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**A NOTE ON THE TOPOLOGY OF THE COMPLEMENTS
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A NOTE ON THE TOPOLOGY OF THE COMPLEMENTS OF FIBER-TYPE LINE ARRANGEMENTS IN $\mathbb{C}P^2$

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We prove that $B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is a $K(\pi, 1)$ space, where π is the mapping class group of an $(n+1)$ -punctured sphere. As a consequence we derive that the center-projecting braid monodromy of a fiber-type projective line arrangement determines the diffeomorphic type of its complement.

1. Introduction

A complex arrangement of hyperplanes \mathcal{A} is a finite collection of \mathbb{C} -linear subspaces of dimension $n - 1$ in \mathbb{C}^n . Denote by $M(\mathcal{A}) = \mathbb{C}^n - \bigcup\{H : H \in \mathcal{A}\}$ the complement of \mathcal{A} . The theory of arrangements of hyperplanes is not only closely related to singularity theory, algebraic geometry and hypergeometric function theory, but also has its own interesting questions. For example, one of the central problems is to find the relationship between the topological structure and combinatorial structure of an arrangement. In other words, one wants to understand the topological properties of $M(\mathcal{A})$ and how to classify the arrangements according to their combinatorics. To study such problems, mathematicians have developed many techniques, for example, the lattice-isotopy theorem and braid monodromy method which will be used in this paper. The lattice-isotopy theorem was used in [Jiang and Yau 1994; Wang and Yau 2005; 2007; 2008; Yau and Ye 2009] to derive the structures of so-called nice arrangements and prove that their differential structures are determined by their combinatorics. Braid monodromy method has been widely used to study the topology of complements of plane algebraic curves and line arrangements; see, for example, [Moishezon 1981; Cohen and Suciu 1997; Dung 1999; Kulikov and Taikher 2000; Cohen 2001; Artal Bartolo et al. 2003; 2007]. However, there are still many kinds of arrangements for which we are far from understanding the relationship between the topology and combinatorics. This is true even in the case of a fiber-type projective line arrangement, that is, the projectivization of a fiber-type hyperplane arrangement in \mathbb{C}^3 . Cohen [2001] studied the structure and properties of the fundamental group of the complement of

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a fiber-type arrangement. He showed that the Whitehead group of the fundamental group of the complement of a fiber-type arrangement is trivial, which was conjectured by Aravinda, Farrell and Roushon [2000]. He also proved the conjecture by Xicoténcatl [1997] on the structure of the Lie algebra associated to the lower central series of the fundamental group. Besides that, we still don't know whether the combinatorics of a line arrangement determines the topology of its complement.

It is well known that fiber-type projective line arrangements are the same as supersolvable projective line arrangements (see, for example, [Orlik and Terao 1992]). Moreover, Jiang et al. [2001] studied the geometric characterization of supersolvable line arrangements in $\mathbb{C}\mathbb{P}^2$. They showed that any fiber-type line arrangement in $\mathbb{C}\mathbb{P}^2$ has a center through which every multiple point of the arrangement has a line in the arrangement passing. The complement of a fiber-type projective line arrangement is a locally trivial fiber bundle with punctured sphere as base and fibers. It is a natural question how to classify the complements of fiber-type line arrangements in $\mathbb{C}\mathbb{P}^2$ by center-projecting braid monodromies (see Definition/Construction 4.1). One of the applications of such braid monodromies is that the fundamental group of a fiber-type projective line arrangement is isomorphic to the semidirect product of free groups $\mathbf{F}_m \rtimes_{\phi} \mathbf{F}_n$, where ϕ is the center-projecting braid monodromy [Cohen 2001]. The purpose of this paper is to use this center-projecting braid monodromy to study the topology of the complement.

It is well known that the braid monodromy determines the homotopy type of the complement of an algebraic curve [Libgober 1986]. In this paper, we prove that for a fiber-type projective line arrangement its center-projecting braid monodromy determines even the diffeomorphic type of its complement, consequently, determines the homotopy type.

