Pacific Journal of Mathematics

A NOTE ON THE TOPOLOGY OF THE COMPLEMENTS OF FIBER-TYPE LINE ARRANGEMENTS IN $\mathbb{CP}^2$

SHENG-LI TAN, STEPHEN S.-T. YAU AND FEI YE

Volume 251 No. 1 May 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 251, No. 1, 2011

A NOTE ON THE TOPOLOGY OF THE COMPLEMENTS OF FIBER-TYPE LINE ARRANGEMENTS IN $\mathbb{CP}^2$

SHENG-LI TAN, STEPHEN S.-T. YAU AND FEI YE

We prove that $B\mathrm{Diff}_+(S^2, \{x_1, \ldots, x_{n+1}\})$ is a $K(\pi, 1)$ space, where π is the mapping class group of an $(n+1)$-punctured sphere. As a consequence we derive that the center-projecting braid monodromy of a fiber-type projective line arrangement determines the diffeomorphic type of its complement.

1. Introduction

A complex arrangement of hyperplanes $\mathscr{A}$ is a finite collection of $\mathbb{C}$-linear subspaces of dimension $n-1$ in $\mathbb{C}^n$. Denote by $M(\mathscr{A}) = \mathbb{C}^n - \bigcup\{H : H \in \mathscr{A}\}$ the complement of $\mathscr{A}$. The theory of arrangements of hyperplanes is not only closely related to singularity theory, algebraic geometry and hypergeometric function theory, but also has its own interesting questions. For example, one of the central problems is to find the relationship between the topological structure and combinatorial structure of an arrangement. In other words, one wants to understand the topological properties of $M(\mathscr{A})$ and how to classify the arrangements according to their combinatorics. To study such problems, mathematicians have developed many techniques, for example, the lattice-isotopy theorem and braid monodromy method which will be used in this paper. The lattice-isotopy theorem was used in [Jiang and Yau 1994; Wang and Yau 2005; 2007; 2008; Yau and Ye 2009] to derive the structures of so-called nice arrangements and prove that their differential structures are determined by their combinatorics. Braid monodromy method has been widely used to study the topology of complements of plane algebraic curves and line arrangements; see, for example, [Moishezon 1981; Cohen and Suciu 1997; Dung 1999; Kulikov and Taĭkher 2000; Cohen 2001; Artal Bartolo et al. 2003; 2007]. However, there are still many kinds of arrangements for which we are far from understanding the relationship between the topology and combinatorics. This is true even in the case of a fiber-type projective line arrangement, that is, the projectivization of a fiber-type hyperplane arrangement in $\mathbb{C}^3$. Cohen [2001] studied the structure and properties of the fundamental group of the complement of

MSC2000: primary 14N20, 52C35; secondary 57R22, 37E30.
Keywords: fiber-type line arrangement, braid monodromy, differentiable fiber bundle.

a fiber-type arrangement. He showed that the Whitehead group of the fundamental group of the complement of a fiber-type arrangement is trivial, which was conjectured by Aravinda, Farrell and Roushon [2000]. He also proved the conjecture by Xicoténcatl [1997] on the structure of the Lie algebra associated to the lower central series of the fundamental group. Besides that, we still don't know whether the combinatorics of a line arrangement determines the topology of its complement.

It is well known that fiber-type projective line arrangements are the same as supersolvable projective line arrangements (see, for example, [Orlik and Terao 1992]). Moreover, Jiang et al. [2001] studied the geometric characterization of supersolvable line arrangements in $\mathbb{CP}^2$. They showed that any fiber-type line arrangement in $\mathbb{CP}^2$ has a center through which every multiple point of the arrangement has a line in the arrangement passing. The complement of a fiber-type projective line arrangement is a locally trivial fiber bundle with punctured sphere as base and fibers. It is a natural question how to classify the complements of fiber-type line arrangements in $\mathbb{CP}^2$ by center-projecting braid monodromies (see Definition/Construction 4.1). One of the applications of such braid monodromies is that the fundamental group of a fiber-type projective line arrangement is isomorphic to the semidirect product of free groups $\mathbf{F}_m \rtimes_\phi \mathbf{F}_n$, where ϕ is the center-projecting braid monodromy [Cohen 2001]. The purpose of this paper is to use this center-projecting braid monodromy to study the topology of the complement.

It is well known that the braid monodromy determines the homotopy type of the complement of an algebraic curve [Libgober 1986]. In this paper, we prove that for a fiber-type projective line arrangement its center-projecting braid monodromy determines even the diffeomorphic type of its complement, consequently, determines the homotopy type.

