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**CHERN NUMBERS AND THE INDICES OF SOME ELLIPTIC
DIFFERENTIAL OPERATORS**

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Libgober and Wood proved that the Chern number $c_1 c_{n-1}$ of a compact complex manifold of dimension n can be determined by its Hirzebruch χ_y -genus. Inspired by the idea of their proof, we show that, for compact, spin, almost-complex manifolds, more Chern numbers can be determined by the indices of some twisted Dirac and signature operators. As a byproduct, we get a divisibility result of certain characteristic number for such manifolds. Using our method, we give a direct proof of the result of Libgober and Wood, which was originally proved by induction.

1. Introduction and main results

Suppose (M, J) is a compact, almost-complex $2n$ -manifold with a given almost complex structure J . This J makes the tangent bundle of M into a n -dimensional complex vector bundle T_M . Let $c_i(M, J) \in H^{2i}(M; \mathbb{Z})$ be the i -th Chern class of T_M . Suppose we have a formal factorization of the total Chern class as follows:

$$1 + c_1(M, J) + \cdots + c_n(M, J) = \prod_{i=1}^n (1 + x_i),$$

i.e., x_1, \dots, x_n are formal Chern roots of T_M . The Hirzebruch χ_y -genus of (M, J) , $\chi_y(M, J)$, is defined by

$$\chi_y(M, J) = \left(\prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}} \right) [M].$$

Here $[M]$ is the fundamental class of the orientation of M induced by J and y is an indeterminate. If J is specified, we simply denote $\chi_y(M, J)$ by $\chi_y(M)$.

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When the almost complex structure J is integrable (equivalently, when M is an n -dimensional compact complex manifold), $\chi_y(M)$ can be obtained by

$$\chi^p(M) = \sum_{q=0}^n (-1)^q h^{p,q}(M), \quad \chi_y(M) = \sum_{p=0}^n \chi^p(M) \cdot y^p,$$

where $h^{p,q}(\cdot)$ is the Hodge number of type (p, q) . This is given by the Hirzebruch–Riemann–Roch Theorem, proved in [Hirzebruch 1966] for projective manifolds and in [Atiyah and Singer 1968] in the general case.

The formula

$$(1-1) \quad \sum_{p=0}^n \chi^p(M) \cdot y^p = \left(\prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}} \right) [M]$$

implies that $\chi^p(M)$, the index of the Dolbeault complex, can be expressed as a rational combination of some Chern numbers of M . Conversely, we can address the following question.

Question 1.1. For an n -dimensional compact complex manifold M , given a partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$ of weight n , can the Chern number $c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_l} [M]$ be determined by $\chi^p(M)$, or more generally by the indices of some other elliptic differential operators?

For the simplest case $c_n[M]$, the answer is affirmative and well-known [Hirzebruch 1966, Theorem 15.8.1]:

$$c_n[M] = \chi_y(M)|_{y=-1} = \sum_{p=0}^n (-1)^p \chi^p(M).$$

The next-to-simplest case is the Chern number $c_1 c_{n-1} [M]$. The answer here is also affirmative, as was proved by Libgober and Wood [1990, pp. 141–143]:

$$(1-2) \quad \sum_{p=2}^n (-1)^p \binom{p}{2} \chi^p(M) = \frac{n(3n-5)}{24} c_n[M] + \frac{1}{12} c_1 c_{n-1} [M].$$

The idea of their proof is quite enlightening: expanding both sides of (1-1) at $y = -1$ and comparing the coefficients of the term $(y+1)^2$, one gets (1-2).

Inspired by this idea, in this paper we consider twisted Dirac operators and signature operators on compact, *spin*, *almost-complex* manifolds and show that the Chern numbers c_n , $c_1 c_{n-1}$, $c_1^2 c_{n-2}$ and $c_2 c_{n-2}$ of such manifolds can be determined by the indices of these operators.

Remark 1.2. Equation (1-2) was also obtained later by Salamon [1996, p. 144], who applied this result extensively to hyper-Kähler manifolds.