Main Theorem. *Let \mathcal{A}_1 and \mathcal{A}_2 be two fiber-type projective line arrangements. If they have the same center-projecting braid monodromies, then their complements are diffeomorphic.*

The key ingredient of the proof is Proposition 3.1. It shows that the classifying space of the structure group of the complement, the orientation-preserving diffeomorphism group $\text{Diff}_+(S^2, \{x_1, x_2, \dots, x_{n+1}\})$ of S^2 fixing the set $\{x_1, x_2, \dots, x_{n+1}\}$, is a $K(\pi, 1)$ space, where π is the mapping class group of a punctured sphere. Morita [1987] explained that $B\text{Diff}_0(\Sigma_g)$, where $\text{Diff}_0(\Sigma_g)$ is the subgroup of diffeomorphisms of a Riemann surface Σ_g which can be deformed to the identity, is contractible for $g \geq 2$, using a result of Earle and Eells [1967]. However, in our case, Earle and Eells' result is not applicable.

2. The complements of fiber-type line arrangements in $\mathbb{C}\mathbb{P}^2$

We begin by recalling some definitions which one can find in [Orlik 1992].

Definition 2.1. A hyperplane arrangement \mathcal{A} is called *strictly linear fibered* if, after a suitable linear change of coordinates, the restriction of the projection of $M(\mathcal{A})$ to the first $(n - 1)$ coordinates is a fiber bundle projection with base the complement $M(\mathcal{B})$ of an arrangement \mathcal{B} in $\mathbb{C}^{(n-1)}$, and fiber the complement C_* of finitely many points in \mathbb{C}^3 .

Definition 2.2. A 1-arrangement \mathcal{A}_1 of finitely points in \mathbb{C} is fiber-type. An n -arrangement is fiber-type if it is strictly linear fibered over a fiber-type $(n - 1)$ -arrangement. A fiber-type projective line arrangement \mathcal{A}^* in $\mathbb{C}P^2$ is the projectivization of a fiber-type 3-arrangement \mathcal{A}_3 in \mathbb{C} .

Definition 2.3. Let \mathcal{A}^* be an arrangement in $\mathbb{C}P^2$ and c be a point in the lattice $L(\mathcal{A}^*)$. The point c is called a center of \mathcal{A}^* if for any multiple point p of \mathcal{A}^* there is a line l in \mathcal{A}^* connecting c and p .

Let \mathcal{A}^* be a fiber-type projective line arrangement with complement $M(\mathcal{A}^*)$. We now recall some geometric characterizations of fiber-type line arrangements.

Theorem 2.4 [Terao 1986]. *An arrangement \mathcal{A} is fiber-type if and only if $L(\mathcal{A})$ is supersolvable.*

Theorem 2.5 [Jiang et al. 2001]. *Let \mathcal{A} be a 3-arrangement. The lattice $L(\mathcal{A})$ is a supersolvable if and only if the projectivization \mathcal{A}^* has a center.*

Using the above two theorems, the structure of the complements of fiber-type projective line arrangements can be characterized as follows.

Remark 2.6 [Jiang et al. 2001]. Let c be the center of \mathcal{A}^* . After a suitable linear transformation, we may assume that $c = (0 : 1 : 0)$ and that one of the lines passing through c is the line at infinity, $z = 0$. We can view $M(\mathcal{A}^*)$ as a subset of \mathbb{C}^2 . Assume that the lines passing c are defined by the equations

$$z = 0, \quad x = k_1z, \quad \dots, \quad x = k_mz,$$

and the rest of the lines in \mathcal{A}^* are

$$y = a_1x + b_1z, \quad \dots, \quad y = a_nx + b_nz.$$

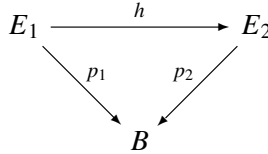
Therefore, $M(\mathcal{A}^*)$ is a fiber bundle over base $X = \mathbb{C}P^1 - \{k_1, \dots, k_m, \infty\}$ and with fibers $F_x = \mathbb{C}P^1 - \{a_1x + b_1, \dots, a_nx + b_n, \infty\}$, $x \in X$, under the first coordinate projection $\mathbb{C}^2 \rightarrow \mathbb{C}$. Moreover, this fiber bundle admits a structure group $\text{Diff}_+(S^2, \{x_1, \dots, x_n, x_{n+1}\})$.