Main Theorem. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two fiber-type projective line arrangements. If they have the same center-projecting braid monodromies, then their complements are diffeomorphic.*

The key ingredient of the proof is Proposition 3.1. It shows that the classifying space of the structure group of the complement, the orientation-preserving diffeomorphism group $\mathrm{Diff}_+(S^2, \{x_1, x_2, \ldots, x_{n+1}\})$ of S^2 fixing the set $\{x_1, x_2, \ldots, x_{n+1}\}$, is a $K(\pi, 1)$ space, where π is the mapping class group of a punctured sphere. Morita [1987] explained that $B\mathrm{Diff}_0(\Sigma_g)$, where $\mathrm{Diff}_0(\Sigma_g)$ is the subgroup of diffeomorphisms of a Riemann surface Σ_g which can be deformed to the identity, is contractible for $g \geq 2$, using a result of Earle and Eells [1967]. However, in our case, Earle and Eells' result is not applicable.

2. The complements of fiber-type line arrangements in $\mathbb{CP}^2$

We begin by recalling some definitions which one can find in [Orlik 1992].

Definition 2.1. A hyperplane arrangement $\mathcal{A}$ is called *strictly linear fibered* if, after a suitable linear change of coordinates, the restriction of the projection of $M(\mathcal{A})$ to the first $(n-1)$ coordinates is a fiber bundle projection with base the complement $M(\mathcal{B})$ of an arrangement $\mathcal{B}$ in $\mathbb{C}^{(n-1)}$, and fiber the complement C_* of finitely many points in $\mathbb{C}^3$.

Definition 2.2. A 1-arrangement $\mathcal{A}_1$ of finitely points in $\mathbb{C}$ is fiber-type. An n-arrangement is fiber-type if it is strictly linear fibered over a fiber-type $(n-1)$-arrangement. A fiber-type projective line arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$ is the projectivization of a fiber-type 3-arrangement $\mathcal{A}_3$ in $\mathbb{C}$.

Definition 2.3. Let $\mathcal{A}^*$ be an arrangement in $\mathbb{CP}^2$ and c be a point in the lattice $L(\mathcal{A}^*)$. The point c is called a center of $\mathcal{A}^*$ if for any multiple point p of $\mathcal{A}^*$ there is a line l in $\mathcal{A}^*$ connecting c and p.

Let $\mathcal{A}^*$ be a fiber-type projective line arrangement with complement $M(\mathcal{A}^*)$. We now recall some geometric characterizations of fiber-type line arrangements.

Theorem 2.4 [Terao 1986]. *An arrangement $\mathcal{A}$ is fiber-type if and only if $L(\mathcal{A})$ is supersolvable.*

Theorem 2.5 [Jiang et al. 2001]. *Let $\mathcal{A}$ be a 3-arrangement. The lattice $L(\mathcal{A})$ is a supersolvable if and only if the projectivization $\mathcal{A}^*$ has a center.*

Using the above two theorems, the structure of the complements of fiber-type projective line arrangements can be characterized as follows.

Remark 2.6 [Jiang et al. 2001]. Let c be the center of $\mathcal{A}^*$. After a suitable linear transformation, we may assume that $c = (0:1:0)$ and that one of the lines passing through c is the line at infinity, $z = 0$. We can view $M(\mathcal{A}^*)$ as a subset of $\mathbb{C}^2$. Assume that the lines passing c are defined by the equations

$$z = 0, \quad x = k_1 z, \quad \dots, \quad x = k_m z,$$

and the rest of the lines in $\mathcal{A}^*$ are

$$y = a_1 x + b_1 z, \quad \dots, \quad y = a_n x + b_n z.$$

Therefore, $M(\mathcal{A}^*)$ is a fiber bundle over base $X = \mathbb{CP}^1 - \{k_1, \dots, k_m, \infty\}$ and with fibers $F_x = \mathbb{CP}^1 - \{a_1 x + b_1, \dots, a_n x + b_n, \infty\}$, $x \in X$, under the first coordinate projection $\mathbb{C}^2 \to \mathbb{C}$. Moreover, this fiber bundle admits a structure group $\mathrm{Diff}_+(S^2, \{x_1, \dots, x_n, x_{n+1}\})$.

