

*Pacific
Journal of
Mathematics*

**A CONSTANT MEAN CURVATURE ANNULUS
TANGENT TO TWO IDENTICAL SPHERES IS DELAUNEY**

SUNG-HO PARK

Volume 251 No. 1

May 2011

A CONSTANT MEAN CURVATURE ANNULUS TANGENT TO TWO IDENTICAL SPHERES IS DELAUNEY

SUNG-HO PARK

We show that a compact embedded annulus of constant mean curvature in \mathbb{R}^3 tangent to two spheres of the same radius along its boundary curves and having nonvanishing Gaussian curvature is part of a Delaunay surface. In particular, if the annulus is minimal, it is part of a catenoid. We also show that a compact embedded annulus of constant mean curvature with negative meeting a sphere tangentially and a plane at a constant contact angle $\geq \pi/2$ (in the case of positive Gaussian curvature) or $\leq \pi/2$ (in the negative case) is part of a Delaunay surface. Thus, if the contact angle is $\geq \pi/2$ and the annulus is minimal, it is part of a catenoid.

Delaunay surfaces are rotational surfaces (surfaces of revolution) of constant mean curvature in \mathbb{R}^3 . Besides cylinders and spheres, they are divided into unduloids, nodoids, and (allowing the case of zero mean curvature in the definition, for convenience) the catenoid, recognized long ago [Bonnet 1860] as the only nonplanar minimal surface of rotation in \mathbb{R}^3 .

Thus a Delaunay surface meets every plane perpendicular to the axis of rotation under a constant angle. Conversely, if a compact surface of constant mean curvature meets two parallel planes in constant contact angles, it is part of a Delaunay surface. This can be proved by using Alexandrov's moving plane argument [Alexandrov 1962; Hopf 1989] with planes perpendicular to the parallel planes.

A compact immersed minimal annulus meeting two parallel planes in constant contact angles is also part of a catenoid. This result is not true when the constant mean curvature is nonzero: Wente [1995] constructed examples of immersed constant mean curvature annuli in a slab or in a ball meeting the boundary planes or the boundary sphere perpendicularly. Compared to the above first case, we may ask whether a compact minimal annulus or a compact embedded constant mean curvature annulus meeting two spheres in constant contact angles is part of a catenoid or of a plane. In [Park and Pyo \geq 2011], it is shown that if a compact embedded minimal annulus meets two concentric spheres perpendicularly then the minimal annulus is part of a plane.

Supported by the Hankuk University of Foreign Studies Research Fund of 2011.
MSC2000: 53A10.

Keywords: minimal annulus, contact angle, sphere, parallel surface.

In this paper, we show that a compact embedded constant mean curvature annulus \mathcal{A} in \mathbb{R}^3 meeting two spheres S_1 and S_2 of the same radius ρ tangentially and having nonvanishing Gaussian curvature K is part of a Delaunay surface. More precisely, depending on the values of K and the mean curvature H we have three cases: (i) $K < 0$ and $H > -1/\rho$, in which case \mathcal{A} is part of a unduloid if $H < 0$, part of a catenoid if $H = 0$ and part of a nodoid if $H > 0$, (ii) $K > 0$ and $-1/\rho < H < -1/2\rho$, in which case \mathcal{A} is part of a unduloid, and (iii) $K > 0$ and $H < -1/\rho$, in which case \mathcal{A} is part of a nodoid. In the first two cases, \mathcal{A} stays outside of the balls B_1 and B_2 bounded by S_1 and S_2 . If (iii) holds, then $\mathcal{A} \subset B_1 \cap B_2$.

We also show that a compact embedded constant mean curvature annulus \mathcal{B} in \mathbb{R}^3 with negative (respectively, positive) Gaussian curvature meeting a unit sphere tangentially and a plane in constant contact angle $\geq \pi/2$ (respectively, $\leq \pi/2$) is part of a Delaunay surface. In particular, a compact embedded minimal annulus in \mathbb{R}^3 meeting a sphere tangentially and a plane in constant contact angle $\geq \pi/2$ is part of a catenoid.

To prove Theorems 3.1 and 3.2, we use the $-\rho$ -parallel surface $\tilde{\mathcal{A}}$ of \mathcal{A} (respectively, $\tilde{\mathcal{B}}$ of \mathcal{B}), that is, the parallel surface of \mathcal{A} (respectively, of \mathcal{B}) with distance ρ in the direction to the centers of the spheres. We use Alexandrov's moving plane argument [Alexandrov 1962; Hopf 1989] to prove that $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are rotational. Since $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are the parallel surfaces of \mathcal{A} and \mathcal{B} respectively, \mathcal{A} and \mathcal{B} are also rotational and, hence, are part of a Delaunay surface or part of a catenoid.

