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UNIVERSAL INEQUALITIES FOR THE EIGENVALUES OF THE BIHARMONIC OPERATOR ON SUBMANIFOLDS

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We establish universal inequalities for the eigenvalues of the clamped plate problem on compact submanifolds of Euclidean space, of spheres and of real, complex and quaternionic projective spaces. We prove similar results for the biharmonic operator on domains of Riemannian manifolds that admit spherical eigenmaps (this includes compact homogeneous Riemannian spaces) and finally on domains of hyperbolic space.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension n and let Δ be the Laplacian operator on M.

We will be concerned with the following eigenvalue problem for the Dirichlet biharmonic operator, called the clamped plate problem:

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where Ω is a bounded domain in M, Δ^2 is the biharmonic operator in M and ν is the outward unit normal. It is well known that the eigenvalues of this problem form a countable family $0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$.

For the case when $M = \mathbb{R}^n$, Payne, Pólya and Weinberger [1956] established the following inequality, for each $k \ge 1$:

$$\lambda_{k+1} - \lambda_k \le \frac{8(n+2)}{n^2 k} \sum_{i=1}^{k} \lambda_i .$$

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Implicit in [Payne et al. 1956], as noticed by Ashbaugh [1999], is the better inequality

$$\lambda_{k+1} - \lambda_k \le \frac{8(n+2)}{n^2k^2}\Big(\sum_{i=1}^{k} \lambda_i^{1/2}\Big)^2. \tag{1-2}$$

Later, Hile and Yeh [1984] extended ideas from earlier work on the Laplacian by Hile and Protter [1980] and proved the better bound

$$\frac{n^2k^{3/2}}{8(n+2)} \le \Big(\sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1}-\lambda_i}\Big)\Big(\sum_{i=1}^{k} \lambda_i\Big)^{1/2}.$$

Implicit in their work is the stronger inequality

$$\frac{n^2k^2}{8(n+2)} \le \Big(\sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1}-\lambda_i}\Big)\Big(\sum_{i=1}^{k} \lambda_i^{1/2}\Big),$$

which was proved independently by Hook [1990] and Chen and Qian [1990]; see also [Chen and Qian 1993a; 1993b; 1994].

Cheng and Yang [2006] obtained the bound

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) \le \Big(\frac{8(n+2)}{n^2}\Big)^{1/2} \sum_{i=1}^{k}\big(\lambda_i(\lambda_{k+1}-\lambda_i)\big)^{1/2}. \tag{1-3}$$

Very recently, Cheng, Ichikawa and Mametsuka [2009b] obtained an inequality for eigenvalues of Laplacian with any order l on a bounded domain in $\mathbb{R}^n$. In particular, they showed that for $l = 2$,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{8(n+2)}{n^2} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i. \tag{1-4}$$

For the case when $M = \mathbb{S}^n$, Wang and Xia [2007] showed that

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n}\Big(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\big(n^2+(2n+4)\lambda_i^{1/2}\big)\Big)^{1/2} \times \Big(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\;(n^2+4\lambda_i^{1/2})\Big)^{1/2}, \tag{1-5}$$

from which they deduced, using a variant of Chebyshev's inequality,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n^2} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\big(2(n+2)\lambda_i^{1/2}+n^2\big)(4\lambda_i^{1/2}+n^2). \tag{1-6}$$

This last inequality was also obtained by a different method by Cheng, Ichikawa and Mametsuka [2009a].

On the other hand, Wang and Xia [2007] also considered the problem (1-1) on domains of an n-dimensional complete minimal submanifold M of $\mathbb{R}^m$ and proved

$$\begin{aligned}(1\text{-}7)\quad &\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\ &\leq \left(\frac{8(n+2)}{n^2}\right)^{1/2}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{1/2}\right)^{1/2}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{1/2}\right)^{1/2},\end{aligned}$$

from which they deduced the following generalization of inequality (1-4) to minimal Euclidean submanifolds:

$$(1\text{-}8)\qquad \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{8(n+2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

Recently, Cheng, Ichikawa and Mametsuka [2010] extended this last inequality to any complete Riemannian submanifold M in $\mathbb{R}^m$ and showed

$$(1\text{-}9)\quad \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{1}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\big(n^2\delta+2(n+2)\lambda_i^{1/2}\big)(n^2\delta+4\lambda_i^{1/2}),$$

with

$$\delta = \sup_\Omega |H|^2,$$

where H is the mean curvature of M.