Definition 2.7. Let $\mathcal{A}^*$ be a fiber-type projective line arrangement in $\mathbb{CP}^2$. Let $c = (0:1:0)$ be the center of $\mathcal{A}^*$. Denote by $\mathrm{St}(c)$ the set of lines in $\mathcal{A}^*$ passing through c. Define the subarrangement associated to $\mathcal{A}^*$ as $\mathcal{B} = \mathcal{A}^* - \mathrm{St}(c)$.

Note that $\mathcal{B}$ can be viewed as an affine arrangement in $\mathbb{C}^2 = \mathbb{CP}^2 - L_\infty$. We will construct the braid monodromy of $\mathcal{B}$ related to $\mathcal{A}^*$ in Section 4.

3. Classification of the complements of fiber-type line arrangements in $\mathbb{CP}^2$ as fiber bundles

For any differentiable fiber bundle with fiber F, let the group $\mathrm{Diff}_+(F)$ be its structure group, the group generated by all orientation preserving diffeomorphisms of F equipped with topology. It is well-known that $\mathrm{Diff}_+(F)$ is also a manifold. Two differentiable fiber bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are isomorphic if there exists an diffeomorphism $h : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

The following natural bijection is a well-known fact:

$$\{\text{isomorphism class of differentiable fiber bundles over } X\} \cong [X, B\mathrm{Diff}_+(F)],$$

where $[X, B\mathrm{Diff}_+(F)]$ is the set of homotopy classes of differentiable maps from X to the classifying space $B\mathrm{Diff}_+(F)$.

Note that the homotopy classes of continuous maps and that of differential maps are canonically the same (see Corollary 3.8.18 in [Conlon 2001]). So the classification of differentiable fiber bundles over X with structure group $\mathrm{Diff}_+(F)$ lies in the set of homotopy classes of continuous maps $X \to B\mathrm{Diff}_+(F)$.

It is well known from obstruction theory (see for example Theorem 11 on page 428 in [Spanier 1981]) that if $B\mathrm{Diff}_+(F)$ is a $K(\pi, 1)$ space, then

$$[X, B\mathrm{Diff}_+(F)] \cong \mathrm{hom}_{\mathrm{conj}}(\pi_1(X), \pi_1(B\mathrm{Diff}_+(F))$$

where $\mathrm{hom}_{\mathrm{conj}}$ means the conjugacy classes of homomorphisms. Two homomorphisms f and g are in the same conjugacy class if and only if there is an inner automorphism a of the target group such that $f = a \circ g \circ a^{-1}$. In the following, we will show that the classifying space of $B\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is a $K(\pi, 1)$ space and the fundamental group is nothing but the mapping class group of an $(n+1)$-punctured sphere, which is the group $\pi_0(\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}))$ of path components of $\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$; see, for example, Chapter 4 in [Birman 1974].

Proposition 3.1. *$B\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is a $K(\pi, 1)$ space. Moreover,*

$$\pi_1(B\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})) = \pi_0(\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}))$$

is the mapping class group of an $(n+1)$-punctured sphere.

Proof. Let $\mathrm{Diff}_+(S^2, x_1, \dots, x_{n+1})$ be the subgroup of $\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ consisting of diffeomorphisms fixing the base points x_i, $i = 1, \dots, n+1$. Then $\mathrm{Diff}_+(S^2, x_1, \dots, x_{n+1})$ is a normal subgroup in

$$\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$$

with the symmetric group $\mathfrak{S}_{n+1}$ as its quotient. On the classifying space level, it follows the fibration

$$B\mathrm{Diff}_+(S^2, x_1, \dots, x_{n+1}) \to B\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}) \to B\mathfrak{S}_{n+1};$$

see [Piccinini and Spreafico 1998, Theorem 6.1]. Since $\mathfrak{S}_{n+1}$ is a discrete group, $\pi_i(\mathfrak{S}_{n+1}) = 0$ for $i \geq 1$. Then

$$\pi_i(B\mathfrak{S}_{n+1}) \cong \pi_{i-1}(\mathfrak{S}_{n+1}) = 0$$

for $i \geq 2$, which implies that $B\mathfrak{S}_{n+1}$ is a $K(\mathfrak{S}_{n+1}, 1)$-space. The advantage of working with $\mathrm{Diff}_+(S^2, x_1, x_2, \dots, x_{n+1})$ is that we can take x_{n+1} to be the point at ∞ and identify

$$\mathrm{Diff}_+(S^2, x_1, \dots, x_{n+1}) \cong \mathrm{Diff}_+(S^2 - \{\infty\}, x_1, \dots, x_n)$$

