2\pi]$ of positive capacity. Suppose that for each $\theta \in E$, there is a real analytic function F_θ defined in a neighborhood Q_θ of $\Gamma_\theta \cap \Omega$ in $\mathbb{R}^2$ such that f and F_θ coincide on $\Gamma_\theta \cap \Omega$. Then there is an analytic function F defined in some neighborhood U of the origin that coincides with f on $U \cap \Lambda$, where $\Lambda := \bigcup_{\theta\in E} \Gamma_\theta$.*

Proof. The proof is similar to that of Theorem 2.1. Let

$$g_\theta(x, y) := g(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

Then $g_\theta(x, h(x))$ is convergent for each $\theta \in E$. By Theorem 1.6, g is convergent. □

Corollary 2.3. *Let f be a C^∞ function defined in a neighborhood of* 0 *in* $\mathbb{R}^2$, *and let $\Gamma = \{(t, \phi(t))\}$ be an analytic curve passing through or ending at the origin in* $\mathbb{R}^2$. *Suppose that for each $\theta \in [0, 2\pi]$, the restriction of f to Γ_θ extends to a real analytic function F_θ in a neighborhood Q_θ of the origin. Then f is analytic in a neighborhood of the origin.*

Remark 2.4. We can see from the proofs that in Theorem 2.1, Theorem 2.2, and Corollary 2.3 the hypothesis on f can be weakened to f having a Taylor series at the origin in the sense that there are numbers a_{ij} such that for each positive integer n,

$$f(x, y) - \sum_{i+j\le n} a_{ij}x^i y^j = o((x^2 + y^2)^{n/2}).$$

3. Examples

Here we show that the restrictions in our main theorems are necessary.

Example 3.1. P. Lelong [1951] proved that if E is a set with $\operatorname{cap} E = 0$ then one can find a divergent power series $g(x, y)$ such that for all $s \in E$, $g(x, sx)$ is convergent. For completeness we present here a construction of such an example. Since $\operatorname{cap} E = 0$, there is a sequence of positive numbers (δ_n) with $\lim \delta_n = 0$, and a sequence of polynomials $(P_n(x))$ with $\max_{x\in E} |P(x)| \le \delta_n^n$, where

$$P_n(x) = \sum_{j=0}^{n} b_{nj}x^{n-j},$$

with $b_{n0} = 1$. Let

$$a_{ij} = \delta_{i+j}^{-(i+j)} b_{i+j,i} \quad \text{and} \quad g(x, y) = \sum a_{ij}x^i y^j.$$

Then

$$g(x, sx) = \sum \delta_n^{-n} P_n(s)x^n.$$

For $s \in E$ we have $|\delta_n^{-n} P_n(s)| \le 1$, so $g(x, sx)$ is convergent. Note that $a_{0j} = \delta_j^{-j}$, which obviously implies that g is divergent, since $\lim \delta_j = 0$.

Example 3.2. We show that the condition in Theorem 1.1 that $h(x)$ is not a monomial of the form $b_k x^k$ with $\sigma k - \tau = 0$ cannot be dispensed with. Let σ, k be positive integers, and $\phi \in \mathbb{C}[[x]]$ a divergent series with $\phi(0) = 0$. Let $g(x, y) = \phi(x^k) - \phi(y)$ and $h(x) = x^k$. Then g is divergent; but $g(s^\sigma x, s^{\sigma k} h(x)) = 0$ for each $s \in \mathbb{C}$.

Example 3.3. We show that the hypothesis in Theorem 1.1 that $h(x)$ is convergent cannot be dispensed with when $\sigma\tau \le 0$. (By Theorem 1.2 that hypothesis can be dispensed with when $\sigma\tau > 0$.) The example also shows that Theorem 1.2 fails for $\sigma\tau \le 0$.

Suppose that $\tau \le 0$, $\sigma > 0$. Let $u(x) = x + \cdots$ be a divergent series. Let $h(x), \phi(x)$ be the series satisfying $\phi(u(x)) = x$ and $x^{|\tau|}h(x)^\sigma = u(x^{\sigma+|\tau|})$. Then ϕ, h are divergent. Let $f(x, y) = \phi(x^{|\tau|}y^\sigma)$. Then f is divergent; but

$$f(s^\sigma x, s^\tau h(x)) = x^{\sigma+|\tau|} \quad \text{for each } s \in \mathbb{C} \setminus \{0\}.$$

Now we consider the case where $\sigma = 0$, $\tau = 1$. Let $h(x) = x + \cdots$ be a divergent series, and let $\phi(x)$ be the series satisfying $h(x)\phi(x) = x^2$. Then ϕ is divergent. Let $f(x, y) = \phi(x)y$. Then f is divergent; but $f(x, sh(x)) = sx^2$ for each $s \in \mathbb{C}$.

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