Let M be a compact, *almost-complex* $2n$ -manifold. We still use x_1, \dots, x_n to denote the corresponding formal Chern roots of the n -dimensional complex vector bundle T_M . Suppose E is a complex vector bundle over M . Set

$$\begin{aligned}\hat{A}(M, E) &:= \left(\text{ch}(E) \cdot \prod_{i=1}^n \frac{x_i/2}{\sinh(x_i/2)} \right) [M], \\ L(M, E) &:= \left(\text{ch}(E) \cdot \prod_{i=1}^n \frac{x_i(1+e^{-x_i})}{1-e^{-x_i}} \right) [M],\end{aligned}$$

where $\text{ch}(E)$ is the Chern character of E . The celebrated Atiyah–Singer index theorem [Hirzebruch et al. 1992, pp. 74–81] states that $L(M, E)$ is the index of the signature operator twisted by E and when M is *spin*, $\hat{A}(M, E)$ is the index of the Dirac operator twisted by E .

Definition 1.3. Set

$$A_y(M) := \sum_{p=0}^n \hat{A}(M, \Lambda^p(T_M^*)) \cdot y^p \quad \text{and} \quad L_y(M) := \sum_{p=0}^n L(M, \Lambda^p(T_M^*)) \cdot y^p,$$

where $\Lambda^p(T_M^*)$ denotes the p -th exterior power of the dual of T_M .

Our main result is the following:

Theorem 1.4. *Let M be a compact, almost-complex manifold.*

$$\begin{aligned}(1) \quad & \sum_{p=0}^n (-1)^p \hat{A}(M, \Lambda^p(T_M^*)) = c_n[M], \\ & \sum_{p=1}^n (-1)^p \cdot p \cdot \hat{A}(M, \Lambda^p(T_M^*)) = \frac{1}{2}(nc_n[M] + c_1c_{n-1}[M]), \\ & \sum_{p=2}^n (-1)^p \binom{p}{2} \hat{A}(M, \Lambda^p(T_M^*)) = \left(\frac{n(3n-5)}{24}c_n + \frac{3n-2}{12}c_1c_{n-1} + \frac{1}{8}c_1^2c_{n-2} \right) [M]; \\ (2) \quad & \sum_{p=0}^n (-1)^p L(M, \Lambda^p(T_M^*)) = 2^n c_n[M], \\ & \sum_{p=1}^n (-1)^p \cdot p \cdot L(M, \Lambda^p(T_M^*)) = 2^{n-1}(nc_n[M] + c_1c_{n-1}[M]), \\ & \sum_{p=2}^n (-1)^p \binom{p}{2} L(M, \Lambda^p(T_M^*)) \\ & \quad = 2^{n-2} \left(\frac{n(3n-5)}{6}c_n + \frac{3n-2}{3}c_1c_{n-1} + c_1^2c_{n-2} - c_2c_{n-2} \right) [M].\end{aligned}$$

Corollary 1.5. (1) *The Chern numbers $c_n[M]$, $c_1c_{n-1}[M]$ and $c_1^2c_{n-2}[M]$ can be determined by $A_y(M)$.*

(2) *The characteristic numbers $c_n[M]$, $c_1c_{n-1}[M]$ and $c_1^2c_{n-2}[M] - c_2c_{n-2}[M]$ can be determined by $L_y(M)$.*

(3) *When M is a spin manifold, the Chern numbers $c_n[M]$, $c_1c_{n-1}[M]$, $c_1^2c_{n-2}[M]$ and $c_2c_{n-2}[M]$ can be expressed by using linear combinations of the indices of some twisted Dirac and signature operators.*

As remarked in [Libgober and Wood 1990, p. 143], it was shown by Milnor [1960] that every complex cobordism class contains a non-singular algebraic variety. Milnor also showed that two almost-complex manifolds are complex cobordant if and only if they have the same Chern numbers. Hence Libgober and Wood's result implies that the characteristic number

$$\frac{n(3n-5)}{24}c_n[N] + \frac{1}{12}c_1c_{n-1}[N]$$

is always an integer for any compact, *almost-complex* $2n$ -manifold N .

Note that the right-hand side of the third equality in Theorem 1.4 is

$$\left(\frac{n(3n-5)}{24}c_n[M] + \frac{1}{12}c_1c_{n-1}[M]\right) + \frac{1}{8}\left(2(n-1)c_1c_{n-1}[M] + c_1^2c_{n-2}[M]\right).$$

Corollary 1.6. *For a compact, spin, almost-complex manifold M , the integer*

$$2(n-1)c_1c_{n-1}[M] + c_1^2c_{n-2}[M]$$

is divisible by 8.