with the group $\mathrm{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)$ of diffeomorphisms of $\mathbb{R}^2$ that keep the n points $x_1, \dots, x_n$ fixed. The later is a better known group. Following from the well-known criterion for classifying spaces [Steenrod 1999, Theorem 19.4; Cohen 1998, Proposition 2.15], we have another fibration

$$\mathrm{Diff}_+(\mathbb{R}^2)/\,\mathrm{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n) \to B\mathrm{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n) \xrightarrow{f} B\mathrm{Diff}_+(\mathbb{R}^2),$$

where f is defined by forgetting the n points. Consider the configuration space $F_n(\mathbb{R}^2)$ of n points in $\mathbb{R}^2$:

$$F_n(\mathbb{R}^2) = \{\, (x_1, \dots, x_n) \mid x_i \in \mathbb{R}^2 \text{ for } i = 1, 2, \dots, n \text{ and } x_i \neq x_j \text{ if } i \neq j \,\}.$$

It is easy to see that the fiber $\mathrm{Diff}_+(\mathbb{R}^2)/\,\mathrm{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)$ equals $F_n(\mathbb{R}^2)$, which can be considered as the quotient of the flowing homomorphism

$$\begin{aligned} \mathrm{Diff}_+(\mathbb{R}^2) &\to F_n(\mathbb{R}^2) \\ h &\mapsto (h(x_1), \dots, h(x_n)). \end{aligned}$$

It is well known that the configuration space $F_n(\mathbb{R}^2)$ is a $K(\pi, 1)$-space and its fundamental group is a braid group. On the other hand, by Theorem 1 in [Friberg 1973], $\mathrm{Diff}_+(\mathbb{R}^2)$ has the same homotopy type as SO(2), which is homeomorphic to the circle S^1. So $\pi_1(\mathrm{Diff}_+(\mathbb{R}^2)) \cong \pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ and $\pi_i(\mathrm{Diff}_+(\mathbb{R}^2)) \cong \pi_i(\mathrm{SO}(2)) = 0$ for $i \geq 2$. Hence to prove that

$$\pi_i(B\mathrm{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)) = 0$$

for $i \geq 2$, by using the long exact sequence of the fibration

$$\begin{array}{ccccc} \pi_i(F_n(\mathbb{R}^2)) & \longrightarrow & \pi_i(B\mathrm{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)) & \longrightarrow & \pi_i(B\mathrm{Diff}_+(\mathbb{R}^2)) \\ & & & & \wr\| \\ & & & & \pi_{i-1}(\mathrm{Diff}_+(\mathbb{R}^2)) \\ & & & & \wr\| \\ & & & & \pi_{i-1}(\mathrm{SO}(2)), \end{array}$$

it is enough to prove that the boundary map ∂ in the diagram

$$\begin{tikzcd} \pi_2(B\mathrm{Diff}_+(\mathbb{R}^2)) \arrow[r, "\partial"] \arrow[d, "\cong"'] & \pi_1(F_n(\mathbb{R}^2)) \\ \pi_1(\mathrm{Diff}_+(\mathbb{R}^2)) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is injective. The map $\pi_1(\mathrm{Diff}_+(\mathbb{R}^2)) \xrightarrow{\varphi} \pi_1(F_n(\mathbb{R}^2))$ can be identified with the induced homomorphism given by

$$\begin{aligned} \mathrm{Diff}_+(\mathbb{R}^2) &\to F_n(\mathbb{R}^2) \\ h &\mapsto (h(x_1), \dots, h(x_n)). \end{aligned}$$

From this interpretation, it is easy to see that a generator of $\pi_1(\mathrm{Diff}_+(\mathbb{R}^2))$ is mapped to a nontrivial element in $\pi_1(F_{n+1}(\mathbb{R}^2))$. Thus ∂ is injective and hence $B\mathrm{Diff}_+(S^2, x_1, \dots, x_{n+1})$ is a $K(\pi, 1)$-space. So $B\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is also a $K(\pi, 1)$-space and

$$\pi_1(B\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})) = \pi_0(\mathrm{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}))$$

is the mapping class group of an $(n+1)$-punctured sphere. □

It follows immediately that:

Theorem 3.2. *Let $B = S^2 \backslash \{k_1, k_2, \dots, k_{m+1}\}$ and $F = S^2 \backslash \{x_1, x_2, \dots, x_{n+1}\}$. The isomorphic classes of differentiable fiber bundles over B with fiber F and structure group $G = \mathrm{Diff}_+(F)$ are in one-to-one correspondence with the conjugacy classes of homomorphisms from $\pi_1(B)$ to $\pi_1(BG) = \mathcal{M}^n$, where $\mathcal{M}^n$ is the mapping class group of an n-punctured sphere.*

4. Application of braid monodromy

Before we prove our Main Theorem, we will give the definition of center-projecting braid monodromy of a fiber-type projective line arrangement and some useful results [Cohen and Suciu 1997; Dung 1999; Artal Bartolo et al. 2003].

