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Libgober and Wood proved that the Chern number c_1c_{n-1} of a compact complex manifold of dimension n can be determined by its Hirzebruch χ_y-genus. Inspired by the idea of their proof, we show that, for compact, spin, almost-complex manifolds, more Chern numbers can be determined by the indices of some twisted Dirac and signature operators. As a byproduct, we get a divisibility result of certain characteristic number for such manifolds. Using our method, we give a direct proof of the result of Libgober and Wood, which was originally proved by induction.

1. Introduction and main results

Suppose (M, J) is a compact, almost-complex $2n$-manifold with a given almost complex structure J. This J makes the tangent bundle of M into a n-dimensional complex vector bundle T_M. Let $c_i(M, J) \in H^{2i}(M; \mathbb{Z})$ be the i-th Chern class of T_M. Suppose we have a formal factorization of the total Chern class as follows:

$$1 + c_1(M, J) + \cdots + c_n(M, J) = \prod_{i=1}^{n}(1 + x_i),$$

i.e., $x_1, \ldots, x_n$ are formal Chern roots of T_M. The Hirzebruch χ_y-genus of (M, J), $\chi_y(M, J)$, is defined by

$$\chi_y(M, J) = \left(\prod_{i=1}^{n} \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}\right)[M].$$

Here $[M]$ is the fundamental class of the orientation of M induced by J and y is an indeterminate. If J is specified, we simply denote $\chi_y(M, J)$ by $\chi_y(M)$.

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When the almost complex structure J is integrable (equivalently, when M is an n-dimensional compact complex manifold), $\chi_y(M)$ can be obtained by

$$\chi^p(M)=\sum_{q=0}^{n}(-1)^q h^{p,q}(M),\ \ \chi_y(M)=\sum_{p=0}^{n}\chi^p(M)\cdot y^p,$$

where $h^{p,q}(\,\cdot\,)$ is the Hodge number of type (p,q). This is given by the Hirzebruch–Riemann–Roch Theorem, proved in [Hirzebruch 1966] for projective manifolds and in [Atiyah and Singer 1968] in the general case.

The formula

$$\sum_{p=0}^{n}\chi^p(M)\cdot y^p=\left(\prod_{i=1}^{n}\frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}}\right)[M] \tag{1-1}$$

implies that $\chi^p(M)$, the index of the Dolbeault complex, can be expressed as a rational combination of some Chern numbers of M. Conversely, we can address the following question.

Question 1.1. For an n-dimensional compact complex manifold M, given a partition $\lambda=\lambda_1\lambda_2\cdots\lambda_l$ of weight n, can the Chern number $c_{\lambda_1}c_{\lambda_2}\cdots c_{\lambda_l}[M]$ be determined by $\chi^p(M)$, or more generally by the indices of some other elliptic differential operators?

For the simplest case $c_n[M]$, the answer is affirmative and well-known [Hirzebruch 1966, Theorem 15.8.1]:

$$c_n[M]=\chi_y(M)|_{y=-1}=\sum_{p=0}^{n}(-1)^p\chi^p(M).$$

The next-to-simplest case is the Chern number $c_1c_{n-1}[M]$. The answer here is also affirmative, as was proved by Libgober and Wood [1990, pp. 141–143]:

$$\sum_{p=2}^{n}(-1)^p\binom{p}{2}\chi^p(M)=\frac{n(3n-5)}{24}c_n[M]+\frac{1}{12}c_1c_{n-1}[M]. \tag{1-2}$$

The idea of their proof is quite enlightening: expanding both sides of (1-1) at $y=-1$ and comparing the coefficients of the term $(y+1)^2$, one gets (1-2).

Inspired by this idea, in this paper we consider twisted Dirac operators and signature operators on compact, *spin, almost-complex* manifolds and show that the Chern numbers c_n, c_1c_{n-1}, $c_1^2c_{n-2}$ and c_2c_{n-2} of such manifolds can be determined by the indices of these operators.

Remark 1.2. Equation (1-2) was also obtained later by Salamon [1996, p. 144], who applied this result extensively to hyper-Kähler manifolds.