Definition 2.7. Let \mathcal{A}^* be a fiber-type projective line arrangement in $\mathbb{C}P^2$. Let $c = (0 : 1 : 0)$ be the center of \mathcal{A}^* . Denote by $\text{St}(c)$ the set of lines in \mathcal{A}^* passing through c . Define the subarrangement associated to \mathcal{A}^* as $\mathcal{B} = \mathcal{A}^* - \text{St}(c)$.

Note that \mathcal{B} can be viewed as an affine arrangement in $\mathbb{C}^2 = \mathbb{C}P^2 - L_\infty$. We will construct the braid monodromy of \mathcal{B} related to \mathcal{A}^* in Section 4.

3. Classification of the complements of fiber-type line arrangements in $\mathbb{C}\mathbb{P}^2$ as fiber bundles

For any differentiable fiber bundle with fiber F , let the group $\text{Diff}_+(F)$ be its structure group, the group generated by all orientation preserving diffeomorphisms of F equipped with topology. It is well-known that $\text{Diff}_+(F)$ is also a manifold. Two differentiable fiber bundles $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$ are isomorphic if there exists an diffeomorphism $h : E_1 \rightarrow E_2$ such that the following diagram commutes:



The following natural bijection is a well-known fact:

$$\{ \text{isomorphism class of differentiable fiber bundles over } X \} \cong [X, B\text{Diff}_+(F)],$$

where $[X, B\text{Diff}_+(F)]$ is the set of homotopy classes of differentiable maps from X to the classifying space $B\text{Diff}_+(F)$.

Note that the homotopy classes of continuous maps and that of differential maps are canonically the same (see Corollary 3.8.18 in [Conlon 2001]). So the classification of differentiable fiber bundles over X with structure group $\text{Diff}_+(F)$ lies in the set of homotopy classes of continuous maps $X \rightarrow B\text{Diff}_+(F)$.

It is well known from obstruction theory (see for example Theorem 11 on page 428 in [Spanier 1981]) that if $B\text{Diff}_+(F)$ is a $K(\pi, 1)$ space, then

$$[X, B\text{Diff}_+(F)] \cong \text{hom}_{\text{conj}}(\pi_1(X), \pi_1(B\text{Diff}_+(F)))$$

where hom_{conj} means the conjugacy classes of homomorphisms. Two homomorphisms f and g are in the same conjugacy class if and only if there is an inner automorphism a of the target group such that $f = a \circ g \circ a^{-1}$. In the following, we will show that the classifying space of $B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is a $K(\pi, 1)$ space and the fundamental group is nothing but the mapping class group of an $(n+1)$ -punctured sphere, which is the group $\pi_0(\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}))$ of path components of $\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$; see, for example, Chapter 4 in [Birman 1974].

Proposition 3.1. *$B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is a $K(\pi, 1)$ space. Moreover,*

$$\pi_1(B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})) = \pi_0(\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}))$$

is the mapping class group of an $(n+1)$ -punctured sphere.

