BLOCKS OF THE CATEGORY OF CUSPIDAL $\mathfrak{sp}_{2n}$-MODULES

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In this paper we show that every block of the category of cuspidal generalized weight modules with finite dimensional generalized weight spaces over the Lie algebra $sp_{2n}(\mathbb{C})$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1, t_2, \ldots, t_n]]$-modules.

1. Introduction and description of the results

Fix the ground field to be the complex numbers. Fix $n \in \{2, 3, \ldots\}$ and consider the symplectic Lie algebra $sp_{2n} =: g$ with a fixed Cartan subalgebra $\mathfrak{h}$ and root space decomposition

$$g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\Delta$ denotes the corresponding root system. For a $g$-module $V$ and $\lambda \in \mathfrak{h}^*$ set

$$V_\lambda := \{ v \in V : h \cdot v = \lambda(h)v \text{ for any } h \in \mathfrak{h} \},$$

$$V^\lambda := \{ v \in V : (h - \lambda(h))^k \cdot v = 0 \text{ for any } h \in \mathfrak{h} \text{ and } k \gg 0 \}.$$

A $g$-module $V$ is called

- a weight module provided that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$;
- a generalized weight module provided that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$;
- a cuspidal module provided that for any $\alpha \in \Delta$ the action of any nonzero element from $\mathfrak{g}_\alpha$ on $V$ is bijective.

If $V$ is a generalized weight module, then the set $\{ \lambda \in \mathfrak{h}^* : V_\lambda \neq 0 \}$ is called the support of $V$ and is denoted by $\text{supp}(V)$.

Denote by $\hat{\mathcal{C}}$ the full subcategory in $g$-mod that consists of all cuspidal generalized weight modules with finite dimensional generalized weight spaces, and by $\mathcal{C}$ the full subcategory of $\hat{\mathcal{C}}$ consisting of all weight modules. Understanding the categories $\mathcal{C}$ and $\hat{\mathcal{C}}$ is a classical problem in the representation theory of Lie algebras. The first major step towards the solution of this problem was made in [Mathieu 2000], where all simple objects in $\hat{\mathcal{C}}$ were classified. Britten et al. [2004]


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showed that the category $\mathcal{C}$ is semisimple, hence completely understood. The aim of the present note is to describe the category $\hat{\mathcal{C}}$.

Apart from $\mathfrak{sp}_{2n}$, cuspidal weight modules with finite dimensional weight spaces exist only for the Lie algebra $\mathfrak{sl}_n$ [Fernando 1990]. In the latter case, simple objects in the corresponding category $\hat{\mathcal{C}}$ are classified in [Mathieu 2000], the category $\mathcal{C}$ is described in [Grantcharov and Serganova 2010] (see also [Mazorchuk and Stroppel 2011]), and the category $\hat{\mathcal{C}}$ is described in [Mazorchuk and Stroppel 2011]. Taking all these results into account, the present paper completes the study of cuspidal generalized weight modules with finite dimensional generalized weight spaces over semisimple finite dimensional Lie algebras.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. The action of $Z(\mathfrak{g})$ on any object from $\hat{\mathcal{C}}$ is locally finite. Using this and the standard support arguments gives the following block decomposition of $\hat{\mathcal{C}}$:

$$\hat{\mathcal{C}} \cong \bigoplus_{\chi : Z(\mathfrak{g}) \to \mathbb{C}} \hat{\mathcal{C}}_{\chi, \xi},$$

where $\hat{\mathcal{C}}_{\chi, \xi}$ consists of all $V$ such that $\text{supp}(V) \subset \xi$ and $(z - \chi(z))^k \cdot v = 0$ for all $v \in V$, $z \in Z(\mathfrak{g})$ and $k \gg 0$. Set

$$\mathcal{C}_{\chi, \xi} := \mathcal{C} \cap \hat{\mathcal{C}}_{\chi, \xi}.$$ 

From [Mathieu 2000, Section 9] it follows that each nontrivial $\hat{\mathcal{C}}_{\chi, \xi}$ contains a unique (up to isomorphism) simple object. In particular, $\hat{\mathcal{C}}_{\chi, \xi}$ is indecomposable, hence a block. From this and [Britten et al. 2004] we thus get that every nontrivial block $\mathcal{C}_{\chi, \xi}$ is equivalent to the category of finite dimensional $\mathbb{C}$-modules. Our main result is the following:

**Theorem 1.** Every nontrivial block $\hat{\mathcal{C}}_{\chi, \xi}$ is equivalent to the category of finite dimensional $\mathbb{C}[\llbracket t_1, t_2, \ldots, t_n \rrbracket]$-modules.

To prove Theorem 1 we use and further develop the technique of extension of the module structure from a Lie subalgebra, originally developed in [Mazorchuk and Stroppel 2011] for the study of categories of singular and nonintegral cuspidal generalized weight $\mathfrak{sl}_n$-modules. The proof of Theorem 1 is given in Section 4. In Section 2 we recall the standard reduction to the case of the so-called simple completely pointed modules (that is, simple weight cuspidal modules for which all nontrivial weight spaces are one-dimensional) and a realization of such modules using differential operators. In Section 3 we define a functor from the category of finite dimensional $\mathbb{C}[\llbracket t_1, t_2, \ldots, t_n \rrbracket]$-modules to any block $\hat{\mathcal{C}}_{\chi, \xi}$ containing a simple completely pointed module. In Section 4 we prove that this functor is an equivalence of categories. In Section 5 we present some consequences of our main result. In particular, we recover the main result of [Britten et al. 2004] stated above.
2. Completely pointed simple cuspidal weight modules

A weight $\mathfrak{g}$-module $V$ is called pointed provided that $\dim V_\lambda = 1$ for some $\lambda \in \mathfrak{h}^*$. If $V$ is a pointed simple cuspidal weight $\mathfrak{g}$-module, then, obviously, all nontrivial weight spaces of $V$ are one-dimensional, in which case one says that $V$ is completely pointed (see [Britten et al. 2004]). It is enough to consider blocks with completely pointed simple modules because of the following:

**Lemma 2.** All nontrivial blocks of $\mathcal{C}$ are equivalent.

**Proof.** In the case of the category $\mathcal{C}$, this is proved in [Britten et al. 2004, Lemma 2]. The same argument works in the case of the category $\mathcal{\hat{C}}$. □

We recall the explicit realization of completely pointed simple cuspidal modules from [Britten and Lemire 1987]. Denote by $W_n$ the $n$-th Weyl algebra, that is, the algebra of differential operators with polynomial coefficients in variables $x_1, x_2, \ldots, x_n$. The algebra $W_n$ is generated by $x_i$ and $\partial/\partial x_i$, $i = 1, \ldots, n$, which satisfy the relations $[\partial/\partial x_i, x_j] = \delta_{i,j}$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the vectors of the standard basis in $\mathbb{C}^n$. Identify $\mathbb{C}^n$ with $\mathfrak{h}^*$ such that $\Delta$ becomes the following standard root system of type $C_n$:

$$\{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq n\}.$$

Then

$$H = H_n = \{2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n-1}\}$$

is a basis of $\Delta$. Fix a basis of $\mathfrak{g}$ of the form

$$C := \{X_{\pm \varepsilon_i \pm \varepsilon_j} : 1 \leq i < j \leq n\} \cup \{X_{\pm 2\varepsilon_i} : i = 1, 2, \ldots, n\} \cup \{H_\alpha : \alpha \in H\}$$

such that the following map defines an injective Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra associated with $W_n$:

$$X_{\varepsilon_i - \varepsilon_j} \mapsto x_i \frac{\partial}{\partial x_j}, \quad 1 \leq i \neq j \leq n,$$

$$X_{\varepsilon_i + \varepsilon_j} \mapsto x_i x_j, \quad i, j = 1, 2, \ldots, n,$$

$$X_{-\varepsilon_i - \varepsilon_j} \mapsto \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}, \quad i, j = 1, 2, \ldots, n,$$

$$H_{\varepsilon_i + \varepsilon_j} \mapsto x_{i+1} \frac{\partial}{\partial x_{i+1}} - x_i \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n-1,$$

$$H_{2\varepsilon_1} \mapsto \frac{1}{2} \left(x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} x_1\right).$$

Set

$$B := \{(b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n : b_1 + b_2 + \cdots + b_n \in 2\mathbb{Z}\}.$$

For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ define $N(a)$ to be the linear span of

$$\{x^b := x_1^{a_1+b_1} x_2^{a_2+b_2} \cdots x_n^{a_n+b_n} : b \in B\}.$$
First define an action of the elements from $C$ on $N(a)$ using the formulae from (1) as follows:

\[
\begin{align*}
X_{\epsilon_i - \epsilon_j} x^b &= (a_j + b_j)x^{b + \epsilon_i - \epsilon_j} & 1 \leq i \neq j \leq n, \\
X_{\epsilon_i + \epsilon_j} x^b &= x^{b + \epsilon_i + \epsilon_j} & i, j = 1, 2, \ldots, n, \\
X_{-\epsilon_i - \epsilon_j} x^b &= (a_i + b_i)(a_j + b_j)x^{b - \epsilon_i - \epsilon_j} & 1 \leq i \neq j \leq n, \\
X_{-2\epsilon_i} x^b &= (a_i + b_i)(a_i + b_i - 1)x^{b - 2\epsilon_i} & i = 1, 2, \ldots, n, \\
H_{\epsilon_i + 1 - \epsilon_i} x^b &= (a_{i+1} + b_{i+1} - a_i - b_i)x^b & i = 1, 2, \ldots, n - 1, \\
H_{2\epsilon_i} x^b &= \frac{1}{2}(2a_1 + 2b_1 + 1)x^b.
\end{align*}
\]

(2)

**Theorem 3** [Britten and Lemire 1987]. (i) For every $a \in \mathbb{C}^n$ the formulae in (2) define on $N(a)$ the structure of a completely pointed weight $\mathfrak{g}$-module.

(ii) If $a_i \not\in \mathbb{Z}$ for all $i = 1, \ldots, n$, then the module $N(a)$ is simple and cuspidal.

(iii) Every completely pointed simple cuspidal $\mathfrak{g}$-module is isomorphic to $N(a)$ for some $a \in \mathbb{C}^n$ such that $a_i \not\in \mathbb{Z}$, $i = 1, \ldots, n$.

### 3. The functor $\mathbf{F}$

This section is similar to [Mazorchuk and Stroppel 2011, Section 3.1]. Fix $a \in \mathbb{C}^n$ such that $a_i \not\in \mathbb{Z}$, $i = 1, \ldots, n$. Let $\hat{C}_a$ denote the block of $\hat{C}$ containing $N(a)$. The category $\hat{C}_a$ is closed under extensions. Denote the category of finite dimensional $\mathbb{C}[t_1, t_2, \ldots, t_n]$-modules by $\mathbb{C}[t_1, t_2, \ldots, t_n]$-mod. For $V \in \mathbb{C}[t_1, t_2, \ldots, t_n]$-mod denote by $T_i$ the linear operator describing the action of $t_i$ on $V$. Set $0 = (0, 0, \ldots, 0) \in B$.