1. Constant mean curvature annulus meeting spheres tangentially

In the following, we may assume that the spheres have radius 1. Let \mathcal{A} be a compact embedded annulus with constant mean curvature H meeting two unit spheres S_1 and S_2 tangentially along the boundary curves γ_1 and γ_2 . We fix the unit normal N of \mathcal{A} in such a way that N points away from the center of S_i along each γ_i . Let $Y : A(1, R) \rightarrow \mathbb{R}^3$ be a conformal parametrization of \mathcal{A} from an annulus $A(1, R) = \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} \leq R\}$. We define X by $X = Y \circ \exp$ on the strip $B = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \log R\}$. Then X is periodic with period 2π . Let $z = u + iv$ and $\lambda^2 := |X_u|^2 = |X_v|^2$ with $\lambda > 0$.

Let h_{ij} , $i, j = 1, 2$, be the coefficients of the second fundamental form of X with respect to N . Note that the Hopf differential $\phi(z) dz^2 = (h_{11} - h_{22} - 2ih_{12}) dz^2$ is holomorphic for constant mean curvature surfaces [Hopf 1989]. The theorem of Joachimsthal [do Carmo 1976] says that γ_1 and γ_2 are curvature lines of \mathcal{A} . Hence $h_{12} \equiv 0$ on $u = 0$ and $u = \log R$. Since h_{12} is harmonic and periodic, we have $h_{12} \equiv 0$ on B . This implies that z is a conformal curvature coordinate and $h_{11} - h_{22}$ is constant [McCuan 1997]. Let $c = h_{11} - h_{22}$. If \mathcal{A} is minimal, then we

have $K < 0$ and $c = 2h_{11} > 0$ by the choice of N . When $H = -1$, \mathcal{A} is part of the unit sphere $S_1 = S_2$ by the boundary comparison principle for the mean curvature operator [Gilbarg and Trudinger 2001]. We assume that $H \neq -1$ in the following. The principal curvatures of \mathcal{A} are

$$(1) \quad \kappa_1 = H + \frac{c}{2\lambda^2} \quad \text{and} \quad \kappa_2 = H - \frac{c}{2\lambda^2}.$$

We use for γ_1 and γ_2 the parametrizations $\gamma_1(v) = X(0, v)$ and $\gamma_2(v) = X(\log R, v)$, for $v \in [0, 2\pi)$. In the following, we assume that \mathcal{A} has nonzero Gaussian curvature.

Lemma 1.1. *Each $\gamma_i(v)$, $i = 1, 2$, has constant speed $\sqrt{c/2(1+H)}$ and κ_2 is -1 on γ_1 and γ_2 . As spherical curves, γ_1 and γ_2 are convex. On $\mathcal{A} \setminus \partial\mathcal{A}$, we have $\lambda^2 < c/2(1+H)$ when $K < 0$ and $\lambda^2 > c/2(1+H)$ when $K > 0$.*

Proof. The curvature vector of $\gamma_1(v)$ is

$$(2) \quad \begin{aligned} \vec{\kappa} &= \frac{1}{|X_v|} \frac{d}{dv} \left(\frac{X_v}{|X_v|} \right) = \frac{1}{|X_v|^2} X_{vv} - \frac{X_v}{|X_v|^4} (X_v \cdot X_{vv}) \\ &= \frac{1}{\lambda^2} \left(-\frac{\lambda_u}{\lambda} X_u + h_{22} N \right). \end{aligned}$$

Let the center of S_1 be the origin of \mathbb{R}^3 . Since \mathcal{A} is tangential to S_1 along γ_1 , we have $N(0, v) = X(0, v) = \gamma_1(v)$ on γ_1 . Since γ_1 is on the unit sphere S_1 , the curvature vector $\vec{\kappa}$ of γ_1 satisfies $(\vec{\kappa} \cdot \gamma_1)(v) = -1$. Hence we have $\kappa_2 = h_{22}/\lambda^2 = -1$ on γ_1 . Since $\lambda^2 = |\gamma_{1v}|^2$ on γ_1 , we have $|\gamma_{1v}| = \sqrt{c/2(1+H)}$ from (1). By choosing the center of S_2 as the origin of \mathbb{R}^3 , we get the results for γ_2 .

The Gaussian curvature K satisfies

$$\Delta \log \lambda = -K\lambda^2,$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$. We can rewrite this equation as

$$(3) \quad \lambda \Delta \lambda = |\nabla \lambda|^2 - K\lambda^4.$$

Since $\lambda_v(0, v) = 0$ and $\lambda_v(\log R, v) = 0$ and $K \neq 0$, λ does not have interior maximum when $K < 0$, and does not have interior minimum when $K > 0$. Since $\lambda^2 = c/2(1+H)$ on γ_1 and γ_2 , it follows that $\lambda^2 < c/2(1+H)$ on $\mathcal{A} \setminus \partial\mathcal{A}$ when $K < 0$ and $\lambda^2 > c/2(1+H)$ when $K > 0$. Moreover we have $\lambda_u \leq 0$ on $u = 0$ and $\lambda_u \geq 0$ on $u = \log R$ when $K < 0$ and $\lambda_u \geq 0$ on $u = 0$ and $\lambda_u \leq 0$ on $u = \log R$ when $K > 0$. Since $X_u/|X_u| \in TS_i$ is perpendicular to γ_i , the geodesic curvature of γ_i as a spherical curve is $\vec{\kappa} \cdot (X_u/|X_u|) = -\lambda_u/\lambda^2$. Hence γ_1 and γ_2 are convex as spherical curves. □

Remark 1.2. If $\lambda^2 \equiv c/2(1+H)$ on \mathcal{A} , then $K \equiv 0$ and \mathcal{A} is part of a cylinder.