Proof. Let $\text{Diff}_+(S^2, x_1, \dots, x_{n+1})$ be the subgroup of $\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ consisting of diffeomorphisms fixing the base points x_i , $i = 1, \dots, n+1$. Then $\text{Diff}_+(S^2, x_1, \dots, x_{n+1})$ is a normal subgroup in

$$\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$$

with the symmetric group \mathfrak{S}_{n+1} as its quotient. On the classifying space level, it follows the fibration

$$B\text{Diff}_+(S^2, x_1, \dots, x_{n+1}) \rightarrow B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}) \rightarrow B\mathfrak{S}_{n+1};$$

see [Piccinini and Spreafico 1998, Theorem 6.1]. Since \mathfrak{S}_{n+1} is a discrete group, $\pi_i(\mathfrak{S}_{n+1}) = 0$ for $i \geq 1$. Then

$$\pi_i(B\mathfrak{S}_{n+1}) \cong \pi_{i-1}(\mathfrak{S}_{n+1}) = 0$$

for $i \geq 2$, which implies that $B\mathfrak{S}_{n+1}$ is a $K(\mathfrak{S}_{n+1}, 1)$ -space. The advantage of working with $\text{Diff}_+(S^2, x_1, x_2, \dots, x_{n+1})$ is that we can take x_{n+1} to be the point at ∞ and identify

$$\text{Diff}_+(S^2, x_1, \dots, x_{n+1}) \cong \text{Diff}_+(S^2 - \{\infty\}, x_1, \dots, x_n)$$

with the group $\text{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)$ of diffeomorphisms of \mathbb{R}^2 that keep the n points x_1, \dots, x_n fixed. The later is a better known group. Following from the well-known criterion for classifying spaces [Steenrod 1999, Theorem 19.4; Cohen 1998, Proposition 2.15], we have another fibration

$$\text{Diff}_+(\mathbb{R}^2)/\text{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n) \rightarrow B\text{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n) \xrightarrow{f} B\text{Diff}_+(\mathbb{R}^2),$$

where f is defined by forgetting the n points. Consider the configuration space $F_n(\mathbb{R}^2)$ of n points in \mathbb{R}^2 :

$$F_n(\mathbb{R}^2) = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}^2 \text{ for } i = 1, 2, \dots, n \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

It is easy to see that the fiber $\text{Diff}_+(\mathbb{R}^2)/\text{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)$ equals $F_n(\mathbb{R}^2)$, which can be considered as the quotient of the flowing homomorphism

$$\begin{aligned} \text{Diff}_+(\mathbb{R}^2) &\rightarrow F_n(\mathbb{R}^2) \\ h &\mapsto (h(x_1), \dots, h(x_n)). \end{aligned}$$

It is well known that the configuration space $F_n(\mathbb{R}^2)$ is a $K(\pi, 1)$ -space and its fundamental group is a braid group. On the other hand, by Theorem 1 in [Friberg 1973], $\text{Diff}_+(\mathbb{R}^2)$ has the same homotopy type as $\text{SO}(2)$, which is homeomorphic to the circle S^1 . So $\pi_1(\text{Diff}_+(\mathbb{R}^2)) \cong \pi_1(\text{SO}(2)) = \mathbb{Z}$ and $\pi_i(\text{Diff}_+(\mathbb{R}^2)) \cong \pi_i(\text{SO}(2)) = 0$ for $i \geq 2$. Hence to prove that

$$\pi_i(B\text{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)) = 0$$

for $i \geq 2$, by using the long exact sequence of the fibration

$$\begin{array}{ccccc} \pi_i(F_n(\mathbb{R}^2)) & \longrightarrow & \pi_i(B\text{Diff}_+(\mathbb{R}^2, x_1, \dots, x_n)) & \longrightarrow & \pi_i(B\text{Diff}_+(\mathbb{R}^2)) \\ & & & & \parallel \\ & & & & \pi_{i-1}(\text{Diff}_+(\mathbb{R}^2)) \\ & & & & \parallel \\ & & & & \pi_{i-1}(\text{SO}(2)), \end{array}$$

it is enough to prove that the boundary map ∂ in the diagram

$$\begin{array}{ccc} \pi_2(B\text{Diff}_+(\mathbb{R}^2)) & \xrightarrow{\partial} & \pi_1(F_n(\mathbb{R}^2)) \\ \cong \downarrow & \nearrow \varphi & \\ \pi_1(\text{Diff}_+(\mathbb{R}^2)) & & \end{array}$$

is injective. The map $\pi_1(\text{Diff}_+(\mathbb{R}^2)) \xrightarrow{\varphi} \pi_1(F_n(\mathbb{R}^2))$ can be identified with the induced homomorphism given by

$$\begin{aligned} \text{Diff}_+(\mathbb{R}^2) &\rightarrow F_n(\mathbb{R}^2) \\ h &\mapsto (h(x_1), \dots, h(x_n)). \end{aligned}$$

