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INEQUALITIES FOR THE NAVIER AND DIRICHLET EIGENVALUES OF ELLIPTIC OPERATORS

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This paper studies eigenvalues of elliptic operators on a bounded domain in a Euclidean space. We obtain lower bounds for the eigenvalues of elliptic operators of higher orders with Navier boundary condition. We also prove lower bounds and universal inequalities of Payne–Pólya–Weinberger–Yang type for the eigenvalues of second order elliptic equations in divergence form with Dirichlet boundary condition.

1. Introduction

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Let Δ be the Laplacian of $\mathbb{R}^n$ and consider the fixed membrane or Dirichlet eigenvalue problem

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1-1}$$

Let

$$0 < \lambda_1 < \lambda_2 \leq \cdots \to \infty$$

denote the eigenvalues (repeated with multiplicity) of the problem (1-1). Weyl's asymptotic formula [1912] tells us that

$$\lambda_k \sim C(n)\left(\frac{k}{|\Omega|}\right)^{2/n} \quad \text{as } k \to \infty, \tag{1-2}$$

where $|\Omega|$ is the volume of Ω and $C(n) = (2\pi)^2 \omega_n^{-2/n}$ with ω_n being the volume of the unit ball in $\mathbb{R}^n$. Pólya [1961] showed that for any "plane covering domain" Ω in $\mathbb{R}^2$ (those that tile $\mathbb{R}^2$) this asymptotic relation is a one-sided inequality (his proof

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also works for $\mathbb{R}^n$-covering domains) and conjectured, for any domain $\Omega \subset \mathbb{R}^n$, the inequality

$$\lambda_k \geq C(n)\left(\frac{k}{|\Omega|}\right)^{2/n} \quad \text{for all } k. \tag{1-3}$$

Li and Yau [1983] showed the lower bound

$$\sum_{i=1}^{k} \lambda_i \geq \frac{nkC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n}, \tag{1-4}$$

which yields an individual lower bound on λ_k in the form

$$\lambda_k \geq \frac{nC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n}. \tag{1-5}$$

Similar bounds for eigenvalues with Neumann boundary condition were proved in [Kröger 1992; 1994; Laptev 1997]. It was pointed out in [Laptev and Weidl 2000] that (1-4) also follows from an earlier result of Berezin [1972] by using the Legendre transformation. Melas [2003] gave an improvement of (1-4):

$$\sum_{i=1}^{k} \lambda_i \geq \frac{nkC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n} + d_n k \frac{|\Omega|}{I(\Omega)}, \tag{1-6}$$

where the constant d_n depends only on the dimension and

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x-a|^2 \, dx$$

is the "moment of inertia" of Ω.

In this paper, we study eigenvalues of elliptic operators of higher orders with Navier boundary condition and of second order elliptic equations in divergence form with Dirichlet boundary condition and prove lower bounds for them which are similar to the inequality (1-6). We will also prove universal inequalities of Yang type for the Dirichlet eigenvalues of second order equations in divergence form. The first two results concern eigenvalues with Navier boundary condition.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let l be a positive integer. Consider the eigenvalue problem*

$$\begin{cases} (-\Delta)^l u = \lambda u \quad \text{in } \Omega, \ u \in C^\infty(\Omega), \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \cdots = \Delta^{l-1} u|_{\partial\Omega} = 0. \end{cases} \tag{1-7}$$

Let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$$

be the eigenvalues of (1-7) *and denote by* $\mu_1, \dots, \mu_n$ *the first* n *nonzero eigenvalues of the Neumann problem*

$$\begin{cases} -\Delta v = \mu v \quad \text{in } \Omega, \\ (\partial v/\partial \nu)|_{\partial\Omega} = 0, \end{cases} \tag{1-8}$$

where ν *is the unit outward normal vector field along* $\partial\Omega$. *Then*

$$\sum_{j=1}^{k} \lambda_j^{1/l} \geq \frac{nkC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n} + \frac{d(n)k}{\sum_{i=1}^{n} \mu_i^{-1}}. \tag{1-9}$$

Here $d(n)$ *is a positive constant depending only on* n.

Theorem 1.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* l *be a fixed positive integer. Let* L *be the elliptic operator given by*

$$Lu = \sum_{m=1}^{l} a_m(-\Delta)^m u, \quad u \in C^\infty(\Omega),$$

where the a_m *are constants with* $a_m \geq 0$, $1 \leq m \leq l$, *and* $a_l = 1$. *Consider the eigenvalue problem*

$$\begin{cases} Lu = \Lambda u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \cdots = \Delta^{l-1} u|_{\partial\Omega} = 0. \end{cases} \tag{1-10}$$

Let

$$0 < \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_k \leq \cdots \to \infty$$

be the eigenvalues of (1-10). *Then*

$$\Lambda_k \geq \sum_{m=1}^{l} a_m \left(\frac{nC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n} + \frac{d(n)}{\sum_{i=1}^{n} \mu_i^{-1}}\right)^m, \tag{1-11}$$

where $\mu_1, \dots, \mu_n$ *are the first* n *nonzero Neumann eigenvalues of* Ω *and* $d(n)$ *is a positive constant depending on* n.

Our next results are about second order equations in divergence form with Dirichlet boundary condition. Firstly, we have a Li–Yau type inequality.