The goal of Section 2 of this article is to study the relation between eigenvalues of the biharmonic operator and the local geometry of Euclidean submanifolds M of arbitrary codimension. The approach is based on an algebraic formula (see Theorem 2.3) we proved in [Ilias and Makhoul 2010]. This approach is useful for the unification and for the generalization of all the results in the literature. In fact, using this general algebraic inequality, we obtain (see Theorem 2.4) the inequality

$$\begin{aligned}(1\text{-}10)\quad \sum_{i=1}^{k} f(\lambda_i) \leq \frac{1}{n}&\left(\sum_{i=1}^{k} g(\lambda_i)\big(2(n+2)\lambda_i^{1/2}+n^2\delta\big)\right)^{1/2}\\ &\times\left(\sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1}-\lambda_i)}(4\lambda_i^{1/2}+n^2\delta)\right)^{1/2},\end{aligned}$$

where f and g are two functions satisfying some functional conditions (see Definition 2.1), $\delta=\sup_\Omega |H|^2$ and H is the mean curvature of M. The family of such pairs of functions is large. And particular choices for f and g lead to the known results. For instance, if we take $f(x)=g(x)=(\lambda_{k+1}-x)^2$, then (1-10) becomes

$$\text{(1-11)}\quad \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n}\Big(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\big(2(n+2)\lambda_i^{1/2}+n^2\delta\big)\Big)^{1/2} \times \Big(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(4\lambda_i^{1/2}+n^2\delta)\Big)^{1/2},$$

which gives easily (see Remark 2.2) inequality (1-9) of Cheng, Ichikawa and Mametsuka [2010].

In Section 3 we consider the case of manifolds admitting spherical eigenmaps and obtain similar results. As a consequence, we obtain universal inequalities for the clamped plate problem on domains of any compact homogeneous Riemannian manifold.

In Section 4, we show how one can easily obtain, from the algebraic techniques used in the previous sections, universal inequalities for eigenvalues of (1-1) on domains of hyperbolic space $\mathbb{H}^n$.

All our results hold if we add a potential to Δ^2 (that is, Δ^2+q where q is a smooth potential). For instance, in this case instead of inequality (1-10), we obtain

$$\text{(1-12)}\quad \sum_{i=1}^{k} f(\lambda_i) \le \frac{1}{n}\Big(\sum_{i=1}^{k} g(\lambda_i)\big(2(n+2)\bar\lambda_i^{1/2}+n^2\delta\big)\Big)^{1/2} \times \Big(\sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1}-\lambda_i)}(4\bar\lambda_i^{1/2}+n^2\delta)\Big)^{1/2},$$

where $\bar\lambda_i = \lambda_i - \inf_\Omega q$.

Finally, the case of the clamped problem with weight

$$\text{(1-13)}\quad \begin{cases} \Delta^2 u = \lambda\,\rho\, u & \text{in } \Omega,\\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

can be easily treated with minor changes.

2. Euclidean submanifolds

Before stating the main result of this section, we introduce a family of pairs of functions and a theorem obtained in [Ilias and Makhoul 2010], which will play an essential role in the proofs of all our results.

Definition 2.1. Let $\lambda \in \mathbb{R}$. A pair (f, g) of functions defined on $]-\infty, \lambda[$ belongs to $\Im_\lambda$ if f and g are positive and, for any distinct $x,\ y \in\,]-\infty, \lambda[$,

$$\text{(2-1)}\quad \Big(\frac{f(x)-f(y)}{x-y}\Big)^2 + \Big(\frac{\big(f(x)\big)^2}{g(x)(\lambda-x)} + \frac{\big(f(y)\big)^2}{g(y)(\lambda-y)}\Big)\Big(\frac{g(x)-g(y)}{x-y}\Big) \le 0.$$

Remark 2.2. This definition of the family $\mathfrak{I}_\lambda$ differs slightly from that given in [Ilias and Makhoul 2010], but all the results there are still valid.

A direct consequence of our definition is that g must be nonincreasing.

If we multiply f and g of $\mathfrak{I}_\lambda$ by positive constants, the resulting functions are also in $\mathfrak{I}_\lambda$. In the case where f and g are differentiable, one can easily deduce from (2-1) the necessary condition

$$\big((\ln f(x))'\big)^2 \le \frac{-2}{\lambda - x}(\ln g(x))'.$$

This last condition helps us to find many pairs (f, g) satisfying the conditions of Definition 2.1, for example,

$$\begin{gathered}\{(1, (\lambda - x)^\alpha) \mid \alpha \ge 0\},\\ \{((\lambda - x), (\lambda - x)^\beta) \mid \beta \ge \tfrac{1}{2}\},\\ \{((\lambda - x)^\delta, (\lambda - x)^\delta) \mid 0 < \delta \le 2\}.\end{gathered}$$

Let $\mathcal{H}$ be a complex Hilbert space with scalar product $\langle\cdot\,,\cdot\rangle$ and corresponding norm $\|\cdot\|$. For any two operators A and B, we denote by $[A, B]$ their commutator, defined by $[A, B] = AB - BA$.

