MULTIGRADED FUJITA APPROXIMATION

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The original Fujita approximation theorem states that the volume of a big divisor $D$ on a projective variety $X$ can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of $X$. One can also formulate it in terms of graded linear series as follows: Let $W_\bullet = \{W_k\}$ be the complete graded linear series associated to a big divisor $D$, where

$$W_k = H^0(X, \mathcal{O}_X(kD)).$$

For each fixed positive integer $p$, define $W_\bullet^{(p)}$ to be the graded linear sub-series of $W_\bullet$ generated by $W_p$:

$$W_m^{(p)} = \begin{cases} 0 & \text{if } p \nmid m, \\ \text{Image}(S^k W_p \to W_{kp}) & \text{if } m = kp. \end{cases}$$

Then the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \to \infty$. We will show that, under this formulation, the Fujita approximation theorem can be generalized to the case of multigraded linear series.

1. Introduction

Let $X$ be an irreducible variety of dimension $d$ over an algebraically closed field $K$, and let $D$ be a (Cartier) divisor on $X$. When $X$ is projective, the following limit, which measures how fast the dimension of the section space $H^0(X, \mathcal{O}_X(mD))$ grows, is called the volume of $D$:

$$\text{vol}(D) = \text{vol}_X(D) = \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$ 

One says that $D$ is big if $\text{vol}(D) > 0$. It turns out that the volume is an interesting numerical invariant of a big divisor [Lazarsfeld 2004a, Section 2.2.C], and it plays a key role in several recent works in birational geometry [Tsuji 2000; Boucksom et al. 2004; Hacon and McKernan 2006; Takayama 2006].

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When $D$ is ample, one can show that $\text{vol}(D) = D^d$, the self-intersection number of $D$. This is no longer true for a general big divisor $D$, since $D^d$ may even be negative. However, Fujita [1994] showed that the volume of a big divisor can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of $X$. This theorem, known as Fujita approximation, has several implications for the properties of volumes, and is also a crucial ingredient in [Boucksom et al. 2004] (see [Lazarsfeld 2004b, Section 11.4] for more details).

Lazarsfeld and Mustață [2009] (henceforth [LM]) recently obtained, among other things, a generalization of Fujita approximation to graded linear series. Recall that a graded linear series $W_\bullet = \{W_k\}$ on a (not necessarily projective) variety $X$ associated to a divisor $D$ consists of finite dimensional vector subspaces $W_k \subseteq H^0(X, \mathcal{O}_X(kD))$ for each $k \geq 0$, with $W_0 = K$, such that

$$W_k \cdot W_\ell \subseteq W_{k+\ell}$$

for all $k, \ell \geq 0$. Here the product on the left denotes the image of $W_k \otimes W_\ell$ under the multiplication map $H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(\ell D)) \to H^0(X, \mathcal{O}_X((k+\ell)D))$. In order to state the Fujita approximation for $W_\bullet$, they defined, for each fixed positive integer $p$, a graded linear series $W_\bullet^{(p)}$ which is the subgraded linear series of $W_\bullet$ generated by $W_p$:

$$W_m^{(p)} = \begin{cases} 0 & \text{if } p \nmid m, \\ \text{Im}(S^k W_p \to W_{kp}) & \text{if } m = kp. \end{cases}$$

Then under mild hypotheses, they showed that the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \to \infty$. See [LM, Theorem 3.5] for the precise statement, as well as [LM, Remark 3.4] for how this is equivalent to the original statement of Fujita when $X$ is projective and $W_\bullet$ is the complete graded linear series associated to a big divisor $D$ (that is, $W_k = H^0(X, \mathcal{O}_X(kD))$ for all $k \geq 0$).

The goal of this note is to generalize the Fujita approximation theorem to multi-graded linear series. We will adopt the following notation from [LM, Section 4.3]: Let $D_1, \ldots, D_r$ be divisors on $X$. For $\vec{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, write $\vec{m}D = \sum m_i D_i$, and put $|\vec{m}| = \sum |m_i|$.

**Definition.** A multi-graded linear series $W_\bullet$ on $X$ associated to the $D_i$ consists of finite-dimensional vector subspaces $W_k \subseteq H^0(X, \mathcal{O}_X(\vec{k}D))$. 
for each $\vec{k} \in \mathbb{N}^r$, with $W_0 = K$, such that

$$W_{\vec{k}} \cdot W_{\vec{m}} \subseteq W_{\vec{k} + \vec{m}},$$

where the multiplication on the left denotes the image of $W_{\vec{k}} \otimes W_{\vec{m}}$ under the natural map

$$H^0(X, \mathcal{O}_X(\vec{k}D)) \otimes H^0(X, \mathcal{O}_X(\vec{m}D)) \to H^0(X, \mathcal{O}_X((\vec{k} + \vec{m})D)).$$

Given $\vec{a} \in \mathbb{N}^r$, denote by $W_{\vec{a}, \bullet}$ the singly graded linear series associated to the divisor $\vec{a}D$ given by the subspaces $W_{k\vec{a}} \subseteq H^0(X, \mathcal{O}_X(k\vec{a}D))$. Then put

$$\text{vol}_{W_{\vec{a}, \bullet}}(\vec{a}) = \text{vol}(W_{\vec{a}, \bullet})$$