Theorem 1.3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* V *be a nonnegative continuous function on* Ω. *Consider the eigenvalue problem*

$$\begin{cases} -\sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(a_{\alpha\beta}(x)\frac{\partial u}{\partial x_\beta}\right) + V(x)u = \lambda u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1-12}$$

Assume that there is a positive number ξ_0 such that the symmetric matrix $[a_{\alpha\beta}]$ satisfies $[a_{\alpha\beta}] \ge \xi_0 I$ in the sense of quadratic forms throughout Ω, where I is the identity matrix of order n. Let

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty$$

be the eigenvalues of (1-12). Then

$$\sum_{j=1}^{k} \lambda_j \ge \xi_0 k \left(\frac{nC(n)}{n+2} \left(\frac{k}{|\Omega|} \right)^{2/n} + \frac{d(n)}{\sum_{i=1}^{n} \mu_i^{-1}} + \frac{V_0}{\xi_0} \right). \tag{1-13}$$

Here $V_0 = \inf_{x\in\Omega} V(x)$, $\mu_1, \dots, \mu_n$ and $d(n)$ are as in Theorem 1.1.

We then prove a universal inequality of Payne–Pólya–Weinberger–Yang type [Payne et al. 1956; Yang 1991] for an eigenvalue problem more general than (1-12).

Theorem 1.4. *Let Ω be a connected bounded domain in $\mathbb{R}^n$ and let V be a nonnegative continuous function on Ω with $V_0 = \inf_{x\in\Omega} V(x)$. Let ρ be a continuous function on Ω satisfying $\rho_1 \le \rho(x) \le \rho_2$ for all $x \in \Omega$, for some positive constants ρ_1 and ρ_2. Assume also that there are positive numbers ξ_1 and ξ_2 such that the symmetric matrix $[a_{\alpha\beta}]$ satisfies $[a_{\alpha\beta}] \ge \xi_1 I$ in the sense of quadratic forms and $\sum_{\alpha=1}^{n} a_{\alpha\alpha} \le n\xi_2$ throughout Ω. Consider the eigenvalue problem*

$$\begin{cases} -\displaystyle\sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(a_{\alpha\beta}(x) \frac{\partial u}{\partial x_\beta} \right) + V(x)u = \lambda \rho u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1-14}$$

Let

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty$$

be the eigenvalues of (1-14). Then

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4\xi_2 \rho_2^2}{n\xi_1 \rho_1^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i - \frac{V_0}{\rho_2} \right). \tag{1-15}$$

Remark 1.5. Universal inequalities of Payne–Pólya–Weinberger-Yang type for eigenvalues of elliptic operators on Riemannian manifolds have been studied recently by many mathematicians. One can find various interesting results in this direction, for example, in [Ashbaugh 1999; 2002; Ashbaugh and Benguria 1993a; 1993b; Ashbaugh and Hermi 2004; Cheng and Yang 2005; 2006a; 2006b; 2006c; 2007; El Soufi et al. 2007; Harrell 1993; Harrell and Michel 1994; Harrell and Stubbe 1997; Harrell and Yıldırım Yolcu 2009; Hile and Protter 1980; Hook 1990; Laptev 1997; Levitin and Parnovski 2002; Sun et al. 2008; Wang and Xia 2007a; 2007b; 2008; 2010a; 2010b; 2010c; 2011].

2. An auxiliary result

Before proving 1.1–1.3, we show the following fact.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $w_1, \dots, w_k : \Omega \to \mathbb{R}$ be smooth functions satisfying*

$$w_i|_{\partial\Omega} = 0 \quad \textit{and} \quad \int_\Omega w_i(x) w_j(x)\, dx = \delta_{ij} \quad \textit{for } i, j = 1, \dots, k. \tag{2-1}$$

Then

$$\sum_{j=1}^{k} \int_\Omega |\nabla w_j(x)|^2\, dx \ge \frac{nkC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n} + \frac{d(n)k}{\sum_{i=1}^{n} \mu_i^{-1}}, \tag{2-2}$$

where $d(n)$ is a computational positive constant depending only on n and the μ_i are the first n nonzero Neumann eigenvalues of the Laplacian of Ω.

Proof. Let $v_1, \dots, v_n$ be orthonormal eigenfunctions corresponding to $\mu_1, \dots, \mu_n$:

$$-\Delta v_i = \mu_i v_i \quad \text{in } \Omega, \qquad \frac{\partial v_i}{\partial \nu}\Big|_{\partial\Omega} = 0, \qquad \int_\Omega v_i v_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, n.$$

By a translation of the origin and a suitable rotation of axes, we can assume, using [Ashbaugh and Benguria 1993b, p. 563], that

$$\int_\Omega x_i\, dx = 0 \quad \text{for } i = 1, \dots, n,$$
$$\int_\Omega x_j v_i\, dx = 0 \quad \text{for } j = 2, \dots, n,\ i = 1, \dots, j-1.$$

It then follows from inequality (2.8) in the same paper that

$$\sum_{i=1}^{n} \frac{1}{\mu_i} \ge \frac{\int_\Omega |x|^2 dx}{|\Omega|}. \tag{2-3}$$

By a simple rearrangement argument, we have

$$\int_\Omega |x|^2\, dx \ge \frac{n}{n+2}|\Omega|\left(\frac{|\Omega|}{\omega_n}\right)^{2/n}. \tag{2-4}$$