For $b \in B$ consider a copy $V^b$ of $V$. Define

\[
FV := \bigoplus_{b \in B} V^b.
\]

Define the action of elements from $C$ on the vector space $FV$ in the following way: for $v \in V^b$ set

\[
\begin{align*}
X_{\epsilon_i - \epsilon_j} v &= (T_j + (a_j + b_j) \text{Id}_V)v & \in V^{b + \epsilon_i - \epsilon_j}, \\
X_{\epsilon_i + \epsilon_j} v &= v & \in V^{b + \epsilon_i + \epsilon_j}, \\
X_{-\epsilon_i - \epsilon_j} v &= (T_i + (a_i + b_i) \text{Id}_V)(T_j + (a_j + b_j) \text{Id}_V)v & \in V^{b - \epsilon_i - \epsilon_j}, \\
X_{-2\epsilon_i} v &= (T_i + (a_i + b_i) \text{Id}_V)(T_i + (a_i + b_i - 1) \text{Id}_V)v & \in V^{b - 2\epsilon_i}, \\
H_{\epsilon_i + 1 - \epsilon_i} v &= (T_{i+1} - T_i + (a_{i+1} + b_{i+1} - a_i - b_i) \text{Id}_V)v & \in V^b, \\
H_{2\epsilon_i} v &= \frac{1}{2}(2T_1 + (2a_1 + 2b_1 + 1) \text{Id}_V)v & \in V^b,
\end{align*}
\]

(3)
where $i$ and $j$ are as in the respective row of (2). For a homomorphism $f : V \to W$ of $\mathbb{C}[\![t_1, t_2, \ldots, t_n]\!]$-modules denote by $Ff$ the diagonally extended linear map from $FV$ to $FW$, that is, for every $b \in B$ and $v \in V^b$, set

$$Ff(v) = f(v) \in W^b.$$  

### Proposition 4

(i) The formulae of (3) define on $FV$ the structure of a $\mathfrak{g}$-module.

(ii) Every $V^b$ is a generalized weight space of $FV$. Moreover, for $b \neq b'$ the weights of $V^b$ and $V^{b'}$ are different.

(iii) The module $FV$ belongs to $\hat{\mathfrak{c}}_a$.

(iv) Formulae (3) and (4) turn $F$ into a functor

$$F : \mathbb{C}[\![t_1, t_2, \ldots, t_n]\!]\text{-mod} \to \hat{\mathfrak{c}}_a.$$

(v) The functor $F$ is exact, faithful and full.

**Proof.** Consider the $\mathfrak{g}$-module $N(a)$ for $a$ as above. Then, for every $b$, the defining relations of $\mathfrak{g}$ (in terms of elements from $C$) applied to $x^b$ can be written as some polynomial equations in the $a_i$. Since (2) defines a $\mathfrak{g}$-module for any $a$ by Theorem 3(i), these equations hold for any $a$, that is, they are actually formal identities in the $a_i$. Now write

$$T_j + (a_j + b_j) \text{Id}_V = A_j + B_j,$$

a sum of matrices, where $A_j = T_j + a_j \text{Id}_V$ and $B_j = b_j \text{Id}_V$. All $A_i$ and $B_j$ commute with each other and with all the $T_i$. For a fixed $b$, the defining relations for $\mathfrak{g}$ on $FV$ reduce to our formal identities (in the $A_i$) and hence are satisfied. This proves claim (i). Claim (ii) follows from the last two lines in (3) and the fact that all the $T_i$ are nilpotent (hence zero is the only eigenvalue).

As $f$ commutes with all $T_i$, the map $Ff$ commutes with the action of all elements from $\mathfrak{c}$ and hence defines a homomorphism of $\mathfrak{g}$-modules. By construction we also have $F(f \circ f') = Ff \circ Ff'$, which implies claim (iv).

By construction, $F$ is exact and faithful. It sends the simple one-dimensional $\mathbb{C}[\![t_1, t_2, \ldots, t_n]\!]$-module to $N(a)$ (as in this case all $T_i = 0$ and hence (3) gives (2)), which is an object of the category $\hat{\mathfrak{c}}_a$ closed under extensions. Claim (iii) follows.

To complete the proof of claim (v) we are left to show that $F$ is full. Let $\varphi : FV \to FW$ be a $\mathfrak{g}$-homomorphism. Then $\varphi$ commutes with the action of all elements from $\mathfrak{h}$. Using claim (ii), we get that $\varphi$ induces, by restriction, a linear map $f : V = V^0 \to W = W^0$. As $\varphi$ commutes with all $H_{\xi_{i+1}-\xi_i}$, the map $f$ commutes with all operators $T_{i+1} - T_i$. As $\varphi$ commutes with $H_{2\xi_1}$, the map $f$ commutes with $T_1$. It follows that $f$ is a homomorphism of $\mathbb{C}[\![t_1, t_2, \ldots, t_n]\!]$-modules. This yields $\varphi = Ff$ and thus the functor $F$ is full. This completes the proof of claim (v) and of the whole proposition. \qed
4. Proof of Theorem 1

Because of Lemma 2 it is enough to fix one particular block and show there that \( F \) is an equivalence. Thus, we may assume that \( a_i + a_j \notin \mathbb{Z} \) for all \( i, j \) (in particular, \( a_i \notin \mathbb{Z} \) for all \( i \)). According to Proposition 4, we are only left to show that \( F \) is dense (that is, essentially surjective). We establish the density of \( F \) by induction on \( n \). We first prove the induction step and then the basis of the induction, which is the case \( n = 2 \).

Denote by \( \lambda \) the weight of \( x^0 \in N(a) \) (see Proposition 4(ii)). Let \( M \in \hat{\mathfrak{c}}_{a} \). Set \( V := M_{\lambda} \) and denote by \( M' \) the \( a \)-module \( U(a)V \).

4.1. Reduction to the case \( n = 2 \). The main result of this section is the following:

**Proposition 5.** If the functor \( F \) is dense for \( n = 2 \), then it is dense for any \( n \geq 2 \).

**Proof.** Assume that \( n > 2 \) and that the functor \( F \) is dense in the case of the algebra \( \mathfrak{sp}_{2n-2} \). Realize \( \mathfrak{sp}_{2n-2} \) as the subalgebra \( a \) of \( \mathfrak{g} \) corresponding to the subset \( H_{n-1} \subset \mathfrak{h} \) of simple roots.