2. The -1 -parallel surface

The -1 -parallel surface $\tilde{\mathcal{A}}$ of \mathcal{A} is defined by

$$\tilde{X} = X - N.$$

The image of γ_1 (respectively, of γ_2) in $\tilde{\mathcal{A}}$ is a point corresponding to the center of S_1 (respectively, of S_2). We denote the centers of S_1 and S_2 by O and O_2 for simplicity. We fix the unit normal \tilde{N} of $\tilde{\mathcal{A}}$ to be N . Since $z = u + iv$ is a curvature coordinate of X , we have

$$(4) \quad \tilde{X}_u = \left(1 + \frac{h_{11}}{\lambda^2}\right) X_u \quad \text{and} \quad \tilde{X}_v = \left(1 + \frac{h_{22}}{\lambda^2}\right) X_v.$$

Since $\kappa_2 = -1$ on γ_i by Lemma 1.1, \tilde{X} is singular for $u = 0$ and $u = \log R$. By Lemma 1.1, we have $\lambda^2 \neq c/2(1+H)$ on $\mathcal{A} \setminus \partial\mathcal{A}$, which implies that $1 + \kappa_2 \neq 0$ on $\mathcal{A} \setminus \partial\mathcal{A}$. When $K < 0$, we have $\kappa_1 > 0$ on $\mathcal{A} \setminus \partial\mathcal{A}$. Hence \tilde{X} is regular for $0 < u < \log R$ and we have $H > -1$.

Now suppose that $K > 0$. Since $\kappa_2 = -1$ on γ_i by Lemma 1.1, we have $\kappa_1 < 0$ and $H < -1/2$. We consider two cases separately: $H < -1$ and $-1 < H < -1/2$. If $H < -1$, then $c < 0$ from $\lambda^2 = c/2(1+H) > 0$ on γ_i . Hence we have $\kappa_1 < -1$, which implies that \tilde{X} is regular for $0 < u < \log R$. If $-1 < H < -1/2$, then we must have $c > 0$. This implies that $1 + \kappa_1 \neq 0$. Otherwise we have $0 < 2\lambda^2(1+H) = -c$, which contradicts $c > 0$. Hence \tilde{X} is regular for $0 < u < \log R$.

Remark 2.1. When $K < 0$ or $K > 0$ and $-1 < H < -1/2$, \mathcal{A} stays outside of the balls B_1 and B_2 bounded by S_1 and S_2 . If $K > 0$ and $H < -1$, then $\mathcal{A} \subset B_1 \cap B_2$.

Lemma 2.2. *The mean curvature \tilde{H} and the Gaussian curvature \tilde{K} of $\tilde{\mathcal{A}}$ satisfies $(1+H)\tilde{K} = (1+2H)\tilde{H} - H$. On $\tilde{\mathcal{A}} \setminus \{O, O_2\}$, we have the following:*

- (i) *If $K < 0$ and $H > -1$, then $\tilde{\kappa}_1 > 0$, $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$.*
- (ii) *If $K > 0$ and $-1 < H < -1/2$, then $0 < c/2\lambda^2(1+H) < \min\{1, -H/(1+H)\}$, $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1+H)$ and $\tilde{H} < H/(1+H)$.*
- (iii) *If $K > 0$ and $H < -1$, then $0 < c/2\lambda^2(1+H) < 1$, $\tilde{\kappa}_1 > (1+2H)/2(1+H)$, $\tilde{\kappa}_2 > H/(1+H)$ and $\tilde{H} > H/(1+H)$.*

Proof. Since

$$\tilde{h}_{12} = N \cdot \tilde{X}_{uv} = \left(1 + \frac{h_{11}}{\lambda^2}\right) (N \cdot X_{uv}) = 0,$$

(u, v) is a curvature coordinate (not conformal) for $\tilde{\mathcal{A}}$ except for O and O_2 . We have

$$\tilde{h}_{11} = N \cdot \tilde{X}_{uu} = \left(1 + \frac{h_{11}}{\lambda^2}\right) h_{11}, \quad \tilde{h}_{22} = N \cdot \tilde{X}_{vv} = \left(1 + \frac{h_{22}}{\lambda^2}\right) h_{22}.$$