Extend each w_i to $\mathbb{R}^n$ by letting $w_i(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. For a function g on $\mathbb{R}^n$, we will denote by $\mathcal{F}(g)$ the Fourier transformation of g. For any $z \in \mathbb{R}^n$, we have, by definition,

$$\mathcal{F}(w_j)(z) = (2\pi)^{-n/2} \int_\Omega e^{-i\langle x, z\rangle} w_j(x)\, dx. \tag{2-5}$$

Since $\{w_j\}_{j=1}^k$ is an orthonormal set in $L^2(\Omega)$, the Bessel inequality gives

$$\sum_{j=1}^{k} |\mathcal{F}(w_j)(z)|^2 \le (2\pi)^{-n} \int_\Omega |e^{-i\langle x,z\rangle}|^2\, dx = (2\pi)^{-n}|\Omega|. \tag{2-6}$$

For each $q = 1, \dots, n$ and $j = 1, \dots, k$, since w_j vanishes on $\partial\Omega$, one gets from the divergence theorem that

$$\begin{aligned} z_q \mathcal{F}(w_j)(z) &= i(2\pi)^{-n/2} \int_\Omega \frac{\partial e^{-i\langle x,z\rangle}}{\partial x_q} w_j(x)\, dx \\ &= -i(2\pi)^{-n/2} \int_\Omega \frac{\partial w_j(x)}{\partial x_q} e^{-i\langle x,z\rangle}\, dx = -i\mathcal{F}\Big(\frac{\partial w_j}{\partial z_q}\Big)(z). \end{aligned} \tag{2-7}$$

It then follows from the Plancherel formula that

$$\begin{aligned} \int_{\mathbb{R}^n} |z|^2 |\mathcal{F}(w_j)(z)|^2\, dz &= \int_{\mathbb{R}^n} \sum_{q=1}^{n} \Big|\mathcal{F}\Big(\frac{\partial w_j}{\partial z_q}\Big)(z)\Big|^2\, dz \\ &= \int_\Omega \sum_{q=1}^{n} \Big(\frac{\partial w_j}{\partial x_q}\Big)^2\, dx = \int_\Omega |\nabla w_j(x)|^2\, dx. \end{aligned} \tag{2-8}$$

Since

$$\nabla(\mathcal{F}(w_j))(z) = (2\pi)^{-n/2} \int_\Omega (-ixe^{-i\langle x,z\rangle} w_j(x))\, dx,$$

we have

$$\sum_{j=1}^{k} |\nabla(\mathcal{F}(w_j))(z)|^2 \le (2\pi)^{-n} \int_\Omega |ixe^{-i\langle x,z\rangle}|^2\, dx = (2\pi)^{-n} \int_\Omega |x|^2\, dx. \tag{2-9}$$

Set

$$G(z) = \sum_{j=1}^{k} |\mathcal{F}(w_j)(z)|^2.$$

Then $0 \le G(z) \le (2\pi)^{-n}|\Omega|$ and

$$\begin{aligned} |\nabla G(z)| &\le 2\Big(\sum_{j=1}^{k} |\mathcal{F}(w_j)(z)|^2\Big)^{1/2} \Big(\sum_{j=1}^{k} |\nabla(\mathcal{F}(w_j))(z)|^2\Big)^{1/2} \\ &\le 2(2\pi)^{-n}\Big(|\Omega| \int_\Omega |x|^2\, dx\Big)^{1/2} \end{aligned} \tag{2-10}$$

for every $z \in \mathbb{R}^n$. We also have

$$\int_{\mathbb{R}^n} G(z)\, dz = \sum_{j=1}^{k} \int_\Omega w_j(x)^2\, dx = k. \tag{2-11}$$

Let $G^*(z) = g(|z|)$ be the decreasing spherical rearrangement of G. By approximation, we may assume that $g : [0, +\infty) \to [0, (2\pi)^{-n}|\Omega|]$ is absolutely continuous. Setting $\alpha(t) = |\{G^* > t\}| = |\{G > t\}|$, we have

$$\alpha(g(s)) = \omega_n s^n, \tag{2-12}$$

which implies that $n\omega_n s^{n-1} = \alpha'(g(s))g'(s)$ for almost every s. The coarea formula [Chavel 1984] tells us that

$$\alpha'(t) = -\int_{\{G=t\}} \frac{1}{|\nabla G|}\, dA_t. \tag{2-13}$$

Set $\eta = 2(2\pi)^{-n}\big(|\Omega| \int_\Omega |x|^2\, dx\big)^{1/2}$; then one infers from (2-10) and the isoperimetric inequality that

$$-\alpha'(g(s)) \geq \eta^{-1}\text{area}(\{G = g(s)\}) \geq \eta^{-1} n\omega_n s^{n-1}, \tag{2-14}$$

and so

$$-\eta \leq g'(s) \leq 0 \tag{2-15}$$

for almost every s. It follows from (2-11) that

$$k = \int_{\mathbb{R}^n} G(z)\, dz = \int_{\mathbb{R}^n} G^*(z)\, dz = n\omega_n \int_0^\infty s^{n-1} g(s)\, ds \tag{2-16}$$

and, by (2-8),

$$\begin{aligned}\sum_{j=1}^k \int_\Omega |\nabla w_j|^2 = \int_{\mathbb{R}^n} |z|^2 G(z)\, dz \geq \int_{\mathbb{R}^n} |z|^2 G^*(z)\, dz \\ = n\omega_n \int_0^\infty s^{n+1} g(s)\, ds,\end{aligned} \tag{2-17}$$

since $z \to |z|^2$ is radial and increasing.