Let \( Y_1, Y_2, \ldots, Y_n \) be the linear operators representing the action of the elements \( H_{2\varepsilon_1}, H_{\varepsilon_2-\varepsilon_1}, H_{\varepsilon_3-\varepsilon_2}, \ldots, H_{\varepsilon_n-\varepsilon_{n-1}} \) on \( V \), respectively. Set

\[
T_1 := Y_1 - \frac{1}{2}(2a_1 + 1) \text{Id}_V, \\
T_2 := Y_2 + T_1 - (a_2 - a_1) \text{Id}_V, \\
T_3 := Y_3 + T_2 - (a_3 - a_2) \text{Id}_V, \\
\vdots \\
T_n := Y_n + T_{n-1} - (a_n - a_{n-1}) \text{Id}_V.
\]

The \( T_i \) are obviously pairwise commuting nilpotent linear operators.

The module \( M' \) is a cuspidal generalized weight \( a \)-module with finite dimensional weight spaces. Moreover, as all composition subquotients of \( M \) are of the form \( N(a) \), all composition subquotients of \( M' \) are of the form \( N(a)' \), the latter being a completely pointed simple cuspidal \( a \)-module. By our inductive assumption, the functor \( F \) is dense in the case of the algebra \( a \). Hence \( M' \cong N' := \bigoplus_b V^b \), where \( b \in \mathbb{B} \) is such that \( b_n = 0 \), and the action of \( a \) on \( N' \) is given by (3).

**Lemma 6.** There is a unique (up to isomorphism) \( g \)-module \( Q \in \hat{\mathfrak{c}}_a \) such that \( Q' = N' \) and which gives the linear operator \( T_n \) when computed using (5).

**Proof.** The existence statement is clear, so we need only to show uniqueness. Assume that \( Q \in \hat{\mathfrak{c}}_a \) is such that \( Q' = N' \) and the formulae in (5) applied to \( Q \) produce the linear operator \( T_n \). Since \( a_n \notin \mathbb{Z} \), the endomorphism \( T_n + (a_n + b_n) \text{Id}_V \) is invertible for all \( b_n \in \mathbb{Z} \). As the action of \( X_{\varepsilon_n-\varepsilon_{n-1}} \) on \( Q \) is bijective, we can fix a weight basis in \( Q \) such that both the \( a \)-action on \( Q' = N' \) and the action of \( X_{\varepsilon_n-\varepsilon_{n-1}} \) on the whole \( Q \) is given by (3). As \( n > 2 \), the elements \( X_{\pm 2\varepsilon_1} \) commute.
with \(X_{\epsilon_n - \epsilon_{n-1}}\) and hence their action extends uniquely to the whole of \(Q\) using this commutativity. This holds similarly for all elements \(X_{\pm (\epsilon_i - \epsilon_{i-1})}\), \(i < n - 1\), and for the element \(X_{\epsilon_{n-2} - \epsilon_{n-1}}\). This leaves us with the elements \(X_{\epsilon_{n-1} - \epsilon_{n-2}}\) and \(X_{\epsilon_{n-1} - \epsilon_n}\). The simple roots \(\epsilon_{n-1} - \epsilon_{n-2}\) and \(\epsilon_n - \epsilon_{n-1}\) corresponding to the elements \(X_{\epsilon_{n-1} - \epsilon_{n-2}}\) and \(X_{\epsilon_n - \epsilon_{n-1}}\) generate a root system of type \(A_2\) (this corresponds to the algebra \(\mathfrak{sl}_3\)). Lemmas 21 and 22 of [Mazorchuk and Stroppel 2011] prove that the actions of \(X_{\epsilon_{n-1} - \epsilon_{n-2}}\) and \(X_{\epsilon_n - \epsilon_{n-1}}\) extend uniquely to \(Q\). This completes the proof of Lemma 6. □

The module \(FV\) obviously satisfies \((FV)' = N'\) and defines the linear operator \(T_n\) when computed using (5). Hence Lemma 6 implies \(M \cong FV\). Since \(M \in \hat{\mathcal{C}}_a\) was arbitrary, the functor \(F\) is dense, completing the proof of Proposition 5. □

4.2. Base of the induction: some \(\mathfrak{sl}_2\)-theory as preparation. In this section we will recall (and slightly improve) some classical \(\mathfrak{sl}_2\)-theory. For details see [Mazorchuk 2010]. Consider the Lie algebra \(\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})\) with standard basis

\[
e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \(V\) be a finite dimensional vector space and \(A\) and \(B\) be two commuting linear operators on \(V\). For \(i \in \mathbb{Z}\) denote by \(V^{(i)}\) a copy of \(V\) and consider the vector space \(\overline{V} := \bigoplus_{i \in \mathbb{Z}} V^{(i)}\) (a direct sum of copies of \(V\) indexed by \(i\)). Define the actions of \(e, f\) and \(h\) on \(\overline{V}\) as follows: for \(v \in V^{(i)}\) set

\[
\begin{align*}
v &= (P - i \text{Id}_V)v \quad \in V^{(i+1)}, \\
v &= (Q + i \text{Id}_V)v \quad \in V^{(i-1)}, \\
v &= (Q - P + 2i \text{Id}_V)v \in V^{(i)}.
\end{align*}
\]

This can be depicted as follows (here right arrows represent the action of \(e\), left arrows represent the action of \(f\) and loops represent the action of \(h\)):

\[
\cdots \xrightarrow{\text{Id}_V} V^{(-1)} \xrightarrow{P + \text{Id}_V} V^{(0)} \xrightarrow{P} V^{(1)} \xrightarrow{P - \text{Id}_V} \cdots
\]

\[
\cdots \xrightarrow{Q - \text{Id}_V} V^{(-1)} \xrightarrow{Q + \text{Id}_V} V^{(0)} \xrightarrow{Q - P} V^{(1)} \xrightarrow{Q - P + 2 \text{Id}_V} \cdots
\]

\[
\cdots \xrightarrow{Q - P - 2 \text{Id}_V} V^{(-1)} \xrightarrow{Q - P} V^{(0)} \xrightarrow{Q - P + 2 \text{Id}_V} \cdots
\]

\[
\cdots \xrightarrow{Q - 2 \text{Id}_V} V^{(-1)} \xrightarrow{Q - P} V^{(0)} \xrightarrow{Q - P + \text{Id}_V} \cdots
\]

Proposition 7. (i) The formulae in (6) define on \(\overline{V}\) the structure of a generalized weight \(\mathfrak{sl}_2\)-module with finite dimensional generalized weight spaces.