The principal curvatures of $\tilde{\mathcal{A}}$ are

$$\tilde{\kappa}_1 = \frac{\kappa_1}{1 + \kappa_1} = \frac{H/(1 + H) + (c/2\lambda^2(1 + H))}{1 + (c/2\lambda^2(1 + H))},$$

$$\tilde{\kappa}_2 = \frac{\kappa_2}{1 + \kappa_2} = \frac{H/(1 + H) - (c/2\lambda^2(1 + H))}{1 - (c/2\lambda^2(1 + H))}.$$

From $\kappa_1 + \kappa_2 = 2H$, we have $H = \frac{\tilde{H} - \tilde{K}}{1 - 2\tilde{H} - \tilde{K}}$ or $(1 + H)\tilde{K} = (1 + 2H)\tilde{H} - H$. It is straightforward to see that

$$\tilde{H} = \frac{H/(1 + H) - (c/2\lambda^2(1 + H))^2}{1 - (c/2\lambda^2(1 + H))^2}.$$

Note that $\kappa_2 < 0$ on \mathcal{A} . First suppose that $K < 0$. Then we have $\kappa_1 > 0$, which implies that $\tilde{\kappa}_1 = \kappa_1/(1 + \kappa_1) > 0$. Since $c/2\lambda^2(1 + H) > 1$ by Lemma 1.1, we have $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$.

When $K > 0$, we have $\kappa_1 = H + c/2\lambda^2 < 0$. If $-1 < H < -1/2$, then we have $c > 0$ because $\lambda^2 = c/2(1 + H) > 0$ on γ_i . It follows that $c/2\lambda^2(1 + H) < -H/(1 + H)$. By Lemma 1.1, we also have $c/2\lambda^2(1 + H) < 1$. Therefore $0 < c/2\lambda^2(1 + H) < \min\{1, -H/(1 + H)\}$. It is easy to see that $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1 + H) < 0$ and $\tilde{H} < H/(1 + H) < 0$.

When $K > 0$ and $H < -1$, we have $c < 0$ and $0 < c/2\lambda^2(1 + H) < 1$. It is straightforward to see that $\tilde{\kappa}_1 > (1 + 2H)/(1 + H)$, $\tilde{\kappa}_2 > H/(1 + H)$ and $\tilde{H} > H/(1 + H)$. □

This lemma says that $\tilde{\mathcal{A}}$ is a linear Weingarten surface with two singular points O and O_2 and is positively curved outside O and O_2 .

Lemma 2.3. $\tilde{\mathcal{A}}$ is embedded.

Proof. Let $\nu(v) = (X_u/|X_u|)(0, v)$. Note that ν is a closed curve in the unit sphere S_1 . We claim that ν is convex as a spherical curve. Otherwise, there is a great circle η intersecting the image of ν at no less than 3 points $\nu(v_1), \dots, \nu(v_n)$. (It is possible that ν maps an interval $(v_a, v_b) \subset [0, 2\pi)$ into a single point. We choose the v_i 's in such a way that ν maps no two v_i 's to the same point.) Each $\nu(v_i)$ determines a great circle $\mathbb{S}_{v_i}^1 \subset S_1$ contained in the plane perpendicular to $\nu(v_i)$. At each $\gamma_1(v_i)$, γ_1 is tangent to $\mathbb{S}_{v_i}^1$. Since η and $\mathbb{S}_{v_i}^1$ are perpendicular, γ_1 cannot be convex when $n \geq 3$. Hence ν intersect every geodesic of S_1 at no more than two points. This shows that ν is convex as a spherical curve. Similarly, $(X_u/|X_u|)(\log R, v)$ is also convex as a spherical curve.

Since $\tilde{\mathcal{A}}$ is a parallel surface of \mathcal{A} , the tangent cone $\text{Tan}(O, \tilde{\mathcal{A}})$ of $\tilde{\mathcal{A}}$ at O is the cone formed by rays from O through ν . Since ν is a convex spherical curve, $\text{Tan}(O, \tilde{\mathcal{A}})$ is convex. This shows that a small neighborhood of O in $\tilde{\mathcal{A}}$ is embedded

and nonnegatively curved as a metric space [Alexandrov 1948]. Similarly, there is a neighborhood of O_2 in $\tilde{\mathcal{A}}$ which is embedded and nonnegatively curved as a metric space.

Hadamard showed that a closed surface S in \mathbb{R}^3 with strictly positive Gaussian curvature is the boundary of a convex body [Hopf 1989]. In particular, S is embedded. Alexandrov [1948] generalized Hadamard’s theorem to nonnegatively curved metric spaces. Since $\tilde{\mathcal{A}}$ is a nonnegatively curved closed metric space, $\tilde{\mathcal{A}}$ is embedded. □

Remark 2.4. We have $\nu_v = (\lambda_u/\lambda^2)X_v$. At points where $\lambda_u \neq 0$, the curvature vector of ν is

$$\vec{\kappa}_\nu = \frac{1}{\lambda_u} \left(-\frac{\lambda_u}{\lambda} X_u + h_{22} N \right).$$

The geodesic curvature of ν as a spherical curve $\vec{\kappa}_\nu \cdot N = h_{22}/\lambda_u$.