Example 1.7. The total Chern class of the complex projective space $\mathbb{C}P^n$ is given by $c(\mathbb{C}P^n) = (1 + g)^{n+1}$, where g is the standard generator of $H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$. $\mathbb{C}P^n$ is spin if and only if n is odd, as $c_1(\mathbb{C}P^n) = (n+1)g$. Suppose $n = 2k + 1$. Then

$$2(n-1)c_1c_{n-1}[\mathbb{C}P^n] + c_1^2c_{n-2}[\mathbb{C}P^n] = 8(k+1)^2(k(2k+1) + \frac{1}{3}k(k+1)(2k+1)).$$

It is easy to check that $\mathbb{C}P^4$ does not satisfy this divisibility result.

2. Proof of the main result

Lemma 2.1. *Let M be a compact, almost-complex manifold. Then:*

$$A_y(M) = \left(\prod_{i=1}^n \left(\frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot e^{-x_i(1+y)/2} \right) \right) [M],$$

$$L_y(M) = \left(\prod_{i=1}^n \left(\frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot (1 + e^{-x_i(1+y)}) \right) \right) [M].$$

Proof. From

$$c(T_M) = \prod_{i=1}^n (1 + x_i)$$

we have (see, for example, [Hirzebruch et al. 1992, p. 11])

$$c(\Lambda^p(T_M^*)) = \prod_{1 \leq i_1 < \dots < i_p \leq n} (1 - (x_{i_1} + \dots + x_{i_p})).$$

Hence

$$ch(\Lambda^p(T_M^*))y^p = \sum_{1 \leq i_1 < \dots < i_p \leq n} e^{-(x_{i_1} + \dots + x_{i_p})} y^p = \sum_{1 \leq i_1 < \dots < i_p \leq n} \left(\prod_{j=1}^p y e^{-x_{i_j}} \right).$$

Therefore

$$\sum_{p=0}^n ch(\Lambda^p(T_M^*))y^p = \sum_{p=0}^n \left(\sum_{1 \leq i_1 < \dots < i_p \leq n} \left(\prod_{j=1}^p y e^{-x_{i_j}} \right) \right) = \prod_{i=1}^n (1 + y e^{-x_i}).$$

So

$$\begin{aligned} (2-1) \quad A_y(M) &= \sum_{p=0}^n \hat{A}(M, \Lambda^p(T_M^*)) \cdot y^p \\ &= \left(\left(\sum_{p=0}^n ch(\Lambda^p(T_M^*))y^p \right) \cdot \prod_{i=1}^n \frac{x_i/2}{\sinh(x_i/2)} \right) [M] \\ &= \left(\prod_{i=1}^n \left((1 + y e^{-x_i}) \cdot \frac{x_i/2}{\sinh(x_i/2)} \right) \right) [M] \\ &= \left(\prod_{i=1}^n \left(\frac{x_i(1 + y e^{-x_i})}{1 - e^{-x_i}} \cdot e^{-x_i/2} \right) \right) [M]. \end{aligned}$$

Since for the evaluation only the homogeneous component of degree n in the x_i enters, then we obtain an additional factor $(1 + y)^n$ if we replace x_i by $x_i(1 + y)$ in (2-1). We therefore obtain

$$\begin{aligned} A_y(M) &= \left(\frac{1}{(1 + y)^n} \prod_{i=1}^n \left(\frac{x_i(1 + y)(1 + y e^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot e^{-x_i(1+y)/2} \right) \right) [M] \\ &= \left(\prod_{i=1}^n \left(\frac{x_i(1 + y e^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot e^{-x_i(1+y)/2} \right) \right) [M]. \end{aligned}$$

Similarly,

$$\begin{aligned}
 L_y(M) &= \left(\prod_{i=1}^n \left((1 + ye^{-x_i}) \cdot \frac{x_i(1 + e^{-x_i})}{1 - e^{-x_i}} \right) \right) [M] \\
 &= \left(\frac{1}{(1+y)^n} \prod_{i=1}^n \left(\frac{x_i(1+y)(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot (1 + e^{-x_i(1+y)}) \right) \right) [M] \\
 &= \left(\prod_{i=1}^n \left(\frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot (1 + e^{x_i(1+y)}) \right) \right) [M]. \quad \square
 \end{aligned}$$