(ii) Every cuspidal generalized weight \(\mathfrak{sl}_2\)-module with finite dimensional generalized weight spaces is isomorphic to \(\overline{V}\) for some \(V\) with \(P\) and \(Q\) as above.

(iii) The action of the Casimir element \(c := (h + 1)^2 + 4fe\) on \(\overline{V}\) is given by the linear operator \((P + Q + \text{Id}_V)^2\).
Let $\mathbb{C}^2$ denote the natural $\mathfrak{sl}_2$-module (the unique two-dimensional simple $\mathfrak{sl}_2$-module). Then the linear operator $(c - (P + Q + 2 \text{Id}_V)^2)(c - (P + Q)^2)$ annihilates the $\mathfrak{sl}_2$-module $\mathbb{C}^2 \otimes \overline{V}$.

Let $\mathbb{C}^3$ denote the unique three-dimensional simple $\mathfrak{sl}_2$-module. Then the linear operator $(c - (P + Q + 3 \text{Id}_V)^2)(c - (P + Q + \text{Id}_V)^2)(c - (P + Q - \text{Id}_V)^2)$ annihilates the $\mathfrak{sl}_2$-module $\mathbb{C}^3 \otimes \overline{V}$.

**Proof.** The fact that $\overline{V}$ is an $\mathfrak{sl}_2$-module is checked by a direct computation. That $\overline{V}$ is a generalized weight module follows from the fact that the action of $h$ on $\overline{V}$ preserves (by (6)) each $V^i$ and hence is locally finite. Since the category of generalized weight modules is closed under extensions, to prove that $\overline{V}$ has finite dimensional generalized weight spaces it is enough to consider the case when $h$ has a unique eigenvalue on $V^{(0)}$, say $\lambda$. However, in this case $h$ has a unique eigenvalue on $V^i$, namely $\lambda + 2i$, which implies that $\overline{V}^\lambda = V$ is finite dimensional. Claim (i) follows.

To prove Claim (iii) we observe that the action of $e$ on $V^i$ is given by

$$(Q - P + (2i + 1) \text{Id}_V)^2 + 4(Q + (i + 1) \text{Id}_V)(P - i \text{Id}_V) = (P + Q + \text{Id}_V)^2.$$ 

Claim (ii) can be found with all details in [Mazorchuk 2010, Chapter 3].

To prove claim (iv) choose a basis $\{v_1, \ldots, v_k\}$ in $V$, which gives rise to a basis $\{v_1^{(i)}, \ldots, v_k^{(i)}, i \in \mathbb{Z}\}$ in $\overline{V}$. Choose the standard basis $\{e_1, e_2\}$ in $\mathbb{C}^2$. Since $he_1 = e_1$, $he_2 = -e_2$ and $h$ acts by $Q - P + 2i \text{Id}_V$ on $V^{(i)}$, we obtain that $h$ acts by $Q - P + (2i + 1) \text{Id}_V$ on the vector space $W^{(i)}$ with basis

$$\{e_1 \otimes v_1^{(i)}, \ldots, e_1 \otimes v_1^{(i)}, e_2 \otimes v_1^{(i+1)}, \ldots, e_2 \otimes v_1^{(i+1)}\}.$$

We have $\mathbb{C}^2 \otimes \overline{V} \cong \bigoplus_{i \in \mathbb{Z}} W^{(i)}$ and one easily computes that in the above basis the actions of $e$ and $f$ on $\mathbb{C}^2 \otimes \overline{V}$ are given by the following picture:

$$\cdots \overline{W}^{(-1)} \overline{W}^{(0)} \overline{W}^{(1)} \cdots$$

$$\begin{pmatrix} P + \text{Id} & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 0 & P \\ P - \text{Id} & 0 \end{pmatrix} \begin{pmatrix} Q + \text{Id} & 0 \\ 0 & Q + 2 \text{Id} \end{pmatrix} \begin{pmatrix} P + \text{Id} & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 0 & P \\ P - \text{Id} & 0 \end{pmatrix} \begin{pmatrix} Q + \text{Id} & 0 \\ 0 & Q + 2 \text{Id} \end{pmatrix} \cdots$$

The action of $c$ on $W^{(0)}$ is now easily computed to be given by the linear operator

$$G := \begin{pmatrix} (Q - P + 2 \text{Id})^2 + 4(Q + \text{Id})P \\ 4P \end{pmatrix} \begin{pmatrix} 4(Q + \text{Id}) \\ (Q - P + 2 \text{Id})^2 + 4(Q + 2 \text{Id})(P - \text{Id}) + 4 \text{Id} \end{pmatrix}.$$ 

The characteristic polynomial of $G$ is

$$\chi_G(\lambda) = (\lambda - (P + Q + 2 \text{Id})^2)(\lambda - (P + Q)^2).$$

Claim (iv) now follows from the Cayley–Hamilton theorem.

We have an isomorphism of $\mathfrak{sl}_2$-modules as follows: $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^3 \oplus \mathbb{C}$ (here $\mathbb{C}$ is the trivial module), and hence claim (v) follows applying claim (iv) twice.
Alternatively, one could do a direct calculation, similar to the proof of (iii). The proposition follows.