Let $f_i(x) = a_i x + b_i$, $1 \le i \le n$, be the linear functions of the lines not passing through the center c of a fiber-type projective line arrangement. Define

$$f : \mathbb{C} \setminus \{k_1, \dots, k_m\} \to F_n(\mathbb{C})$$

to be the map $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$.

Definition/Construction 4.1. Let $\mathcal{A}^*$ be a fiber-type line arrangement in $\mathbb{CP}^2$ with center c and let $\mathcal{B}$ be the subarrangement associated to $\mathcal{A}^*$. Choose the projection from the complement $M(\mathcal{B})$ in $\mathbb{C}^2$ to $\mathbb{C}$ so that it coincides with the projection from $M(\mathcal{A}^*)$ to a $\mathbb{CP}^1$ through the center c. Let $\infty, k_1, k_2, \dots, k_m$ be the points in $\mathbb{CP}^1$ that are the projective images of the lines in $\mathcal{A}^*$ passing through c. The braid monodromy of $\mathcal{B}$ is the homomorphism $\varphi : \pi_1(\mathbb{C} \setminus \{k_1, \dots, k_m\}) \to B_n$ induced by the map f, where B_n is the braid group of n strings (see [Birman 1974]) and n is the number of the lines in $\mathcal{A}^*$ not passing through the center c. Such a braid monodromy is called the *center-projecting braid monodromy* of the fiber-type line arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$.

One can easily check that the braid monodromy of $\mathcal{B}$ coincides with the monodromy of the fiber bundle $M(\mathcal{A}^*)$.

This fact about the bundle structure of $M(\mathcal{A}^*)$ is a theorem of Cohen [2001]:

Theorem 4.2. *The complement of $\mathcal{A}^*$ with the natural bundle structure is equivalent to the pullback of the bundle of configuration spaces $p_{n+1} : F_{n+1}(\mathbb{C}) \to F_n(\mathbb{C})$ via f.*

The next corollary follows immediately from Theorem 4.2 and Proposition 3.1.

Corollary 4.3. *Let $g : \pi_1(B) \to \mathcal{M}^n$ be a classifying morphism representing the isomorphism class of the bundle $M(\mathcal{A}^*) \to B$ and $q : B_n \to \mathcal{M}^n$ be the classifying morphism representing the isomorphism class the fiber bundle $F_{n+1}(\mathbb{C}) \to F_n(\mathbb{C})$. Then g factors through q via the center-projecting braid monodromy φ.*

Proof. Let $G = \mathrm{Diff}_+(S^2, \{x_1, \dots, x_n, x_{n+1}\})$ be the structure group of the bundle $M(\mathcal{A}^*) \to B$. Let $g' : B \to BG$ be a differentiable map which induces the map g and $q' : F_n(\mathbb{C}) \to BG$ be a differentiable map which induces the map q. Then we have the following bundle isomorphisms: $g'^* EG \cong M(\mathcal{A}^*) \cong f^*(F_{n+1}(\mathbb{C})) \cong f^*(q'^*(EG)) = (q' \circ f)^*(EG)$, where BG is the classifying space of G and EG is the universal fiber bundle over BG. Then $q' \circ f$ and g' are representing the same bundle. Therefore $g = q \circ \varphi$, because the braid monodromy φ is induced by the map f. $\square$

Denote by $\mathbf{F}_m$ the free group generated by m elements.

Definition 4.4. Let $\psi_1, \psi_2 : \pi_1(\mathbf{C} \setminus \{k_1, \dots, k_m\}) = \mathbf{F}_m \to B_n$ be the center-projecting braid monodromies of $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ respectively. We say that $\mathcal{A}_1^*$ and

$\mathscr{A}_2^*$ have the same braid monodromy if there exists an element $\rho \in B_n$ such that $\psi_2(\alpha) = \rho \cdot \psi_1(\alpha) \cdot \rho^{-1}$ for any $\alpha \in \mathbf{F}_m$.