3. Main results

We use Alexandrov’s moving plane argument [Alexandrov 1962; Hopf 1989] to prove the theorems.

Theorem 3.1. *A compact embedded constant mean curvature annulus \mathcal{A} with non-vanishing Gaussian curvature meeting two spheres S_1 and S_2 of the same radius tangentially is part of a Delaunay surface. In particular, if \mathcal{A} is minimal, then \mathcal{A} is part of a catenoid.*

Proof. We suppose that the radius of S_1 and S_2 is 1. By Lemma 2.2 and Lemma 2.3, $\tilde{\mathcal{A}}$ is a compact embedded surface with two singular points O and O_2 and satisfying $(1 + H)\tilde{K} = (1 + 2H)\tilde{H} - H$ at regular points. A small neighborhood of a regular point of $\tilde{\mathcal{A}}$ can be represented as the graph of a function $f(x, y)$ satisfying

$$(5) \quad 2(1 + H)(f_{xx}f_{yy} - f_{xy}^2) + 2H(1 + f_x^2 + f_y^2)^2 \\ = (1 + 2H)((1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy})(1 + f_x^2 + f_y^2)^{1/2}.$$

This equation can be rewritten as

$$(6) \quad \det(2(1 + H)D^2 f + A(Df)) = W^4,$$

where

$$A(Df) = -(1 + 2H) \begin{pmatrix} (1 + f_x^2)W & f_x f_y W \\ f_x f_y W & (1 + f_y^2)W \end{pmatrix} \quad \text{and} \quad W = \sqrt{1 + f_x^2 + f_y^2}.$$

Equation (6) is elliptic with respect to f if $2(1 + H)D^2 f + A(Df)$ is positive definite. Since $\det(2(1 + H)D^2 f + A(Df)) = W^4 > 0$, this happens if

$$(7) \quad \text{Tr}(2(1 + H)D^2 f + A(Df)) = 2(1 + H)\Delta f - (1 + 2H)(2 + f_x^2 + f_y^2)W$$

is strictly positive.

First we consider the case $K < 0$. Since $\tilde{H} > 1$ by Lemma 2.2, we have

$$(8) \quad \Delta f + f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} > 2W^{3/2},$$

for f representing $\tilde{\mathcal{A}}$. We may assume that f is defined on $B(0, \epsilon) \subset T_p \tilde{\mathcal{A}}$ so that $\nabla f(0) = \vec{0}$ and $D^2 f$ is diagonal. For sufficiently small $\epsilon = \epsilon(p)$, (8) implies that (7) is strictly positive. Hence (6) is elliptic with respect to f representing $\tilde{\mathcal{A}}$.

When $-1 < H < -1/2$, (7) is automatically satisfied.

Now we consider the case $K > 0$ and $H < -1$. Since $\tilde{H} > H/(1 + H)$ by Lemma 2.2, we have

$$(9) \quad \Delta f + f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} > \frac{2H}{1+H} W^{3/2}.$$

Assuming that f is defined on $B(0, \epsilon) \subset T_p \tilde{\mathcal{A}}$ with $\nabla f(0) = \vec{0}$ and $D^2 f$ is diagonal, (9) implies that

$$\Delta f - \frac{1+2H}{2(1+H)} (2 + f_x^2 + f_y^2) W$$

is strictly positive for sufficiently small ϵ . So $\det(-2(1+H)D^2 f - A(Df)) = W^4$ is elliptic for f representing $\tilde{\mathcal{A}}$. The ellipticity of (6) for f representing $\tilde{\mathcal{A}}$ enables us to use the maximum principle and the boundary point lemma [Gilbarg and Trudinger 2001].

Since $\tilde{\mathcal{A}}$ is convex and embedded, we can use Alexandrov’s moving plane argument [Alexandrov 1962; Hopf 1989] to show that $\tilde{\mathcal{A}}$ is rotational as follows. Let Π_θ be the plane containing the line segment $\overline{OO_2} \subset \mathbb{R}^3$ and making angle θ with a fixed vector \vec{E} which is perpendicular to $\overline{OO_2}$. Fix a positive constant L such that each plane Π_θ^L that is parallel to Π_θ with distance L from Π_θ does not meet $\tilde{\mathcal{A}}$ for all θ . Let Π_θ^l be the plane between Π_θ^L and Π_θ with distance l from Π_θ . When Π_θ^l intersects $\tilde{\mathcal{A}}$, we reflect the Π_θ^L side part of $\tilde{\mathcal{A}}$ about Π_θ^l . Denote this reflected surface by $\tilde{\mathcal{A}}_{l,\theta}^{\text{ref}}$. As we decrease l from L , there might be a first $l_\theta \geq 0$ for which $\tilde{\mathcal{A}}_{l_\theta,\theta}^{\text{ref}}$ is tangent to $\tilde{\mathcal{A}}$ at an interior point or at a boundary point of $\partial \tilde{\mathcal{A}}_{l_\theta,\theta}^{\text{ref}}$. We call this point the *first touch point*. If there is no nonnegative l with the first touch point, we repeat the process for $\Pi_{\theta+\pi}^L$ to find $l_{\theta+\pi}$, which must be positive. At the first touch point, we apply the comparison principles for (5) to see that the part of $\tilde{\mathcal{A}}$ in the Π_θ side and $\tilde{\mathcal{A}}_{l_\theta,\theta}^{\text{ref}}$ are identical and, hence, $l_\theta = 0$. This implies that Π_θ is a symmetry plane for $\tilde{\mathcal{A}}$. Since θ can be chosen arbitrarily, $\tilde{\mathcal{A}}$ should be rotational and, hence, \mathcal{A} is also rotational. Since the Delaunay surfaces and the catenoid are the only nonplanar rotational minimal and constant mean curvature surfaces, \mathcal{A} is part of a Delaunay surface or part of a catenoid. \square