The statement of Proposition 7(ii) is a special case of a more general result of Gabriel and Drozd describing blocks of the category of (generalized) weight \( \mathfrak{sl}_2 \)-modules, in particular, simple weight \( \mathfrak{sl}_2 \)-modules (see [Drozd 1983; Dixmier 1996, 7.8.16]). The statements of Proposition 7(iv) and (v) are \( \mathfrak{sl}_2 \)-refinements of a theorem of Kostant [1975, Theorem 5.1] describing possible (generalized) central characters of the tensor product of a finite dimensional module with an infinite dimensional module.

4.3. **The case \( n = 2 \).** Assume now that \( n = 2 \). We have \( a_1, a_2, a_1 + a_2 \notin \mathbb{Z} \). Let \( \mathfrak{a} \) denote the Lie subalgebra of \( \mathfrak{g} \) generated by \( X_{\pm (\varepsilon_2 - \varepsilon_1)} \). The algebra \( \mathfrak{a} \) is isomorphic to \( \mathfrak{sl}_2 \).

Let \( M \in \widehat{\mathfrak{g}}_a \). Denote by \( \lambda \) the weight of \( x^0 \in N(a) \) and set \( V := M_\lambda \). Let \( Y_1 \) and \( Y_2 \) be the linear operators representing the actions of the elements \( H_{\varepsilon_2 - \varepsilon_1} \) and \( C := (H_{\varepsilon_2 - \varepsilon_1} + 1)^2 + 4X_{\varepsilon_1 - \varepsilon_2} \) on \( V \). The element \( C \) is a Casimir element for \( \mathfrak{a} \). In particular, the operators \( Y_1 \) and \( Y_2 \) commute. Our first observation is the following:

**Lemma 8.** The action of \( C \) on \( V \) is invertible and hence has a square root.

**Proof.** From (2) we have that \( C \) acts on \( x^0 \) by

\[
(a_2 - a_1 + 1)^2 + 4(a_2 + 1)a_1 = (a_1 + a_2 + 1)^2.
\]

Since \( a_1 + a_2 \notin \mathbb{Z} \) by our assumptions, \( x^0 \) is an eigenvector of \( C \) with a nonzero eigenvalue. As the module \( M \) has a composition series with subquotients isomorphic to \( N(a) \), the complex number \( (a_1 + a_2 + 1)^2 \neq 0 \) is the only eigenvalue of \( C \) on \( V \). The claim follows.

Consider the \( \mathfrak{a} \)-module \( M' := U(\mathfrak{a})M_\lambda \). Let \( Y'_2 \) denote any square root of \( Y_2 \), which is a polynomial in \( Y_2 \) (it exists by Lemma 8). So \( Y'_2 \) commutes with \( Y_1 \). Set

\[
T_1 := \frac{Y'_2 - Y_1 - \text{Id}_V}{2} - a_1 \text{Id}_V, \quad T_2 := \frac{Y'_2 + Y_1 - \text{Id}_V}{2} - a_2 \text{Id}_V.
\]

Then \( T_1 \) and \( T_2 \) are two commuting nilpotent linear operators (it is easy to check that 0 is the unique eigenvalue for both \( T_1 \) and \( T_2 \)), hence define on \( V \) the structure of a \( \mathbb{C}[[t_1, t_2]] \)-module. The aim of this section is to establish an isomorphism \( FV \cong M \), which would complete the proof of Theorem 1.

Set \( R' := U(\mathfrak{a})(FV)_\lambda \). A direct computation using (3) shows that \( H_{\varepsilon_2 - \varepsilon_1} \) and \( C \) act on \( (FV)_\lambda = V^0 \) as the linear operators \( Y_1 \) and \( Y_2 \), respectively. As any cuspidal generalized weight \( \mathfrak{a} \)-module is uniquely determined by the actions of \( H_{\varepsilon_2 - \varepsilon_1} \) and \( C \) (see [Drozd 1983; Mazorchuk 2010, 3.7] for full details), it follows that \( M' \cong R' \). The isomorphism \( FV \cong M \) now follows from the next proposition:
\textbf{Proposition 9.} There is at most one (up to isomorphism) \( \mathfrak{g} \)-module \( R \in \mathcal{E}_a \) such that \( U(a)R_\lambda = R' \).

\textit{Proof.} Let \( R \in \mathcal{E}_a \) be such that \( U(a)R_\lambda = R' \). Choose a weight basis in \( R \) such that the action of \( a \) on \( R' \) and the action of \( X_{2\xi_1} \) on \( R \) is given by (3) (in other words these actions coincide with the corresponding actions on \( \mathcal{F} \mathcal{V} \)). Since \( X_{\xi_1-\xi_2} \) commutes with \( X_{2\xi_1} \), it follows that the action of \( X_{\xi_1-\xi_2} \) on \( R \) is also given by (3).

It is left to show that the action of \( X_{\xi_2-\xi_1} \) extends uniquely from \( R' \) to \( R \) and then that there is a unique way to define the action of \( X_{-2\xi_1} \). This will be done in the Lemmata 10 and 11 below. \qed

\textbf{Lemma 10.} There is a unique way to extend the action of \( X_{\xi_2-\xi_1} \) from \( R' \) to \( R \).

\textit{Proof.} We first show that for every \( k \in \{1, 2, \ldots \} \), the action of \( X_{\xi_2-\xi_1} \) extends uniquely from \( X_{2\xi_1}^{k-1} R' \) to \( X_{2\xi_1}^k R' \) (here \( X_{2\xi_1}^0 R' = R' \)).