Let M be a compact, *almost-complex* $2n$-manifold. We still use $x_1, \dots, x_n$ to denote the corresponding formal Chern roots of the n-dimensional complex vector bundle T_M. Suppose E is a complex vector bundle over M. Set

$$\hat{A}(M, E) := \left(\operatorname{ch}(E) \cdot \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)}\right)[M],$$
$$L(M, E) := \left(\operatorname{ch}(E) \cdot \prod_{i=1}^{n} \frac{x_i(1+e^{-x_i})}{1-e^{-x_i}}\right)[M],$$

where $\operatorname{ch}(E)$ is the Chern character of E. The celebrated Atiyah–Singer index theorem [Hirzebruch et al. 1992, pp. 74–81] states that $L(M, E)$ is the index of the signature operator twisted by E and when M is *spin*, $\hat{A}(M, E)$ is the index of the Dirac operator twisted by E.

Definition 1.3. Set

$$A_y(M) := \sum_{p=0}^{n} \hat{A}(M, \Lambda^p(T_M^*)) \cdot y^p \quad \text{and} \quad L_y(M) := \sum_{p=0}^{n} L(M, \Lambda^p(T_M^*)) \cdot y^p,$$

where $\Lambda^p(T_M^*)$ denotes the p-th exterior power of the dual of T_M.

Our main result is the following:

Theorem 1.4. *Let M be a compact, almost-complex manifold.*

(1)
$$\sum_{p=0}^{n} (-1)^p \hat{A}(M, \Lambda^p(T_M^*)) = c_n[M],$$
$$\sum_{p=1}^{n} (-1)^p \cdot p \cdot \hat{A}(M, \Lambda^p(T_M^*)) = \frac{1}{2}\big(nc_n[M] + c_1c_{n-1}[M]\big),$$
$$\sum_{p=2}^{n} (-1)^p \binom{p}{2} \hat{A}(M, \Lambda^p(T_M^*)) = \Big(\frac{n(3n-5)}{24}c_n + \frac{3n-2}{12}c_1c_{n-1} + \frac{1}{8}c_1^2c_{n-2}\Big)[M];$$

(2)
$$\sum_{p=0}^{n} (-1)^p L(M, \Lambda^p(T_M^*)) = 2^n c_n[M],$$
$$\sum_{p=1}^{n} (-1)^p \cdot p \cdot L(M, \Lambda^p(T_M^*)) = 2^{n-1}\big(nc_n[M] + c_1c_{n-1}[M]\big),$$
$$\sum_{p=2}^{n} (-1)^p \binom{p}{2} L(M, \Lambda^p(T_M^*))$$
$$= 2^{n-2}\Big(\frac{n(3n-5)}{6}c_n + \frac{3n-2}{3}c_1c_{n-1} + c_1^2c_{n-2} - c_2c_{n-2}\Big)[M].$$

Corollary 1.5. (1) *The Chern numbers* $c_n[M]$, $c_1c_{n-1}[M]$ *and* $c_1^2c_{n-2}[M]$ *can be determined by* $A_y(M)$.

(2) *The characteristic numbers* $c_n[M]$, $c_1c_{n-1}[M]$ *and* $c_1^2c_{n-2}[M]-c_2c_{n-2}[M]$ *can be determined by* $L_y(M)$.

(3) *When* M *is a spin manifold, the Chern numbers* $c_n[M]$, $c_1c_{n-1}[M]$, $c_1^2c_{n-2}[M]$ *and* $c_2c_{n-2}[M]$ *can be expressed by using linear combinations of the indices of some twisted Dirac and signature operators.*

As remarked in [Libgober and Wood 1990, p. 143], it was shown by Milnor [1960] that every complex cobordism class contains a non-singular algebraic variety. Milnor also showed that two almost-complex manifolds are complex cobordant if and only if they have the same Chern numbers. Hence Libgober and Wood's result implies that the characteristic number

$$\frac{n(3n-5)}{24}c_n[N]+\frac{1}{12}c_1c_{n-1}[N]$$

is always an integer for any compact, *almost-complex* $2n$-manifold N.

Note that the right-hand side of the third equality in Theorem 1.4 is

$$\Big(\frac{n(3n-5)}{24}c_n[M]+\frac{1}{12}c_1c_{n-1}[M]\Big)+\frac{1}{8}\Big(2(n-1)c_1c_{n-1}[M]+c_1^2c_{n-2}[M]\Big).$$

Corollary 1.6. *For a compact, spin, almost-complex manifold* M, *the integer*

$$2(n-1)c_1c_{n-1}[M]+c_1^2c_{n-2}[M]$$

is divisible by 8.