(assuming that this quantity is finite). It will also be convenient for us to consider $W_{\vec{a}, \bullet}$ when $\vec{a} \in \mathbb{Q}^r_{\geq 0}$, given by

$$W_{\vec{a}, k} = \begin{cases} W_{k\vec{a}} & \text{if } k\vec{a} \in \mathbb{N}^r, \\ 0 & \text{otherwise}. \end{cases}$$

Our multigraded Fujita approximation, similar to the singly graded version, is going to state that (under suitable conditions) the volume of $W_{\bullet}$ can be approximated by the volume of the following finitely generated submultigraded linear series of $W_{\bullet}$:

**Definition.** Given a multigraded linear series $W_{\bullet}$ and a positive integer $p$, define $W_{\bullet}^{(p)}$ to be the submultigraded linear series of $W_{\bullet}$ generated by all $W_{\vec{m}}$ with $|\vec{m}| = p$, or concretely,

$$W_{\vec{m}}^{(p)} = \begin{cases} 0 & \text{if } p \nmid |\vec{m}|, \\ \sum_{|\vec{m}_1| = p, \ldots, |\vec{m}_k| = p} W_{\vec{m}_1} \cdots W_{\vec{m}_k} & \text{if } |\vec{m}| = kp. \end{cases}$$

We now state our multigraded Fujita approximation when $W_{\bullet}$ is a complete multigraded linear series, since this is the case of most interest and allows for a more streamlined statement. The Remark on page 335 points out what assumptions on $W_{\bullet}$ are actually needed in the proof.

**Theorem.** Let $X$ be an irreducible projective variety of dimension $d$, and let $D_1, D_2, \ldots, D_r$ be big divisors on $X$. Let $W_{\bullet}$ be the complete multigraded linear series associated to the $D_1$, namely

$$W_{\vec{m}} = H^0(X, \mathcal{O}_X(\vec{m}D))$$
for each \( \vec{m} \in \mathbb{N}^r \). Then given any \( \varepsilon > 0 \), there exists an integer \( p_0 = p_0(\varepsilon) \) having the property that if \( p \geq p_0 \), then

\[
|1 - \frac{\text{vol}_{\vec{w}^{(p)}_{\vec{m}}}(\vec{a})}{\text{vol}_{\vec{w}_{\vec{m}}}(\vec{a})}| < \varepsilon
\]

for all \( \vec{a} \in \mathbb{N}^r \).

\[\text{2. Proof of the Theorem}\]

The main tool in our proof is the theory of Okounkov bodies developed systematically in [Lazarsfeld and Mustaţă 2009]. Given a graded linear series \( \vec{W} \) on a \( d \)-dimensional variety \( X \), its Okounkov body \( \Delta(\vec{W}) \) is a convex body in \( \mathbb{R}^d \) that encodes many asymptotic invariants of \( \vec{W} \), the most prominent one being the volume of \( \vec{W} \), which is precisely \( d! \) times the Euclidean volume of \( \Delta(\vec{W}) \). The idea first appeared in Okounkov’s papers [1996; 2003] in the case of complete linear series of ample line bundles on a projective variety. Later it was further developed and applied to much more general graded linear series by Lazarsfeld and Mustaţă [2009] and also independently by Kaveh and Khovanskii [2008; 2009].

\[\text{Proof of the Theorem.}\]

Let \( T = \{(a_1, \ldots, a_r) \in \mathbb{R}^r_{\geq 0} \mid a_1 + \cdots + a_r = 1\} \), and let \( T_{\mathbb{Q}} \) be the set of all points in \( T \) with rational coordinates. The fraction inside (1) is invariant under scaling of \( \vec{a} \) due to homogeneity, hence it is enough to prove (1) for \( \vec{a} \in T_{\mathbb{Q}} \).

Let \( \Delta(\vec{W}) \subseteq \mathbb{R}^d \times \mathbb{R}^r \) be the global Okounkov cone of \( \vec{W} \) as in [LM, Theorem 4.19], and let \( \pi : \Delta(\vec{W}) \to \mathbb{R}^r \) be the projection map. For each \( \vec{a} \in T \), write \( \Delta(\vec{W})_{\vec{a}} \) for the fiber \( \pi^{-1}(\vec{a}) \). Define in a similar fashion the convex cone \( \Delta(\vec{W}^{(p)}_{\vec{a}}) \) and the convex bodies \( \Delta(\vec{W}^{(p)}_{\vec{a}})_{\vec{a}} \). By [LM, Theorem 4.19],

\[
\Delta(\vec{W}_{\vec{a}})_{\vec{a}} = \Delta(\vec{W}^{(p)}_{\vec{a}})_{\vec{a}} \quad \text{for all } \vec{a} \in T_{\mathbb{Q}}.
\]