Consider the following picture:

\begin{equation}
\begin{array}{c}
\bullet \\
\downarrow \quad X \quad \downarrow \\
Q \\
\downarrow \quad 1 \quad \downarrow \\
\bullet \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\bullet \\
\downarrow \quad P+1 \quad \downarrow \\
Q \\
\downarrow \quad P \quad \downarrow \\
\bullet \\
\end{array}
\end{equation}

Here bullets are weight spaces with some fixed bases. The lower row is a part of \( X_{2\xi_1}^{k-1} R' \) where the \( a \)-action is already known by induction. The bases in the weight spaces in the lower row are chosen such that the action of \( a \) in the lower row is given by (3). The upper row is a part of \( X_{2\xi_1}^k R' \) where the \( a \)-action is to be determined. Arrows pointing up indicate the action of \( X_{2\xi_1} \). The bases of the weight spaces in the upper row are chosen such that the action of \( X_{2\xi_1} \) is given by the operator \( \text{Id}_V \) (as in (3)). Left arrows indicate the action of \( X_{\xi_1-\xi_2} \). The latter commutes with the action of \( X_{2\xi_1} \) and hence is given by the same linear operator in each column. Right arrows indicate the action of \( X_{\xi_2-\xi_1} \) (which is known for \( X_{2\xi_1}^{k-1} R' \) and is to be determined for \( X_{2\xi_1}^k R' \)). The part to be determined is given by the dashed arrow. Labels \( P \) and \( Q \) represent coefficients (which are linear operators on \( V \)) appearing in the corresponding parts of formulae (3). Note that \( P \) and \( Q \) commute. The action of \( X_{\xi_2-\xi_1} \) on \( X_{2\xi_1}^k R' \) which is to be determined is given by some unknown linear operator \( X \).

From \( H_{\xi_2-\xi_1} = [X_{\xi_2-\xi_1}, X_{\xi_1-\xi_2}] \) we see that the action of \( H_{\xi_2-\xi_1} \) on the middle weight space in the lower row is given by \( Q - P \). Using \( [H_{\xi_2-\xi_1}, X_{2\xi_1}] = -2X_{2\xi_1} \) we get that \( H_{\xi_2-\xi_1} \) acts on the right dot of the upper row via \( Q - P - 2 \). Using \( [H_{\xi_2-\xi_1}, X_{\xi_1-\xi_2}] = -2X_{\xi_1-\xi_2} \) we get that \( H_{\xi_2-\xi_1} \) acts on the left dot of the upper row via \( Q - P - 4 \). So the action of \( C \) on the upper row is given by \( (Q - P - 3)^2 + 4XQ \).
The action of $C$ on the lower row is given by $(Q - P - 1)^2 + 4(P + 1)Q = (Q + P + 1)^2$.

The elements $X_{2\varepsilon_1}$, $X_{2\varepsilon_2}$ and $X_{\varepsilon_1+\varepsilon_1}$ form a weight basis of a simple three-dimensional $\alpha$-module $\mathbb{C}^3$ with respect to the adjoint action of $\alpha$. Hence the upper row of our picture is a subsingleton of the tensor product of the lower row and $\mathbb{C}^3$.

Therefore, from Proposition 7(v) we obtain that the linear operator

$$(C - (Q + P - 1)^2)(C - (Q + P + 1)^2)(C - (Q + P + 3)^2)$$

annihilates the upper row. A direct computation using (3) shows that the action of the operators $C - (Q + P - 1)^2$ and $C - (Q + P + 1)^2$ on the part $X_{2\varepsilon_1}^k N(\alpha)'$ of the module $N(\alpha)$ is invertible. As the $\mathfrak{g}$-module we are working with must have a composition series with subquotients $N(\alpha)$, it follows that the action of both $C - (Q + P - 1)^2$ and $C - (Q + P + 1)^2$ on $X_{2\varepsilon_1}^k R'$ is invertible. Hence $C - (Q + P + 3)^2$ annihilates $X_{2\varepsilon_1}^k R'$, which gives the equation

$$(Q - P - 3)^2 + 4XQ = (Q + P + 3)^2.$$  

This equation has a unique solution, namely $X = Q + 3$, which gives the required extension.

Similarly one shows that for $k \in \{-1, -2, \ldots\}$, the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k+1} R'$ to $X_{2\varepsilon_1}^k R'$ (here again $X_{2\varepsilon_1}^0 R' = R'$).

\[ \square \]

**Lemma 11.** There is a unique way to define the action of $X_{-2\varepsilon_1}$ on $N$.

**Proof.** To determine this action of $X_{-2\varepsilon_1}$ on $N$ we consider the following extension of the picture (7) with the same notation as in the proof of Lemma 10:

\[ \begin{array}{c}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
P+3 & P+2 & P+1 \\
Q & Q+1 & Q+2 \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1 \\
u & v & w \\
\downarrow & \downarrow & \downarrow \\
P+3 & P+2 & P+1 \\
Q & Q+1 & Q+2 \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1 \\
x & y & z \\
\downarrow & \downarrow & \downarrow \\
Q & Q+1 & Q+2 \\
\end{array} \]

Here all right arrows, representing the action of $X_{\varepsilon_2-\varepsilon_1}$, are now determined by Lemma 10 and we have to figure out the down arrows, representing the action of $X_{-2\varepsilon_1}$. The two dotted arrows will be used later on in the proof.

Consider the $\mathfrak{sl}_2$-subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ generated by $e := X_{2\varepsilon_1}$ and $f := X_{-2\varepsilon_1}$. Set $h := [e, f]$. Denote by $Z$ the action of $h$ in the leftmost weight space of the middle
row. Then $Z = x - u$. The element $h$ commutes with both $h$ and $H_{e_2 - e_1}$. Therefore, by (3), the operator $Z$ commutes with both $T_1$ and $T_2$ and hence with both $P$ and $Q$.

The algebra $c$ has the quadratic Casimir element $C_c$, whose action on the $c$-module given by the leftmost column of our picture is given by $x + f(Z)$, where $f$ is some polynomial of degree two. From (3) it follows that the unique eigenvalue of this action is nonzero, in particular, $x + f(Z)$ is invertible. Let $x'$ be a fixed square root $x + f(Z)$, which is a polynomial in $x + f(Z)$.