We used the embeddedness of \mathcal{A} to prove that $\tilde{\mathcal{A}}$ is embedded. Whether there is a nonembedded minimal or constant mean curvature annulus meeting two unit

spheres tangentially is an interesting question. Moreover we raise the following questions.

- (1) Is a compact immersed minimal annulus or a compact embedded minimal or constant mean curvature surface meeting a sphere perpendicularly or in constant contact angles part of a catenoid or part of a Delaunay surface? Nitsche showed that an immersed disk type minimal or constant mean curvature surface meeting a sphere in constant contact angle is either a flat disk or a spherical cap [Nitsche 1985].
- (2) Is a compact immersed minimal annulus or a compact embedded minimal or constant mean curvature surface meeting two spheres in constant contact angles part of a catenoid or a plane or part of a Delaunay surface?
- (3) Is a compact immersed minimal or constant mean curvature annulus or a compact embedded minimal or constant mean curvature surface meeting a sphere and a plane in constant contact angles part of a catenoid or part of a Delaunay surface? We give an affirmative answer to this problem in a special case in the following.

Theorem 3.2. *A compact embedded constant mean curvature annulus \mathcal{B} with negative (respectively, positive) Gaussian curvature meeting a sphere tangentially and a plane in constant contact angle $\geq \pi/2$ (respectively, $\leq \pi/2$) is part of a Delaunay surface. In particular, if \mathcal{B} is minimal and the constant contact angle is $\geq \pi/2$ then \mathcal{B} is part of a catenoid.*

The angle is measured between the outward conormal of \mathcal{B} and the outward conormal of the bounded domain in Π bounded by the boundary curve. Since the proof of this theorem is similar to that of Theorem 3.1, we omit some previously proved details.

Proof. Denote the sphere by S_2 and the plane by Π . We may assume that the radius of S_2 is 1. Let α be the constant contact angle between \mathcal{B} and Π . If $\alpha = \pi/2$, then we can reflect \mathcal{B} about Π to get a constant mean curvature annulus meeting two unit spheres tangentially. Hence \mathcal{B} is part of a catenoid or a Delaunay surface by Theorem 3.1.

In the following, we assume that $\alpha \neq \pi/2$. As in the case for \mathcal{A} in Section 1, there is a conformal parametrization X of \mathcal{B} from a strip $\{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \log R\}$ for which $z = u + iv$ is a curvature coordinate. We fix the normal N of \mathcal{B} to point away from the center of S_2 . Let $c_1(v) = X(0, v)$ be on Π and $c_2(v) = X(\log R, v)$ be on S_2 with $\partial X_3/\partial u > 0$ along c_1 . As in Lemma 1.1, c_2 has constant speed $\sqrt{c/2(1+H)}$ and $\kappa_2 = -1$ along c_2 . Since $K \neq 0$ on \mathcal{B} and $z = u + iv$ is a curvature coordinate, we have $\kappa_2 < 0$ on c_1 . The curvature of c_1 is $|\vec{\kappa}| = -\kappa_2/\sin \alpha > 0$, which shows that c_1 is locally convex. Since c_1 is a Jordan curve, it is convex.

First, we assume that $K < 0$ and $\alpha > \pi/2$. Since $(\vec{\kappa}/|\vec{\kappa}|) \cdot (X_u/X_u) = \cos \alpha < 0$ on c_1 , it follows from (2) that $\lambda_u > 0$ on c_1 . Since $\lambda_v(\log R, v) = 0$ (see Lemma 1.1), it follows from (3) that $\lambda_u \geq 0$ on c_2 . Otherwise, λ will have an interior maximum, which contradicts (3). Hence we have $\lambda^2 < c/2(1 + H)$ on $\mathcal{B} \setminus c_2$. Note that $\kappa_1 > 0$ and $\kappa_2 < 0$ in \mathcal{B} . From $\lambda_u \leq 0$ on c_2 , we see that c_2 is convex as a spherical curve (see Lemma 1.1). Arguing as in the proof of Lemma 2.3, we see that $(X_u/|X_u|)(\log R, v)$ is also convex as a spherical curve.