From this interpretation, it is easy to see that a generator of $\pi_1(\text{Diff}_+(\mathbb{R}^2))$ is mapped to a nontrivial element in $\pi_1(F_{n+1}(\mathbb{R}^2))$. Thus ∂ is injective and hence $B\text{Diff}_+(S^2, x_1, \dots, x_{n+1})$ is a $K(\pi, 1)$ -space. So $B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})$ is also a $K(\pi, 1)$ -space and

$$\pi_1(B\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\})) = \pi_0(\text{Diff}_+(S^2, \{x_1, \dots, x_{n+1}\}))$$

is the mapping class group of an $(n+1)$ -punctured sphere. □

It follows immediately that:

Theorem 3.2. *Let $B = S^2 \setminus \{k_1, k_2, \dots, k_{m+1}\}$ and $F = S^2 \setminus \{x_1, x_2, \dots, x_{n+1}\}$. The isomorphic classes of differentiable fiber bundles over B with fiber F and structure group $G = \text{Diff}_+(F)$ are in one-to-one correspondence with the conjugacy classes of homomorphisms from $\pi_1(B)$ to $\pi_1(BG) = \mathcal{M}^n$, where \mathcal{M}^n is the mapping class group of an n -punctured sphere.*

4. Application of braid monodromy

Before we prove our [Main Theorem](#), we will give the definition of center-projecting braid monodromy of a fiber-type projective line arrangement and some useful results [[Cohen and Suciu 1997](#); [Dung 1999](#); [Artal Bartolo et al. 2003](#)].

Let $f_i(x) = a_i x + b_i$, $1 \leq i \leq n$, be the linear functions of the lines not passing through the center c of a fiber-type projective line arrangement. Define

$$f : \mathbb{C} \setminus \{k_1, \dots, k_m\} \rightarrow F_n(\mathbb{C})$$

to be the map $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$.

Definition/Construction 4.1. Let \mathcal{A}^* be a fiber-type line arrangement in $\mathbb{C}\mathbb{P}^2$ with center c and let \mathcal{B} be the subarrangement associated to \mathcal{A}^* . Choose the projection from the complement $M(\mathcal{B})$ in \mathbb{C}^2 to \mathbb{C} so that it coincides with the projection from $M(\mathcal{A}^*)$ to a $\mathbb{C}\mathbb{P}^1$ through the center c . Let $\infty, k_1, k_2, \dots, k_m$ be the points in $\mathbb{C}\mathbb{P}^1$ that are the projective images of the lines in \mathcal{A}^* passing through c . The braid monodromy of \mathcal{B} is the homomorphism $\varphi : \pi_1(\mathbb{C} \setminus \{k_1, \dots, k_m\}) \rightarrow B_n$ induced by the map f , where B_n is the braid group of n strings (see [Birman 1974]) and n is the number of the lines in \mathcal{A}^* not passing through the center c . Such a braid monodromy is called the *center-projecting braid monodromy* of the fiber-type line arrangement \mathcal{A}^* in $\mathbb{C}\mathbb{P}^2$.

One can easily check that the braid monodromy of \mathcal{B} coincides with the monodromy of the fiber bundle $M(\mathcal{A}^*)$.

This fact about the bundle structure of $M(\mathcal{A}^*)$ is a theorem of Cohen [2001]:

Theorem 4.2. *The complement of \mathcal{A}^* with the natural bundle structure is equivalent to the pullback of the bundle of configuration spaces $p_{n+1} : F_{n+1}(\mathbb{C}) \rightarrow F_n(\mathbb{C})$ via f .*

The next corollary follows immediately from Theorem 4.2 and Proposition 3.1.