Theorem 2.3. *Let $A : \mathfrak{D} \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator defined on a dense domain $\mathfrak{D}$, which is semibounded below and has a discrete spectrum*

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots.$$

Let

$$\{T_p : \mathfrak{D} \to \mathcal{H}\}_{p=1}^n$$

be a collection of skew-symmetric operators and

$$\{B_p : T_p(\mathfrak{D}) \to \mathcal{H}\}_{p=1}^n$$

a collection of symmetric operators, leaving $\mathfrak{D}$ invariant. Denote by

$$\{u_i\}_{i=1}^\infty$$

a basis of orthonormal eigenvectors of A, u_i corresponding to λ_i. Let $k \ge 1$ and assume that $\lambda_{k+1} > \lambda_k$. Then, for any (f, g) in $\mathfrak{I}_{\lambda_{k+1}}$

$$\begin{aligned}\text{(2-2)}\quad \Big(\sum_{i=1}^k\sum_{p=1}^n f(\lambda_i)\big\langle [T_p, B_p]u_i, u_i\big\rangle\Big)^2 &\\ \le 4\Big(\sum_{i=1}^k\sum_{p=1}^n g(\lambda_i)\big\langle [A, B_p]u_i, B_pu_i\big\rangle\Big)&\\ \times\Big(\sum_{i=1}^k\sum_{p=1}^n \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)}\|T_pu_i\|^2\Big).&\end{aligned}$$

Our first result is the following application of this inequality to the eigenvalues of the clamped plate problem (1-1) on a domain of a Euclidean submanifold:

Theorem 2.4. *Let $X : M \to \mathbb{R}^m$ be an isometric immersion of an n-dimensional Riemannian manifold M in $\mathbb{R}^m$. Let Ω be a bounded domain of M and consider the clamped plate problem* (1-1) *on Ω. Then for any $k \geq 1$ such that $\lambda_{k+1} > \lambda_k$ and for any (f, g) in $\Im_{\lambda_{k+1}}$, we have*

$$\sum_{i=1}^{k} f(\lambda_i) \leq \frac{2}{n}\Big(\sum_{i=1}^{k} g(\lambda_i)\big(2(n+2)\lambda_i^{1/2}+n^2\delta\big)\Big)^{1/2} \times \Big(\sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1}-\lambda_i)}\Big(\lambda_i^{1/2}+\frac{n^2}{4}\delta\Big)\Big)^{1/2}, \tag{2-3}$$

where $\delta = \sup_\Omega |H|^2$ and H be the mean curvature vector field of the immersion X (*that is, which is given by $\frac{1}{n}$* trace h, *where h is the second fundamental form of X*).

Proof. We apply inequality (2-2) of Theorem 2.3 with $A = \Delta^2$, $B_p = X_p$ and $T_p = [\Delta, X_p]$, $p = 1, \dots, m$, where $X_1, \dots, X_m$ are the components of the immersion X. This gives

$$\Big(\sum_{i=1}^{k}\sum_{p=1}^{m} f(\lambda_i)\big\langle\big[[\Delta, X_p], X_p\big]u_i, u_i\big\rangle_{L^2}\Big)^2 \leq 4\Big(\sum_{i=1}^{k}\sum_{p=1}^{m} g(\lambda_i)\big\langle [\Delta^2, X_p]u_i, X_pu_i\big\rangle_{L^2}\Big) \times \Big(\sum_{i=1}^{k}\sum_{p=1}^{m} \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1}-\lambda_i)}\|[\Delta, X_p]u_i\|^2_{L^2}\Big), \tag{2-4}$$

where u_i are the L^2-normalized eigenfunctions. First we have, for $p = 1, \dots, m$,

$$[\Delta^2, X_p]u_i = \Delta^2 X_p u_i + 2\nabla\Delta X_p.\nabla u_i + 2\Delta(\nabla X_p\cdot\nabla u_i) + 2\Delta X_p\Delta u_i + 2\nabla X_p.\nabla\Delta u_i.$$

Thus

$$\begin{aligned}\langle [\Delta^2, X_p]u_i, X_pu_i\rangle_{L^2} &= \int_\Omega u_i^2 X_p\Delta^2 X_p + 2\int_\Omega X_pu_i\nabla\Delta X_p.\nabla u_i + 2\int_\Omega X_pu_i\Delta(\nabla X_p\cdot\nabla u_i) \\ &\quad + 2\int_\Omega X_pu_i\Delta X_p\Delta u_i + 2\int_\Omega X_pu_i\nabla X_p.\nabla\Delta u_i \\ &= \int_\Omega \Delta X_p\Delta(X_pu_i^2) - 2\int_\Omega \operatorname{div}(X_pu_i\nabla u_i)\Delta X_p + 2\int_\Omega \Delta(X_pu_i)\nabla X_p\cdot\nabla u_i \\ &\quad + 2\int_\Omega X_p\Delta X_pu_i\Delta u_i - 2\int_\Omega \operatorname{div}(X_pu_i\nabla X_p)\Delta u_i.\end{aligned}$$