When $K > 0$ and $\alpha < \pi/2$, we have $(\vec{\kappa}/|\vec{\kappa}|) \cdot (X_u/|X_u|) = \cos \alpha > 0$ on c_1 . Hence $\lambda_u < 0$ on c_1 . Since $\lambda_v(\log R, v) = 0$, it follows from (3) that λ does not have interior minimum. Then we have $\lambda_u \leq 0$ on c_2 and $\lambda^2 > c/2(1 + H)$ on $\mathcal{B} \setminus c_2$. Note that $\kappa_1 < 0$ and $\kappa_2 < 0$ in \mathcal{B} . From $\lambda_u \leq 0$ on c_2 , it follows that c_2 is convex as a spherical curve. Moreover $(X_u/|X_u|)(\log R, v)$ is convex as a spherical curve (see Lemma 2.3).

Let $\tilde{\mathcal{B}}$ be the -1 -parallel surface of \mathcal{B} . As in Section 2, we can show that $\tilde{\mathcal{B}}$ is regular except for O_2 : the image of c_2 , and $H > -1$ when $K < 0$ and $H < -1/2$ when $K > 0$. As in Lemma 2.2, we see that mean curvature \tilde{H} and the Gaussian curvature \tilde{K} of $\tilde{\mathcal{B}}$ satisfies $(1 + H)\tilde{K} = (1 + 2H)\tilde{H} - H$ and (i) if $K < 0$ and $H > -1$, then $\tilde{\kappa}_1 > 0$, $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$, (ii) if $K > 0$ and $-1 < H < -1/2$, then $0 < c/2\lambda^2(1 + H) < \min\{1, -H/(1 + H)\}$, $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1 + H)$ and $\tilde{H} < H/(1 + H)$, and (iii) if $K > 0$ and $H < -1$, then $0 < c/2\lambda^2(1 + H) < 1$, $\tilde{\kappa}_1 > (1 + 2H)/2(1 + H)$, $\tilde{\kappa}_2 > H/(1 + H)$ and $\tilde{H} > H/(1 + H)$.

The convexity of $(X_u/|X_u|)(\log R, v)$ as a spherical curve implies that there is a neighborhood of O_2 in $\tilde{\mathcal{B}}$ which is embedded and nonnegatively curved as a metric space. Let $\tilde{\Pi}$ be the plane parallel to Π and containing \tilde{c}_1 . The curvature of \tilde{c}_1 is $|\tilde{\kappa}_2|/\sin \alpha$, which does not vanish. Hence \tilde{c}_1 is locally convex. Using the orthogonal projection onto $\tilde{\Pi}$, \tilde{c}_1 may be considered as a $(\sin \alpha)$ -parallel curve of c_1 in $\tilde{\Pi}$. Hence \tilde{c}_1 is also a convex Jordan curve.

Suppose that $K < 0$ and $\alpha > \pi/2$. Since $\kappa_1 > 0$, \tilde{X}_u is a positive multiple of X_u by (4). The positivity of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ implies that $\tilde{\mathcal{B}}$ meets $\tilde{\Pi}$ in constant angle $\pi - \alpha$. Suppose that $K > 0$ and $\alpha < \pi/2$. If $-1 < H < -1/2$, then we have $c > 0$ and $\kappa_1 > -1$. Hence \tilde{X}_u is a positive multiple of X_u by (4). The negativity of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ implies that $\tilde{\mathcal{B}}$ meets $\tilde{\Pi}$ in constant angle α . When $K > 0$ and $H < -1$, we have $c < 0$ and $\kappa_1 < -1$. Hence \tilde{X}_u is negative multiple of X_u by (4). In this case, $\tilde{\mathcal{B}}$ lies below $\tilde{\Pi}$ and $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are both positive. It is straightforward to see that $\tilde{\mathcal{B}}$ meets $\tilde{\Pi}$ in constant angle α .

Let $\check{\mathcal{B}}$ be the singular surface obtained from $\tilde{\mathcal{B}}$ by attaching the disk in $\tilde{\Pi}$ bounded by \tilde{c}_1 to $\tilde{\mathcal{B}}$. Since $\tilde{\mathcal{B}}$ meets $\tilde{\Pi}$ in acute angle, $\check{\mathcal{B}}$ is a nonnegatively curved metric space. By Alexandrov's generalization [1948] of Hadamard's theorem, $\check{\mathcal{B}}$ is the boundary of a convex body. Therefore $\check{\mathcal{B}}$ is embedded. Note again that \tilde{H} , \tilde{K} , $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ satisfy the statements of Lemma 2.2. Hence (5) is elliptic for functions

representing $\tilde{\mathcal{B}}$ locally. We can apply Alexandrov's moving plane argument to $\tilde{\mathcal{B}}$ using planes perpendicular to $\tilde{\Pi}$ as in the proof of Theorem 3.1 to see that $\tilde{\mathcal{B}}$ is rotational. Hence \mathcal{B} is rotational and, as a result, is part of a Delaunay surface or part of a catenoid. \square