The elements $X_{e_2 - e_1}$ and $X_{e_2 + e_1}$ form a basis of a simple two-dimensional $c$-module with respect to the adjoint action. Using Proposition 7(iv) and arguments similar to those used in the proof of Lemma 10, we get that $C_c - (x' + 1)^2$ or $C_c - (x' - 1)^2$ annihilates the middle column (the sign depends on the original choice of $x'$). The middle column equals $X_{e_2 - e_1}$ applied to the leftmost column.

Similarly, the elements $X_{e_1 - e_2}$ and $X_{-e_2 + e_1}$ form a basis of a simple two-dimensional $c$-module with respect to the adjoint action. Applying the same arguments as in the previous paragraph we get that $C_c - (x')^2$ annihilates any vector of the form $X_{e_1 - e_2}X_{e_2 - e_1}v$, where $v$ is from the leftmost column. This implies that the actions of $C_c$ and $X_{e_1 - e_2}X_{e_2 - e_1}$ and thus the actions of $C_c$ and $C$ on the leftmost column commute. As the action of $H$ commutes with the action of $C$, we thus obtain that $x$ commutes with the action of $C$. This implies that $x$ commutes with $T_1 + T_2$. As it obviously commutes with $T_1 - T_2$, we get that $x$ commutes with both $T_1$ and $T_2$ and hence with both $P$ and $Q$.

Similarly one shows that $y$, $u$, $v$ and $w$ commute with both $P$ and $Q$. From the commutativity of $X_{e_2 - e_1}$ and $X_{-2e_1}$ we get the conditions

$$y(P + 1) = (P - 1)x, \quad V(P + 3) = (P + 1)u, \quad w(P + 2)(P + 3) = P(P + 1)u.$$  

Here everything commutes by the above and $P + 1$, $P + 2$ and $P + 3$ are invertible (as $X_{e_2 - e_1}$ acts bijectively). Therefore

$$y = (P - 1)(P + 1)^{-1}x, \quad v = (P + 1)(P + 3)^{-1}u, \quad w = P(P + 1)(P + 3)^{-1}(P + 2)^{-1}u.$$  

This implies that $y$, $v$ and $w$ are uniquely determined by $x$ and $u$.

Since the actions of both $X_{e_2 - e_1}$ and $X_{2e_1}$ are completely determined, we can compute the action of $X_{2e_2}$ and see that it is given (similarly to the action of $X_{2e_1}$) by $Id_V$ (this is depicted by the dotted arrows in the picture). As $X_{-2e_2}$ and $X_{2e_2}$ commute, we obtain that $w = x$, that is,

$$x = P(P + 1)(P + 3)^{-1}(P + 2)^{-1}u.$$  

Therefore the only parameter left for now is $u$.

On the one hand, the action of the element $h$ on the middle dot of the second row is given by $y - v = (P - 1)(P + 1)^{-1}x - (P + 1)(P + 3)^{-1}u$. On the other hand, from $[h, X_{e_2 - e_1}] = 4X_{e_2 - e_1}$ we have that this action equals $Z + 4 = x - u + 4$.
This gives us the equation

\[(P - 1)(P + 1)^{-1}x - (P + 1)(P + 3)^{-1}u = x - u + 4.\]

Using (9) and (8) we get the equation

\[
\frac{P(P - 1)}{(P + 2)(P + 3)}u + \frac{P + 1}{P + 3}u = \frac{P(P + 1)}{(P + 2)(P + 3)}u - u + 4.
\]

This is a linear equation with nonzero coefficients and thus it has a unique solution, namely \(u = (P + 3)(P + 2)\). Hence \(u\) is uniquely defined. The claim of the lemma follows.

\[\Box\]

5. Consequences

**Corollary 12.** Let \(a \in \mathbb{C}^n\) be such that \(a_i \not\in \mathbb{Z}\) and \(a_i + a_j \not\in \mathbb{Z}\) for all \(i\) and \(j\). Let \(M \in \hat{\mathcal{C}}\) and \(\lambda \in \text{supp}(M)\). Denote by \(U_0\) the centralizer of \(h\) in \(U(\mathfrak{g})\). Then for any \(A, B \in U_0\) the actions of \(A\) and \(B\) on \(M_\lambda\) commute.

**Proof.** By Proposition 4, we may assume that \(M \cong FV\). For the module \(FV\) the claim follows from the formulae in (3). \[\Box\]

**Corollary 13.** For any simple weight cuspidal \(\mathfrak{g}\)-module \(L\) with finite dimensional weight spaces we have \(\dim \text{Ext}^1_{\mathfrak{g}}(L, L) = n\).

**Proof.** This follows from Theorem 1 and the observation that a similar equality is true for the unique simple \(\mathbb{C}[\![t_1, t_2, \ldots, t_n]\!]-\)module. \[\Box\]

We also recover the main result of [Britten et al. 2004]:

**Corollary 14.** The category of all weight cuspidal \(\mathfrak{g}\)-modules is semisimple.

**Proof.** By [Britten et al. 2004, Lemma 2], all blocks of the category of weight cuspidal \(\mathfrak{g}\)-modules are equivalent. Hence it is enough to prove the claim for the block containing \(N(a)\) for some \(a \in \mathbb{C}^n\) such that \(a_i + a_j \not\in \mathbb{Z}\) for all \(i, j\). From (3) it follows that the module \(FV\) is weight if and only if all operators \(T_i\) are semisimple, hence zero. Therefore from Theorem 1 we get that the block of the category of weight cuspidal modules is equivalent to the category of finite dimensional modules over \(\mathbb{C}[\![t_1, t_2, \ldots, t_n]\!]/(t_1 - 0, t_2 - 0, \ldots, t_n - 0) \cong \mathbb{C}\). The claim follows. \[\Box\]

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