A straightforward calculation gives

$$\langle [\Delta^2, X_p]u_i, X_p u_i\rangle_{L^2} = 4\int_\Omega u_i \Delta X_p \nabla X_p \cdot \nabla u_i + \int_\Omega (\Delta X_p)^2 u_i^2 + 4\int_\Omega (\nabla X_p \cdot \nabla u_i)^2 - 2\int_\Omega |\nabla X_p|^2 u_i \Delta u_i. \tag{2-5}$$

Since X is an isometric immersion, we have

$$nH = (\Delta X_1, \dots, \Delta X_m) \tag{2-6}$$

and

$$\sum_{p=1}^m u_i \Delta X_p \nabla X_p \cdot \nabla u_i = 0, \quad \sum_{p=1}^m (\nabla X_p \cdot \nabla u_i)^2 = |\nabla u_i|^2. \tag{2-7}$$

Incorporating these identities in (2-5) and summing on p from 1 to m, we obtain

$$\begin{aligned}
\sum_{p=1}^m \langle [\Delta^2, X_p]u_i, X_p u_i\rangle_{L^2} &= 4\int_\Omega |\nabla u_i|^2 - 2n\int_\Omega u_i \Delta u_i + n^2 \int_\Omega |H|^2 u_i^2 \\
&= 2(n+2)\int_\Omega u_i(-\Delta u_i) + n^2\int_\Omega |H|^2 u_i^2 \\
&\le 2(n+2)\Big(\int_\Omega (-\Delta u_i)^2\Big)^{1/2}\Big(\int_\Omega u_i^2\Big)^{1/2} + n^2\int_\Omega |H|^2 u_i^2 && \text{(2-8)}\\
&= 2(n+2)\lambda_i^{1/2} + n^2\int_\Omega |H|^2 u_i^2 \\
&\le 2(n+2)\lambda_i^{1/2} + n^2\delta, && \text{(2-9)}
\end{aligned}$$

where the Cauchy–Schwarz inequality gave (2-8) and where $\delta = \sup_\Omega |H|^2$.

On the other hand, we have

$$[\Delta, X_p]u_i = 2\nabla X_p \cdot \nabla u_i + u_i \Delta X_p.$$

Then

$$\begin{aligned}
\sum_{p=1}^m \|[\Delta, X_p]u_i\|_{L^2}^2 &= \sum_{p=1}^m \int_\Omega (2\nabla X_p \cdot \nabla u_i + u_i \Delta X_p)^2 \\
&= 4\sum_{p=1}^m \int_\Omega (\nabla X_p \cdot \nabla u_i)^2 + 4\sum_{p=1}^m \int_\Omega u_i \Delta X_p \nabla X_p \cdot \nabla u_i \\
&\quad + \sum_{p=1}^m \int_\Omega (\Delta X_p)^2 u_i^2.
\end{aligned}$$

Using the identities (2-6) and (2-7), we obtain

$$\begin{aligned}
\text{(2-10)}\qquad \sum_{p=1}^{m} \|[\Delta, X_p]u_i\|_{L^2}^2 &= 4\int_\Omega |\nabla u_i|^2 + n^2 \int_\Omega |H|^2 u_i^2 \\
&= 4\int_\Omega (-\Delta u_i)\cdot u_i + n^2 \int_\Omega |H|^2 u_i^2 \\
&\le 4\Big(\int_\Omega (-\Delta u_i)^2\Big)^{1/2}\Big(\int_\Omega u_i^2\Big)^{1/2} + n^2\delta \\
&= 4\lambda_i^{1/2} + n^2\delta.
\end{aligned}$$

A direct calculation gives

$$\begin{aligned}
\big\langle [[\Delta, X_p], X_p]u_i, u_i\big\rangle_{L^2} &= \int_\Omega \big(\Delta(X_p^2 u_i) - 2X_p\Delta(X_p u_i) + X_p^2\Delta u_i\big)u_i \\
&= 2\int_\Omega |\nabla X_p|^2 u_i^2.
\end{aligned}$$

Therefore

$$\text{(2-11)}\qquad \sum_{p=1}^{m} \big\langle [[\Delta, X_p], X_p]u_i, u_i\big\rangle_{L^2} = 2\sum_{p=1}^{m}\int_\Omega |\nabla X_p|^2 u_i^2 = 2n.$$