We next apply the following lemma to the function g, with $A = \dfrac{k}{n\omega_n}$ and

$$\eta = 2(2\pi)^{-n}\left(|\Omega| \int_\Omega |x|^2\, dx\right)^{1/2}:$$

Lemma [Melas 2003]. *Let $n \geq 1$ and $\eta,\ A > 0$ and let $h : [0, +\infty) \to [0, +\infty)$ be a decreasing and absolutely continuous function such that*

$$-\eta \leq h'(s) \leq 0 \quad \text{and} \quad \int_0^\infty s^{n-1} h(s)\, ds = A.$$

Then

$$\int_0^\infty s^{n+1} h(s)\, ds \geq \frac{(nA)^{(n+2)/n}}{n+2} h(0)^{-2/n} + \frac{Ah(0)^2}{6(n+2)\eta^2}. \tag{2-18}$$

After applying the lemma and using (2-17), we infer that

$$\sum_{j=1}^{k}\int_{\Omega}|\nabla w_j(x)|^2\,dx \geq \frac{n}{n+2}\,\omega_n^{-2/n}k^{1+2/n}g(0)^{-2/n}+\frac{kg(0)^2}{6(n+2)\eta^2} \geq \frac{n}{n+2}\,\omega_n^{-2/n}k^{1+2/n}g(0)^{-2/n}+\frac{\tau kg(0)^2}{(n+2)\eta^2}, \tag{2-19}$$

where τ is any constant with $0<\tau\leq\frac{1}{6}$. From (2-4) we know that

$$\eta \geq 2(2\pi)^{-n}\Big(\frac{n}{n+2}\Big)^{1/2}\omega_n^{-1/n}|\Omega|^{(n+1)/n}. \tag{2-20}$$

Observe that $0<g(0)\leq(2\pi)^{-n}|\Omega|$. Let $\tau=\tau(n)$ be the constant given by

$$\tau=\min\Big\{\frac{1}{6},\,\frac{16\pi^2 n}{(n+2)\omega_n^{4/n}}\Big\}.$$

Then one can see by using (2-20) that the function

$$\beta(t)=\frac{n}{n+2}\omega_n^{-2/n}k^{1+2/n}t^{-2/n}+\frac{\tau kt^2}{(n+2)\eta^2}$$

satisfies

$$\beta'((2\pi)^{-n}|\Omega|)\leq 0,$$

and so β is decreasing on $(0,\,(2\pi)^{-n}|\Omega|]$. Hence, choosing $d(n)=\dfrac{\tau}{4(n+2)}$, we have

$$\begin{aligned}\sum_{j=1}^{k}\int_{\Omega}|\nabla w_j(x)|^2\,dx &\geq \beta(g(0))\geq\beta((2\pi)^{-n}|\Omega|)\\ &=\frac{n}{n+2}\,\omega_n^{-2/n}k^{1+2/n}((2\pi)^{-n}|\Omega|)^{-2/n}+\frac{\tau k((2\pi)^{-n}|\Omega|)^2}{(n+2)\eta^2}\\ &=\frac{n}{n+2}\Big(\frac{2\pi}{\omega_n^{1/n}}\Big)^2k^{1+2/n}|\Omega|^{-2/n}+\frac{d(n)k|\Omega|}{\int_\Omega|x|^2\,dx}.\end{aligned} \tag{2-21}$$

Substituting (2-3) into (2-21), one gets (2-2). completing the proof of Theorem 2.1. □

3. Proof of the main results

Proof of Theorem 1.1. Let $\{u_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions corresponding to $\{\lambda_i\}_{i=1}^k$:

$$(-\Delta)^l u_i = \lambda_i u_i \quad \text{in } \Omega, \qquad u_i|_{\partial\Omega} = \Delta u_i|_{\partial\Omega} = \cdots = \Delta^{l-1} u_i|_{\partial\Omega} = 0,$$
$$\int_\Omega u_i u_j = \delta_{ij} \quad \text{for } i, j = 1, \ldots, k.$$

We show that for each $s = 1, \ldots, l$ and $i = 1, \ldots, k$,

$$0 \le \int_\Omega u_i (-\Delta)^s u_i \le \lambda_i^{s/l}. \tag{3-1}$$

When $l = 1$, (3-1) holds trivially, so assume $l > 1$. when $s \in \{1, \ldots, l\}$ is even, we have from the divergence theorem that

$$\int_\Omega u_i (-\Delta)^s u_i = \int_\Omega u_i \Delta^s u_i = \int_\Omega (\Delta^{s/2} u_i)^2 \ge 0.$$

On the other hand, if $s \in \{1, \ldots, l\}$ is odd,

$$\begin{aligned}\int_\Omega u_i (-\Delta)^s u_i &= -\int_\Omega u_i \Delta^s u_i \\ &= -\int_\Omega \Delta^{(s-1)/2} u_i \Delta(\Delta^{(s-1)/2} u_i) = \int_\Omega |\nabla(\Delta^{(s-1)/2} u_i)|^2 \ge 0.\end{aligned}$$

Thus the first inequality in (3-1) holds.

We claim now that for any $s = 1, \ldots, l-1$,

$$\left(\int_\Omega u_i (-\Delta)^s u_i\right)^{s+1} \le \left(\int_\Omega u_i (-\Delta)^{s+1} u_i\right)^s. \tag{3-2}$$

Since

$$\left(\int_\Omega u_i \Delta u_i\right)^2 \le \int_\Omega u_i^2 \int_\Omega (\Delta u_i)^2 = \int_\Omega u_i \Delta^2 u_i,$$

we know that (3-2) holds when $s = 1$.