Example 1.7. The total Chern class of the complex projective space $\mathbb{C}P^n$ is given by $c(\mathbb{C}P^n)=(1+g)^{n+1}$, where g is the standard generator of $H^2(\mathbb{C}P^n;\mathbb{Z})\cong\mathbb{Z}$. $\mathbb{C}P^n$ is spin if and only if n is odd, as $c_1(\mathbb{C}P^n)=(n+1)g$. Suppose $n=2k+1$. Then

$$2(n-1)c_1c_{n-1}[\mathbb{C}P^n]+c_1^2c_{n-2}[\mathbb{C}P^n]=8(k+1)^2\big(k(2k+1)+\tfrac{1}{3}k(k+1)(2k+1)\big).$$

It is easy to check that $\mathbb{C}P^4$ does not satisfy this divisibility result.

2. Proof of the main result

Lemma 2.1. *Let* M *be a compact, almost-complex manifold. Then*:

$$A_y(M)=\bigg(\prod_{i=1}^{n}\Big(\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot e^{-x_i(1+y)/2}\Big)[M],$$

$$L_y(M)=\bigg(\prod_{i=1}^{n}\Big(\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{-x_i(1+y)})\Big)\bigg)[M].$$

Proof. From

$$c(T_M) = \prod_{i=1}^{n} (1 + x_i)$$

we have (see, for example, [Hirzebruch et al. 1992, p. 11])

$$c(\Lambda^p(T_M^*)) = \prod_{1 \le i_1 < \cdots < i_p \le n} \big(1 - (x_{i_1} + \cdots + x_{i_p})\big).$$

Hence

$$ch(\Lambda^p(T_M^*))y^p = \sum_{1 \le i_1 < \cdots < i_p \le n} e^{-(x_{i_1} + \cdots + x_{i_p})} y^p = \sum_{1 \le i_1 < \cdots < i_p \le n} \bigg(\prod_{j=1}^{p} y e^{-x_{i_j}} \bigg).$$

Therefore

$$\sum_{p=0}^{n} ch(\Lambda^p(T_M^*))y^p = \sum_{p=0}^{n} \bigg(\sum_{1 \le i_1 < \cdots < i_p \le n} \bigg(\prod_{j=1}^{p} y e^{-x_{i_j}} \bigg) \bigg) = \prod_{i=1}^{n} (1 + y e^{-x_i}).$$

So

$$\begin{aligned} \text{(2-1)} \qquad A_y(M) &= \sum_{p=0}^{n} \hat{A}(M, \Lambda^p(T_M^*)) \cdot y^p \\ &= \bigg(\bigg(\sum_{p=0}^{n} ch(\Lambda^p(T_M^*))y^p \bigg) \cdot \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)} \bigg)[M] \\ &= \bigg(\prod_{i=1}^{n} \bigg((1 + y e^{-x_i}) \cdot \frac{x_i/2}{\sinh(x_i/2)} \bigg) \bigg)[M] \\ &= \bigg(\prod_{i=1}^{n} \bigg(\frac{x_i(1 + y e^{-x_i})}{1 - e^{-x_i}} \cdot e^{-x_i/2} \bigg) \bigg)[M]. \end{aligned}$$

Since for the evaluation only the homogeneous component of degree n in the x_i enters, then we obtain an additional factor $(1+y)^n$ if we replace x_i by $x_i(1+y)$ in (2-1). We therefore obtain

$$\begin{aligned} A_y(M) &= \bigg(\frac{1}{(1+y)^n} \prod_{i=1}^{n} \bigg(\frac{x_i(1+y)(1 + y e^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot e^{-x_i(1+y)/2} \bigg) \bigg)[M] \\ &= \bigg(\prod_{i=1}^{n} \bigg(\frac{x_i(1 + y e^{-x_i(1+y)})}{1 - e^{-x_i(1+y)}} \cdot e^{-x_i(1+y)/2} \bigg) \bigg)[M]. \end{aligned}$$

Similarly,

$$\begin{aligned}
L_y(M) &= \left(\prod_{i=1}^{n}\left((1+ye^{-x_i})\cdot\frac{x_i(1+e^{-x_i})}{1-e^{-x_i}}\right)\right)[M] \\
&= \left(\frac{1}{(1+y)^n}\prod_{i=1}^{n}\left(\frac{x_i(1+y)(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{-x_i(1+y)})\right)\right)[M] \\
&= \left(\prod_{i=1}^{n}\left(\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{x_i(1+y)})\right)\right)[M]. \qquad \square
\end{aligned}$$