Corollary 4.3. *Let $g : \pi_1(B) \rightarrow \mathcal{M}^n$ be a classifying morphism representing the isomorphism class of the bundle $M(\mathcal{A}^*) \rightarrow B$ and $q : B_n \rightarrow \mathcal{M}^n$ be the classifying morphism representing the isomorphism class the fiber bundle $F_{n+1}(\mathbb{C}) \rightarrow F_n(\mathbb{C})$. Then g factors through q via the center-projecting braid monodromy φ .*

Proof. Let $G = \text{Diff}_+(S^2, \{x_1, \dots, x_n, x_{n+1}\})$ be the structure group of the bundle $M(\mathcal{A}^*) \rightarrow B$. Let $g' : B \rightarrow BG$ be a differentiable map which induces the map g and $q' : F_n(\mathbb{C}) \rightarrow BG$ be a differentiable map which induces the map q . Then we have the following bundle isomorphisms: $g'^* EG \cong M(\mathcal{A}^*) \cong f^*(F_{n+1}(\mathbb{C})) \cong f^*(q'^*(EG)) = (q' \circ f)^*(EG)$, where BG is the classifying space of G and EG is the universal fiber bundle over BG . Then $q' \circ f$ and g' are representing the same bundle. Therefore $g = q \circ \varphi$, because the braid monodromy φ is induced by the map f . \square

Denote by \mathbf{F}_m the free group generated by m elements.

Definition 4.4. Let $\psi_1, \psi_2 : \pi_1(\mathbb{C} \setminus \{k_1, \dots, k_m\}) = \mathbf{F}_m \rightarrow B_n$ be the center-projecting braid monodromies of \mathcal{A}_1^* and \mathcal{A}_2^* respectively. We say that \mathcal{A}_1^* and

\mathcal{A}_2^* have the same braid monodromy if there exists an element $\rho \in B_n$ such that $\psi_2(\alpha) = \rho \cdot \psi_1(\alpha) \cdot \rho^{-1}$ for any $\alpha \in \mathbf{F}_m$.

Main Theorem. *Let \mathcal{A}_1^* and \mathcal{A}_2^* be two fiber-type projective line arrangements. If they have the same center-projecting braid monodromies, then their complements $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ are diffeomorphic.*

Proof. By Remark 2.6, the complements of the two fiber-type line arrangements are fiber bundles. Since they have the same center-projecting braid monodromy, they have the same base, fiber and structure group. By Theorem 3.2, we know that the isomorphism classes of such fiber bundles over same base with same fiber and structure group are in one-to-one correspondence with the homomorphisms $\pi_1(S^2 \setminus \{x_1, \dots, x_{m+1}\}) \rightarrow \mathcal{M}^n$ up to conjugation. By Corollary 4.3, the isomorphism class of the complement of a fiber-type projective line arrangement as a fiber bundle is determined by the braid monodromy. Let the homomorphism $q : B_n \rightarrow \mathcal{M}^n$ be a representative of the isomorphism class of the bundle of configurations $F_{n+1}(\mathbb{C}) \rightarrow F_n(\mathbb{C})$. If $\psi_1, \psi_2 : \pi_1(\mathbb{C} \setminus \{k_1, \dots, k_m\}) = \mathbf{F}_m \rightarrow B_n$ are the same center-projecting braid monodromies associated to \mathcal{A}_1^* and \mathcal{A}_2^* respectively, then there exists a $\rho \in B_n$ such that $\psi_2(\alpha) = \rho \cdot \psi_1(\alpha) \cdot \rho^{-1}$ for any $\alpha \in \mathbf{F}_m$. Thus $q \circ \psi_2(\alpha) = q(\rho) \cdot (q \circ \psi_1(\alpha)) \cdot (q(\rho))^{-1}$ for any $\alpha \in \mathbf{F}_m$. This implies that $q \circ \psi_1$ and $q \circ \psi_2$ determine the same isomorphism class. By the definition of isomorphism of differentiable fiber bundles, any two members in the isomorphism class have diffeomorphic total spaces. This proves the theorem. \square