Although [LM, Theorem 4.19] requires \( \vec{a} \) to be in the relative interior of \( T \), here we know that (2) holds even for those \( \vec{a} \) in the boundary of \( T \) because the big cone of \( X \) is open and \( \vec{W} \) was assumed to be the complete multigraded linear series. By the singly graded Fujita approximation, \( \text{vol}(\vec{W}_{\vec{a}}) \) can be approximated arbitrarily closely by \( \text{vol}(\vec{W}^{(p)}_{\vec{a}}) \) if \( p \) is sufficiently large. (Here by \( \vec{W}^{(p)}_{\vec{a}} \) we mean \( \vec{W}^{(p)} \) restricted to the \( \vec{a} \) direction, which certainly contains \( (\vec{W}^{(p)}_{\vec{a}})_{\vec{a}} \).) Hence given any finite subset \( S \subset T_{\mathbb{Q}} \) and any \( \varepsilon' > 0 \), we have

\[
\text{vol}(\Delta(\vec{W}^{(p)}_{\vec{a}})) \geq \text{vol}(\Delta(\vec{W}_{\vec{a}})) - \varepsilon' \quad \text{for all } \vec{a} \in S
\]
as soon as \( p \) is sufficiently large.

Because the function \( \vec{a} \mapsto \text{vol}(\Delta(\vec{W}_{\vec{a}})) \) is uniformly continuous on \( T \), given any \( \varepsilon' > 0 \), we can partition \( T \) into a union of polytopes with disjoint interiors
$T = \bigcup T_i$, in such a way that the vertices of each $T_i$ all have rational coordinates, and on each $T_i$ we have a constant $M_i$ such that

$$(3) \quad M_i \leq \text{vol}(\Delta(W_\bullet \cdot \vec{a})) \leq M_i + \varepsilon' \quad \text{for all } \vec{a} \in T_i.$$  

Let $S$ be the set of vertices of all the $T_i$. Then as we saw in the end of the previous paragraph, as soon as $p$ is sufficiently large we have

$$(4) \quad \text{vol}(\Delta(W_\bullet(\cdot p) \cdot \vec{a})) \geq \text{vol}(\Delta(W_\bullet \cdot \vec{a})) - \varepsilon' \quad \text{for all } \vec{a} \in S.$$  

We claim that this implies

$$(5) \quad \text{vol}(\Delta(W_\bullet(\cdot p) \cdot \vec{a})) \geq \text{vol}(\Delta(W_\bullet \cdot \vec{a})) - 2\varepsilon' \quad \text{for all } \vec{a} \in T_Q.$$  

To show this, it suffices to verify it on each of the $T_i$. Let $\vec{v}_1, \ldots, \vec{v}_k$ be the vertices of $T_i$. Then each $\vec{a} \in T_i$ can be written as a convex combination of the vertices: $\vec{a} = \sum t_j \vec{v}_j$ where each $t_j \geq 0$ and $\sum t_j = 1$. Since $\Delta(W_\bullet(\cdot p))$ is convex, we have

$$\Delta(W_\bullet(\cdot p) \cdot \vec{a}) \supseteq \sum t_j \Delta(W_\bullet(\cdot p) \cdot \vec{v}_j),$$

where the sum on the right means the Minkowski sum. By (3) and (4), the volume of each $\Delta(W_\bullet(\cdot p) \cdot \vec{v}_j)$ is at least $M_i - \varepsilon'$, hence by the Brunn–Minkowski inequality [Kaveh and Khovanskii 2008, Theorem 5.4], we have

$$\text{vol}(\Delta(W_\bullet(\cdot p) \cdot \vec{a})) \geq M_i - \varepsilon' \quad \text{for all } \vec{a} \in T_i \cap T_Q.$$  

This combined with (3) shows that (5) is true on $T_i \cap T_Q$, hence it is true on $T_Q$ since the $T_i$ cover $T$.

Since (1) follows from (5) by choosing a suitable $\varepsilon'$, the proof is complete.  

**Remark.** In the statement of the Theorem we assume that $W_\bullet$ is the complete multigraded linear series associated to big divisors. But in fact since the main tool we used in the proof is the theory of Okounkov bodies established in [Lazarsfeld and Mustaţă 2009], in particular [LM, Theorem 4.19], the really indispensable assumptions on $W_\bullet$ are the same as those in [LM] (which they called Conditions ($A'$) and ($B'$), or ($C'$)). The only place in the proof where we invoke that we are working with a complete multigraded linear series is the sentence right after (2), where we want to say that (2) holds not only in the relative interior of $T$ but also in its boundary. Hence if $W_\bullet$ is only assumed to satisfy Conditions ($A'$) and ($B'$), or ($C'$), then given any $\varepsilon > 0$ and any compact set $C$ contained in $T \cap \text{int}(\text{supp}(W_\bullet))$, there exists an integer $p_0 = p_0(C, \varepsilon)$ such that if $p \geq p_0$ then

$$\text{vol}_{W_\bullet(p)}(\vec{a}) > \text{vol}_{W_\bullet}(\vec{a}) - \varepsilon$$

for all $\vec{a} \in C \cap T_Q$.  

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References


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