Lemma 2.2. *Set $z = 1 + y$. We have*

$$\begin{aligned}
A_y(M) &= \left(\prod_{i=1}^{n}\left((1+x_i)-(x_i+\tfrac{1}{2}x_i^2)z+\left(\tfrac{11}{24}x_i^2+\tfrac{1}{8}x_i^3\right)z^2+\cdots\right)\right)[M], \\
L_y(M) &= \left(\prod_{i=1}^{n}\left(2(1+x_i)-(2x_i+x_i^2)z+\left(\tfrac{7}{6}x_i^2+\tfrac{1}{2}x_i^3\right)z^2+\cdots\right)\right)[M].
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}} &= -x_iy+\frac{x_i(1+y)}{1-e^{-x_i(1+y)}} = -x_i(z-1)+\frac{x_iz}{1-e^{-x_iz}} \\
&= -x_i(z-1)+\left(1+\tfrac{1}{2}x_iz+\tfrac{1}{12}x_i^2z^2+\cdots\right) \\
&= (1+x_i)-\tfrac{1}{2}x_iz+\tfrac{1}{12}x_i^2z^2+\cdots .
\end{aligned}$$

So we have

$$\begin{aligned}
A_y(M) &= \left(\prod_{i=1}^{n}\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot e^{-x_i(1+y)/2}\right)[M] \\
&= \left(\prod_{i=1}^{n}\left((1+x_i)-\tfrac{1}{2}x_iz+\tfrac{1}{12}x_i^2z^2+\cdots\right)\left(1-\tfrac{1}{2}x_iz+\tfrac{1}{8}x_i^2z^2+\cdots\right)\right)[M] \\
&= \left(\prod_{i=1}^{n}\left((1+x_i)-(x_i+\tfrac{1}{2}x_i^2)z+\left(\tfrac{11}{24}x_i^2+\tfrac{1}{8}x_i^3\right)z^2+\cdots\right)\right)[M].
\end{aligned}$$

Similarly,

$$\begin{aligned}
L_y(M) &= \left(\prod_{i=1}^{n}\left(\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{-x_i(1+y)})\right)\right)[M] \\
&= \left(\prod_{i=1}^{n}\left((1+x_i)-\tfrac{1}{2}x_iz+\tfrac{1}{12}x_i^2z^2+\cdots\right)\left(2-x_iz+\tfrac{1}{2}x_i^2z^2+\cdots\right)\right)[M] \\
&= \left(\prod_{i=1}^{n}\left(2(1+x_i)-(2x_i+x_i^2)z+\left(\tfrac{7}{6}x_i^2+\tfrac{1}{2}x_i^3\right)z^2+\cdots\right)\right)[M]. \qquad \square
\end{aligned}$$

Let $f(x_1,\dots,x_n)$ be a symmetric polynomial in $x_1,\dots,x_n$. Then $f(x_1,\dots,x_n)$ can be expressed in terms of $c_1,\dots,c_n$ in a unique way. We use $h(f(x_1,\dots,x_n))$ to denote the homogeneous component of degree n in $f(x_1,\dots,x_n)$. For instance, when $n=3$,

$$\begin{aligned}h(x_1+x_2+x_3+x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2)\\ &= x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2\\ &= (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)-3x_1x_2x_3=c_1c_2-3c_3.\end{aligned}$$

The next lemma is a crucial technical ingredient in the proof of our main result.

Lemma 2.3.

(1) $h_1 := h\Big(\sum_{i=1}^{n}\Big(x_i\prod_{j\neq i}(1+x_j)\Big)\Big)=nc_n.$

(2) $h_{11} := h\Big(\sum_{1\le i<j\le n}\Big(x_ix_j\prod_{k\neq i,j}(1+x_k)\Big)\Big)=\dfrac{n(n-1)}{2}c_n.$

(3) $h_2 := h\Big(\sum_{i=1}^{n}\Big(x_i^2\prod_{j\neq i}(1+x_j)\Big)\Big)=-nc_n+c_1c_{n-1}.$

(4) $h_{12} := h\Big(\sum_{1\le i<j\le n}\Big((x_i^2x_j+x_ix_j^2)\prod_{k\neq i,j}(1+x_k)\Big)\Big)=(n-2)(-nc_n+c_1c_{n-1}).$

(5) $h_{22} := h\Big(\sum_{1\le i<j\le n}\Big(x_i^2x_j^2\prod_{k\neq i,j}(1+x_k)\Big)\Big)=\dfrac{n(n-3)}{2}c_n-(n-2)c_1c_{n-1}+c_2c_{n-2}.$

(6) $h_3 := h\Big(\sum_{i=1}^{n}\Big(x_i^3\prod_{j\neq i}(1+x_j)\Big)\Big)=nc_n-c_1c_{n-1}+c_1^2c_{n-2}-2c_2c_{n-2}.$

Now we can complete the proof of Theorem 1.4; we postpone the proof of Lemma 2.3 to the end of this section.