Lemma 2.2. *Set $z = 1 + y$. We have*

$$\begin{aligned}
 A_y(M) &= \left(\prod_{i=1}^n \left((1 + x_i) - (x_i + \frac{1}{2}x_i^2)z + (\frac{11}{24}x_i^2 + \frac{1}{8}x_i^3)z^2 + \dots \right) \right) [M], \\
 L_y(M) &= \left(\prod_{i=1}^n \left(2(1 + x_i) - (2x_i + x_i^2)z + (\frac{7}{6}x_i^2 + \frac{1}{2}x_i^3)z^2 + \dots \right) \right) [M].
 \end{aligned}$$

$$\begin{aligned}
 \text{Proof. } \frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} &= -x_i y + \frac{x_i(1+y)}{1 - e^{-x_i(1+y)}} = -x_i(z-1) + \frac{x_i z}{1 - e^{-x_i z}} \\
 &= -x_i(z-1) + \left(1 + \frac{1}{2}x_i z + \frac{1}{12}x_i^2 z^2 + \dots \right) \\
 &= (1 + x_i) - \frac{1}{2}x_i z + \frac{1}{12}x_i^2 z^2 + \dots
 \end{aligned}$$

So we have

$$\begin{aligned}
 A_y(M) &= \left(\prod_{i=1}^n \frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot e^{-x_i(1+y)/2} \right) [M] \\
 &= \left(\prod_{i=1}^n \left((1 + x_i) - \frac{1}{2}x_i z + \frac{1}{12}x_i^2 z^2 + \dots \right) \left(1 - \frac{1}{2}x_i z + \frac{1}{8}x_i^2 z^2 + \dots \right) \right) [M] \\
 &= \left(\prod_{i=1}^n \left((1 + x_i) - (x_i + \frac{1}{2}x_i^2)z + (\frac{11}{24}x_i^2 + \frac{1}{8}x_i^3)z^2 + \dots \right) \right) [M].
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 L_y(M) &= \left(\prod_{i=1}^n \left(\frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot (1 + e^{-x_i(1+y)}) \right) \right) [M] \\
 &= \left(\prod_{i=1}^n \left((1 + x_i) - \frac{1}{2}x_i z + \frac{1}{12}x_i^2 z^2 + \dots \right) (2 - x_i z + \frac{1}{2}x_i^2 z^2 + \dots) \right) [M] \\
 &= \left(\prod_{i=1}^n \left(2(1 + x_i) - (2x_i + x_i^2)z + (\frac{7}{6}x_i^2 + \frac{1}{2}x_i^3)z^2 + \dots \right) \right) [M]. \quad \square
 \end{aligned}$$

Let $f(x_1, \dots, x_n)$ be a symmetric polynomial in x_1, \dots, x_n . Then $f(x_1, \dots, x_n)$ can be expressed in terms of c_1, \dots, c_n in a unique way. We use $h(f(x_1, \dots, x_n))$ to denote the homogeneous component of degree n in $f(x_1, \dots, x_n)$. For instance, when $n = 3$,

$$\begin{aligned} h(x_1 + x_2 + x_3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2) \\ = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 \\ = (x_1 + x_2 + x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3) - 3x_1 x_2 x_3 = c_1 c_2 - 3c_3. \end{aligned}$$

The next lemma is a crucial technical ingredient in the proof of our main result.

Lemma 2.3.

$$\begin{aligned} (1) \quad h_1 &:= h\left(\sum_{i=1}^n \left(x_i \prod_{j \neq i} (1 + x_j)\right)\right) = n c_n. \\ (2) \quad h_{11} &:= h\left(\sum_{1 \leq i < j \leq n} \left(x_i x_j \prod_{k \neq i, j} (1 + x_k)\right)\right) = \frac{n(n-1)}{2} c_n. \\ (3) \quad h_2 &:= h\left(\sum_{i=1}^n \left(x_i^2 \prod_{j \neq i} (1 + x_j)\right)\right) = -n c_n + c_1 c_{n-1}. \\ (4) \quad h_{12} &:= h\left(\sum_{1 \leq i < j \leq n} \left((x_i^2 x_j + x_i x_j^2) \prod_{k \neq i, j} (1 + x_k)\right)\right) = (n-2)(-n c_n + c_1 c_{n-1}). \\ (5) \quad h_{22} &:= h\left(\sum_{1 \leq i < j \leq n} \left(x_i^2 x_j^2 \prod_{k \neq i, j} (1 + x_k)\right)\right) = \frac{n(n-3)}{2} c_n - (n-2)c_1 c_{n-1} + c_2 c_{n-2}. \\ (6) \quad h_3 &:= h\left(\sum_{i=1}^n \left(x_i^3 \prod_{j \neq i} (1 + x_j)\right)\right) = n c_n - c_1 c_{n-1} + c_1^2 c_{n-2} - 2c_2 c_{n-2}. \end{aligned}$$

Now we can complete the proof of Theorem 1.4; we postpone the proof of Lemma 2.3 to the end of this section.