To conclude, we simply use the estimates (2-9), (2-10) and (2-11) together with inequality (2-4). □

Remarks 2.5. • As indicated in the end of the introduction, Theorem 2.4 holds for a general operator $\Delta^2 + q$, where q is a smooth potential. Indeed, this is an immediate consequence of the fact that $[\Delta^2 + q, X_p] = [\Delta^2, X_p]$ and the entire proof of Theorem 2.4 works in this situation. The only modification is in the estimation of the term $\int_\Omega |\nabla u_i|^2$. In this case, letting $\bar{\lambda}_i = \lambda_i - \inf_\Omega q$, we have

$$\int_\Omega |\nabla u_i|^2 \le \Big(\int_\Omega (-\Delta u_i)^2\Big)^{1/2}\Big(\int_\Omega u_i^2\Big)^{1/2} = \Big(\lambda_i - \int_\Omega q u_i^2\Big)^{1/2} \le (\bar{\lambda}_i)^{1/2}.$$

Taking into account this modification in inequalities (2-8) and (2-10), we obtain inequality (1-12).

• If $f(x) = g(x) = (\lambda_{k+1} - x)^2$, then inequality (2-3) extends inequality (1-7) of Wang and Xia [2007] to any Riemannian submanifolds of $\mathbb{R}^m$. By using a Chebyshev inequality (for instance the one of [Cheng et al. 2009b, Lemma 1]), inequality (1-9) of Cheng, Ichikawa and Mametsuka [2010] can be easily deduced from inequality (2-3).

• If $f(x) = g(x)^2 = (\lambda_{k+1} - x)$, then inequality (2-3) generalizes inequality (1-3) of Cheng and Yang [2006] to the case of Euclidean submanifolds.

Using the standard embeddings of the rank one compact symmetric spaces in a Euclidean space (see for instance [El Soufi et al. 2009, Lemma 3.1] for the values of $|H|^2$ of these embeddings), we can extend easily the previous theorem to domains or submanifolds of these symmetric spaces and obtain:

Theorem 2.6. *Let $\overline{M}$ be the sphere $\mathbb{S}^m$, the real projective space $\mathbb{R}P^m$, the complex projective space $\mathbb{C}P^m$ or the quaternionic projective space $\mathbb{Q}P^m$ endowed with their respective metrics. Let (M, g) be a compact Riemannian manifold of dimension n and let $X : M \to \overline{M}$ be an isometric immersion of mean curvature H. Consider the clamped plate problem on a bounded domain Ω of M. For any $k \geq 1$ such that $\lambda_{k+1} > \lambda_k$ and for any $(f, g) \in \Im_{\lambda_{k+1}}$, we have*

$$\sum_{i=1}^{k} f(\lambda_i) \leq \frac{2}{n}\Big(\sum_{i=1}^{k} g(\lambda_i)\big(2(n+2)\lambda_i^{1/2}+n^2\delta'\big)\Big)^{1/2} \times\Big(\sum_{i=1}^{k}\frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1}-\lambda_i)}\Big(\lambda_i^{1/2}+\frac{n^2}{4}\delta'\Big)\Big)^{1/2}, \tag{2-12}$$

where

$$\delta' = \sup(|H|^2 + d(n)), \quad \textit{where } d(n) = \begin{cases} 1 & \textit{if } \overline{M} = \mathbb{S}^m, \\ 2(n+1)/n & \textit{if } \overline{M} = \mathbb{R}P^m, \\ 2(n+2)/n & \textit{if } \overline{M} = \mathbb{C}P^m, \\ 2(n+4)/n & \textit{if } \overline{M} = \mathbb{Q}P^m. \end{cases}$$

Remarks 2.7. • As in [El Soufi et al. 2009, Remark 3.2], in some special geometrical situations, the constant $d(n)$ in the inequality of Theorem 2.6 can be replaced by a sharper one. For instance, when $\overline{M} = \mathbb{C}P^m$ and

- M is odd-dimensional, then $d(n)$ can be replaced by $d'(n) = (2/n)(n+2-1/n)$,
- $X(M)$ is totally real, then $d(n)$ can be replaced by $d'(n) = 2(n+1)/n$.

• When $f(x) = g(x) = (\lambda_{k+1} - x)^2$, and $\overline{M}$ is a sphere, (2-12) generalizes to submanifolds inequality (1-5) established by Wang and Xia for spherical domains.

• As for Theorem 2.4, the result of Theorem 2.6 holds for a more general operator $\Delta^2 + q$, with the same modification (that is, $\bar{\lambda}_i^{1/2}$ instead of $\lambda_i^{1/2}$).