Main Theorem. *Let $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ be two fiber-type projective line arrangements. If they have the same center-projecting braid monodromies, then their complements $M(\mathscr{A}_1^*)$ and $M(\mathscr{A}_2^*)$ are diffeomorphic.*

Proof. By Remark 2.6, the complements of the two fiber-type line arrangements are fiber bundles. Since they have the same center-projecting braid monodromy, they have the same base, fiber and structure group. By Theorem 3.2, we know that the isomorphism classes of such fiber bundles over same base with same fiber and structure group are in one-to-one correspondence with the homomorphisms $\pi_1(S^2 \setminus \{x_1, \dots, x_{m+1}\}) \to \mathscr{M}^n$ up to conjugation. By Corollary 4.3, the isomorphism class of the complement of a fiber-type projective line arrangement as a fiber bundle is determined by the braid monodromy. Let the homomorphism $q : B_n \to \mathscr{M}^n$ be a representative of the isomorphism class of the bundle of configurations $F_{n+1}(\mathbb{C}) \to F_n(\mathbb{C})$. If $\psi_1, \psi_2 : \pi_1(\mathbf{C} \setminus \{k_1, \dots, k_m\}) = \mathbf{F}_m \to B_n$ are the same center-projecting braid monodromies associated to $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ respectively, then there exists a $\rho \in B_n$ such that $\psi_2(\alpha) = \rho \cdot \psi_1(\alpha) \cdot \rho^{-1}$ for any $\alpha \in \mathbf{F}_m$. Thus $q \circ \psi_2(\alpha) = q(\rho) \cdot (q \circ \psi_1(\alpha)) \cdot (q(\rho))^{-1}$ for any $\alpha \in \mathbf{F}_m$. This implies that $q \circ \psi_1$ and $q \circ \psi_2$ determine the same isomorphism class. By the definition of isomorphism of differentiable fiber bundles, any two members in the isomorphism class have diffeomorphic total spaces. This proves the theorem. $\square$

Combined with a theorem of Jiang and Yau [1993], our Main Theorem implies that the center-projecting braid monodromy of a fiber-type projective line arrangement determines its lattice. In fact:

Theorem 4.5 [Cohen and Suciu 1997]. *The braid monodromy of a line arrangement determines its lattice.*

The braid monodromies they considered are generic braid monodromies, that is, projecting from a generic point such that each fiber of the projection contains at most one singularity. However, their method seems also work for nongeneric cases. In fact, when there is more than one singularity in a fiber, the images of the local braid monodromies still record the twists of the braids which reflect the intersecting of lines.

Example 4.6. The complements of any two line arrangements $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ of six lines with four triple points and three nodes are diffeomorphic. Clearly, any triple point can be viewed as a center for such an arrangement. Assume that the line at infinity passes through the center. After removing the center, the subarrangement in $\mathbb{C}^2$ contains three lines, the three solid lines in Figure 1, and the braid monodromy is uniquely determined. In fact, the center-projecting braid monodromies of $\mathscr{A}_1^*$

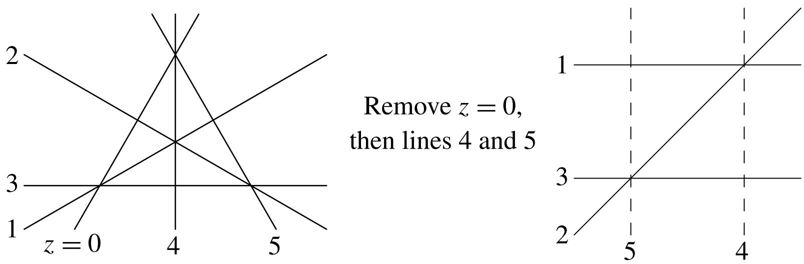

Figure 1. Arrangement of six lines with four triple points and three nodes and its associated subarrangement.

and $\mathcal{A}_2^*$ coincide with the generic braid monodromy of arrangement of 3 lines. Let ξ_1 and ξ_2 be two circles centered at x_1 and x_2, in the base $B = \mathbb{C} \setminus \{x_1, x_2\}$, where x_1 and x_2 are the projections of lines 4 and 5 respectively. Assume that ξ_1 and ξ_2 have a tangent point between x_1 and x_2. Then the fundamental group of the base B is $\pi_1(B) = \langle \xi_1, \xi_2 \rangle$. It is easy to see that the braid monodromy of arrangement of 3 lines as shown in Figure 1 is uniquely determined up to conjugacy by the images of ξ_1 and ξ_2 which are the monodromy generators σ_1^2 (the image of ξ_1) and σ_2^2 (the image of ξ_2), where σ_1 and σ_2 are the two generators of the braid group $\mathbf{B}_3$ on 3 strings as shown in Figure 2 (see, for example, [Cohen and Suciu 1997] on how to calculate braid monodromy generators in general). Hence by our theorem, the complements $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ are diffeomorphic.