Suppose that (3-2) is true for $s - 1$, that is,

$$\left(\int_\Omega u_i (-\Delta)^{s-1} u_i\right)^s \le \left(\int_\Omega u_i (-\Delta)^s u_i\right)^{s-1}. \tag{3-3}$$

When s is even, we have

$$\begin{aligned}\int_\Omega u_i (-\Delta)^s u_i &= \int_\Omega \Delta^{s/2-1} u_i \Delta(\Delta^{s/2} u_i) = -\int_\Omega \nabla(\Delta^{s/2-1} u_i) \nabla(\Delta^{s/2} u_i) \\ &\le \left(\int_\Omega |\nabla(\Delta^{s/2-1} u_i)|^2\right)^{1/2} \left(\int_\Omega |\nabla(\Delta^{s/2} u_i)|^2\right)^{1/2} \\ &= \left(-\int_\Omega \Delta^{s/2-1} u_i \Delta^{s/2} u_i\right)^{1/2} \left(-\int_\Omega \Delta^{s/2} u_i \Delta^{s/2+1} u_i\right)^{1/2} \\ &= \left(\int_\Omega u_i (-\Delta)^{s-1} u_i\right)^{1/2} \left(\int_\Omega u_i (-\Delta)^{s+1} u_i\right)^{1/2}.\end{aligned}$$

On the other hand, when s is odd,

$$\begin{aligned}\int_\Omega u_i(-\Delta)^s u_i &= \int_\Omega (-\Delta)^{(s-1)/2} u_i (-\Delta)^{(s+1)/2} u_i \\ &\le \left(\int_\Omega \left((-\Delta)^{(s-1)/2} u_i\right)^2\right)^{1/2} \left(\int_\Omega \left((-\Delta)^{(s+1)/2} u_i\right)^2\right)^{1/2} \\ &= \left(\int_\Omega u_i(-\Delta)^{s-1} u_i\right)^{1/2} \left(\int_\Omega u_i(-\Delta)^{s+1} u_i\right)^{1/2}.\end{aligned}$$

Thus we always have

$$\int_\Omega u_i(-\Delta)^s u_i \le \left(\int_\Omega u_i(-\Delta)^{s-1} u_i\right)^{1/2} \left(\int_\Omega u_i(-\Delta)^{s+1} u_i\right)^{1/2}. \tag{3-4}$$

Substituting (3-3) into (3-4), we know that (3-2) is true for s. Using (3-2) repeatedly, we get

$$\int_\Omega u_i(-\Delta)^s u_i \le \left(\int_\Omega u_i(-\Delta)^{s+1} u_i\right)^{s/(s+1)} \le \cdots \le \left(\int_\Omega u_i(-\Delta)^{l} u_i\right)^{s/l} = \lambda_i^{s/l}.$$

Thus the second inequality in (3-1) is also true. Consequently,

$$\sum_{j=1}^k \int_\Omega |\nabla u_j|^2 = \sum_{j=1}^k \int_\Omega u_j(-\Delta u_j) \le \sum_{j=1}^k \lambda_j^{1/l} \tag{3-5}$$

which implies (1-9) by applying Theorem 2.1 to the functions $u_1, \dots, u_k$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $\{u_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions of the problem (1-11) corresponding to $\{\lambda_i\}_{i=1}^k$:

$$Lu_i = \Lambda u_i \quad \text{in } \Omega, \qquad u_i|_{\partial\Omega} = \Delta u_i|_{\partial\Omega} = \cdots = \Delta^{l-1} u_i|_{\partial\Omega} = 0,$$
$$\int_\Omega u_i u_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, k.$$

Denote by $\{\lambda_i\}_{i=1}^k$ the first k fixed membrane eigenvalues of the Laplacian of Ω corresponding to the orthonormal eigenfunctions $\{v_i\}_{i=1}^k$:

$$-\Delta v_i = \lambda_i v_i \quad \text{in } \Omega, \qquad v_i|_{\partial\Omega} = 0, \qquad \int_\Omega v_i v_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, k.$$

Let $w = \sum_{j=1}^k \alpha_j u_j \neq 0$ be such that

$$\int_\Omega w v_i = 0 \quad \text{for } i = 1, \dots, k-1. \tag{3-6}$$

Such an element w exists because $\{\alpha_j \mid 1 \le j \le k\}$ is a nontrivial solution of a system of $k-1$ linear equations

$$\sum_{j=1}^{k} \alpha_j \int_\Omega u_j v_i = 0, \quad 1 \le i \le k-1, \tag{3-7}$$

in k unknowns. Also assume without loss of generality that

$$\int_\Omega w^2 = 1. \tag{3-8}$$

Then it follows from the Rayleigh–Ritz inequality that

$$\lambda_k \le \int_\Omega w(-\Delta w). \tag{3-9}$$

By using the arguments similar to those in the proof of (3-2), we have

$$\left(\int_\Omega w(-\Delta)^j w\right)^{j+1} \le \left(\int_\Omega w(-\Delta)^{j+1} w\right)^j, \quad j = 1, \dots, l-1. \tag{3-10}$$