3. Manifolds admitting spherical eigenmaps

In this section, as before, we let (M, g) be a Riemannian manifold and Ω be a bounded domain of M. A map $X : (M, g) \to \mathbb{S}^{m-1}$ is called an eigenmap if its components $X_1, X_2, \ldots, X_m$ are all eigenfunctions associated to the same eigenvalue λ of the Laplacian of (M, g). This is equivalent to say that the map X is a harmonic map from (M, g) into $\mathbb{S}^{m-1}$ with constant energy λ (that is, $\sum_{p=1}^{m} |\nabla X_p|^2 = \lambda$). The most important examples of such manifolds M are the compact homogeneous

Riemannian manifolds. In fact, they admit eigenmaps for all the positive eigenvalues of their Laplacian; see [Li 1980].

Theorem 3.1. *Let λ be an eigenvalue of the Laplacian of (M, g) and suppose that (M, g) admits an eigenmap X associated to this eigenvalue λ. Let Ω be a bounded domain of M and consider the clamped plate problem* (1-1) *on Ω. For any $k \geq 1$ such that $\lambda_{k+1} > \lambda_k$ and for any $(f, g) \in \Im_{\lambda_{k+1}}$, we have*

$$\sum_{i=1}^{k} f(\lambda_i) \leq \Big(\sum_{i=1}^{k} g(\lambda_i)(\lambda + 6\lambda_i^{1/2})\Big)^{1/2} \Big(\sum_{i=1}^{k} \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)} (\lambda + 4\lambda_i^{1/2})\Big)^{1/2}. \tag{3-1}$$

Proof. As in the proof of Theorem 2.4, we apply Theorem 2.3 with $A = \Delta^2$, $B_p = X_p$ and $T_p = [\Delta, X_p]$, $p = 1, \ldots, m$, to obtain

$$\Big(\sum_{i=1}^{k}\sum_{p=1}^{m} f(\lambda_i)\langle [[\Delta, X_p], X_p]u_i, u_i\rangle_{L^2}\Big)^2 \leq 4\Big(\sum_{i=1}^{k}\sum_{p=1}^{m} g(\lambda_i)\langle [\Delta^2, X_p]u_i, X_p u_i\rangle_{L^2}\Big) \times \Big(\sum_{i=1}^{k}\sum_{p=1}^{m} \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)} \|[\Delta, X_p]u_i\|_{L^2}^2\Big), \tag{3-2}$$

where $\{u_i\}_{i=1}^{\infty}$ is a complete L^2-orthonormal basis of eigenfunctions of Δ^2 associated to $\{\lambda_i\}_{i=1}^{\infty}$. As in (2-11), and using the equality

$$\sum_{p=1}^{m} |\nabla X_p|^2 = \lambda,$$

we have

$$\sum_{p=1}^{m} \big\langle [[\Delta, X_p], X_p]u_i, u_i\big\rangle_{L^2} = 2\sum_{p=1}^{m} \int_\Omega |\nabla X_p|^2 u_i^2 = 2\lambda. \tag{3-3}$$

We further have

$$\begin{aligned}\sum_{p=1}^{m} \|[\Delta, X_p]u_i\|_{L^2}^2 &= \sum_{p=1}^{m} \int_\Omega \big([\Delta, X_p]u_i\big)^2 \\ &= 4\int_\Omega \sum_{p=1}^{m} (\nabla X_p \cdot \nabla u_i)^2 + \int_\Omega \sum_{p=1}^{m} (\Delta X_p)^2 u_i^2 + 4\int_\Omega \sum_{p=1}^{m} u_i \Delta X_p \nabla X_p \cdot \nabla u_i\end{aligned}$$

Applying Cauchy–Schwarz and the equalities

$$\sum_{p=1}^{m} X_p^2 = 1 \quad \text{and} \quad X_p = -\lambda X_p,$$

we then obtain

$$\begin{aligned}
&\sum_{p=1}^{m} \|[\Delta, X_p]u_i\|_{L^2}^2 \\
&\qquad \le 4\int_\Omega \sum_{p=1}^{m} |\nabla X_p|^2\,|\nabla u_i|^2 + \lambda^2 \int_\Omega \Big(\sum_{p=1}^{m} X_p^2\Big) u_i^2 - 2\lambda \int_\Omega u_i \nabla\Big(\sum_{p=1}^{m} X_p^2\Big)\cdot \nabla u_i \\
&\qquad = 4\lambda \int_\Omega (-\Delta u_i)u_i + \lambda^2 \le 4\lambda \Big(\int_\Omega (-\Delta u_i)^2\Big)^{1/2} \Big(\int_\Omega u_i^2\Big)^{1/2} + \lambda^2 \\
&\qquad = 4\lambda\lambda_i^{1/2} + \lambda^2.
\end{aligned}$$