Proof. From Lemma 2.2, the constant term in $A_y(M)$ is

$$\left(\prod_{i=1}^n (1 + x_i)\right)[M] = c_n[M].$$

The coefficient of z is

$$\begin{aligned} \left(\sum_{i=1}^n \left(-\left(x_i + \frac{1}{2}x_i^2\right) \prod_{j \neq i} (1 + x_j)\right)\right)[M] &= (-h_1 - \frac{1}{2}h_2)[M] \\ &= -\frac{1}{2}(n c_n[M] + c_1 c_{n-1}[M]). \end{aligned}$$

The coefficient of z^2 is

$$\begin{aligned} & \left(\sum_{i=1}^n \left(\left(\frac{11}{24}x_i^2 + \frac{1}{8}x_i^3 \right) \prod_{j \neq i} (1+x_j) \right) + \sum_{1 \leq i < j \leq n} \left((x_i + \frac{1}{2}x_i^2)(x_j + \frac{1}{2}x_j^2) \prod_{k \neq i, j} (1+x_k) \right) \right) [M] \\ &= \left(\frac{11}{24}h_2 + \frac{1}{8}h_3 + h_{11} + \frac{1}{2}h_{12} + \frac{1}{4}h_{22} \right) [M] \\ &= \left(\frac{n(3n-5)}{24}c_n + \frac{3n-2}{12}c_1c_{n-1} + \frac{1}{8}c_1^2c_{n-2} \right) [M]. \end{aligned}$$

Similarly, for $L_y(M)$, the constant term is

$$\left(2^n \prod_{i=1}^n (1+x_i) \right) [M] = 2^n c_n [M].$$

The coefficient of z is

$$\begin{aligned} & \left(\sum_{i=1}^n \left(-(2x_i + x_i^2) \prod_{j \neq i} 2(1+x_j) \right) \right) [M] = (-2^n h_1 - 2^{n-1} h_2) [M] \\ &= -2^{n-1} (nc_n [M] + c_1 c_{n-1} [M]). \end{aligned}$$

The coefficient of z^2 is

$$\begin{aligned} & \left(\sum_{i=1}^n \left(\left(\frac{7}{6}x_i^2 + \frac{1}{2}x_i^3 \right) \prod_{j \neq i} 2(1+x_j) \right) + \sum_{1 \leq i < j \leq n} \left((2x_i + x_i^2)(2x_j + x_j^2) \prod_{k \neq i, j} 2(1+x_k) \right) \right) [M] \\ &= \left(\frac{7 \cdot 2^{n-2}}{3} h_2 + 2^{n-2} h_3 + 2^n h_{11} + 2^{n-1} h_{12} + 2^{n-2} h_{22} \right) [M] \\ &= 2^{n-2} \left(\frac{n(3n-5)}{6} c_n + \frac{3n-2}{3} c_1 c_{n-1} + c_1^2 c_{n-2} - c_2 c_{n-2} \right) [M]. \quad \square \end{aligned}$$

Proof of Lemma 2.3. In the following proof, \hat{x}_i means deleting x_i . Parts (1) and (2) are quite obvious. For (3),

$$\begin{aligned} h_2 &= \sum_{i=1}^n \left(h \left(x_i^2 \prod_{j \neq i} (1+x_j) \right) \right) = \sum_{i=1}^n \left(x_i \sum_{j \neq i} x_1 \cdots \hat{x}_j \cdots x_n \right) = \sum_{i=1}^n (x_i c_{n-1} - c_n) \\ &= -nc_n + c_1 c_{n-1}. \end{aligned}$$

For (4),

$$\begin{aligned} h_{12} &= \sum_{1 \leq i < j \leq n} \left(h \left((x_i^2 x_j + x_i x_j^2) \prod_{k \neq i, j} (1+x_k) \right) \right) \\ &= \sum_{1 \leq i < j \leq n} \left((x_i + x_j) \sum_{k \neq i, j} x_1 \cdots \hat{x}_k \cdots x_n \right) \\ &= (n-2) \sum_{i=1}^n \left(x_i \sum_{k \neq i} x_1 \cdots \hat{x}_k \cdots x_n \right) = (n-2) h_2 = (n-2)(-nc_n + c_1 c_{n-1}). \end{aligned}$$