Hence

$$\int_\Omega w(-\Delta w) \le \left(\int_\Omega w(-\Delta)^s w\right)^{1/s}, \quad s = 1, \dots, l, \tag{3-11}$$

which, combined with (3-9), gives

$$\lambda_k^s \le \int_M w(-\Delta)^s w, \quad s = 1, 2, \dots, l.$$

Thus we have

$$\begin{aligned} a_1\lambda_k + a_2\lambda_k^2 + \cdots + a_{l-1}\lambda_k^{l-1} + \lambda_k^l &\\ \le \int_\Omega w\big(a_1(-\Delta) + a_2(-\Delta)^2 + \cdots + a_{l-1}(-\Delta)^{l-1} + (-\Delta)^l\big)w &\\ = \int_\Omega wLw = \sum_{i,j=1}^{k} \alpha_i\alpha_j \int_\Omega u_i L u_j = \sum_{i,j=1}^{k} \alpha_i\alpha_j \int_\Omega u_i \Lambda_j u_j &\\ = \sum_{i,j=1}^{k} \alpha_i\alpha_j \Lambda_j \delta_{ij} = \sum_{i=1}^{k} \alpha_i^2 \Lambda_i \le \Lambda_k, & \end{aligned} \tag{3-12}$$

where, in the last equality, we have used the fact that

$$\sum_{i=1}^{k} \alpha_i^2 = \int_\Omega w^2 = 1.$$

It is easy to see by taking $l = 1$ in (1-9) that

$$\lambda_k \geq \frac{nC(n)}{n+2}\left(\frac{k}{|\Omega|}\right)^{2/n} + \frac{d(n)}{\sum_{i=1}^n \mu_i^{-1}}. \tag{3-13}$$

Substituting (3-13) into (3-12), we get (1-11). Theorem 1.2 follows. □

Proof of Theorem 1.3. Let $\{u_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions of the problem (1-12) corresponding to $\{\lambda_i\}_{i=1}^k$:

$$-\sum_{\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha}\left(a_{\alpha\beta}(x)\frac{\partial u_i}{\partial x_\beta}\right) + V(x)u_i = \lambda_i u_i \quad \text{in } \Omega, \tag{3-14}$$

$$u_i|_{\partial\Omega} = 0, \quad \int_\Omega u_i u_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, k. \tag{3-15}$$

Multiplying (3-14) by u_i, integrating over Ω, and using the divergence theorem and the inequalities $V \geq V_0$ and $[a_{\alpha\beta}] \geq \xi_0 I$, we obtain

$$\begin{aligned}\lambda_i &= \int_\Omega \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_i}{\partial x_\beta} + V(x)u_i^2\right)\\ &\geq \int_\Omega \xi_0 \sum_{\alpha=1}^n \left(\frac{\partial u_i}{\partial x_\alpha}\right)^2 + V_0\int_\Omega u_i^2 = \xi_0\int_\Omega |\nabla u_i|^2 + V_0,\end{aligned}$$

which gives

$$\sum_{i=1}^k \int_\Omega |\nabla u_i|^2 \leq \frac{1}{\xi_0}\left(\sum_{i=1}^k \lambda_i - kV_0\right). \tag{3-16}$$

Observing (3-15), one gets (1-13) by using Theorem 2.1 applied to $u_1, \dots, u_k$. Theorem 1.3 follows. □

Proof of Theorem 1.4. Let $x_1, \dots, x_n$ be the coordinate functions on $\mathbb{R}^n$. For a function $f : \Omega \to \mathbb{R}$, set $f_{,\alpha} = \partial f/\partial x_\alpha$, $\alpha = 1, \dots, n$. Let $\{u_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions of the problem (1-14) corresponding to $\{\lambda_i\}_{i=1}^k$:

$$-\sum_{\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha}\left(a_{\alpha\beta}(x)\frac{\partial u_i}{\partial x_\beta}\right) + V(x)u_i = \lambda_i \rho u_i \quad \text{in } \Omega,$$

$$u_i|_{\partial\Omega} = 0, \quad \int_\Omega u_i u_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, k.$$

For each $\alpha = 1, \dots, n$ and $i = 1, \dots, k$, following [Payne et al. 1956], consider the functions $\phi_{\alpha i} : \Omega \to \mathbb{R}$ given by

$$\phi_{\alpha i} = x_\alpha u_i - \sum_{j=1}^k r_{\alpha ij} u_j, \tag{3-17}$$

where

$$r_{\alpha ij} = \int_\Omega \rho x_\alpha u_i u_j. \tag{3-18}$$

Since $\phi_{\alpha i}|_{\partial\Omega} = 0$ and

$$\int_\Omega \rho u_j \phi_{\alpha i} = 0 \quad \text{for } i, j = 1, \dots, k \text{ and } \alpha = 1, \dots, n,$$