Similarly, we infer from (2-5) that

$$\begin{aligned}
&\sum_{p=1}^{m} \langle\, [\Delta^2, X_p]u_i, X_p u_i\rangle_{L^2} \\
&\quad = \lambda^2 \int_\Omega u_i^2 - \lambda \int_\Omega \nabla\Big(\sum_{p=1}^{m} X_p^2\Big)\cdot \nabla u_i^2 + 4\sum_{p=1}^{m}\int_\Omega (\nabla X_p \cdot \nabla u_i)^2 + 2\lambda \int_\Omega (-\Delta u_i)u_i \\
&\quad \le \lambda^2 + 4\int_\Omega \sum_{p=1}^{m} |\nabla X_p|^2\,|\nabla u_i|^2 + 2\lambda \Big(\int_\Omega (-\Delta u)^2\Big)^{1/2} \Big(\int_\Omega u_i^2\Big)^{1/2} \\
&\quad \le \lambda^2 + 4\lambda\lambda_i^{1/2} + 2\lambda\lambda_i^{1/2} \\
&\quad = \lambda^2 + 6\lambda\lambda_i^{1/2}.
\end{aligned}$$

Incorporating these two bounds, together with (3-3), in inequality (3-2) gives the theorem. □

Corollary 3.2. *Let (M, g) be a compact homogeneous Riemannian manifold without boundary and let λ_1 be the first nonzero eigenvalue of its Laplacian. Then the inequality* (3-1) *of Theorem 3.1 holds with $\lambda = \lambda_1$.*

Remark 3.3. As before, one can get a similar result for the operator $\Delta^2 + q$.

4. Domains in hyperbolic space

We turn next to the case of a domain Ω of hyperbolic space. It is easy to establish a universal inequality for eigenvalues of the clamped plate problem (1-1) on Ω in the vein of the preceding ones. Unfortunately, until now we have not succeeded in obtaining a simple generalization for the case of domains of hyperbolic submanifolds. In what follows, we take the half-space model for $\mathbb{H}^n$, that is,

$$\mathbb{H}^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

with the standard metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{x_n^2}.$$

In terms of the coordinates $(x_i)_{i=1}^n$, the Laplacian of $\mathbb{H}^n$ is given by

$$\Delta = x_n^2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_j} + (2-n)x_n \frac{\partial}{\partial x_n}.$$

Theorem 4.1. *For any $k \geq 1$ such that $\lambda_{k+1} > \lambda_k$, the eigenvalues λ_i of the clamped problem* (1-1) *on the bounded domain Ω of $\mathbb{H}^n$ must satisfy for any $(f, g) \in \Im_{\lambda_{k+1}}$,*

$$\sum_{i=1}^{k} f(\lambda_i) \leq \Biggl(\sum_{i=1}^{k} g(\lambda_i)(6\lambda_i^{1/2} - (n-1)^2)\Biggr)^{1/2} \times \Biggl(\sum_{i=1}^{k} \biggl(\frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)}\biggr)(4\lambda_i^{1/2} - (n-1)^2)\Biggr)^{1/2}. \tag{4-1}$$

Proof. Theorem 2.3 remains valid for $A = \Delta^2$, $B_p = F = \ln x_n$ and $T_p = [\Delta, F]$, for all $p = 1, \dots, n$. Thus, denoting by u_i the eigenfunction corresponding to λ_i, we have

$$\Biggl(\sum_{i=1}^{k} f(\lambda_i) \bigl\langle [[\Delta, F], F]u_i, u_i \bigr\rangle_{L^2}\Biggr)^2 \leq 4\Biggl(\sum_{i=1}^{k} g(\lambda_i) \langle\, [\Delta^2, F]u_i, Fu_i \rangle_{L^2}\Biggr) \times \Biggl(\sum_{i=1}^{k} \biggl(\frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)}\biggr) \|[\Delta, F]u_i\|_{L^2}^2\Biggr). \tag{4-2}$$

We start with the calculation of

$$\begin{aligned} \bigl\langle [[\Delta, F], F]u_i, u_i \bigr\rangle_{L^2} &= \int_\Omega \bigl([\Delta, F](Fu_i) - F[\Delta, F]u_i\bigr)u_i \\ &= \int_\Omega \bigl(\Delta(F^2 u_i) - 2F\Delta(Fu_i) + F^2 \Delta u_i\bigr)u_i. \end{aligned}$$

Note that

$$\Delta F = 1 - n \quad \text{and} \quad |\nabla F|^2 = 1. \tag{4-3}$$

Thus a direct calculation gives

$$\big\langle [[\Delta, F], F]u_i, u_i\big\rangle_{L^2} = 2\int_\Omega |\nabla F|^2 u_i^2 = 2. \tag{4-4}$$