References

- [Alexandrov 1948] A. D. Alexandrov, *Vnutrenniaia geometriia vypuklykh poverkhnostei*, OGIZ, Moscow-Leningrad, 1948. Translated as *Die innere Geometrie der konvexen FlAachen*, Akad. Verl., Berlin, 1955. MR 10,619c
- [Alexandrov 1962] A. D. Alexandrov, "Uniqueness theorems for surfaces in the large, V", *Amer. Math. Soc. Transl. (2)* **21** (1962), 412–416. MR 27 #698e Zbl 0119.16603
- [Bonnet 1860] O. Bonnet, "Mémoire sur l'emploi d'un nouveau système de variables dans l'étude des surfaces courbes", *J. Math. Pures Appl. (2)* **5** (1860), 153–266.
- [do Carmo 1976] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, N.J., 1976. MR 52 #15253 Zbl 0326.53001
- [Gilbarg and Trudinger 2001] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Math., Springer, Berlin, 2001. Reprint of the 1998 edition. MR 2001k:35004 Zbl 1042.35002
- [Hopf 1989] H. Hopf, *Differential geometry in the large*, 2nd ed., Lecture Notes in Math. **1000**, Springer, Berlin, 1989. Notes taken by P. Lax and J. W. Gray, With a preface by S. S. Chern, With a preface by K. Voss. MR 90f:53001 Zbl 0669.53001
- [McCuan 1997] J. McCuan, "Symmetry via spherical reflection and spanning drops in a wedge", *Pacific J. Math.* **180**:2 (1997), 291–323. MR 98m:53013 Zbl 0885.53009
- [Nitsche 1985] J. C. C. Nitsche, "Stationary partitioning of convex bodies", *Arch. Rational Mech. Anal.* **89**:1 (1985), 1–19. MR 86j:53013 Zbl 0572.52005
- [Park and Pyo \geq 2011] S. Park and J. Pyo, "Embedded minimal surfaces meeting 1 or 2 spheres in constant angle 0 or $\pi/2$ ", In preparation.
- [Wente 1995] H. C. Wente, "Tubular capillary surfaces in a convex body", pp. 288–298 in *Advances in geometric analysis and continuum mechanics* (Stanford, CA, 1993), edited by P. Concus and K. Lancaster, Int. Press, Cambridge, MA, 1995. MR 96j:53009 Zbl 0854.53012

Received February 7, 2010.

SUNG-HO PARK
 GRADUATE SCHOOL OF EDUCATION
 HANKUK UNIVERSITY OF FOREIGN STUDIES
 270 IMUN-DONG
 DONGDAEMUN-GU
 SEOUL 130-791
 SOUTH KOREA
 sunghopark@hufs.ac.kr

PACIFIC JOURNAL OF MATHEMATICS

<http://www.pjmath.org>

Founded in 1951 by

E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

V. S. Varadarajan (Managing Editor)

Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jonathan Rogawski
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
jonr@math.ucla.edu

PRODUCTION

pacific@math.berkeley.edu

Silvio Levy, Scientific Editor

Matthew Cargo, Senior Production Editor

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US \$420/year for the electronic version, and \$485/year for print and electronic.

Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

A NON-PROFIT CORPORATION

Typeset in L^AT_EX

Copyright ©2011 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 251 No. 1 May 2011

An analogue of the Cartan decomposition for p -adic symmetric spaces of split p -adic reductive groups	1
PATRICK DELORME and VINCENT SÉCHERRE	
Unital quadratic quasi-Jordan algebras	23
RAÚL FELIPE	
The Dirichlet problem for constant mean curvature graphs in $\mathbb{H} \times \mathbb{R}$ over unbounded domains	37
ABIGAIL FOLHA and SOFIA MELO	
Osgood–Hartogs-type properties of power series and smooth functions	67
BUMA L. FRIDMAN and DAOWEI MA	
Twisted Cappell–Miller holomorphic and analytic torsions	81
RUNG-TZUNG HUANG	
Generalizations of Agol’s inequality and nonexistence of tight laminations	109
THILO KUESSNER	
Chern numbers and the indices of some elliptic differential operators	173
PING LI	
Blocks of the category of cuspidal \mathfrak{sp}_{2n} -modules	183
VOLODYMYR MAZORCHUK and CATHARINA STROPPEL	
A constant mean curvature annulus tangent to two identical spheres is Delauney	197
SUNG-HO PARK	
A note on the topology of the complements of fiber-type line arrangements in $\mathbb{C}\mathbb{P}^2$	207
SHENG-LI TAN, STEPHEN S.-T. YAU and FEI YE	
Inequalities for the Navier and Dirichlet eigenvalues of elliptic operators	219
QIAOLING WANG and CHANGYU XIA	
A Beurling–Hörmander theorem associated with the Riemann–Liouville operator	239
XUECHENG WANG	



0030-8730(201105)251:1;1-E