Proof. From Lemma 2.2, the constant term in $A_y(M)$ is

$$\Big(\prod_{i=1}^{n}(1+x_i)\Big)[M]=c_n[M].$$

The coefficient of z is

$$\begin{aligned}\Big(\sum_{i=1}^{n}\Big(-\big(x_i+\tfrac12x_i^2\big)\prod_{j\neq i}(1+x_j)\Big)\Big)[M] &= \big(-h_1-\tfrac12h_2\big)[M]\\ &= -\tfrac12\big(nc_n[M]+c_1c_{n-1}[M]\big).\end{aligned}$$

The coefficient of z^2 is

$$\Big(\sum_{i=1}^{n}\Big(\big(\tfrac{11}{24}x_i^2+\tfrac{1}{8}x_i^3\big)\prod_{j\neq i}(1+x_j)\Big)+\sum_{1\le i<j\le n}\Big((x_i+\tfrac{1}{2}x_i^2)(x_j+\tfrac{1}{2}x_j^2)\prod_{k\neq i,j}(1+x_k)\Big)\Big)[M]$$
$$=\big(\tfrac{11}{24}h_2+\tfrac{1}{8}h_3+h_{11}+\tfrac{1}{2}h_{12}+\tfrac{1}{4}h_{22}\big)[M]$$
$$=\Big(\frac{n(3n-5)}{24}c_n+\frac{3n-2}{12}c_1c_{n-1}+\frac{1}{8}c_1^2c_{n-2}\Big)[M].$$

Similarly, for $L_y(M)$, the constant term is

$$\Big(2^n\prod_{i=1}^{n}(1+x_i)\Big)[M]=2^nc_n[M].$$

The coefficient of z is

$$\Big(\sum_{i=1}^{n}\Big(-(2x_i+x_i^2)\prod_{j\neq i}2(1+x_j)\Big)\Big)[M]=(-2^nh_1-2^{n-1}h_2)[M]$$
$$=-2^{n-1}(nc_n[M]+c_1c_{n-1}[M]).$$

The coefficient of z^2 is

$$\Big(\sum_{i=1}^{n}\Big(\big(\tfrac{7}{6}x_i^2+\tfrac{1}{2}x_i^3\big)\prod_{j\neq i}2(1+x_j)\Big)+\sum_{1\le i<j\le n}\Big((2x_i+x_i^2)(2x_j+x_j^2)\prod_{k\neq i,j}2(1+x_k)\Big)\Big)[M]$$
$$=\Big(\frac{7\cdot 2^{n-2}}{3}h_2+2^{n-2}h_3+2^nh_{11}+2^{n-1}h_{12}+2^{n-2}h_{22}\Big)[M]$$
$$=2^{n-2}\Big(\frac{n(3n-5)}{6}c_n+\frac{3n-2}{3}c_1c_{n-1}+c_1^2c_{n-2}-c_2c_{n-2}\Big)[M].\quad\square$$

Proof of Lemma 2.3. In the following proof, $\hat{x_i}$ means deleting x_i. Parts (1) and (2) are quite obvious. For (3),

$$h_2=\sum_{i=1}^{n}\Big(h\Big(x_i^2\prod_{j\neq i}(1+x_j)\Big)\Big)=\sum_{i=1}^{n}\Big(x_i\sum_{j\neq i}x_1\cdots\hat{x_j}\cdots x_n\Big)=\sum_{i=1}^{n}(x_ic_{n-1}-c_n)$$
$$=-nc_n+c_1c_{n-1}.$$

For (4),

$$h_{12}=\sum_{1\le i<j\le n}\Big(h\Big((x_i^2x_j+x_ix_j^2)\prod_{k\neq i,j}(1+x_k)\Big)\Big)$$
$$=\sum_{1\le i<j\le n}\Big((x_i+x_j)\sum_{k\neq i,j}x_1\cdots\hat{x_k}\cdots x_n\Big)$$
$$=(n-2)\sum_{i=1}^{n}\Big(x_i\sum_{k\neq i}x_1\cdots\hat{x_k}\cdots x_n\Big)=(n-2)h_2=(n-2)(-nc_n+c_1c_{n-1}).$$