Combined with a theorem of Jiang and Yau [1993], our Main Theorem implies that the center-projecting braid monodromy of a fiber-type projective line arrangement determines its lattice. In fact:

Theorem 4.5 [Cohen and Suciu 1997]. *The braid monodromy of a line arrangement determines its lattice.*

The braid monodromies they considered are generic braid monodromies, that is, projecting from a generic point such that each fiber of the projection contains at most one singularity. However, their method seems also work for nongeneric cases. In fact, when there is more than one singularity in a fiber, the images of the local braid monodromies still record the twists of the braids which reflect the intersecting of lines.

Example 4.6. The complements of any two line arrangements \mathcal{A}_1^* and \mathcal{A}_2^* of six lines with four triple points and three nodes are diffeomorphic. Clearly, any triple point can be viewed as a center for such an arrangement. Assume that the line at infinity passes through the center. After removing the center, the subarrangement in \mathbb{C}^2 contains three lines, the three solid lines in Figure 1, and the braid monodromy is uniquely determined. In fact, the center-projecting braid monodromies of \mathcal{A}_1^*

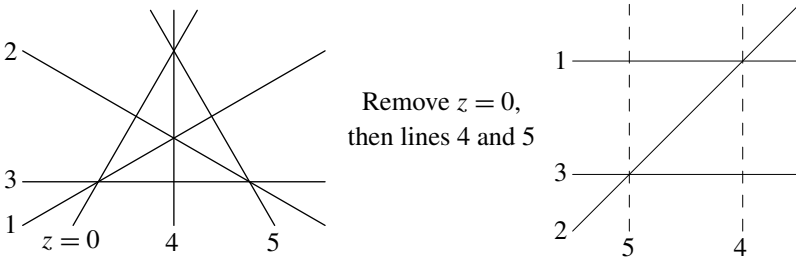


Figure 1. Arrangement of six lines with four triple points and three nodes and its associated subarrangement.

and \mathcal{A}_2^* coincide with the generic braid monodromy of arrangement of 3 lines. Let ξ_1 and ξ_2 be two circles centered at x_1 and x_2 , in the base $B = \mathbb{C} \setminus \{x_1, x_2\}$, where x_1 and x_2 are the projections of lines 4 and 5 respectively. Assume that ξ_1 and ξ_2 have a tangent point between x_1 and x_2 . Then the fundamental group of the base B is $\pi_1(B) = \langle \xi_1, \xi_2 \rangle$. It is easy to see that the braid monodromy of arrangement of 3 lines as shown in Figure 1 is uniquely determined up to conjugacy by the images of ξ_1 and ξ_2 which are the monodromy generators σ_1^2 (the image of ξ_1) and σ_2^2 (the image of ξ_2), where σ_1 and σ_2 are the two generators of the braid group \mathbf{B}_3 on 3 strings as shown in Figure 2 (see, for example, [Cohen and Suciu 1997] on how to calculate braid monodromy generators in general). Hence by our theorem, the complements $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ are diffeomorphic.

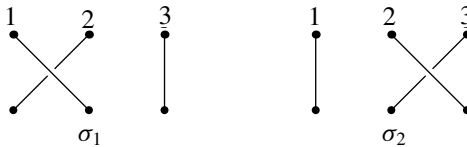


Figure 2. Braid generators of \mathbf{B}_3 .

Remark 4.7. The arrangement in the example above is well studied in many aspects. For example, it has been shown in a recent paper [Nazir and Yoshinaga 2010] that the moduli space of line arrangements of six lines with four triple points and three nodes is irreducible, so is connected. In fact, it is easy to see that line arrangements of six lines with four triple points and three nodes are of simple C_3 type in the sense of Nazir and Yoshinaga.

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