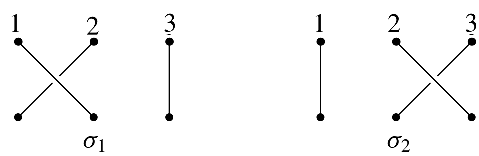

Figure 2. Braid generators of $\mathbf{B}_3$.

Remark 4.7. The arrangement in the example above is well studied in many aspects. For example, it has been shown in a recent paper [Nazir and Yoshinaga 2010] that the moduli space of line arrangements of six lines with four triple points and three nodes is irreducible, so is connected. In fact, it is easy to see that line arrangements of six lines with four triple points and three nodes are of simple C_3 type in the sense of Nazir and Yoshinaga.

Acknowledgement

We gratefully acknowledge professor Ronnie Lee for many useful discussions on Proposition 3.1.

References

[Aravinda et al. 2000] C. S. Aravinda, F. T. Farrell, and S. K. Roushon, "Algebraic K-theory of pure braid groups", *Asian J. Math.* **4**:2 (2000), 337–343. MR 2002a:19002 Zbl 0980.19001

[Artal Bartolo et al. 2003] E. Artal Bartolo, J. Carmona Ruber, and J. I. Cogolludo Agustín, "Braid monodromy and topology of plane curves", *Duke Math. J.* **118** (2003), 261–278. MR 2004k:14015 Zbl 1058.14053

[Artal Bartolo et al. 2007] E. Artal Bartolo, J. Carmona Ruber, and J. I. Cogolludo Agustín, "Effective invariants of braid monodromy", *Trans. Amer. Math. Soc.* **359**:1 (2007), 165–183. MR 2007f:14005 Zbl 1109.14013

[Birman 1974] J. S. Birman, *Braids, links, and mapping class groups*, Annals of Math. Studies **82**, Princeton University Press, Princeton, N.J., 1974. MR 51 #11477 Zbl 0305.57013

[Cohen 1998] R. L. Cohen, "The topology of fiber bundles", Lecture notes, Stanford University, 1998.

[Cohen 2001] D. C. Cohen, "Monodromy of fiber-type arrangements and orbit configuration spaces", *Forum Math.* **13**:4 (2001), 505–530. MR 2002i:52016 Zbl 1018.20032

[Cohen and Suciu 1997] D. C. Cohen and A. I. Suciu, "The braid monodromy of plane algebraic curves and hyperplane arrangements", *Comment. Math. Helv.* **72**:2 (1997), 285–315. MR 98f:52012 Zbl 0959.52018

[Conlon 2001] L. Conlon, *Differentiable manifolds*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser, Boston, 2001. MR 2002b:58001 Zbl 0994.57001

[Dung 1999] N. V. Dung, "Braid monodromy of complex line arrangements", *Kodai Math. J.* **22**:1 (1999), 46–55. MR 2000e:14039 Zbl 0954.14014

[Earle and Eells 1967] C. J. Earle and J. Eells, "The diffeomorphism group of a compact Riemann surface", *Bull. Amer. Math. Soc.* **73** (1967), 557–559. MR 35 #3705 Zbl 0196.09402

[Friberg 1973] B. Friberg, "A topological proof of a theorem of Kneser", *Proc. Amer. Math. Soc.* **39** (1973), 421–426. MR 47 #9657 Zbl 0273.57017

[Jiang and Yau 1993] T. Jiang and S. S.-T. Yau, "Topological invariance of intersection lattices of arrangements in $\mathbf{C}P^2$", *Bull. Amer. Math. Soc.* (*N.S.*) **29**:1 (1993), 88–93. MR 94b:52022 Zbl 0847.52011

[Jiang and Yau 1994] T. Jiang and S. S.-T. Yau, "Diffeomorphic types of the complements of arrangements of hyperplanes", *Compositio Math.* **92**:2 (1994), 133–155. MR 95e:32042 Zbl 0828.57018

[Jiang et al. 2001] T. Jiang, S. S.-T. Yau, and L.-Y. Yeh, "Simple geometric characterization of supersolvable arrangements", *Rocky Mountain J. Math.* **31**:1 (2001), 303–312. MR 2001m:55047 Zbl 1008.32015