On the other hand, using again the identities of (4-3), we obtain

$$\begin{aligned}\|[\Delta, F]u_i\|_{L^2}^2 &= \int_\Omega (\Delta F u_i + 2\nabla F\cdot\nabla u_i)^2\\ &= \int_\Omega (\Delta F)^2 u_i^2 + 4\int_\Omega (\nabla F\cdot\nabla u_i)^2 + 4\int_\Omega \Delta F u_i \nabla F\cdot\nabla u_i\\ &= (1-n)^2 + 4\int_\Omega (\nabla F\cdot\nabla u_i)^2 + 4(1-n)\int_\Omega u_i\nabla F\cdot\nabla u_i.\end{aligned} \tag{4-5}$$

But

$$\int_\Omega u_i\nabla F\cdot\nabla u_i = -\int_\Omega u_i\nabla F\cdot\nabla u_i - \int_\Omega u_i^2\Delta F,$$

hence

$$\int_\Omega u_i\nabla F\cdot\nabla u_i = \frac{n-1}{2}. \tag{4-6}$$

Then we infer from (4-3), (4-5) and (4-6) that

$$\begin{aligned}\|[\Delta, F]u_i\|_{L^2}^2 &\le -(n-1)^2 + 4\int_\Omega |\nabla F|^2\,|\nabla u_i|^2\\ &= -(n-1)^2 + 4\int_\Omega |\nabla u_i|^2 = -(n-1)^2 + 4\int_\Omega u_i(-\Delta u_i)\\ &\le -(n-1)^2 + 4\Big(\int_\Omega u_i^2\Big)^{1/2}\Big(\int_\Omega (-\Delta u_i)^2\Big)^{1/2}\\ &= 4\lambda_i^{1/2} - (n-1)^2.\end{aligned} \tag{4-7}$$

Now,

$$\begin{aligned}[\Delta^2, F]u_i &= \Delta^2(Fu_i) - F\Delta^2 u_i = \Delta(\Delta F u_i + 2\nabla F\cdot\nabla u_i + F\Delta u_i) - F\Delta^2 u_i\\ &= 2(1-n)\Delta u_i + 2\Delta(\nabla F\cdot\nabla u_i) + 2\nabla F\cdot\nabla\Delta u_i;\end{aligned} \tag{4-8}$$

thus

$$\begin{aligned}&\langle [\Delta^2, F]u_i, Fu_i\rangle_{L^2}\\ &\quad= 2(1-n)\int_\Omega Fu_i\Delta u_i + 2\int_\Omega Fu_i\Delta(\nabla F\cdot\nabla u_i) + 2\int_\Omega Fu_i\nabla F\cdot\nabla\Delta u_i\\ &\quad= 2(1-n)\int_\Omega Fu_i\Delta u_i + 2\int_\Omega \Delta(Fu_i)\nabla F\cdot\nabla u_i - 2\int_\Omega \operatorname{div}(Fu_i\nabla F)\Delta u_i\\ &\quad= 2\int_\Omega \Delta F u_i\nabla F\cdot\nabla u_i + 4\int_\Omega (\nabla F\cdot\nabla u_i)^2 - 2\int_\Omega |\nabla F|^2 u_i\Delta u_i.\end{aligned}$$

We infer from (4-3) and (4-6) that

$$\begin{aligned}\text{(4-9)}\quad \langle [\Delta^2, F]u_i, Fu_i\rangle_{L^2} &\le -(n-1)^2 + 4\int_\Omega |\nabla F|^2\,|\nabla u_i|^2 + 2\int_\Omega u_i(-\Delta u_i)\\ &= -(n-1)^2 + 6\int_\Omega u_i(-\Delta u_i)\\ &\le 6\Bigl(\int_\Omega u_i^2\Bigr)^{1/2}\Bigl(\int_\Omega (-\Delta u_i)^2\Bigr)^{1/2} - (n-1)^2\\ &= 6\lambda_i^{1/2} - (n-1)^2.\end{aligned}$$

Inequality (4-2) along with (4-4), (4-7) and (4-9) gives the theorem. □

Remarks 4.2. • It will be interesting to look for an extension of Theorem 4.1 to domains of hyperbolic submanifolds.

• Our method works for any bounded domain Ω of a Riemannian manifold admitting a function such that $|\nabla h|$ is constant and $|\Delta h| \le C$, where C is a constant.

• As before, we have the same statement as in Theorem 4.1 for the operator $\Delta^2 + q$; it suffices to replace $\lambda_i^{1/2}$ by $\bar{\lambda}_i^{1/2}$.

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