it follows from the Rayleigh–Ritz inequality that

$$\begin{aligned} \lambda_{k+1} \int_\Omega \rho \phi_{\alpha i}^2 &\le \int_\Omega \phi_{\alpha i} \Big(-\sum_{\beta,\gamma=1}^n (a_{\beta\gamma} \phi_{\alpha i,\gamma})_{,\beta} + V \phi_{\alpha i} \Big) \\ &= \int_\Omega \phi_{\alpha i} \Big(-\sum_{\beta,\gamma=1}^n (a_{\beta\gamma} (x_\alpha u_i)_{,\gamma})_{,\beta} + V x_\alpha u_i - \sum_{j=1}^k r_{\alpha ij} \lambda_j \rho u_j \Big) \\ &= \int_\Omega \phi_{\alpha i} \Big(-\sum_{\beta,\gamma=1}^n (a_{\beta\gamma} (x_\alpha u_i)_{,\gamma})_{,\beta} + V x_\alpha u_i \Big) \\ &= \int_\Omega \phi_{\alpha i} \Big(\lambda_i \rho x_\alpha u_i - \sum_{\beta=1}^n ((a_{\alpha\beta} u_i)_{,\beta} + a_{\alpha\beta} u_{i,\beta}) \Big) \\ &= \lambda_i \int_\Omega \rho \phi_{\alpha i}^2 - \int_\Omega \phi_{\alpha i} \Big(\sum_{\beta=1}^n ((a_{\alpha\beta} u_i)_{,\beta} + a_{\alpha\beta} u_{i,\beta}) \Big) \\ &= \lambda_i \int_\Omega \rho \phi_{\alpha i}^2 - \int_\Omega x_\alpha u_i \Big(\sum_{\beta=1}^n ((a_{\alpha\beta} u_i)_{,\beta} + a_{\alpha\beta} u_{i,\beta}) \Big) + \sum_{j=1}^k r_{\alpha ij} s_{\alpha ij}, \end{aligned} \tag{3-19}$$

where

$$s_{\alpha ij} = \int_\Omega \Big(\sum_{\beta=1}^n ((a_{\alpha\beta} u_i)_{,\beta} + a_{\alpha\beta} u_{i,\beta}) \Big) u_j.$$

Multiplying the equation

$$-\sum_{\beta,\gamma=1}^n (a_{\beta\gamma} u_{j,\beta})_{,\gamma} + V u_j = \lambda_j \rho u_j \tag{3-20}$$

by $x_\alpha u_i$, we have

$$-\sum_{\beta,\gamma=1}^n (a_{\beta\gamma} u_{j,\beta})_{,\gamma} x_\alpha u_i + V x_\alpha u_i u_j = \lambda_j \rho x_\alpha u_i u_j. \tag{3-21}$$

Interchanging the roles of i and j, we get

$$-\sum_{\beta\gamma=1}^{n}(a_{\beta\gamma}u_{i,\beta})_{,\gamma}x_\alpha u_j + Vx_\alpha u_i u_j = \lambda_i \rho x_\alpha u_i u_j. \tag{3-22}$$

Subtracting (3-21) from (3-22) and integrating the resulted equation on Ω, we get by using the divergence theorem that

$$\begin{aligned}(\lambda_i-\lambda_j)r_{\alpha ij} &= \sum_{\beta,\gamma=1}^{n}\int_\Omega \big((a_{\beta\gamma}u_{j,\beta})_{,\gamma}x_\alpha u_i - (a_{\beta\gamma}u_{i,\beta})_{,\gamma}x_\alpha u_j\big)\\ &= \sum_{\beta,\gamma=1}^{n}\int_\Omega \big(-a_{\beta\gamma}u_{j,\beta}(x_\alpha u_i)_{,\gamma} + a_{\beta\gamma}u_{i,\beta}(x_\alpha u_j)_{,\gamma}\big)\\ &= \sum_{\beta=1}^{n}\int_\Omega \big(-a_{\alpha\beta}u_{j,\beta}u_i + a_{\alpha\beta}u_{i,\beta}u_j\big)\\ &= \sum_{\beta=1}^{n}\int_\Omega \big((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta}\big)u_j = s_{\alpha ij},\end{aligned} \tag{3-23}$$

which, combined with (3-19), gives

$$\begin{aligned}(\lambda_{k+1}-\lambda_i)\int_\Omega \rho\phi_{\alpha i}^2 &\le -\int_\Omega \phi_{\alpha i}\Big(\sum_{\beta=1}^{n}((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta})\Big)\\ &= -\int_\Omega x_\alpha u_i\Big(\sum_{\beta=1}^{n}((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta})\Big) + \sum_{j=1}^{k}(\lambda_i-\lambda_j)r_{\alpha ij}^2.\end{aligned} \tag{3-24}$$

Set

$$t_{\alpha ij} = \int_\Omega u_j u_{i,\alpha};$$

then $t_{\alpha ij} + t_{\alpha ji} = 0$ and

$$\int_\Omega (-2)\phi_{\alpha i}u_{i,\alpha} = -2\int_\Omega x_\alpha u_i u_{i,\alpha} + 2\sum_{j=1}^{k} r_{\alpha ij}t_{\alpha ij} = \|u_i\|^2 + 2\sum_{j=1}^{k} r_{\alpha ij}t_{\alpha ij}. \tag{3-25}$$

Multiplying (3-25) by $(\lambda_{k+1}-\lambda_i)^2$ and using the Schwarz inequality and (3-24), we get

$$(\lambda_{k+1}-\lambda_i)^2\Big(\|u_i\|^2 + 2\sum_{j=1}^{k} r_{\alpha ij}t_{\alpha ij}\Big)$$