For (5),

$$\begin{aligned}
c_2c_{n-2} &= \Big(\sum_{1\le i<j\le n} x_ix_j\Big)\Big(\sum_{1\le k<l\le n} x_1\cdots\hat{x}_k\cdots\hat{x}_l\cdots x_n\Big)\\
&= \sum_{1\le i<j\le n}\Big(x_ix_j\sum_{1\le k<l\le n} x_1\cdots\hat{x}_k\cdots\hat{x}_l\cdots x_n\Big)\\
&= \sum_{1\le i<j\le n}\Big(x_1x_2\cdots x_n + (x_i^2x_j+x_ix_j^2)\sum_{k\ne i,j} x_1\cdots\hat{x}_k\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n\\
&\qquad\qquad + x_i^2x_j^2\sum_{\substack{1\le k<l\le n\\ k\ne i,j\\ l\ne i,j}} x_1\cdots\hat{x}_k\cdots\hat{x}_l\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n\Big)\\
&= \frac{n(n-1)}{2}c_n + h_{12} + h_{22}.
\end{aligned}$$

Therefore,

$$h_{22} = c_2c_{n-2} - \frac{n(n-1)}{2}c_n - h_{12} = \frac{n(n-3)}{2}c_n - (n-2)c_1c_{n-1} + c_2c_{n-2}.$$

For (6),

$$\begin{aligned}
&(c_1^2-2c_2)c_{n-2}\\
&= \Big(\sum_{i=1}^n x_i^2\Big)\Big(\sum_{1\le j<k\le n} x_1\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n\Big) = \sum_{i=1}^n\Big(x_i^2\sum_{1\le j<k\le n} x_1\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n\Big)\\
&= \sum_{i=1}^n\Big(\Big(x_i^3\sum_{\substack{1\le j<k\le n\\ j\ne i\\ k\ne i}} x_1\cdots\hat{x}_j\cdots\hat{x}_k\cdots\hat{x}_i\cdots x_n\Big) + \Big(x_i^2\sum_{k\ne i} x_1\cdots\hat{x}_k\cdots\hat{x}_i\cdots x_n\Big)\Big)\\
&= h_3 + h_2.
\end{aligned}$$

Hence $h_3 = (c_1^2-2c_2)c_{n-2} - h_2 = nc_n - c_1c_{n-1} + c_1^2c_{n-2} - 2c_2c_{n-2}$. □

3. Concluding remarks

Libgober and Wood's proof [1990, p. 142, Lemma 2.2] of (1-2) is by induction. Here, using our method, we can give a quite direct proof. We have shown that

$$\begin{aligned}
\chi_y(M) &= \Big(\prod_{i=1}^n \frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\Big)[M]\\
&= \Big(\prod_{i=1}^n\Big((1+x_i) - \tfrac{1}{2}x_iz + \tfrac{1}{12}x_i^2z^2 + \cdots\Big)\Big)[M].
\end{aligned}$$

The coefficient of z^2 is

$$\left(\sum_{i=1}^{n}\Big(\frac{1}{12}x_i^2\prod_{j\neq i}(1+x_j)\Big)+\sum_{1\le i<j\le n}\Big(\frac{1}{4}x_ix_j\prod_{k\neq i,j}(1+x_k)\Big)\right)[M]$$
$$=\big(\tfrac{1}{12}h_2+\tfrac{1}{4}h_{11}\big)[M]=\frac{n(3n-5)}{24}c_n[M]+\frac{1}{12}c_1c_{n-1}[M].$$

It is natural to ask what the coefficients are for higher-order terms $(y+1)^p$, for $p\ge 3$. Unfortunately the coefficients become very complicated for such terms. In [Libgober and Wood 1990, pp. 144–145] there is a detailed remark on the coefficients of the higher-order terms of $\chi_y(M)$. Note that the expression of $A_y(M)$ (resp. $L_y(M)$) has an additional factor $e^{-x_i(1+y)/2}$ (resp. $1+e^{x_i(1+y)}$) relative to than that of $\chi_y(M)$. Hence we cannot expect that there are *explicit* expressions of higher-order coefficients similar to Theorem 1.4.

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