[Kulikov and Taĭkher 2000] V. S. Kulikov and M. Taĭkher, "Braid monodromy factorizations and diffeomorphism types", *Izv. Ross. Akad. Nauk Ser. Mat.* **64**:2 (2000), 89–120. MR 2001f:14030 Zbl 1004.14005

[Libgober 1986] A. Libgober, "On the homotopy type of the complement to plane algebraic curves", *J. Reine Angew. Math.* **367** (1986), 103–114. MR 87j:14044 Zbl 0576.14019

[Moishezon 1981] B. G. Moishezon, "Stable branch curves and braid monodromies", pp. 107–192 in *Algebraic geometry* ((Chicago, Ill., 1980)), edited by A. Libgober and P. Wagreich, Lecture Notes in Math. **862**, Springer, Berlin, 1981. MR 83c:14008 Zbl 0476.14005

[Morita 1987] S. Morita, "Characteristic classes of surface bundles", *Invent. Math.* **90**:3 (1987), 551–577. MR 89e:57022 Zbl 0608.57020

[Nazir and Yoshinaga 2010] S. Nazir and M. Yoshinaga, "On the connectivity of the realization spaces of line arrangements", preprint, 2010. arXiv 1009.0202

[Orlik 1992] P. Orlik, "Complements of subspace arrangements", *J. Algebraic Geom.* **1**:1 (1992), 147–156. MR 92h:52014 Zbl 0795.52003

[Orlik and Terao 1992] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grund. der Math. Wissenschaften **300**, Springer, Berlin, 1992. MR 94e:52014 Zbl 0757.55001

[Piccinini and Spreafico 1998] R. A. Piccinini and M. Spreafico, "The Milgram–Steenrod construction of classifying spaces for topological groups", *Exposition. Math.* **16**:2 (1998), 97–130. MR 99f:55022 Zbl 0941.55016

[Spanier 1981] E. H. Spanier, *Algebraic topology*, Springer, New York, 1981. Corrected reprint. MR 83i:55001 Zbl 0477.55001

[Steenrod 1999] N. Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Math., Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks. MR 2000a:55001 Zbl 0942.55002

[Terao 1986] H. Terao, "Modular elements of lattices and topological fibration", *Adv. in Math.* **62**:2 (1986), 135–154. MR 88b:32032 Zbl 0612.05019

[Wang and Yau 2005] S. Wang and S. S.-T. Yau, "Rigidity of differentiable structure for new class of line arrangements", *Comm. Anal. Geom.* **13**:5 (2005), 1057–1075. MR 2007d:32021 Zbl 1115.52010

[Wang and Yau 2007] S. Wang and S. S.-T. Yau, "The diffeomorphic types of the complements of arrangements in $\mathbf{CP}^3$, I, Point arrangements", *J. Math. Soc. Japan* **59**:2 (2007), 423–447. MR 2008i:32041 Zbl 1140.14032

[Wang and Yau 2008] S. Wang and S. S.-T. Yau, "The diffeomorphic types of the complements of arrangements in $\mathbb{CP}^3$, II", *Sci. China Ser. A* **51**:4 (2008), 785–802. MR 2009c:32054 Zbl 1192.14042

[Xicoténcatl 1997] M. Xicoténcatl, *Orbit configuration spaces*, Ph.D. thesis, University of Rochester, 1997.

[Yau and Ye 2009] S. S.-T. Yau and F. Ye, "Diffeomorphic types of complements of nice point arrangements in $\mathbb{CP}^1$", *Sci. China Ser. A* **52**:12 (2009), 2774–2791. MR 2011e:32037 Zbl 1197.14015

Received March 1, 2010.

SHENG-LI TAN
EAST CHINA NORMAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
SHANGHAI 200062
CHINA
sltan@math.ecnu.edu.cn
http://math.ecnu.edu.cn/~sltan/intro.html

STEPHEN S.-T. YAU
UNIVERSITY OF ILLINOIS AT CHICAGO
DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
322 SEO (M/C 249)
851 SOUTH MORGAN STREET
CHICAGO IL 60607
UNITED STATES
yau@uic.edu
http://www.math.uic.edu/~yau/

FEI YE
UNIVERSITY OF ILLINOIS AT CHICAGO
DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
322 SEO (M/C 249)
851 SOUTH MORGAN STREET
CHICAGO 60607
UNITED STATES

fye@math.uic.edu
http://www.math.uic.edu/∼fye

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 251 No. 1 May 2011

0030-8730(201105)251:1;1-E