For (5),

$$\begin{aligned}
 c_2c_{n-2} &= \left(\sum_{1 \leq i < j \leq n} x_i x_j \right) \left(\sum_{1 \leq k < l \leq n} x_1 \cdots \hat{x}_k \cdots \hat{x}_l \cdots x_n \right) \\
 &= \sum_{1 \leq i < j \leq n} \left(x_i x_j \sum_{1 \leq k < l \leq n} x_1 \cdots \hat{x}_k \cdots \hat{x}_l \cdots x_n \right) \\
 &= \sum_{1 \leq i < j \leq n} \left(x_1 x_2 \cdots x_n + (x_i^2 x_j + x_i x_j^2) \sum_{\substack{k \neq i, j \\ 1 \leq k < l \leq n}} x_1 \cdots \hat{x}_k \cdots \hat{x}_l \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n \right. \\
 &\quad \left. + x_i^2 x_j^2 \sum_{\substack{1 \leq k < l \leq n \\ k \neq i, j \\ l \neq i, j}} x_1 \cdots \hat{x}_k \cdots \hat{x}_l \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n \right) \\
 &= \frac{n(n-1)}{2} c_n + h_{12} + h_{22}.
 \end{aligned}$$

Therefore,

$$h_{22} = c_2c_{n-2} - \frac{n(n-1)}{2} c_n - h_{12} = \frac{n(n-3)}{2} c_n - (n-2)c_1c_{n-1} + c_2c_{n-2}.$$

For (6),

$$\begin{aligned}
 &(c_1^2 - 2c_2)c_{n-2} \\
 &= \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{1 \leq j < k \leq n} x_1 \cdots \hat{x}_j \cdots \hat{x}_k \cdots x_n \right) = \sum_{i=1}^n \left(x_i^2 \sum_{1 \leq j < k \leq n} x_1 \cdots \hat{x}_j \cdots \hat{x}_k \cdots x_n \right) \\
 &= \sum_{i=1}^n \left(\left(x_i^3 \sum_{\substack{1 \leq j < k \leq n \\ j \neq i \\ k \neq i}} x_1 \cdots \hat{x}_j \cdots \hat{x}_k \cdots \hat{x}_i \cdots x_n \right) + \left(x_i^2 \sum_{k \neq i} x_1 \cdots \hat{x}_k \cdots \hat{x}_i \cdots x_n \right) \right) \\
 &= h_3 + h_2.
 \end{aligned}$$

Hence $h_3 = (c_1^2 - 2c_2)c_{n-2} - h_2 = nc_n - c_1c_{n-1} + c_1^2c_{n-2} - 2c_2c_{n-2}$. □

3. Concluding remarks

Libgober and Wood’s proof [1990, p. 142, Lemma 2.2] of (1-2) is by induction. Here, using our method, we can give a quite direct proof. We have shown that

$$\begin{aligned}
 \chi_y(M) &= \left(\prod_{i=1}^n \frac{x_i(1 + ye^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \right) [M] \\
 &= \left(\prod_{i=1}^n \left((1 + x_i) - \frac{1}{2}x_iz + \frac{1}{12}x_i^2z^2 + \cdots \right) \right) [M].
 \end{aligned}$$

The coefficient of z^2 is

$$\begin{aligned} & \left(\sum_{i=1}^n \left(\frac{1}{12} x_i^2 \prod_{j \neq i} (1 + x_j) \right) + \sum_{1 \leq i < j \leq n} \left(\frac{1}{4} x_i x_j \prod_{k \neq i, j} (1 + x_k) \right) \right) [M] \\ & = \left(\frac{1}{12} h_2 + \frac{1}{4} h_{11} \right) [M] = \frac{n(3n-5)}{24} c_n [M] + \frac{1}{12} c_1 c_{n-1} [M]. \end{aligned}$$

It is natural to ask what the coefficients are for higher-order terms $(y+1)^p$, for $p \geq 3$. Unfortunately the coefficients become very complicated for such terms. In [Libgober and Wood 1990, pp. 144–145] there is a detailed remark on the coefficients of the higher-order terms of $\chi_y(M)$. Note that the expression of $A_y(M)$ (resp. $L_y(M)$) has an additional factor $e^{-x_i(1+y)/2}$ (resp. $1 + e^{x_i(1+y)}$) relative to than that of $\chi_y(M)$. Hence we cannot expect that there are *explicit* expressions of higher-order coefficients similar to Theorem 1.4.

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