$$
\begin{aligned}
&= (\lambda_{k+1}-\lambda_i)^2 \int_\Omega (-2)\sqrt{\rho}\phi_{\alpha i}\Big(\frac{1}{\sqrt{\rho}}u_{i,\alpha} - \sum_{j=1}^k t_{\alpha ij}\sqrt{\rho}u_j\Big)\\
&\le \delta(\lambda_{k+1}-\lambda_i)^3\|\sqrt{\rho}\phi_{\alpha i}\|^2 + \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega \Big|\frac{1}{\sqrt{\rho}}u_{i,\alpha} - \sum_{j=1}^k t_{\alpha ij}\sqrt{\rho}u_j\Big|^2\\
&= \delta(\lambda_{k+1}-\lambda_i)^3\|\sqrt{\rho}\phi_{\alpha i}\|^2 + \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\Big(\Big\|\frac{1}{\sqrt{\rho}}u_{i,\alpha}\Big\|^2 - \sum_{j=1}^k t_{\alpha ij}^2\Big)\\
&\le \delta(\lambda_{k+1}-\lambda_i)^2\Big(-\int_\Omega x_\alpha u_i\Big(\sum_{\beta=1}^n((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta})\Big) + \sum_{j=1}^k(\lambda_i-\lambda_j)r_{\alpha ij}^2\Big)\\
&\qquad\qquad + \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\Big(\Big\|\frac{1}{\sqrt{\rho}}u_{i,\alpha}\Big\|^2 - \sum_{j=1}^k t_{\alpha ij}^2\Big),
\end{aligned}
$$

where δ is any positive constant. Summing over i and noticing that $r_{\alpha ij} = r_{\alpha ji}$ and $t_{\alpha ij} = -t_{\alpha ji}$, we infer that

$$
\begin{aligned}
&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\|u_i\|^2 - 2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{\alpha ij}t_{\alpha ij}\\
&\le \delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\Big(-\int_\Omega x_\alpha u_i\Big(\sum_{\beta=1}^n((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta})\Big)\Big)\\
&\quad+\sum_{i=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\Big\|\frac{1}{\sqrt{\rho}}u_{i,\alpha}\Big\|^2 - \sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)\delta(\lambda_i-\lambda_j)^2r_{\alpha ij}^2 - \sum_{i,j=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\delta}t_{\alpha ij}^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\|u_i\|^2 &\le \delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\Big(-\int_\Omega x_\alpha u_i\Big(\sum_{\beta=1}^n((a_{\alpha\beta}u_i)_{,\beta}+a_{\alpha\beta}u_{i,\beta})\Big)\Big)\\
&\qquad + \sum_{i=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\Big\|\frac{1}{\sqrt{\rho}}u_{i,\alpha}\Big\|^2.
\end{aligned}
$$

Summing over α, we infer

$$
\begin{aligned}
&n\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\|u_i\|^2\\
&\quad\le \delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\Big(-\sum_{\alpha=1}^n\int_\Omega x_\alpha u_i\Big(\sum_{\beta=1}^n((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta})\Big)\Big)\\
&\qquad\qquad + \sum_{i=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\Big\|\frac{1}{\sqrt{\rho}}|\nabla u_i|\Big\|^2.
\end{aligned}
\tag{3-26}
$$

Since $\int_\Omega \rho u_i^2 = 1$ and $\rho_1 \le \rho(x) \le \rho_2$ for $x \in \Omega$, we have

$$\frac{1}{\rho_2} \le \|u_i\|^2 \le \frac{1}{\rho_1}. \tag{3-27}$$

One gets from the divergence theorem that

$$\begin{aligned} -\sum_{\alpha=1}^n \int_\Omega x_\alpha u_i \Big(\sum_{\beta=1}^n \big((a_{\alpha\beta}u_i)_{,\beta} + a_{\alpha\beta}u_{i,\beta}\big)\Big) \\ &= \int_\Omega \Big(\sum_{\alpha,\beta=1}^n \big(a_{\alpha\beta}u_i(x_\alpha u_i)_{,\beta} - a_{\alpha\beta}u_{i,\beta}x_\alpha u_i\big)\Big) = \int_\Omega \Big(\sum_{\alpha=1}^n a_{\alpha\alpha}\Big)u_i^2 \\ &\le n\xi_2 \int_\Omega u_i^2 \le \frac{n\xi_2}{\rho_1}. \end{aligned} \tag{3-28}$$

Multiplying the equation $-\sum_{\alpha,\beta=1}^n (a_{\alpha\beta}u_{i,\beta})_{,\alpha} + V(x)u_i = \lambda_i \rho u_i$ by u_i and integrating over Ω, we get

$$\lambda_i = \int_\Omega \Big(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x)u_{i,\alpha}u_{i,\beta} + V(x)u_i^2\Big) \ge \int_\Omega \xi_1|\nabla u_i|^2 + \frac{V_0}{\rho_2}, \tag{3-29}$$

which gives

$$\Big\|\frac{1}{\sqrt{\rho}}|\nabla u_i|\Big\|^2 \le \frac{1}{\rho_1}\int_\Omega |\nabla u_i|^2 \le \frac{1}{\rho_1\xi_1}\Big(\lambda_i - \frac{V_0}{\rho_2}\Big). \tag{3-30}$$

Substituting (3-27), (3-28) and (3-30) into (3-26), we infer

$$\frac{n}{\rho_2}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 \le \delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 \cdot \frac{n\xi_2}{\rho_1} + \sum_{i=1}^k \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\cdot\frac{1}{\rho_1\xi_1}\Big(\lambda_i - \frac{V_0}{\rho_2}\Big).$$

Taking

$$\delta = \frac{\Big(\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\Big(\lambda_i - \frac{V_0}{\rho_2}\Big)\Big)^{1/2}}{\Big(\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 n\xi_1\xi_2\Big)^{1/2}}, \tag{3-31}$$

we get (1-15). This completes the proof of Theorem 1.4. □

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